

International Trade and Global Economy

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Abstract

We have elaborated a new method to study international trade based on the theory of economic equilibrium. Here we give developed algorithms to construct equilibrium states. Within the framework of the theory of economic equilibrium, we suggested a mechanism explaining the recession phenomenon by the exchange mechanism breakdown. For this purpose, we introduced the recession level parameter. The paper gives our study of goods exchange nature between countries of the G20. It turned out that in each studied year, the trade between the G20 countries was not in equilibrium. The equilibrium state of trade between the G20 countries was highly degenerated and far from ideal equilibrium. Also, we have calculated the recession level parameter for each equilibrium state and it showed that the international currency was strengthening during 2016-2019.

Keywords

Equilibrium State, Ideal Equilibrium, Recession Phenomenon, Generalized Equilibrium Price Vector, International Trade

1. Introduction

This paper presents the study of international trade and its impact on the economy. In the first section, we introduce basic concepts, formulate the model, and state the problem. Also, we introduce the concept of the ideal state of equilibrium in world trade. As a rule, the world trade is not being in an ideal state of equilibrium, it deviates from it. The deviation can be due to both the state of each of the economies and the tariff restrictions that each of the countries implements to protect against the flow of goods from the outside. This state of equilibrium is usually violated through a monopoly in the supply of a certain type of goods, oligopoly and cartel agreements. The result of the so-called chao-

tic tariff war could be a shift in the world economy towards a state of recession. This paper contains the study of such a deviation from the ideal state of equilibrium. In Section 2, we introduce a mathematical model for international trade and formulate the problem. Section 3 contains our development of mathematical algorithms to solve this problem. For this purpose, we have introduced the concept of a polyhedral cone and suggested an algorithm for the vector to lie in a polyhedral cone. We have proved the theorem describing all positive solutions for a set of equations whose right-hand side is a vector lying in the interior of a polyhedral cone and in the interior of a cone consisting of a subset of linearly independent vectors whose number is equal to the dimension of the cone. To describe all positive solutions of the system of equations with the vector of the right-hand side in the interior of the cone without the above restrictions, we introduced the concept of a generating set of vectors and proved their existence. For a vector in the interior of a polyhedral cone, we obtained a representation in terms of generating vectors. Relying on this, we have described all positive solutions for the set of equations with an arbitrary right-hand side from the interior of the polyhedral cone. In Definitions 7 and 8, we introduce the concept of strict consistency for the supply structure with the demand structure. Lemmas 2 and 3 give sufficient conditions for the representation of the supply matrix in terms of the demand matrix. Under the strict consistency of the supply structure and demand structure, Theorem 4 contains the necessary and sufficient conditions for the existence of the nonnegative solution to a nonlinear set of equations, which are essential conditions for the existence of an equilibrium price vector at which the markets are cleared. Theorem 6 gives an algorithm for constructing of equilibrium price vector. The notion of economy equilibrium is contained in Definition 9. Theorem 5 sets the necessary and sufficient conditions for the existence of economic equilibrium. In Definitions 10 and 11, we formulate the consistency of the supply structure with the demand structure in weak sense. Under the weak consistency of the supply structure and demand structure, Theorem 7 sets the necessary and sufficient conditions for the existence of the nonnegative solution to a nonlinear set of equations, which are essential conditions for the existence of an equilibrium price vector at which the markets are cleared.

Theorem 8 presents an algorithm to construct supply vectors after demand vectors for which the markets are cleared.

Section 4 contains the study of the equilibrium state quality. In Definition 12, we introduce the notion of the degeneracy multiplicity for equilibrium state and the notion of economy recession. Theorem 9 formulates sufficient conditions under which a recession is possible in the world economy.

Section 5 contains the results concerning the problem of the ideal equilibrium state existence. In Theorem 11, we formulate the necessary and sufficient conditions under which the ideal equilibrium state exists. Theorem 12 contains an algorithm to construct the set of supply vectors after the set of demand vectors such that the ideal equilibrium price vector exists.

Section 6 contains a study of international trade between G20 countries. We have calculated the trade balance between countries. The excess demand in current prices recorded the absence of equilibrium in trade between G20 countries. Using Theorem 6, we have calculated the equilibrium price vector in each studied year. It turned out to be highly degenerate for each year. Also, we have introduced an important concept of a generalized equilibrium vector and, on its basis, a recession level parameter.

Section 7 contains the partial analysis of G20 countries' trading.

So, the paper developed a method for studying the exchange of goods between states from the point of view of economic equilibrium. For this, a mathematical model of the exchange of goods between states is proposed, formulated in terms of supply and demand for exchanged goods. The main problem of the existence of economic equilibrium for the proposed exchange model is solved and algorithms for their construction are found. An important role in obtaining these results was played by the concept of a polyhedral cone formed by demand vectors. The problem of a complete description of strictly positive solutions of the system of equations is solved, the matrix of the right part of which is composed of demand column vectors and the right part is the total supply belonging to the interior of the cone formed by the demand vectors. An important concept of the consistency of the supply structure with the demand structure was introduced and sufficient conditions for their implementation are presented.

Under the condition that the supply structure is consistent with the demand structure, the necessary and sufficient conditions for the existence of an equilibrium price vector at which the market is cleared are obtained. The necessary and sufficient conditions for the existence of economic equilibrium are formulated and an algorithm for searching of equilibrium states is proposed. A mechanism for describing the phenomenon of recession is proposed and a parameter is introduced that determines its depth. The conditions for the existence of an ideal equilibrium state are established. A detailed analysis of international trade between the G20 countries is made. The state of equilibrium in the trade of the G20 countries turned out to be highly degenerate and far from ideal equilibrium. The dynamics of the recession parameter showed that the international currency tended to strengthen.

2. Model Formulation and Statement of the Problem

This paper is an application of the economy description concept elaborated in (Gonchar, 2008) to model the international trade between M countries. Every set of goods we describe by a vector $x = (x_1, \dots, x_n)$, where x_i is a quantity of units of the i -th goods, e_i is a unit of its measurement, $x_i e_i$ is the natural quantity of the goods. If p_i is a unit price for the goods $e_i, i = \overline{1, n}$, then $p = (p_1, \dots, p_n)$ is the price vector corresponding to the vector of goods (e_1, \dots, e_n) . The price of goods vector $x = (x_1, \dots, x_n)$ is $\langle p, x \rangle = \sum_{i=1}^n p_i x_i$. The set of possible goods in the considered period of the economy system operation is denoted by S . We assume S is a convex subset of the set R_+^n . As for what fol-

lows only the property of convexity is important, we assume, without loss of generality, that S is a certain n -dimensional parallelepiped that can coincide with R_+^n . Thus, we assume that in the economy system the set of possible goods S is a convex subset of the non-negative orthant R_+^n of n -dimensional arithmetic space R^n , the set of possible prices is a certain cone K_+^n , contained in $R_+^n \setminus \{0\}$, and that can coincide with $R_+^n \setminus \{0\}$.

Definition 1

A set $K_+^n \subseteq R_+^n$ is called a nonnegative cone if together with a point $u \in K_+^n$ the point tu belongs to the set K_+^n for every real $t > 0$. Here and further, $R_+^n \setminus \{0\}$ is a cone formed from the nonnegative orthant R_+^n by ejection of the null vector $\{0\} = \{0, \dots, 0\}$. Further, the cone $R_+^n \setminus \{0\}$ is denoted by R_+^n .

Suppose M countries exchange by n types of goods. Let i_{kj}^s be the quantity of import units for s -th goods from j -th country into k -th country and $p_s i_{kj}^s$ its value, where $p = \{p_s\}_{s=1}^n$ is a price vector. Also, let e_{kj}^s be the quantity of export units for s -th goods from the k -th country into the j -th one and $p_s e_{kj}^s$ is its value. Having these entities, let us introduce the demand $\|c_{sk}^1\|_{s=1, k=1}^{n, M}$, and supply matrices $\|b_{sk}^1\|_{s=1, k=1}^{n, M}$,

$$c_{sk}^1 = \sum_{j=1}^M i_{kj}^s, \quad b_{sk}^1 = \sum_{j=1}^M e_{kj}^s, \quad s = \overline{1, n}, \quad k = \overline{1, M}, \tag{1}$$

and the supply vector $\psi^1 = \{\psi_s^1\}_{s=1}^n$, where

$$\psi_s^1 = \sum_{k=1}^M b_{sk}^1 = \sum_{k, j=1}^M e_{kj}^s, \quad s = \overline{1, n}. \tag{2}$$

The income of the k -th country from its export is

$$D_k(p) = \sum_{s=1}^n p_s b_{sk}^1 = \sum_{j=1}^M \sum_{s=1}^n e_{kj}^s p_s. \tag{3}$$

The conditions for the economy equilibrium is

$$\sum_{k=1}^M c_{sk}^1 \frac{D_k(p)}{\sum_{s=1}^n c_{sk}^1 p_s} \leq \sum_{k, j=1}^M e_{kj}^s = \psi_s^1, \quad s = \overline{1, n}. \tag{4}$$

As the statistical data are given in the cost form, it is convenient to rewrite the set of inequalities (4) in the cost form too

$$\sum_{k=1}^M c_{sk} \frac{D_k}{\sum_{s=1}^n c_{sk}} \leq \psi_s, \quad s = \overline{1, n}, \tag{5}$$

where $c_{sk} = p_s c_{sk}^1$, $\psi_s = p_s \psi_s^1$, $D_k = \sum_{j=1}^M \sum_{s=1}^n p_s e_{kj}^s$. We call the vector $c_k = \{c_{sk}\}_{s=1}^n$, $k = \overline{1, M}$, the demand vector of the k -th country and the vector $b_k = \{b_{sk}\}_{s=1}^n$, $k = \overline{1, M}$, the supply vector of the k -th country.

Definition 2.

We say the exchange by n types of goods in cost form between M countries is at the equilibrium state if the set of inequalities (5) are true. The price vector

$p = \{p_1, \dots, p_n\}$ under which the set of inequalities (5) are true is called the equilibrium price vector.

In what follows, we will use the denotations from (Gonchar, 2008). So, we put $\sum_{j=1}^M p_s e_{kj}^s = b_{sk}$, $s = \overline{1, n}$, $k = \overline{1, M}$ and introduce the property vector of the k -th country $b_k = \{b_{sk}\}_{s=1}^n$, $k = \overline{1, n}$. In these denotations the equilibrium state is written as

$$\sum_{k=1}^M c_{sk} \frac{D_k}{\sum_{s=1}^n c_{sk}} \leq \psi_s, \quad s = \overline{1, n}, \quad (6)$$

where $\psi_s = \sum_{k=1}^M b_{sk}$, $D_k = \sum_{s=1}^n b_{sk}$. As the set of inequalities (6) may be not satisfied by the price vector $p = \{p_1, \dots, p_n\}$, we introduce the relative price vector $p_0 = \{p_1^0, \dots, p_n^0\}$, to provide the equilibrium in the exchange model

$$\sum_{k=1}^M c_{sk} \frac{D_k(p_0)}{\sum_{s=1}^n p_s^0 c_{sk}} \leq \psi_s, \quad s = \overline{1, n}, \quad (7)$$

where $\psi_s = \sum_{k=1}^M b_{sk}$, $D_k(p_0) = \sum_{s=1}^n p_s^0 b_{sk}$. It is evident that if the relative equilibrium price vector $p_0 = \{p_1^0, \dots, p_n^0\}$, satisfying the set of inequalities (7) is such that $p_i^0 = 1, i = \overline{1, n}$, then the price vector $p = \{p_1, \dots, p_n\}$, is an equilibrium price vector. Introduced relative equilibrium vector $p_0 = \{p_1^0, \dots, p_n^0\}$ characterizes the deviation of the exchange model from the equilibrium state.

What means that the exchange of n types of goods in the cost form between M countries is in an equilibrium state and why it is important to investigate the equilibrium states? First, let us mark that if for every country export-import balance is equal zero, then the relative equilibrium vector $p_0 = \{p_1^0, \dots, p_n^0\}$ is such that $p_i^0 = 1, i = \overline{1, n}$, and the price vector $p = \{p_1, \dots, p_n\}$, is an equilibrium price vector, moreover, the set of inequalities (7) becomes the set of equalities. This case is ideal one in the reality. We describe the factors that can violate the ideal equilibrium in the exchange model. In the exchange model, there exist equilibrium states differing from this ideal case. As every country seeks to protect its economy from goods flow from outside, it establishes the customs-tariffs. This leads to the violation of zero export-import balances between the countries. Another factor influencing the violation of zero export-import balance is the state under which the country needs more import than export due to the internal state of economy and vice versa when the country needs more export than import. Such deviations in the export-import balance can lead to the equilibrium states that are far from the ideal equilibrium state. The quality of such equilibrium states can be very different. The main aim is to investigate these equilibrium states. The deformation of the ideal equilibrium state in the exchange model that can arise may also lead to the recession in the world economy. The

above established arguments are very important to investigate these deformed equilibrium states.

3. Algorithms of the Equilibrium States Finding

Let us give a set of definitions useful for what follows.

Definition 3.

By a polyhedral non-negative cone created by a set of vectors $\{a_i, i = \overline{1, t}\}$ of n -dimensional space R^n , we understand the set of vectors

$$d = \sum_{i=1}^t \alpha_i a_i,$$

where $\alpha = \{\alpha_i\}_{i=1}^t$ runs over the set R_+^t .

Definition 4.

The dimension of a non-negative polyhedral cone created by a set of vectors $\{a_i, i = \overline{1, t}\}$ in n -dimensional space R^n is maximum number of linearly independent vectors from the set of vectors $\{a_i, i = \overline{1, t}\}$.

Definition 5.

The vector b belongs to the interior of the non-negative polyhedral r -dimensional cone, $r \leq n$, created by the set of vectors $\{a_1, \dots, a_t\}$ in n -dimensional vector space R^n if a strictly positive vector $\alpha = \{\alpha_i\}_{i=1}^t \in R_+^t$ exists such that

$$b = \sum_{s=1}^t a_s \alpha_s,$$

where $\alpha_s > 0, s = \overline{1, t}$.

Let us give the necessary and sufficient conditions under which a certain vector belongs to the interior of the polyhedral cone.

Lemma 1.

Let $\{a_1, \dots, a_m\}, 1 \leq m \leq n$, be the set of linearly independent vectors in R_+^n . The necessary and sufficient conditions for the vector b to belong to the interior of the non-negative cone K_a^+ created by vectors $\{a_i, i = \overline{1, m}\}$ are the conditions

$$\langle f_i, b \rangle > 0, \quad i = \overline{1, m}, \quad \langle f_i, b \rangle = 0, \quad i = \overline{m+1, n}, \tag{8}$$

where $f_i, i = \overline{1, n}$, is a set of vectors being biorthogonal to the set of linearly independent vectors $\bar{a}_i, i = \overline{1, n}$, and $\bar{a}_i = a_i, i = \overline{1, m}$.

Proof.

It is obvious that the set of biorthogonal vector exists. Really, the vector $f_j, j = \overline{1, n}$, exists solving the set of equations

$$\langle \bar{a}_i, f_j \rangle = \delta_{ij}, \quad i = \overline{1, n}, \tag{9}$$

due to linear independency of vectors $\bar{a}_i, i = \overline{1, n}$.

Necessity. If the vector b belongs to the interior of the non-negative cone K_a^+ , then there exist such numbers $\alpha_i > 0, i = \overline{1, m}$, that

$$b = \sum_{i=1}^m a_i \alpha_i.$$

From here

$$\alpha_i = \langle b, f_i \rangle > 0, \quad i = \overline{1, m}, \quad \langle f_i, b \rangle = 0, \quad i = \overline{m+1, n}.$$

Sufficiency is obvious as α_i is determined by the formula $\alpha_i = \langle b, f_i \rangle$ unambiguously from the representation $b = \sum_{i=1}^n \alpha_i \bar{a}_i$.

The next statement is obvious.

Proposition 1

Let $\{a_1, \dots, a_m\}, 1 \leq m \leq n$, be the set of linearly independent vectors in R_+^n . The necessary and sufficient conditions for the vector b to belong to the non-negative cone K_a^+ created by the set of vectors $\{a_i, i = \overline{1, m}\}$ are the conditions

$$\langle f_i, b \rangle \geq 0, \quad i = \overline{1, m}, \quad \langle f_i, b \rangle = 0, \quad i = \overline{m+1, n}, \tag{10}$$

where $f_i, i = \overline{1, n}$, is a set of vectors being biorthogonal to the set of linearly independent vectors $\bar{a}_i, i = \overline{1, n}$, and $\bar{a}_i = a_i, i = \overline{1, m}$.

From Proposition 1 the representation for the vector b belonging to the cone K_a^+

$$b = \sum_{i=1}^m \alpha_i a_i, \quad \alpha_i \geq 0, \quad i = \overline{1, m}, \tag{11}$$

is valid.

Therefore, to check belonging of the vector b to the interior of the non-negative cone K_a^+ created by vectors $\{a_i, i = \overline{1, m}\}$ one must enlarge the set of m linearly independent vectors $\{a_i, i = \overline{1, m}\}$ up to the set of n linearly independent vectors in R^n , then build the biorthogonal set of vectors $\{f_i, i = \overline{1, n}\}$ for the enlarged set of vectors, and check the conditions of the Proposition 1.

Describe now an algorithm to construct strictly positive solutions to the set of equations

$$\psi = \sum_{i=1}^l C_i y_i, \quad y_i > 0, \quad i = \overline{1, l}, \tag{12}$$

with respect to the vector $y = \{y_i\}_{i=1}^l$ or the same set of equations in coordinate form

$$\sum_{i=1}^l c_{ki} y_i = \psi_k, \quad k = \overline{1, n}, \tag{13}$$

for the vector $\psi = \{\psi_1, \dots, \psi_n\}$ belonging to the interior of the polyhedral cone created by vectors $\{C_i = \{c_{ki}\}_{k=1}^n, i = \overline{1, l}\}$.

Theorem 1. If a certain vector ψ belonging to the interior of a non-negative r -dimensional polyhedral cone created by vectors $\{C_i = \{c_{ki}\}_{k=1}^n, i = \overline{1, l}\}$, is such that there exists a subset of r linearly independent vectors of the set of vectors $\{C_i, i = \overline{1, l}\}$, such that the vector ψ belongs to the interior of the cone created by this subset of vectors, then there exist $l-r+1$ linearly independent

non-negative solutions z_i to the set of Equation (13) such that the set of strictly positive solutions to the set of Equation (13) is given by the formula

$$y = \sum_{i=r}^l \gamma_i z_i, \tag{14}$$

where

$$\begin{aligned} z_i &= \{\langle \psi, f_1 \rangle - \langle C_i, f_1 \rangle y_i^*, \dots, \langle \psi, f_r \rangle - \langle C_i, f_r \rangle y_i^*, 0, \dots, y_i^*, 0, \dots, 0\}, \quad i = \overline{r+1, l}, \\ z_r &= \{\langle \psi, f_1 \rangle, \dots, \langle \psi, f_r \rangle, 0, \dots, 0\}, \\ y_i^* &= \begin{cases} \min_{s \in K_i} \frac{\langle \psi, f_s \rangle}{\langle C_i, f_s \rangle}, & K_i = \{s, \langle C_i, f_s \rangle > 0\}, \\ 1, & \langle C_i, f_s \rangle \leq 0, \forall s = \overline{1, r}, \end{cases} \end{aligned}$$

and the components of the vector $\{\gamma_i\}_{i=r}^l$ satisfy the set of inequalities

$$\begin{aligned} \sum_{i=r}^l \gamma_i &= 1, \quad \gamma_i > 0, \quad i = \overline{r+1, l}, \\ \sum_{i=r+1}^l \langle C_i, f_k \rangle y_i^* \gamma_i &< \langle \psi, f_k \rangle, \quad k = \overline{1, r}. \end{aligned} \tag{15}$$

Proof. The vector ψ belongs to the interior of r -dimensional polyhedral cone, $r \leq n$, and there exist r linearly independent vectors from the set of vectors C_1, \dots, C_l , $l \geq n$, such that ψ is interior for this set of r linearly independent vectors. Without loss of generality, suppose these vectors are C_1, \dots, C_r . If it is not the case, then one can get it by renumbering vectors C_i and components of the vector $y = \{y_1, \dots, y_l\}$, respectively. Therefore, the vector ψ has the representation

$$\psi = \sum_{i=1}^r \alpha_i C_i, \quad \alpha_i > 0, \quad i = \overline{1, r}. \tag{16}$$

Consider the set of equations

$$\sum_{i=1}^l C_i y_i = \psi \tag{17}$$

for the vector $y = \{y_1, \dots, y_l\}$.

Build the set of vectors f_1, \dots, f_n being the set of vectors biorthogonal to the set of linearly independent vectors C_1, \dots, C_r and satisfying the conditions

$$\langle f_i, C_j \rangle = \delta_{ij}, \quad i, j = \overline{1, r}, \quad \langle f_i, C_j \rangle = 0, \quad j = \overline{1, r}, \quad i = \overline{r+1, n},$$

where $\langle x, y \rangle$ is the scalar product of the vectors x, y in R^n . The set of Equations (17) is equivalent to the set of equations

$$\sum_{i=r+1}^l \langle C_i, f_j \rangle y_i + y_j = \langle \psi, f_j \rangle, \quad j = \overline{1, r}, \tag{18}$$

where $\langle \psi, f_i \rangle > 0, i = \overline{1, r}$.

This equivalence holds due to the next equalities $\langle f_i, C_j \rangle = 0, j = \overline{1, l}, i = \overline{r+1, n}$, $\langle \psi, f_i \rangle = 0, i = \overline{r+1, n}$. The last equalities are the consequence of the representa-

tion (16) for the vector ψ . Note that the general solution to the set of equations (18) is

$$y = \left\{ \langle \psi, f_1 \rangle - \sum_{i=r+1}^l \langle C_i, f_1 \rangle y_i, \dots, \langle \psi, f_r \rangle - \sum_{i=r+1}^l \langle C_i, f_r \rangle y_i, y_{r+1}, \dots, y_l \right\}, \quad (19)$$

where the vector $\tilde{y} = \{y_{r+1}, \dots, y_l\}$ runs over the set R^{l-r} .

The vectors $z_i, i = \overline{r, l}$, defined in the Theorem solve the set of Equation (18), their components are non-negative, and they themselves are linearly independent. Build after vectors z_i the vector

$$\bar{y} = \sum_{i=r}^l \gamma_i z_i,$$

where

$$\sum_{i=r}^l \gamma_i = 1.$$

Then

$$\bar{y} = \left\{ \langle \psi, f_1 \rangle - \sum_{i=r+1}^l \langle C_i, f_1 \rangle \gamma_i y_i^*, \dots, \langle \psi, f_r \rangle - \sum_{i=r+1}^l \langle C_i, f_r \rangle \gamma_i y_i^*, \gamma_{r+1} y_{r+1}^*, \dots, \gamma_l y_l^* \right\} \quad (20)$$

satisfies the conditions of the Theorem. Show that there is reciprocal one-to-one correspondence between vectors \bar{y} and \tilde{y} determined by formulae (20) and (19), respectively. To prove this, it is sufficient to prove that every vector $\tilde{y} = \{y_{r+1}, \dots, y_l\} \in R^{l-r}$ has one and only one corresponding set $\gamma_i, i = \overline{r, l}$, such that $\sum_{i=r}^l \gamma_i = 1$. Really, from the linear independence of vectors z_r, z_{r+1}, \dots, z_l it follows that the set of equations

$$\gamma_r z_r + \dots + \gamma_l z_l = y$$

is equivalent to the set of equations

$$\gamma_{r+1} (z_{r+1} - z_r) + \dots + \gamma_l (z_l - z_r) = y - z_r.$$

The vectors $z_{r+1} - z_r, \dots, z_l - z_r$ are linearly independent, therefore we can determine $\gamma_{r+1}, \dots, \gamma_l$ for every vector $\tilde{y} = \{y_{r+1}, \dots, y_l\}$ unambiguously. From the equality $\sum_{i=r}^l \gamma_i = 1$ we determine the number γ_r unambiguously too. It is easy to see that $y_i = \gamma_i y_i^*, i = \overline{r+1, l}$. The solution \bar{y} is strictly positive if $\gamma_i, i = \overline{r, l}$, satisfy the set of inequalities defined in the Theorem 1.

Definition 6. We call a subset of vectors $\{b_1, \dots, b_m\}$ from the set of vectors $\{a_1, \dots, a_t\}$ the generating set of vectors of the r -dimensional cone created by the vectors $\{a_1, \dots, a_t\}$ if it satisfies the conditions:

- 1) every vector $b_i, i = \overline{1, m}$, from the set of vectors $\{b_1, \dots, b_m\}$ does not belong to the cone created by the set of vectors $\{b_1, \dots, b_m\} \setminus \{b_i\}$;
- 2) the cone created by the set of vectors $\{a_1, \dots, a_t\}$ coincides with the cone created by the vectors $\{b_1, \dots, b_m\}$.

Proposition 2.

The generating set of vectors of r -dimensional cone created by a set of vectors $\{a_1, \dots, a_t\}$ exists and contains no less than r vectors.

Proof.

The proof we carry out by induction on the number of vectors. Denote $\{b_1, \dots, b_m\}$ the generating set of vectors for the cone created by the set of vectors $\{a_1, \dots, a_k\}$. Let K_{a_1, \dots, a_k}^+ and K_{b_1, \dots, b_m}^+ be the cones created by the set of vectors $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_m\}$, respectively. In accordance with the Definition 6, for the generating set of vectors the equality $K_{a_1, \dots, a_k}^+ = K_{b_1, \dots, b_m}^+$ holds. Let a_{k+1} be a certain vector out of the cone K_{a_1, \dots, a_k}^+ . The equality $K_{a_1, \dots, a_{k+1}}^+ = K_{b_1, \dots, b_m, a_{k+1}}^+$ takes place. For the vector $b_i, i = \overline{1, m}$, consider the cone $K_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m, a_{k+1}}^+$ created by the set of vectors $\{a_{k+1}\} \cup \{b_1, \dots, b_m\} \setminus \{b_i\}$. Two cases are possible: the vector b_i belongs to the cone $K_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m, a_{k+1}}^+$ or it does not. If b_i belongs to this cone, then we throw out it from the set of vectors $\{b_1, \dots, b_m\} \cup \{a_{k+1}\}$. It is obvious that $K_{b_1, \dots, b_m, a_{k+1}}^+ = K_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m, a_{k+1}}^+$. Further we act analogously. If the vector b_j belongs to the cone $K_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{j-1}, b_{j+1}, \dots, b_m, a_{k+1}}^+$, then we throw out it from the set of vectors $\{a_{k+1}\} \cup \{b_1, \dots, b_m\} \setminus \{b_i\}$. Then the following equalities

$$K_{b_1, \dots, b_m, a_{k+1}}^+ = K_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m, a_{k+1}}^+ = K_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{j-1}, b_{j+1}, \dots, b_m, a_{k+1}}^+$$

hold. Having carried out the finite number of steps, we come to the set of vectors $\{a_{k+1}\} \cup \{b_1^1, \dots, b_m^1\}$ that does not contain none vector $b_j^1, j = \overline{1, m_1}$, belonging to the corresponding cone $K_{b_1^1, \dots, b_{j-1}^1, b_{j+1}^1, \dots, b_m^1, a_{k+1}}^+$. It is obvious that the equality $K_{b_1^1, \dots, b_m^1, a_{k+1}}^+ = K_{b_1, \dots, b_m, a_{k+1}}^+$ takes place. To finish the proof, one has to note that as the basis of induction we can choose any set of r linearly independent vectors from the set of vectors $\{a_i, i = \overline{1, t}\}$.

Proposition 3. A certain vector d belongs to the interior of r -dimensional cone created by vectors $\{a_i, i = \overline{1, t}\}$, if and only if for the vector d the representation

$$d = \sum_{i=1}^m \alpha_i b_i \tag{21}$$

holds, where $\{b_1, \dots, b_m\}$ is the generating set of vectors of the considered cone, $\alpha_i \geq 0$, and among coefficients $\alpha_i, i = \overline{1, m}$, there are no less than r strictly positive numbers.

Proof. Necessity. The cone created by vectors $\{a_i, i = \overline{1, t}\}$ is the union of the cones created by all r linearly independent subsets of vectors $\{b_{i_1}, \dots, b_{i_r}\}$ belonging to the set $\{b_1, \dots, b_m\}$. Therefore, there exists a maximum number of subsets $\{b_{i_1^s}, \dots, b_{i_r^s}\}, s = \overline{1, w}$, of r linearly independent vectors such that the vector d belongs to every cone created by subset of vectors $\{b_{i_1^s}, \dots, b_{i_r^s}\}, s = \overline{1, w}$. According to the Proposition 1, for the vector d belonging to the cone created by vectors $\{a_i, i = \overline{1, t}\}$ there holds the representation

$$d = \sum_{k=1}^r \alpha_{i_k^s} b_{i_k^s}, \quad \alpha_{i_k^s} \geq 0, \quad k = \overline{1, r}, \quad s = \overline{1, w}.$$

From here, the representation

$$d = \frac{1}{W} \sum_{s=1}^w \sum_{k=1}^r \alpha_{i_k^s} b_{i_k^s}.$$

follows. In this representation, the number of different generating vectors entering with strictly positive coefficients is not less than r . Because, if it is not the case, the vector d does not belong to the interior of the cone created by vectors $\{a_i, i = \overline{1, t}\}$.

Sufficiency. If conditions of the Proposition 3 hold, then the vector d is the sum of two vectors, namely, a certain vector x that gets into the interior of the cone created by a subset of r linearly independent vectors of the generating set of vectors and a vector y from the cone created by vectors $\{a_i, i = \overline{1, t}\}$. Let us apply the Theorem 1 to the vector x getting into the interior of the cone created by the subset of r linearly independent vectors from the generating set of vectors, taking for the vector set C_i the vector set a_i for $l = t$. Therefore, the vector x has the representation

$$x = \sum_{s=1}^t \gamma_s a_s,$$

where $\gamma_s > 0, s = \overline{1, t}$. From here it immediately follows that the vector d has the representation

$$d = \sum_{s=1}^t \gamma_s^1 a_s,$$

where $\gamma_s^1 > 0, s = \overline{1, t}$.

The algorithm to check whether the vector d belongs to the interior of the cone created by the vectors $\{a_i, i = \overline{1, t}\}$ consists of building the generating set of vectors $\{b_i, i = \overline{1, m}\}$; for every subset of r linearly independent vectors from the generating set of vectors one must use the Theorem 1 to choose only those subsets of r linearly independent vectors $\{b_{i_k}, k = \overline{1, r}\}$ from the generating set of vectors for which the representations

$$d = \sum_{k=1}^r \delta_{i_k} b_{i_k}, \quad \delta_{i_k} \geq 0, \quad k = \overline{1, r}, \quad (22)$$

are valid.

Two cases are possible:

- 1) There not exists a subset of r linearly independent vectors from the generating set of vectors such that the representation (22) is valid.
- 2) There exists a certain set of subsets of r linearly independent vectors from the generating set of vectors such that for every subset r linearly independent vectors belonging to this set all coefficients of the expansion (22) are non-negative.

In the first case, the vector d does not belong to the interior of the cone created by the vectors $\{a_i, i = \overline{1, t}\}$. In the second one, if the number of different generating vectors appeared in expansions (22) for the vector d with positive coefficients in expansions is not less than r , then the vector d belongs to the interior of the cone created by the set of vectors $\{a_i, i = \overline{1, t}\}$. In the opposite case the vector d does not belong to the interior of the cone created by the vectors

$$\{a_i, i = \overline{1, t}\}.$$

In the next Theorem, we solve the problem to construct the strictly positive solutions to the set of Equation (13) without additional assumption figuring in the Theorem 1. The rank of the set of the vectors $\{C_i = \{c_{ki}\}_{k=1}^n, i = \overline{1, l}\}$ we denote r .

Theorem 2. Let a vector of final consumption ψ belongs to the interior of the cone created by the set of vectors $\{C_i = \{c_{ki}\}_{k=1}^n, i = \overline{1, l}\}$. Then there exists a set of vectors $\psi_s = \{\psi_k^s\}_{k=1}^n, s = \overline{1, 2}$, and a real number $0 \leq \alpha \leq 1$ satisfying conditions:

- 1) every vector $\psi_s = \{\psi_k^s\}_{k=1}^n, s = \overline{1, 2}$, belongs to the interior of a cone created by a set of r linearly independent vectors $\{C_{i_1^s}, \dots, C_{i_r^s}\}$.
- 2) there holds the representation $\psi = \alpha\psi_1 + (1 - \alpha)\psi_2$.

A strictly positive solution to the set of Equation (13) can be written as

$$y = \alpha y_1 + (1 - \alpha) y_2, \quad y_s = \{y_i^s\}_{i=1}^l,$$

where y_s is a strictly positive solution to the set of equations

$$\sum_{i=1}^l c_{ki} y_i^s = \psi_k^s, \quad k = \overline{1, n}, \quad s = \overline{1, 2}, \tag{23}$$

constructed in the Theorem 1.

Proof. Without loss of generality, denote the generating set of vectors for the cone K_{C_1, \dots, C_l}^+ by C_1, \dots, C_m , if it is not the case, we can renumber the set of vectors C_1, \dots, C_l . As the cone K_{C_1, \dots, C_l}^+ has dimension r , let us consider all subsets of vectors $\{C_{i_1^s}, \dots, C_{i_r^s}\}, s = \overline{1, w}$, from the set C_1, \dots, C_m , being linearly independent. It is evident that

$$K_{C_1, \dots, C_l}^+ = \bigcup_{s=1}^w K_{C_{i_1^s}, \dots, C_{i_r^s}}^+,$$

where $K_{C_{i_1^s}, \dots, C_{i_r^s}}^+$ is a nonnegative cone created by the vectors $C_{i_1^s}, \dots, C_{i_r^s}$. Consider two subcones $K_{C_{i_1^{s_1}}, \dots, C_{i_r^{s_1}}}^+$ and $K_{C_{i_1^{s_2}}, \dots, C_{i_r^{s_2}}}^+$ from the cone K_{C_1, \dots, C_l}^+ such that

$$K_{C_{i_1^{s_1}}, \dots, C_{i_r^{s_1}}}^+ \cap K_{C_{i_1^{s_2}}, \dots, C_{i_r^{s_2}}}^+ \neq \emptyset$$

and

$$K_{C_{i_1^{s_1}}, \dots, C_{i_r^{s_1}}}^+ \neq K_{C_{i_1^{s_2}}, \dots, C_{i_r^{s_2}}}^+.$$

If the vector ψ belongs to set

$$K_{C_{i_1^{s_1}}, \dots, C_{i_r^{s_1}}}^+ \cap K_{C_{i_1^{s_2}}, \dots, C_{i_r^{s_2}}}^+$$

and is internal vector for the cone K_{C_1, \dots, C_l}^+ , then there exist two internal vectors ψ_1 and ψ_2 belonging to cones $K_{C_{i_1^{s_1}}, \dots, C_{i_r^{s_1}}}^+$ and $K_{C_{i_1^{s_2}}, \dots, C_{i_r^{s_2}}}^+$, respectively, and such a real number $0 < \alpha < 1$ that

$$\psi = \alpha\psi_1 + (1 - \alpha)\psi_2$$

due to convexity of the cone K_{C_1, \dots, C_l}^+ .

Definition 7. Let $C_i = \{c_{si}\}_{s=1}^n \in R_+^n, i = \overline{1, l}$, be a set of demand vectors and let $b_i = \{b_{si}\}_{s=1}^n \in R_+^n, i = \overline{1, l}$, be a set of supply vectors. We say the structure of supply is agreed with the structure of demand in the strict sense if for the matrix B the representation $B = CB_1$ is true, where the matrix B consists of the vectors $b_i \in R_+^n, i = \overline{1, l}$, as columns, and the matrix C is composed from the vectors $C_i \in R_+^n, i = \overline{1, l}$, as columns. and B_1 is a square nonnegative indecomposable matrix.

Definition 8. Let $C_i = \{c_{si}\}_{s=1}^n \in R_+^n, i = \overline{1, l}$, be a set of demand vectors and let $b_i = \{b_{si}\}_{s=1}^n \in R_+^n, i = \overline{1, l}$, be a set of supply vectors. We say the structure of supply is agreed with the structure of demand in the strict sense of the rank $|I|$ if there exists a subset $I \subseteq N$ such that for the matrix B^I the representation $B^I = C^I B_1^I$ is true, where the matrix B^I consists of the vectors $b_i^I \in R_+^{|I|}, i = \overline{1, l}$, as columns, and the matrix C^I is composed from the vectors $C_i^I \in R_+^{|I|}, i = \overline{1, l}$, as columns, and $B_1^I = |b_{is}^{I,l}|_{i,s \in I}^{|I|}$ is a square nonnegative indecomposable matrix, where $b_i^I = \{b_{ki}\}_{k \in I}$, $C_i^I = \{c_{ki}\}_{k \in I}$ and, moreover, the inequalities

$$\sum_{i=1}^l c_{ki} y_i^I < \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \quad y_i^I = \sum_{s=1}^l b_{is}^{I,l},$$

are valid.

Lemma 2. Suppose the set of supply vectors $b_i = \{b_{si}\}_{s=1}^n \in R_+^n, i = \overline{1, l}$, belongs to the polyhedral cone created by the set of demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$. Then, for the matrix $B = |b_{ki}^{n,l}|_{k=1, i=1}^{n,l}$ created by the columns of vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, the representation

$$B = CB_1 \tag{24}$$

is true, where the matrix $C = |c_{ki}^{n,l}|_{k=1, i=1}^{n,l}$ is created by the columns of vectors $C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, and the matrix $B_1 = |b_{mi}^1|_{m=1, i=1}^l$ is nonnegative. If, in addition, the set of supply vectors $b_i \in R_+^n, i = \overline{1, l}$, belongs to the interior of the polyhedral cone created by the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, the matrix B_1 can be chosen strictly positive.

Proof. The first part of Lemma 2 follows from the second one. If every vector $b_i, i = \overline{1, l}$, belongs to the interior of the polyhedral cone created by vectors $C_i, i = \overline{1, l}$, then due to Theorem 2 there exists such a strictly positive vector $y_i = \{y_{ki}\}_{k=1}^l$ that

$$b_{ki} = \sum_{s=1}^l c_{ks} y_{si}, \quad k = \overline{1, n}.$$

Let us denote $y_{si} = b_{si}^1$, then we obtain

$$b_{ki} = \sum_{s=1}^l c_{ks} b_{si}^1, \quad k = \overline{1, n}, \quad i = \overline{1, l}.$$

This proves Lemma 2.

Lemma 3. Let $b_i \in R_+^n, i = \overline{1, l}$, be a set of supply vectors and let $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, be a set of demand vectors. If for every vector $b_i, i = \overline{1, l}$, there exists a subset of vectors C_{i_1}, \dots, C_{i_k} such that the rank of the set

of vectors C_{i_1}, \dots, C_{i_k} and the set of vectors $b_i, C_{i_1}, \dots, C_{i_k}$ is the same, then for the matrix $B = |b_{ki}|_{k=1, i=1}^{n, l}$ created by the columns of vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, the representation

$$B = CB_1 \tag{25}$$

is true, where the matrix $C = |c_{ki}|_{k=1, i=1}^{n, l}$ is created by the columns of vectors $C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, and the matrix $B_1 = |b_{mi}^1|_{m=1, i=1}^l$ is a square one. If the vector of supply of goods $\sum_{i=1}^l b_i$ belongs to the polyhedral cone created by the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, then only those matrices

$B_1 = |b_{mi}^1|_{m=1, i=1}^l$ in the representation (25) for the matrix B are important for which $\sum_{s=1}^l b_{is}^1 \geq 0, i = \overline{1, l}$. In such a case the representation

$$\sum_{i=1}^l b_i = \sum_{i=1}^l \sum_{s=1}^l b_{is}^1 C_i \tag{26}$$

is true.

Proof. To prove Lemma 3 we need to show the existence of the solutions to the set of equations

$$b_i = \sum_{i=1}^l y_i C_i, \quad i = \overline{1, l}. \tag{27}$$

But for the every fixed vector b_i the set of Equation (27) has a solution due to the Lemma 3 conditions. This proves the first part of Lemma 3. Let us consider the case as $n \geq l$ and the vectors $C_i, i = \overline{1, l}$ are linearly independent. If the vector of supply of goods $\sum_{i=1}^l b_i$ belongs to the polyhedral cone created by the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, then

$$\sum_{i=1}^l b_i = \sum_{i=1}^l y_i C_i, \quad y_i \geq 0, \tag{28}$$

but from the representation (25) $\sum_{i=1}^l b_i = \sum_{i=1}^l y_i^1 C_i$, where $y_i^1 = \sum_{s=1}^l b_{is}^1$, therefore $y_i = \sum_{s=1}^l b_{is}^1, i = \overline{1, l}$. due to linear independence of the vectors $C_i, i = \overline{1, l}$. As $y_i \geq 0$, then in this case Lemma 3 is proved.

If the case $n < l$ is true, then we can come to the case of the demand matrix constructed by vectors-column $C_i^\varepsilon = \{c_{ki}(\varepsilon)\}_{k=1}^l \in R_+^l$, where $c_{ki}(\varepsilon) = c_{ki}$, $k = \overline{1, n}, i \leq n, c_{ki}(\varepsilon) = 0, k = \overline{n+1, l}, i \leq n$, and $C_i^\varepsilon = \{c_{ki}(\varepsilon)\}_{k=1}^l$ where $c_{ki}(\varepsilon) = c_{ki}, k = \overline{1, n}, i > n, c_{ki}(\varepsilon) = \delta_{ki} \varepsilon, k = \overline{n+1, l}, l \geq i > n$. Denote the matrix $C^\varepsilon = |c_{ki}(\varepsilon)|_{k=1, i=1}^{l, l}$ for the sufficiently small $\varepsilon > 0$. Then, the rank of the matrix C^ε is equal l for every sufficiently small $\varepsilon > 0$. Let us put $B^\varepsilon = C^\varepsilon B_1$. Suppose that

$$\sum_{i=1}^l b_i^\varepsilon = \sum_{i=1}^l y_i C_i^\varepsilon, \quad y_i \geq 0, \quad i = \overline{1, l}, \tag{29}$$

but, due to the representation $B^\varepsilon = C^\varepsilon B_1$, from (29) we have

$$\begin{aligned} \sum_{i=1}^l \sum_{s=1}^l c_{ki} b_{is}^1 &= \sum_{i=1}^l y_i c_{ki}, \quad k = \overline{1, n}, \\ \varepsilon y_i &= \varepsilon \sum_{s=1}^l b_{is}^1, \quad i = \overline{n+1, l}. \end{aligned} \tag{30}$$

From (30) we obtain $y_i = \sum_{s=1}^l b_{is}^1 \geq 0$. The last proves Lemma 3.

Let us consider the linear set of equations

$$CX = b, \tag{31}$$

where a matrix C has the dimension $n \times l$, a vector b has the dimension n , $l \geq n$. Without loss of generality, we assume that the rank of the matrix C is equal n , since we consider only those set of equations that are consistent. If it is not so, due to the compatibility of the system of equations, some of the equations can be discarded, leaving only the system of equations in which the rank of the remaining matrix will coincide with the number of equations. Therefore, there is at least one non degenerate minor $|C^{m_1, \dots, m_n}|$ of the matrix C with n columns having indices $m_1 < m_2 < \dots < m_n$, $1 \leq m_i \leq l$, $i = \overline{1, n}$, such that $\det|C^{m_1, \dots, m_n}| \neq 0$. The general solution of the set of Equation (31) can be given as

$$\begin{aligned} X &= f, \quad f(m_k) = X^{m_1, \dots, m_n}(k), \quad k = \overline{1, n}, \\ f(j) &= d_j, \quad j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}, \end{aligned} \tag{32}$$

where

$$X^{m_1, \dots, m_n} = [C^{m_1, \dots, m_n}]^{-1} b - \sum_{s \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}} [C^{m_1, \dots, m_n}]^{-1} C_s d_s, \tag{33}$$

and $[C^{m_1, \dots, m_n}]^{-1}$ is an inverse matrix to the matrix C^{m_1, \dots, m_n} , d_j is a real number, C_s is a s -th column of matrix C , $X^{m_1, \dots, m_n}(k)$ is a k -th component of vector X^{m_1, \dots, m_n} , $f(m_k)$ is a m_k -th component of vector f . Below we give sufficient conditions for the existence of decomposition

$$B = CB_1. \tag{34}$$

For this purpose, it needs to solve l set of equations

$$Cb_i^1 = b_i, \quad i = \overline{1, l}. \tag{35}$$

Theorem 3. Suppose the matrix C has the dimensions $n \times l$ and its rank equals to n , $l \geq n$, the vector $\psi = \sum_{i=1}^l b_i$ belongs to the interior of the cone generated by the column vectors C_i of the matrix C . Moreover, there exists sub cone generated by linearly independent vectors C_{m_1}, \dots, C_{m_n} such that the vector $\psi = \sum_{i=1}^l b_i$ belongs to the interior of the cone generated by the column vec-

tors $C_{m_i}, i = \overline{1, n}$. Then for the matrix B the representation (34) is true such that $\sum_{i=1}^l b_i^l > 0$, where $B_l = |b_1^l, \dots, b_l^l|$.

Proof. The set of Equation (35) has a solution

$$\begin{aligned} b_i^l &= f_i, \quad f_i(m_k) = X_i^{m_1, \dots, m_n}(k), \quad k = \overline{1, n}, \\ f_i(j) &= d_j^i, \quad j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}, \end{aligned} \tag{36}$$

where

$$X_i^{m_1, \dots, m_n} = [C^{m_1, \dots, m_n}]^{-1} b_i - \sum_{s \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}} [C^{m_1, \dots, m_n}]^{-1} C_s d_s^i. \tag{37}$$

Then,

$$\sum_{i=1}^l b_i^l = \begin{cases} [C^{m_1, \dots, m_n}]^{-1} \left(\sum_{i=1}^l b_i \right) (m_k) \\ - \sum_{s \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}} [C^{m_1, \dots, m_n}]^{-1} C_s(m_k) \left(\sum_{i=1}^l d_s^i \right), \quad k = \overline{1, n}, \\ \sum_{i=1}^l d_j^i, \quad j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}. \end{cases} \tag{38}$$

Due to supposition of Theorem 3,

$$[C^{m_1, \dots, m_n}]^{-1} \left(\sum_{i=1}^l b_i \right) (k) > 0, \quad k = \overline{1, n}. \tag{39}$$

Since the real numbers d_j^i are arbitrary, then for every $j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}$ choose d_j^i such that the inequalities

$$\begin{aligned} \sum_{i=1}^l d_j^i &> 0, \quad j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}, \\ [C^{m_1, \dots, m_n}]^{-1} \left(\sum_{i=1}^l b_i \right) (m_k) - \sum_{s \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}} [C^{m_1, \dots, m_n}]^{-1} C_s(m_k) \left(\sum_{i=1}^l d_s^i \right) &> 0, \quad k = \overline{1, n}, \end{aligned}$$

are true. This is possible to do if to choose $\sum_{i=1}^l d_j^i > 0, j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}$, sufficiently small. Really, denoting

$$\begin{aligned} A &= \max_{1 \leq k \leq n, s \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}} \left| [C^{m_1, \dots, m_n}]^{-1} C_s(m_k) \right|, \\ B &= \min_{1 \leq k \leq n} [C^{m_1, \dots, m_n}]^{-1} \left(\sum_{i=1}^l b_i \right) (m_k), \quad \varepsilon = \max_{j \in \{1, 2, \dots, l\} \setminus \{m_1, \dots, m_n\}} \sum_{i=1}^l d_j^i, \end{aligned}$$

we have $0 < A < \infty, 0 < B < \infty, \varepsilon > 0$, and if to put $0 < \varepsilon < \frac{B}{(l-n)A}$ for $l > n$,

then $\sum_{i=1}^l b_i^l > 0$. Theorem 3 is proved.

Theorem 4. Let the structure of supply agree with structure of demand in the strict sense with the supply vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, and the demand

vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, and let $\sum_{s=1}^n c_{si} > 0, i = \overline{1, l}, \sum_{i=1}^l c_{si} > 0, s = \overline{1, n}$.

The necessary and sufficient conditions for the solution existence of the set of equations

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p \rangle}{\langle C_i, p \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{40}$$

relative to the vector p is belonging of the vector $D = \{d_i\}_{i=1}^l$ to the polyhedral cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, where $B = CB_1, B_1 = \{b_{ki}^1\}_{k,i=1}^l$ is a nonnegative indecomposable matrix, the vector $D = \{d_i\}_{i=1}^l$ is a strictly positive solution to the set of equations

$$\sum_{k=1}^l b_{ki}^1 d_k = y_i d_i, \quad y_i = \sum_{k=1}^l b_{ik}^1, \quad i = \overline{1, l}. \tag{41}$$

Proof. Two cases are possible: 1) $n \geq l$ and 2) $n < l$. Consider the first case.

The necessity. First, we assume the vectors $C_i, i = \overline{1, l}$, are linearly independent. Let there exist strictly positive solution $p_0 \in R_+^n$ to the set of Equation (40). As for the matrix B the representation $B = CB_1$ is true, where the matrix B_1 is strictly positive, from the equalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{42}$$

we have the equalities

$$\sum_{i=1}^l c_{ki} \left[\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} - \sum_{k=1}^l b_{ik}^1 \right], \quad k = \overline{1, n}. \tag{43}$$

As the vectors $C_i, i = \overline{1, l}$, are linearly independent, we obtain

$$\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} - \sum_{k=1}^l b_{ik}^1 = 0, \quad i = \overline{1, l}, \tag{44}$$

or

$$\langle b_i, p_0 \rangle - y_i \langle C_i, p_0 \rangle = 0, \quad i = \overline{1, l}. \tag{45}$$

But

$$\langle b_i, p_0 \rangle = \sum_{s=1}^l \langle C_s, p_0 \rangle b_{si}^1. \tag{46}$$

Substituting (46) into (45), we obtain

$$\sum_{s=1}^l \langle C_s, p_0 \rangle b_{si}^1 = y_i \langle C_i, p_0 \rangle, \quad i = \overline{1, l}. \tag{47}$$

Let us put $D = \{\langle C_s, p_0 \rangle\}_{s=1}^l$ then $d_s = \langle C_s, p_0 \rangle > 0, s = \overline{1, l}$. From this it follows that the vector D belongs to the interior of the polyhedral cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$.

Now, suppose the vectors $C_i \in R_+^n, i = \overline{1, l}$, are linearly dependent.

In this case we come to the case 2) $n < l$, at the beginning of the proof. Introduce $l - n$ fictitious goods and let us consider the demand matrix con-

structed by the vectors-columns $C_i^\varepsilon = \{c_{ki}(\varepsilon)\}_{k=1}^l \in R_+^l$, where $c_{ki}(\varepsilon) = c_{ki}$, $k = \overline{1, n}$, $i \leq n$, $c_{ki}(\varepsilon) = 0$, $k = \overline{n+1, l}$, $i \leq n$, and $C_i^\varepsilon = \{c_{ki}(\varepsilon)\}_{k=1}^l$ where $c_{ki}(\varepsilon) = c_{ki}$, $k = \overline{1, n}$, $i > n$, $c_{ki}(\varepsilon) = \delta_{ki}\varepsilon$, $k = \overline{n+1, l}$, $l \geq i > n$. Denote the matrix $C^\varepsilon = \{c_{ki}(\varepsilon)\}_{k=1, i=1}^{l, l}$ for the sufficiently small $\varepsilon > 0$. Then, the rank of the matrix C^ε is equal l for every sufficiently small $\varepsilon > 0$. Let us to put $B^\varepsilon = C^\varepsilon B_1$. Suppose the vector $p_0 = \{p_i^0\}_{i=1}^n \in R_+^n$ solves the problem

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}. \tag{48}$$

Then the vector $p_0^\varepsilon \in R_+^l$ is a solution to the problem

$$\sum_{i=1}^l c_{ki}(\varepsilon) \frac{\langle b_i^\varepsilon, p_0^\varepsilon \rangle}{\langle C_i^\varepsilon, p_0^\varepsilon \rangle} = \sum_{i=1}^l b_{ki}^\varepsilon, \quad k = \overline{1, l}, \tag{49}$$

for every sufficiently small $\varepsilon > 0$, where $p_0^\varepsilon = \{p_i^{0,\varepsilon}\}_{i=1}^l$, $p_i^{0,\varepsilon} = p_i^0$, $i = \overline{1, n}$, $p_i^{0,\varepsilon} = 0$, $i = \overline{n+1, l}$. The equalities (49) can be written in the form

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i^\varepsilon, p_0^\varepsilon \rangle}{\langle C_i^\varepsilon, p_0^\varepsilon \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{50}$$

$$\varepsilon \frac{\langle b_i^\varepsilon, p_0^\varepsilon \rangle}{\langle C_i^\varepsilon, p_0^\varepsilon \rangle} = \varepsilon \sum_{s=1}^l b_{is}^1, \quad i = \overline{n+1, l}. \tag{51}$$

On such a vector p_0^ε we have

$$\frac{\langle b_i^\varepsilon, p_0^\varepsilon \rangle}{\langle C_i^\varepsilon, p_0^\varepsilon \rangle} = \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle}, \quad i = \overline{1, l}. \tag{52}$$

Therefore, the equalities (50), (51) for $\varepsilon > 0$ are written in the form

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{53}$$

$$\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{s=1}^l b_{is}^1, \quad i = \overline{n+1, l}. \tag{54}$$

Taking into account the equalities

$$\sum_{i=1}^l c_{ki} \left[\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} - \sum_{s=1}^l b_{is}^1 \right] = 0, \quad k = \overline{1, n}, \tag{55}$$

and equalities (54), we obtain

$$\sum_{i=1}^n c_{ki} \left[\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} - \sum_{s=1}^l b_{is}^1 \right] = 0, \quad k = \overline{1, n}. \tag{56}$$

As the vectors C_i , $i = \overline{1, n}$ are linearly independent, we obtain

$$\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} - \sum_{s=1}^l b_{is}^1 = 0, \quad i = \overline{1, n}. \tag{57}$$

Due to the equalities $\langle b_i, p_0 \rangle = \sum_{s=1}^l \langle C_i, p_0 \rangle b_{si}^1$, $i = \overline{1, l}$, the equalities

$$\sum_{s=1}^l \langle C_s, p_0 \rangle b_{si}^1 = y_i \langle C_i, p_0 \rangle, \quad i = \overline{1, l}, \tag{58}$$

are true, where $\sum_{s=1}^l b_{is}^1 = y_i, i = \overline{1, l}, \langle C_i, p_0 \rangle = \sum_{s=1}^n c_{si} p_s^0$. The last means the needed.

Let us consider the problem

$$\sum_{s=1}^l b_{si}^1 d_s = y_i d_i, \quad i = \overline{1, l}, \tag{59}$$

relative to the vector $D = \{d_i\}_{i=1}^l$. Due to the matrix B_1 is a strictly positive one, the conjugate problem to the problem (59)

$$\sum_{s=1}^l b_{si}^1 v_s = y_s v_s, \quad i = \overline{1, l}, \tag{60}$$

has strictly positive solution $v = \{v_i\}_{i=1}^l$ with $v_s = 1, s = \overline{1, l}$. If to consider the

matrix $B_2 = \left| \frac{b_{si}^1}{y_s} \right|_{s,i=1}^l$ and introduce the norm of matrix $A = |a_{ij}|_{i,j=1}^l$,

$\|A\| = \max_{1 \leq i \leq l} \sum_{j=1}^l |a_{ij}|$, then $\|B_2\| = 1$. From this it follows that the problem (59) has strictly positive solution, due to Perron-Frobenius Theorem.

Sufficiency. From the fact that the strictly positive vector $D = \{d_i\}_{i=1}^l$ solves the problem (41) and it belongs to the polyhedral cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, then there exists nonnegative vector $p^0 = \{p_s^0\}_{s=1}^n$ such that

$$\sum_{s=1}^n c_{si} p_s^0 = d_i, \quad i = \overline{1, l}. \tag{61}$$

Substituting $d_i, i = \overline{1, l}$, into (41) we obtain

$$\sum_{i=1}^l b_{ik}^1 \sum_{s=1}^n c_{si} p_s^0 = y_k \sum_{s=1}^n c_{sk} p_s^0, \quad i = \overline{1, l}, \tag{62}$$

or

$$\sum_{s=1}^n b_{sk}^1 p_s^0 = y_k \sum_{s=1}^n c_{sk} p_s^0, \quad i = \overline{1, l}, \tag{63}$$

where $y_k = \sum_{i=1}^l b_{ki}^1$. But

$$\sum_{i=1}^l c_{ki} y_i = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}. \tag{64}$$

The equalities (63), (64) give

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{65}$$

where $\langle b_i, p_0 \rangle = \sum_{s=1}^n b_{si} p_s^0, \langle C_i, p_0 \rangle = \sum_{s=1}^n c_{si} p_s^0, i = \overline{1, l}$.

Theorem 4 is proved.

Definition 9. We say the economy system with the set of property vectors

$b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, and the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, is being in the state of equilibrium if there exists such a nonnegative vector $p_0 \in R_+^n$ that the inequalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p \rangle}{\langle C_i, p \rangle} \leq \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{66}$$

are true.

Theorem 5. Suppose the inequalities $\sum_{k=1}^n c_{ki} > 0, i = \overline{1, l}, \sum_{i=1}^l b_{ki} > 0, k = \overline{1, n}, \sum_{i=1}^l c_{ki} > 0, k = \overline{1, n}$, are true. The necessary and sufficient conditions for the existence of equilibrium state in the economy system with the set of supply vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, and the set of demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, are the existence of the nonnegative vector $y = \{y_i\}_{i=1}^l$ and nonempty subset $I \subseteq N, N = \{1, 2, 3, \dots, n\}$, such that the equalities and inequalities

$$\begin{aligned} \sum_{i=1}^l c_{ki} y_i &= \sum_{i=1}^l b_{ki}, \quad k \in I, \\ \sum_{i=1}^l c_{ki} y_i &< \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \end{aligned} \tag{67}$$

are valid and there exists nonnegative vector p_0 solving the set of equations

$$\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = y_i, \quad i = \overline{1, l}. \tag{68}$$

Proof. Necessity. The first supposition of Theorem $\sum_{i=1}^l c_{ki} > 0, k = \overline{1, n}$, means that all goods in the economy system are consumed and the second one $\sum_{k=1}^n c_{ki} > 0, i = \overline{1, l}$, means that the i -th consumer consumes just even if one goods. The third condition $\sum_{i=1}^l b_{ki} > 0, k = \overline{1, n}$, means that the supply of every goods is non zero. If the economy system is at the state of equilibrium, then there exists nonzero vector $p_0 \in R_+^n$ such that

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} \leq \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}. \tag{69}$$

Let us prove that there exists a nonempty set $I \subseteq N$ such that

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k \in I. \tag{70}$$

On the contrary. Suppose that the strict inequalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} < \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{71}$$

are true. Multiplying the k -th inequality by p_k^0 and summing over all k we ob-

tain the inequality

$$\sum_{i=1}^l \langle b_i, p_0 \rangle < \sum_{i=1}^l \langle b_i, p_0 \rangle, \tag{72}$$

which is impossible. So, there exists a nonempty set I such that the equalities (67) are true. Denoting

$$\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = y_i, \quad i = \overline{1, l}, \tag{73}$$

we obtain the needed. The proof of sufficiency is obvious. Really, if the conditions of Theorem 5 are true, then there exists a nonnegative vector $y = \{y_i\}_{i=1}^l$ and nonempty subset $I \subseteq N$, $N = \{1, 2, 3, \dots, n\}$, such that the equalities and inequalities (67) are true and the set of equations (68) has a nonnegative solution p_0 . Substituting $y_i, i = \overline{1, l}$, from (68) into (67) we obtain the needed. Theorem 5 is proved.

The next Theorem is a consequence of Theorem 3.1.3 from (Gonchar, 2008).

Theorem 6. Let $C_i \in R_+^n, i = \overline{1, l}$, be a set of demand vectors and let $b_i \in R_+^n, i = \overline{1, l}$, be a set of supply vectors. If the vectors $C_i \in R_+^n, i = \overline{1, l}$, are strictly positive and $\psi_k = \sum_{i=1}^l b_{ki} > 0, k = \overline{1, n}$, then there exists an equilibrium price vector p_0 such that the inequalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} \leq \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{74}$$

are true.

Proof. Here we give an independent from the proof of Theorem 3.1.3 (Gonchar, 2008) the proof of Theorem 6. Let us introduce on the set

$$P = \left\{ p = \{p_1, \dots, p_n\} \in R_+^n, \sum_{i=1}^n p_i = 1 \right\}$$

an auxiliary map $G^\varepsilon(p) = \{G_k^\varepsilon(p)\}_{k=1}^n$ transforming the set P into itself, where

$$G_k^\varepsilon(p) = \frac{f_k^\varepsilon(p)}{\sum_{i=1}^n f_i^\varepsilon(p)},$$

$$f_k^\varepsilon(p) = \frac{1}{\psi_k} \sum_{i=1}^l \frac{p_k c_{ki} + \varepsilon}{\langle C_i, p \rangle + n\varepsilon} \langle b_i, p \rangle, \quad k = \overline{1, n}. \tag{75}$$

It is evident that $f_k^\varepsilon(p), k = \overline{1, n}$, is a continuous map on the set P due to the conditions of Theorem 6, since

$$\sum_{i=1}^n f_i^\varepsilon(p) \geq \frac{\langle \psi_k, p \rangle}{\max_k \psi_k} \geq \frac{\min_k \psi_k}{\max_k \psi_k} > 0. \tag{76}$$

Due to Brouwer Theorem, there exists the fixed point $p^0(\varepsilon)$ such that

$$\frac{f_k^\varepsilon(p^0(\varepsilon))}{\sum_{i=1}^n f_i^\varepsilon(p^0(\varepsilon))} = p_k^0(\varepsilon), \quad k = \overline{1, n}. \tag{77}$$

Multiplying the left and right hand sides of (77) by ψ_k and summing over k

we obtain

$$\frac{\sum_{k=1}^n \psi_k f_k^\varepsilon(p^0(\varepsilon))}{\sum_{i=1}^n f_i^\varepsilon(p^0(\varepsilon))} = \sum_{k=1}^n \psi_k p_k^0(\varepsilon), \quad k = \overline{1, n}. \tag{78}$$

But

$$\sum_{k=1}^n \psi_k f_k^\varepsilon(p^0(\varepsilon)) = \sum_{k=1}^n \psi_k p_k^0(\varepsilon), \quad k = \overline{1, n}, \tag{79}$$

therefore $\sum_{i=1}^n f_i^\varepsilon(p^0(\varepsilon)) = 1$. The equalities (77) give the equalities

$$\frac{1}{\psi_k} \sum_{i=1}^l \frac{p_k^0(\varepsilon) c_{ki} + \varepsilon}{\langle C_i, p^0(\varepsilon) \rangle + n\varepsilon} \langle b_i, p^0(\varepsilon) \rangle = p_k^0(\varepsilon), \quad k = \overline{1, n}. \tag{80}$$

From the equalities (80) it follows that

$$p_k^0(\varepsilon) \geq \frac{\varepsilon}{\psi_k \left(\max_{k,i} c_{ki} + n\varepsilon \right)} \min_k \psi_k > 0, \quad k = \overline{1, n}. \tag{81}$$

The equalities (80) lead to the inequalities

$$\sum_{i=1}^l \frac{c_{ki}}{\langle C_i, p_k^0(\varepsilon) \rangle + n\varepsilon} \langle b_i, p_k^0(\varepsilon) \rangle \leq \psi_k, \quad k = \overline{1, n}. \tag{82}$$

Using the compactness arguments relative to the sequence of vectors $p^0(\varepsilon)$, belonging to the set P , when ε tends to zero, we obtain the existence of the vector $p_0 \in P$ such that

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} \leq \psi_k, \quad k = \overline{1, n}. \tag{83}$$

Theorem 6 is proved.

Definition 10. Let $C_i \in R_+^n, i = \overline{1, l}$, be a set of demand vectors and let $b_i \in R_+^n, i = \overline{1, l}$, be a set of supply vectors. We say the structure of supply agrees with the structure of demand in the weak sense if for the matrix B the representation $B = CB_1$ is true, where the matrix B consists of the vectors $b_i \in R_+^n, i = \overline{1, l}$, as columns, and the matrix C is composed from the vectors $C_i \in R_+^n, i = \overline{1, l}$, as columns, and B_1 is a square matrix satisfying the conditions

$$\sum_{s=1}^l b_{is}^1 \geq 0, \quad i = \overline{1, l}, \quad B_1 = |b_{is}^1|_{i,s=1}^l. \tag{84}$$

Definition 11. Let $C_i \in R_+^n, i = \overline{1, l}$, be a set of demand vectors and let $b_i \in R_+^n, i = \overline{1, l}$, be a set of supply vectors. We say the structure of supply agrees with the structure of demand in the weak sense of the rank $|I|$ if there exists a subset $I \subseteq N$ such that for the matrix B^I the representation $B^I = C^I B_1^I$ is true, where the matrix B^I consists of the vectors $b_i^I \in R_+^{|I|}, i = \overline{1, l}$, as columns, and the matrix C^I is composed from the vectors $C_i^I \in R_+^n, i = \overline{1, l}$, as columns, and B_1^I is a square matrix, satisfying the conditions

$$\sum_{s=1}^l b_{is}^{1,l} \geq 0, \quad i = \overline{1, l}, \quad B_1^l = \left| b_{is}^{1,l} \right|_{i,s=1}^l, \quad (85)$$

where $b_i^l = \{b_{ki}\}_{k \in I}$, $C_i^l = \{c_{ki}\}_{k \in I}$ and, moreover, the inequalities

$$\sum_{i=1}^l c_{ki} y_i^l < \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \quad y_i^l = \sum_{s=1}^l b_{is}^{1,l}.$$

are valid.

Theorem 7. Let the structure of supply agree with structure of demand in the weak sense with the supply vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, and the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, and let $\sum_{s=1}^n c_{si} > 0, i = \overline{1, l}, \sum_{i=1}^l c_{si} > 0, s = \overline{1, n}$. The necessary and sufficient conditions for the solution existence to the set of equations

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p \rangle}{\langle C_i, p \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \quad (86)$$

is belonging of the vector $D = \{d_i\}_{i=1}^l$ to the polyhedral cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, where $B = CB_1$, $B_1 = \left| b_{ki}^1 \right|_{k,i=1}^l$ is a square matrix, the vector $D = \{d_i\}_{i=1}^l$ is a strictly positive solution to the set of equations

$$\sum_{k=1}^l b_{ki}^1 d_k = y_i d_i, \quad y_i = \sum_{k=1}^l b_{ik}^1 \geq 0, \quad i = \overline{1, l}. \quad (87)$$

Proof. The proof of Theorem 7 is the same as Theorem 4.

Next Theorem 8 is a reformulation of Theorem 7 which gives the possibility to construct the set of supply vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, after the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$ under which there exists the equilibrium price vector clearing the market.

Theorem 8. Let the matrix $F = \|f_{si}\|_{s,i=1}^l$ be such that the strictly positive vector $D = \{d_i\}_{i=1}^l$ belonging to the polyhedral cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, satisfies the set of equations

$$\sum_{k=1}^l f_{ki} d_k = y_i d_i, \quad y_i = \sum_{k=1}^l f_{ik}, \quad i = \overline{1, l}. \quad (88)$$

Suppose the inequalities

$$b_i = a \sum_{s=1}^l C_s (f_{si} - \delta_{si} y_i) + C_i \geq 0, \quad i = \overline{1, l}, \quad a \in R^1, \quad a \neq 0, \quad (89)$$

are true for a certain $a \in R^1$. Then for the supply vectors $b_i \in R_+^n, i = \overline{1, l}$, and the demand vectors $\{C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, the necessary and sufficient conditions for the existence of an equilibrium price vector $p_0 = \{p_i^0\}_{i=1}^n$ satisfying the set of equations

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \quad (90)$$

are the fulfillment of the equalities (88) relative to the vector D .

Proof. The sufficiency. Suppose the vector D belongs to the polyhedral cone

created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, and satisfies the set of equations (88). From the representation (89) for the vectors $b_i, i = \overline{1, l}$, it follows that $B = CB_1$, where $B_1 = a(F - yE) + E$, or $b_{ij}^1 = a(f_{ij} - \delta_{ij}y_i) + \delta_{ij}$. Therefore,

$\sum_{j=1}^l b_{ij}^1 = 1, i = \overline{1, l}$. Further, the equalities

$$\sum_{k=1}^l b_{ki}^1 d_k = d_i, \quad i = \overline{1, l}, \tag{91}$$

are true, since the equalities (88) take place. As $d_i = \langle p_0, C_i \rangle$ we obtain

$$\langle p_0, C_i \rangle = \left\langle p_0, \sum_{k=1}^l C_k b_{ki}^1 \right\rangle = \langle p_0, b_i \rangle, \quad i = \overline{1, l}. \tag{92}$$

The last means that (90) takes place.

The necessity. Let there exists nonnegative solution p_0 to the set of equations (90). From the equality

$$\sum_{i=1}^l (b_i - C_i) = a \sum_{s=1}^l C_s \left(\sum_{i=1}^l f_{si} - \sum_{i=1}^l \delta_{si} y_i \right) = 0 \tag{93}$$

and the equalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l c_{ki}, \quad k = \overline{1, n}, \tag{94}$$

as in the proof of Theorem 4, we obtain

$$\frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = 1, \quad i = \overline{1, l}. \tag{95}$$

From here we have

$$0 = \frac{\langle b_i - C_i, p_0 \rangle}{a} = \sum_{s=1}^l \langle p_0, C_s \rangle (f_{si} - \delta_{si} y_i) = \left(\sum_{s=1}^l d_s f_{si} - d_i y_i \right) = 0, \quad i = \overline{1, l}, \tag{96}$$

where $d_i = \langle p_0, C_i \rangle, i = \overline{1, l}$. So, the vector $D = \{d_i\}_{i=1}^l$ belongs to the polyhedral cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, and satisfies the set of equations (88). Theorem 8 is proved.

Note 1. If the vectors $C_i, i = \overline{1, l}$, are strictly positive, then for every matrix F there exists a real number $a \in R^1, a \neq 0$, such that the inequalities (89) are valid. The last means that having the set of demand vectors $C_i, i = \overline{1, l}$, we can construct the set of supply vectors $b_i, i = \overline{1, l}$, under which the equilibrium price vector exists in correspondence with Theorem 8.

4. Recession Phenomenon Description

In this section, we present the fresh look on the description of recession phenomenon proposed by one of the authors in (Gonchar, 2013; Gonchar, 2019; Gonchar, 2020).

Definition 12. Suppose the economy system is described by demand vectors $C_i \in R_+^n, i = \overline{1, l}$, and supply vectors $b_i \in R_+^n, i = \overline{1, l}$. We say the state of economy equilibrium has multiplicity of degeneracy $|N| - |I|$, if the equilibrium price vector $p_0 \in R_+^n$ satisfies the set of equalities and inequalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k \in I,$$

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} < \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \tag{97}$$

and the solution of these set of equalities and inequalities has multiplicity of degeneracy $|N|-|I|$. We say the economy system is in the state of recession if the multiplicity of degeneracy $|N|-|I|$ of the equilibrium state is sufficiently high to violate the stability of the national currency (Gonchar, 2015a; Gonchar, 2015b).

Theorem 9. Suppose $C_i \in R_+^n, i = \overline{1, l}$, is a set of demand vectors and let $b_i \in R_+^n, i = \overline{1, l}$, be a set of supply vectors in the considered economy system. Let the structure of demand agree with the structure of supply in the weak sens of the rank $|I|$ in correspondence with the Definition 11, where $I \subseteq N$. If the vectors $b_i^l, C_i^l \in R_+^{|I|}, i = \overline{1, l}$, satisfy the conditions of Theorems 5, 7 with $n = |I|$ then the multiplicity of degeneracy of the equilibrium state for the vectors of demand $C_i \in R_+^n, i = \overline{1, l}$, vectos of supply $b_i^0 = \{b_{ki}^0\}_{k=1}^n, b_{ki}^0 = b_{ki}, k \in I, b_{ki}^0 = y_i c_{ki}, k \in N \setminus I$. is not less than $|N|-|I|$.

Proof. Since the conditions of Theorems 5, 7 with $n = |I|$ are true, then there exists the vector $p_0 \in R_+^{|I|}$ such that

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i^l, p_0 \rangle}{\langle C_i^l, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k \in I,$$

$$\sum_{i=1}^l c_{ki} y_i < \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \tag{98}$$

where $y_i = \sum_{k \in I} b_{ki}^1, B^1 = \{b_{ki}^1\}_{k \in I, i = \overline{1, l}}$ and $B^l = C^l B^1$. The columns of matrix B^l are the vectors b_i^l and the columns of matrix C^l are the vectors C_i^l . But

$$\frac{\langle b_i^l, p_0 \rangle}{\langle C_i^l, p_0 \rangle} = y_i, \quad i = \overline{1, l}, \tag{99}$$

due to Theorem 7. This means that the inequalities

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i^l, p_0 \rangle}{\langle C_i^l, p_0 \rangle} = \sum_{i=1}^l b_{ki}, \quad k \in I,$$

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i^l, p_0 \rangle}{\langle C_i^l, p_0 \rangle} < \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \tag{100}$$

are true. From the equalities (99) we obtain

$$\langle b_i^l, p_0 \rangle = \sum_{k \in I} b_{ki} p_k^0, \quad \langle C_i^l, p_0 \rangle = \sum_{k \in I} c_{ki} p_k^0,$$

$$\sum_{k \in I} b_{ki} p_k^0 = y_i \sum_{k \in I} c_{ki} p_k^0. \tag{101}$$

Or,

$$\sum_{k \in I} b_{ki} p_k^0 + y_i \sum_{k \in N \setminus I} c_{ki} p_k^1 = y_i \left[\sum_{k \in I} c_{ki} p_k^0 + \sum_{k \in N \setminus I} c_{ki} p_k^1 \right], \tag{102}$$

where $p_k^1 > 0, k \in N \setminus I$ and are arbitrary ones. Let us introduce $p = \{p_i\}_{i=1}^n$ putting $p_i = p_i^0, i \in I$, $p_i = p_i^1, i \in N \setminus I$, $b_i^0 = \{b_{ki}^0\}_{k=1}^n$, $b_{ki}^0 = b_{ki}, k \in I$, $b_{ki}^0 = y_i c_{ki}, k \in N \setminus I$. Then the equalities (102) are written in the form

$$\langle b_i^0, p \rangle = y_i \langle C_i, p \rangle, \quad i = \overline{1, l}. \tag{103}$$

The last means that the set of equalities and inequalities

$$\begin{aligned} \sum_{i=1}^l c_{ki} \frac{\langle b_i^0, p \rangle}{\langle C_i, p \rangle} &= \sum_{i=1}^l b_{ki}, \quad k \in I, \\ \sum_{i=1}^l c_{ki} \frac{\langle b_i^0, p \rangle}{\langle C_i, p \rangle} &< \sum_{i=1}^l b_{ki}, \quad k \in N \setminus I, \end{aligned} \tag{104}$$

are true. The multiplicity of degeneracy of the equilibrium state is not less than $|N| - |I|$. Theorem 9 is proved.

Below we give the necessary and sufficient conditions for the existence of equilibrium state. Due to Theorem 10, we clarify the sense of multiplicity of degeneracy and recession phenomenon.

Theorem 10. Suppose $C_i \in R_+^n, i = \overline{1, l}$, is a set of demand vectors and let $b_i \in R_+^n, i = \overline{1, l}$, be a set of supply vectors in the considered economy system. Let $\sum_{k=1}^n c_{ki} > 0, \sum_{i=1}^l c_{ki} > 0$. The necessary and sufficient conditions for the existence of the equilibrium price vector p_0 such that

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} \leq \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}, \tag{105}$$

is an existence of nonzero vector $y = \{y_i\}_{i=1}^l, y_i \geq 0, i = \overline{1, l}$, such that

$$\begin{aligned} \bar{\psi} \leq \psi, \quad \bar{\psi} = \{\bar{\psi}_k\}_{k=1}^n, \quad \psi = \{\psi_k\}_{k=1}^n, \quad \bar{\psi}_k = \sum_{i=1}^l c_{ki} y_i, \quad \psi_k = \sum_{i=1}^l b_{ki}, \\ \langle \bar{\psi}, p_0 \rangle = \langle \psi, p_0 \rangle, \quad \langle C_i, p_0 \rangle > 0, \quad i = \overline{1, l}, \end{aligned} \tag{106}$$

and for the vectors $b_i, i = \overline{1, l}$, the representation

$$\begin{aligned} b_i = \bar{b}_i + d_i, \quad \bar{b}_i = y_i \frac{\langle C_i, p_0 \rangle}{\langle \bar{\psi}, p_0 \rangle} \psi, \quad \langle p_0, d_i \rangle = 0, \quad i = \overline{1, l}, \\ \sum_{i=1}^l d_i = 0, \end{aligned} \tag{107}$$

is true.

Proof. Necessity. Let p_0 be an equilibrium price vector, then denoting

$$\bar{\psi} = \sum_{i=1}^l y_i C_i, \quad y_i = \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle}, \quad \bar{b}_i = y_i \frac{\langle C_i, p_0 \rangle}{\langle \bar{\psi}, p_0 \rangle} \psi, \quad i = \overline{1, l}, \tag{108}$$

we have

$$\langle p_0, d_i \rangle = \langle b_i, p_0 \rangle - \langle \bar{b}_i, p_0 \rangle = \langle b_i, p_0 \rangle - y_i \frac{\langle C_i, p_0 \rangle}{\langle \bar{\psi}, p_0 \rangle} \langle \psi, p_0 \rangle. \tag{109}$$

But

$$\langle \bar{\psi}, p_0 \rangle = \sum_{i=1}^l y_i \langle C_i, p_0 \rangle = \sum_{i=1}^l \langle b_i, p_0 \rangle = \langle \psi, p_0 \rangle. \tag{110}$$

Therefore, $\langle p_0, d_i \rangle = 0, i = \overline{1, l}$. At last,

$$\sum_{i=1}^l d_i = \sum_{i=1}^l b_i - \frac{\left\langle \sum_{i=1}^l y_i C_i, p_0 \right\rangle}{\langle \bar{\psi}, p_0 \rangle} \psi = 0, \quad \bar{\psi} \leq \psi. \tag{111}$$

Sufficiency. Let there exist nonzero nonnegative vectors $y = \{y_i\}_{i=1}^l, y_i \geq 0, i = \overline{1, l}$, and $p_0 \in R_n^+$ such that $\bar{\psi} = \sum_{i=1}^l y_i C_i$ satisfies the conditions

$$\bar{\psi} \leq \psi, \quad \langle \bar{\psi}, p_0 \rangle = \langle \psi, p_0 \rangle, \quad \langle C_i, p_0 \rangle > 0 \quad i = \overline{1, l}, \tag{112}$$

and for b_i the representation

$$b_i = \bar{b}_i + d_i = y_i \frac{\langle C_i, p_0 \rangle}{\langle \bar{\psi}, p_0 \rangle} \psi + d_i, \quad i = \overline{1, l}, \tag{113}$$

is true, where $\langle d_i, p_0 \rangle = 0, \sum_{i=1}^l d_i = 0$. Then $\langle b_i, p_0 \rangle = y_i \langle C_i, p_0 \rangle, i = \overline{1, l}$, or

$y_i = \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle}, i = \overline{1, l}$. But $\bar{\psi} \leq \psi$, or

$$\sum_{i=1}^l y_i C_i = \sum_{i=1}^l C_i \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} \leq \psi. \tag{114}$$

Theorem 10 is proved.

If the inequalities (105) are true, then there exists a nonempty subset $I \subseteq N$, where $N = \{1, 2, \dots, n\}$, such that for the index $k \in I$ the inequalities (105) become the equalities. If $I = N$ we say the complete clearing of markets takes place. If $I \subset N$ we say about partial clearing of markets. We call the vector $\bar{\psi}$ the vector of real consumption. In case of partial clearing of markets, the vector of real consumption $\bar{\psi}$ does not coincide with the vector of supply ψ . As a result, the i -th owner of supply vector $b_i = \{b_{ki}\}_{k=1}^n$ gets income $\langle b_i^0, p \rangle$, where $b_i^0 = \{b_{ki}^0\}_{k=1}^n, b_{ki}^0 = b_{ki}, k \in I, b_{ki}^0 = y_i c_{ki}, k \in N \setminus I$, and vector p solves the set of equations

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i^0, p \rangle}{\langle C_i, p \rangle} = \bar{\psi}_k, \quad k = \overline{1, n}, \tag{115}$$

$$\bar{\psi}_k = \sum_{i=1}^l y_i c_{ki}, \quad k = \overline{1, n}, \tag{116}$$

the nonzero nonnegative vector $y = \{y_i\}_{i=1}^l$ solves the set of inequalities

$$\sum_{i=1}^l c_{ki} y_i = \psi_k, \quad k \in I, \tag{117}$$

$$\sum_{i=1}^l c_{ki} y_i < \psi_k, \quad k \in N \setminus I. \tag{118}$$

The price vector p solving the set of equations (115) has the following structure $p = \{p_i\}_{i=1}^n$, $p_i = p_i^0, i \in I$, $p_i = p_i^1, i \in N \setminus I$, where $p_i^1, i \in N \setminus I$ are arbitrary nonnegative real numbers due to Theorem 9.

So, any vector p having the above structure clears the market with the demand vectors $C_i \in R_+^n, i = \overline{1, l}$, and supply vectors $b_i^0 \in R_+^n, i = \overline{1, l}$. But, the set of Equation (115) does not determine uniquely the prices of goods that belong to the set $N \setminus I$ in spite of that the demand for these goods is non zero. The cause is that the consumer needs are completely satisfied on these goods. To determine the prices for goods from the set $N \setminus I$ it needs to remove the degeneracy in the set of Equation (115). For this, it is need to add the infinitely small term removing the degeneracy. This term should take into account the technologies of these goods production and fiscal policy. For example, if the map

$T_i(p) = \{t_{ki}(p)\}_{k=1}^n, t_{ki}(p) = 0, k \in I, i = \overline{1, l}$ takes into account the technologies of production of goods from the set $N \setminus I$ and fiscal policy, then the set of equations

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i^0, p \rangle}{\langle C_i + \varepsilon T_i(p), p \rangle} = \bar{\psi}_k, \quad k = \overline{1, n}, \tag{119}$$

$$\bar{\psi}_k = \sum_{i=1}^l y_i c_{ki}, \quad k = \overline{1, n}, \tag{120}$$

determines the prices for goods from the set $N \setminus I$ under condition that set of equations

$$\langle T_i(p), p \rangle = 0, \quad i = \overline{1, l}, \tag{121}$$

determines the vector $p_1 = \{p_i^1\}_{i \in N \setminus I}$, solving the set of Equation (121). Here $\varepsilon > 0$ and it is very small. Tending $\varepsilon > 0$ to zero, we obtain needed solution. It may happen that the specified procedure is not applicable. In this case, the prices for these goods will be determined by agreements.

In the case, as $I \subset N$, the price vector p solving the set of Equation (115) and taking into account the procedure for determining the ambiguous part of the vector components, stated above, will be called the generalized equilibrium price vector. So, real degeneracy of solutions has the set of Equation (115) and the generalized equilibrium price vector solves the set of Equation (115). The quantity of goods $\psi_k - \bar{\psi}_k, k \in N \setminus I$, does not find a consumer. To characterize this we introduce the parameter of recession level

$$R = \frac{\langle \psi - \bar{\psi}, p \rangle}{\langle \psi, p \rangle},$$

where p is a generalized equilibrium price vector solving the set of Equation (115).

5. Existence of the Ideal Equilibrium State in the International Trade

In this section, we give an application of the result obtained in the previous sec-

tions to the problem of existence of the ideal equilibrium state. The model of international trade is characterized by the supply vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n$ and demand vectors $C_i = \{c_{ki}\}_{k=1}^n \in R_+^n$ satisfying the conditions

$$b_i - C_i = f_i, \quad i = \overline{1, l}, \quad \sum_{i=1}^l f_i = 0. \tag{122}$$

Definition 13. We say the international trade is in the ideal equilibrium state if there exists nonnegative price vector $p_0 = \{p_k^0\}_{k=1}^n \in R_+^n$ such that

$$\langle p_0, b_i - C_i \rangle = 0, \quad \langle p_0, C_i \rangle > 0, \quad i = \overline{1, l}. \tag{123}$$

In the next Theorem we give the necessary and sufficient conditions under which in the international trade there exists the ideal equilibrium state.

Theorem 11. Suppose the supply vectors $b_i = \{b_{ki}\}_{k=1}^n \in R_+^n$ and the demand vectors $C_i = \{c_{ki}\}_{k=1}^n \in R_+^n$ satisfy the conditions (122). If for the matrix B the representation

$$B = CB_1 \tag{124}$$

is true, where $B = \|b_{ki}\|_{k=1, i=1}^{n, l}$, $C = \|c_{ki}\|_{k=1, i=1}^{n, l}$, $B_1 = \|b_{ki}^1\|_{k=1, i=1}^{l, l}$, then the necessary and sufficient conditions for the existence of the ideal equilibrium state is the existence of strictly positive solution $D = \{d_i\}_{i=1}^l$ to the set of equations

$$\sum_{k=1}^l d_k b_{ki}^1 = d_i, \quad i = \overline{1, l}, \tag{125}$$

which belongs to the cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$.

Proof. Necessity. Let the ideal equilibrium state exists, then from the representation (124) we have $b_i = \sum_{s=1}^l C_s b_{si}^1$ or $\langle p_0, b_i \rangle = \sum_{s=1}^l \langle p_0, C_s \rangle b_{si}^1$. Substituting $\langle p_0, b_i \rangle$ into the equalities

$$\langle p_0, b_i \rangle = \langle p_0, C_i \rangle, \quad \langle p_0, C_i \rangle > 0, \quad i = \overline{1, l}, \tag{126}$$

we have

$$\sum_{s=1}^l \langle p_0, C_s \rangle b_{si}^1 = \langle p_0, C_i \rangle, \quad \langle p_0, C_i \rangle > 0, \quad i = \overline{1, l}. \tag{127}$$

Denoting $d_i = \langle p_0, C_i \rangle, i = \overline{1, l}$, we prove the necessity.

Sufficiency. If the conditions of Theorem 11 are true, then from (125) and the fact that vector $D = \{d_i\}_{i=1}^l$ belongs to the cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$, we obtain the existence of nonnegative vector p_0 such that $d_i = \langle p_0, C_i \rangle, i = \overline{1, l}$, and

$$\sum_{s=1}^l \langle p_0, C_s \rangle b_{si}^1 = \langle p_0, C_i \rangle, \quad i = \overline{1, l}, \tag{128}$$

or

$$\langle p_0, b_i \rangle = \langle p_0, C_i \rangle, \quad i = \overline{1, l}. \tag{129}$$

It is evident that

$$\sum_{i=1}^l c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p \rangle} = \sum_{i=1}^l b_{ki}, \quad k = \overline{1, n}. \tag{130}$$

Theorem 11 is proved.

Below we give one method to construct the set of supply vectors having the set of demand vector under which the ideal equilibrium exists.

Theorem 12. Let $C_i = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}$, be a set of demand vectors and let strictly positive vector $D = \{d_i\}_{i=1}^l \in R_+^l$ belongs to the nonnegative cone created by vectors $C_k^T = \{c_{ki}\}_{i=1}^l, k = \overline{1, n}$. If the set of vectors $f_i^1 = \{f_{si}^1\}_{s=1}^l, i = \overline{1, l}$, satisfies conditions

$$\langle d, f_i^1 \rangle = 0, \sum_{s=1}^l f_{si}^1 = 0, \quad i = \overline{1, l}, \tag{131}$$

and the set of vectors $f_i = \sum_{s=1}^l C_s f_{si}^1, i = \overline{1, l}$, is such that $f_i + C_i \geq 0, i = \overline{1, l}$, then for the set of supply vectors $b_i = f_i + C_i, i = \overline{1, l}$, and demand vectors $C_i, i = \overline{1, l}$, there exists an ideal equilibrium.

Proof. Due to conditions of the Theorem, the vector $D = \{d_i\}_{i=1}^l \in R_+^l$ satisfies the set of equations

$$\sum_{s=1}^l f_{si}^1 d_s + d_i = d_i, \quad i = \overline{1, l}. \tag{132}$$

As $d_i = \sum_{k=1}^n c_{ki} p_k^0$ we have

$$\begin{aligned} \langle b_i, p_0 \rangle &= \sum_{k=1}^n b_{ki} p_k^0 = \sum_{s=1}^l f_{si}^1 \sum_{k=1}^n c_{ks} p_k^0 + \sum_{k=1}^n c_{ki} p_k^0 \\ &= \sum_{k=1}^n c_{ki} p_k^0 = \langle C_i, p_0 \rangle, \quad i = \overline{1, l}. \end{aligned} \tag{133}$$

Theorem 12 is proved.

6. International Trade of the G20 Countries.

Below we give an algorithm to study the above stated problem based on Theorems proved above. For the data given we must to verify:

- 1) whether the price vector in the considered economy model is equilibrium one, that is, whether the set of inequalities (5) are valid;
- 2) if it is so, then it needs to establish the degeneracy degree of the equilibrium state. For this it is necessary to find the set I from the Definition 11.

If the considered economy system is not at the equilibrium state, then:

- 1) to verify whether the vector $\sum_{i=1}^l b_i^I$ belongs to the polyhedral cone created by the demand vectors $\{C_i^I = \{c_{ki}\}_{k=1}^n \in R_+^n, i = \overline{1, l}\}$, of the rank I ;
- 2) find out whether there is a consistency of the supply structure with the structure of the choice of rank $|I|$ in correspondence with the Definition 11.

If it is so, to find an equilibrium price vector in correspondence with Theorems 5, 7, 9 and then establish the degeneracy degree of the equilibrium state.

The construction of the matrix B_1^I should be carried out in accordance with

Lemma 3.

In this chapter, we investigate the trade of 19 countries of G20 between themselves. It is convenient to number them as: 1. Argentina, 2. Australia, 3. Brazil, 4. Canada, 5. China, 6. Germany, 7. France, 8. the United Kingdom, 9. Indonesia, 10. India, 11. Italy, 12. Japan, 13. Republic of Korea, 14. Mexico, 15. Russia, 16. Saudi Arabia, 17. Turkey, 18. The United States, 19. South Africa.

These countries trade goods among themselves: 1. Animal, 2. Vegetable, 3. FoodProd, 4. Minerals, 5. Fuels, 6. Chemicals, 7. PlastiRub, 8. HidesSkin, 9. Wood, 10. TextCloth, 11. Footwear, 12. StoneGlas, 13. Metals, 14. MachElec, 15. Transport, 16. Miscellan.

We study the dynamics of the exchange by these goods from 2016 to 2019 (<http://wits.worldbank.org>, <https://data.oecd.org>). The first question that arises is whether the international trade between these countries was in the state of equilibrium. Is the equilibrium state ideal or not. As the statistical data are given in the cost form, we introduce the relative price vector $p_1 = \{p_1^1, \dots, p_n^1\}$, to provide the equilibrium in the exchange model in the form

$$\sum_{k=1}^M c_{sk} \frac{D_k(p_1)}{\sum_{s=1}^n p_s^1 c_{sk}} \leq \psi_s, \quad s = \overline{1, n}, \tag{134}$$

where $\psi_s = \sum_{k=1}^M b_{sk}$, $D_k(p_1) = \sum_{s=1}^n p_s^1 b_{sk}$, M is the number of countries trading among themselves.

The equilibrium state is such that the complete clearing of market takes place in the set of goods I , where the set I consists of three goods: 2016: $I = \{4, 5, 14\}$; 2017: $I = \{4, 5, 14\}$; 2018: $I = \{4, 5, 14\}$; 2019: $I = \{5, 6, 14\}$. The vector $y = \{y_i\}_{i=1}^M$ satisfies the set of inequalities

$$\begin{aligned} \sum_{i=1}^M c_{ki} y_i &= \sum_{i=1}^M b_{ki}, \quad k \in I, \\ \sum_{i=1}^M c_{ki} y_i &< \sum_{i=1}^M b_{ki}, \quad k \in N \setminus I, \end{aligned} \tag{135}$$

Due to Theorem 10, the vector $\bar{y} = \sum_{i=1}^M C_i y_i = \{\bar{y}_k\}_{k=1}^n$ is a vector of real consumption. Equilibrium price vector clearing the market satisfies the set of equations

$$\sum_{i=1}^M c_{ki} \frac{\langle b_i, p_0 \rangle}{\langle C_i, p_0 \rangle} = \sum_{i=1}^M b_{ki}, \quad k \in I. \tag{136}$$

The same vector p_0 also solves the set of equations

$$\sum_{i=1}^M c_{ki} \frac{\langle b_i^0, p_0 \rangle}{\langle C_i, p_0 \rangle} = \bar{y}_k, \quad k = \overline{1, n}, \tag{137}$$

where $b_i^0 = \{b_{ki}^0\}_{k=1}^n$, $b_{ki}^0 = b_{ki}$, $k \in I$, $b_{ki}^0 = c_{ki} y_i$, $k \in N \setminus I$. But the set of equations (137) is degenerate whose degeneracy multiplicity is $|N \setminus I| = |N| - |I|$. The general solution to the set of equations (137) is given by the vector $p_1 = \{p_k^1\}_{k=1}^n$, where $p_k^1 = p_k^0$, $k \in I$, $p_k^1 = p_k$, $k \in N \setminus I$. The components p_k are arbitrary

ones. To remove the arbitrariness of components, it needs to use the procedure after Theorem 10. It is reasonable to put them equal to $p_k = 1, k \in N \setminus I$ that corresponds to the existing current prices in the real trade. As the vector p_0 , solving the set of Equation (136), is determined up to the nonnegative factor $\tau > 0$, it is natural to choose it from the equality $\tau \sum_{k \in I} \psi_k p_k^0 = \sum_{k \in I} \psi_k$, where $\sum_{k \in I} p_k^0 = 1$.

If $I \subset N$, the price vector p_1 solving the set of Equation (137) and taking into account the procedure for determining the ambiguous part of the vector components stated above, will be called the generalized relative equilibrium price vector.

Then the generalized relative equilibrium price vector p_1 takes the form

$$p_1 = \{p_k^1\}_{k=1}^n, \quad p_k^1 = \frac{\sum_{k \in I} \psi_k}{\sum_{k \in I} \psi_k p_k^0} p_k^0, \quad k \in I, \quad p_k^1 = 1, \quad k \in N \setminus I. \quad (138)$$

We introduce the parameter of recession level in the state of generalized relative equilibrium describing by the vector p_1

$$R = \frac{\langle \psi, p_1 \rangle - \langle \bar{\psi}, p_1 \rangle}{\langle \psi, p_1 \rangle} = \frac{\sum_{k \in N \setminus I} (\psi_k - \bar{\psi}_k)}{\sum_{k \in N \setminus I} \psi_k}. \quad (139)$$

Parameter R is the part of goods in the cost form belonging to the set $N \setminus I$ that did not find a consumer.

1. The trade balance of countries in the current prices of 2016, (<http://wits.worldbank.org>, <http://data.oecd.org>):

$$t = \{-12506555.819, -4247750.693999994, 13954511.70100001, -20348178.44799999, 234632443.4900001, 187353444.641, -42234244.03300001, -134066213.167, 1038379.406000004, -75864607.22499998, 26710247.28899999, -5650267.683999983, 17618008.95899998, 14093415.271, 445449.7089999858, -75740653.295, -63314714.982, -848068112.3409998, -14699378.194\}.$$

2. The excess demand in the current prices of 2016 is given by the vector:

$$d = \{-3452090.64224796, 5053384.830972463, -13865582.33848479, 37104759.43572, 103686636.7895926, 29267436.70237225, 3408096.954677969, -110646.2769374326, -5134521.860014513, -41371405.52652189, -7211189.757797152, 9070580.610088348, -11867420.01411062, 127649684.1481535, -194106045.5154499, -38121677.54001272\}.$$

3. The equilibrium price vector of 2016:

$$p_0 = \{0, 0, 0, 0.1044610760732605, 0.6941595181930178, 0, 0, 0, 0, 0, 0, 0, 0.2013794057337217, 0, 0\}$$

4. The excess demand under the equilibrium price vector p_0 is given by the

vector

$$d_1 = \{-17338056.49249968, -20987586.54164383, -16522435.43125352, 0, \\ 0, -34360377.43914306, -21086981.48146927, -7270801.552513503, \\ -22669600.82384092, -68045593.27854195, -11011121.39014155, \\ -12525583.08531252, -58669348.23877645, 0, \\ -236527533.3221221, -89362799.10312033\}.$$

5. The vector y of satisfactions of consumer needs in the equilibrium state:

$$y = \{0.2222900803965602, 1.852564387617036, 0.6951781545307275, \\ 1.569904525752839, 0.9042890505700601, 0.854855714309132, \\ 0.4903516652339595, 0.7280072977426086, 1.72562923938914, \\ 0.2315487084638758, 0.7424012606143664, 0.4959765599064544, \\ 0.7371461292845501, 0.8980955171434071, 4.053300246539547, \\ 0.007744822486138806, 0.1589930101036914, 0.4102487133061255, \\ 0.568621570219266\}.$$

6. The generalized relative equilibrium price vector:

$$p_1 = \{1, 1, 1, 0.3739226981788888, 2.484772412522871, 1, 1, \\ 1, 1, 1, 1, 1, 0.7208458267920281, 1, 1\}.$$

7. Parameter of recession level

$$R = 0.1154218242887561.$$

1. The trade balance of countries in the current prices of 2017,

(<http://wits.worldbank.org>, <http://data.oecd.org>).

$$t = \{-20955418.066, 17548868.49200001, 29779980.45400002, \\ -17191807.28199998, 197522723.5610001, 193794019.425, \\ 3410608.513000003, -57884058.41400002, -123443674.423, \\ -103719934.015, 28451792.65899998, -9783178.784999998, \\ 22622141.578, 18078439.77100002, 4190013.062999986, \\ -73534434.508, -70775689.76800001, -897520360.7180001, \\ -14067155.156\}.$$

2. The excess demand in the current prices of 2017 is given by the vector:

$$d = \{-4669571.905598968, 5938682.461838603, -12829058.6085822, \\ 47555994.54584152, 130380790.441524, 35283593.25050235, \\ -4503508.195131928, 660865.7041778564, -630431.624398760, 5, \\ -40146735.08512312, -6788807.834025413, 13700316.57170004, \\ -15574324.71828502, 117612184.4527521, -201369888.3900037, \\ -64620101.06718785\}.$$

3. The equilibrium price vector of 2017:

$$p_0 = \{0, 0, 0, 0.2080298367573019, 0.5805482232484332, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0.2114219399942648, 0, 0\}.$$

4. The excess demand under the equilibrium price vector p_0 is given by the vec-

tor:

$$d_1 = \{-18946830.99392197, -20704190.8298308, -13036491.80862296, 0, \\ 0, -25711581.8240062, -22420132.4656997, -6497898.935088903, \\ -21191683.87103112, -61743940.53373307, -8385017.411344729, \\ -9173198.185643882, -61524718.76785821, \\ 0, -199861118.9025204, -102038218.8353313\}.$$

5. The vector y of satisfactions of consumer needs in the equilibrium state:

$$y = \{0.1971674346613074, 2.31143911016814, 0.9624626235227182, \\ 1.664488142616343, 0.8186320744288279, 0.8825263191432133, \\ 0.4625738594930592, 0.7376086699759342, 1.54298423732524, \\ 0.2177926319599547, 0.7719377951394272, 0.4991483320000634, \\ 0.7244317108572671, 0.8871842719084082, 3.470531875785658, \\ 0.01497017829432475, 0.1826267000617652, 0.4562457482720676, \\ 0.7725004624585035\}.$$

6. The generalized relative equilibrium price vector:

$$p_1 = \{1, 1, 1, 0.7287563457413587, 2.033738084382344, 1, 1, \\ 1, 1, 1, 1, 1, 0.7406393371327837, 1, 1\}.$$

7. Parameter of recession level:

$$R = 0.0964662047894453.$$

1. The trade balance of countries in the current prices of 2018, (<http://wits.worldbank.org>, <http://data.oecd.org>):

$$t = \{-16619022.138, 33702375.20799999, 25476930.67700002, \\ -15534246.11, 183021043.25999999, 194451460.7840001, \\ -53541585.22699998, -115126675.217, -10330596.57900002, \\ -166023583.307, 13228352.22499998, -28850239.952, \\ 4182609.99199995, 15788055.026, 5732354.11399999, \\ -65674732.445, -62924030.961, -975304847.018, -15796221.222\}$$

says that the ideal equilibrium price vector does not exist, since it exists when the trade balance of all countries equals zero.

2. The excess demand in the current prices of 2018 is given by the vector:

$$d = \{-5108373.236691177, 6640493.985885441, -12372715.70704013, \\ 45915848.29644319, 177134202.1886947, 41909216.3684094, \\ 390269.0902559757, -1942678.578911416, -5130596.317575723, \\ -38150309.19288424, -5560369.386887923, 6332338.063659102, \\ -13219120.68747276, 95712059.52110744, -202539445.4070083, \\ -90010818.9999833\}.$$

that is, the international trade is not in equilibrium state.

3. The equilibrium price vector of 2018:

$$p_0 = \{0, 0, 0, 0.03468980236543773, 0.7011743250347967, \\ 0, 0, 0, 0, 0, 0, 0, 0, 0.2641358725997656, 0, 0\}.$$

4. The excess demand under the equilibrium price vector p_0 is given by the vector:

$$d_1 = \{-20803932.02869892, -9372710.878992409, -9572446.548494428, 0, \\ 0, -6798007.772230029, -12819527.43796217, -6667385.993533991, \\ -22942138.36813404, -53288233.10304466, -3022305.8901780999, \\ -20776706.39643601, -52938784.16174531, 0, \\ -186648289.6868939, -129391141.8017966\}.$$

5. The vector y of satisfactions of consumer needs in the equilibrium state:

$$y = \{0.3085288036658665, 1.625160256186186, 0.9256721379500972, \\ 1.606202190267202, 0.8170690865270578, 0.8701427304152256, \\ 0.4769101654177353, 0.6947864155990097, 1.304124130905562, \\ 0.1733728345636354, 0.6841503531281805, 0.4864315635378446, \\ 0.7233233925484885, 0.8066275354942358, 5.79862373855207, \\ 0.006619545109541444, 0.16927622820105, 0.4925811635185561, \\ 0.4014539385350591\}.$$

6. The generalized relative equilibrium price vector:

$$p_1 = \{1, 1, 1, 0.09850243179294567, 1.990999412424391, 1, 1, \\ 1, 1, 1, 1, 1, 0.7406393371327837, 1, 1\}.$$

7. Parameter of recession level:

$$R = 0.08312669207435437.$$

1. The trade balance of countries in the current prices of 2019, (<http://wits.worldbank.org>, <http://data.oecd.org>):

$$t = \{-1885469.061999998, 55585244.53699998, 20106245.682, \\ -10279496.70399998, 187185674.973, 181699392.9570001, \\ -50332793.06499998, -139286309.793, -10784050.41600002, \\ -96819395.65700001, 26402146.86099998, -32535350.34099999, \\ -15612723.72100003, 24595848.95100003, 45860719.60200001, \\ -74381901.15899999, -50654960.609, -928349383.317, \\ -13046523.298\}$$

says that the ideal equilibrium price vector does not exist, since it exists when the trade balance of all countries equals zero.

2. The excess demand in the current prices of 2019 is given by the vector:

$$d = \{-6141735.428897157, 6063797.88103047, -13255275.47280359, \\ 52251710.58643463, 186548372.3736179, 45306709.17129183, \\ 1369662.094842792, -2720038.084881201, -6618186.654009625, \\ -39113706.13981226, -6366938.219191968, -8778776.358987004, \\ -9391342.416060746, 93239906.79603934, \\ -196064032.3814814, -96330127.74713147\},$$

that is, the international trade is not in equilibrium state.

3. The equilibrium price vector of 2019:

$$p_0 = \{0, 0, 0, 0, 0, 0.6048113803874631, 0.2459536581698094, 0, 0, 0, 0, 0, 0, 0.1492349614427275, 0, 0\}.$$

4. The excess demand under the equilibrium price vector p_0 is given by the vector:

$$d_1 = \{-24543524.88224329, -16112896.08740601, -5305083.896452188, -23305120.01721276, 0, 0, -16612659.20681533, -6997221.787718497, -22035234.51953426, -46699015.99563199, -1412543.337677635, -24560412.55325073, -44736959.8803457, 0, -143445383.5869827, -115120438.4564396\}.$$

5. The vector y of satisfactions of consumer needs in the equilibrium state:

$$y = \{0.3292038600385361, 1.400681305151232, 0.7792293707347417, 1.738892732048001, 0.6448647099381927, 0.988660992070331, 0.6493940665028836, 0.8014038750911935, 1.209334709999378, 0.2866659176874876, 0.6741639849448267, 0.4683230951205403, 0.6340003424659207, 0.5098110072373863, 5.227627294922074, 0.444474001854552, 0.1806457674373209, 0.6128523042643617, 0.3712547326919547\}.$$

6. The generalized relative equilibrium price vector:

$$p_1 = \{1, 1, 1, 1, 2.383266705764021, 0.9691834242627001, 1, 1, 1, 1, 1, 1, 0.5880622066247785, 1, 1\}.$$

7. Parameter of recession level:

$$R = 0.07844142458650168.$$

During 2016-2019, trade relations between 19 countries of the G 20 were in non equilibrium states. The equilibrium state existed in each of the studied years. Each of these equilibrium states was far from ideal equilibrium. Each of the equilibrium states turned out to be highly degenerate. The degeneracy multiplicity was equal 13. An important concept of a generalized equilibrium price vector is introduced defined as a solution to a degenerate system of equations with real consumption. Using the concept of a generalized equilibrium vector, a recession level parameter is introduced. This parameter is a characteristic of the stability for the international exchange currency. The greater its value is, the weaker the international exchange currency is. Between 2016 and 2019, the international currency became more stable from 11.5 percent in 2016 to 7.8 percent in 2019.

7. Partial Analysis of the International Trade of G20 Countries

Below we present an analysis in the form of diagrams of the parts of the demand and supply of the k -th country for the goods exchanged by the G20 countries.

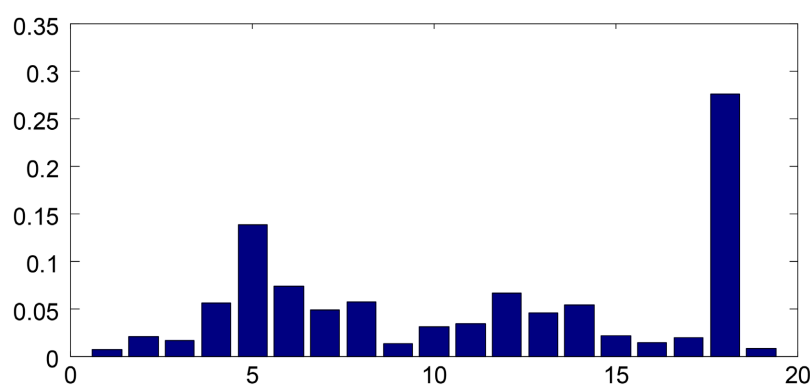
The same analysis is given in the form of diagrams of the supply and demand parts of G20 countries for the k-th type of goods.

The first four diagrams (see **Figures 1-4**) represent parts of the k-th country's demand for the exchanged goods.

We present, in descending order, the parts of demand only for those countries in which these parts are significant. Below each diagram, the parts of the k-th country's demand are given in numerical form. For example, the significant parts of demand for the exchanged goods in 2016-2019 was, in descending order, in countries: USA, China, Germany, Japan, Canada, UK, France, Mexico, Republic of Korea.

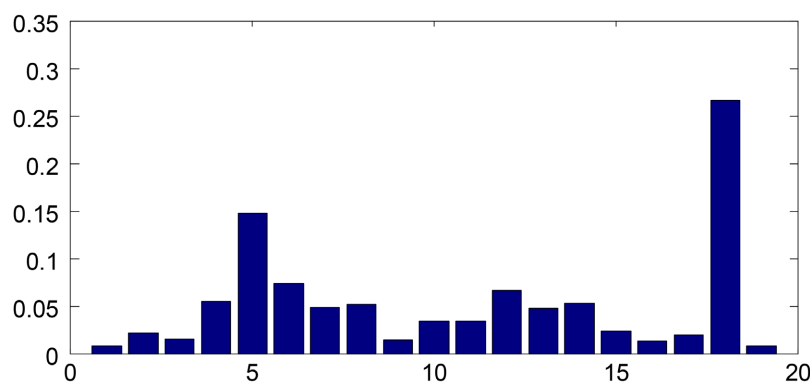
The following four diagrams (see **Figures 5-8**) represent the part of supply by the k-th country of G20 countries.

The last eight diagrams (see **Figures 9-16**) present the part of demand and supply of the k-th goods type by all G20 countries. The following countries have



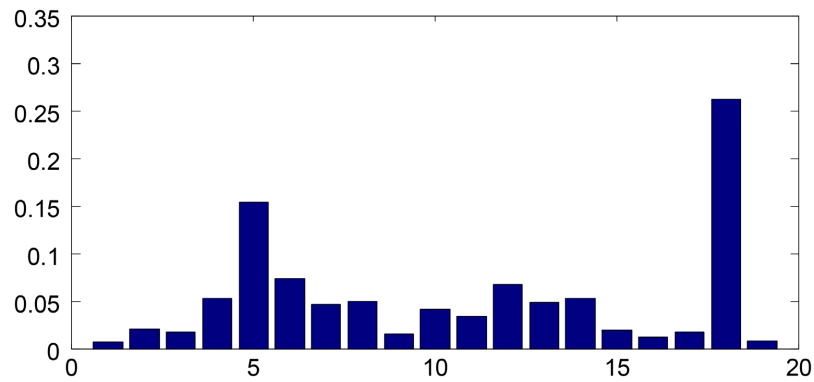
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.007, 2. Australia: 0.02, 3. Brazil: 0.016, 4. Canada: 0.056, 5. China: 0.138, 6. Germany: 0.074, 7. France: 0.048, 8. The United Kingdom: 0.056, 9. Indonesia: 0.013, 10. India: 0.030, 11. Italy: 0.033, 12. Japan: 0.066, 13. Republic of Korea: 0.045, 14. Mexico: 0.053, 15. Russia: 0.021, 16. Saudi Arabia: 0.014, 17. Turkey: 0.019, 18. The United States: 0.275, 19. South Africa: 0.007.

Figure 1. The part of demand of the k-th country for goods of G20 countries, 2016.



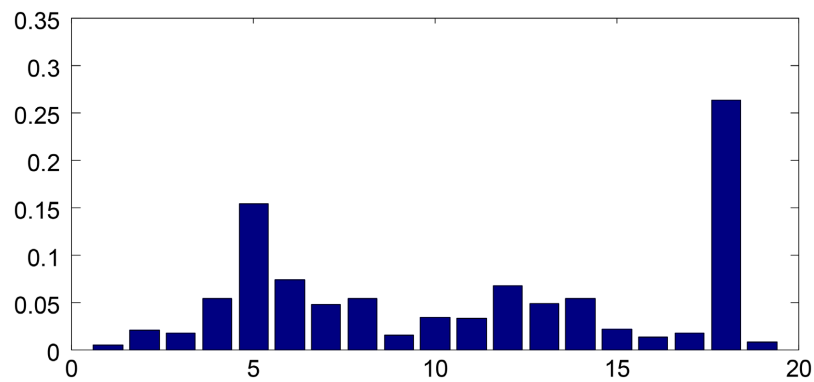
(<http://wits.worldbank.org>, <https://data.oecd.org>). 1. Argentina: 0.007, 2. Australia: 0.021, 3. Brazil: 0.015, 4. Canada: 0.054, 5. China: 0.147, 6. Germany: 0.073, 7. France: 0.048, 8. The United Kingdom: 0.051, 9. Indonesia: 0.013, 10. India: 0.034, 11. Italy: 0.033, 12. Japan: 0.066, 13. Republic of Korea: 0.047, 14. Mexico: 0.052, 15. Russia: 0.023, 16. Saudi Arabia: 0.013, 17. Turkey: 0.019, 18. The United States: 0.266, 19. South Africa: 0.007.

Figure 2. The part of demand of the k-th country for goods of G20 countries, 2017.



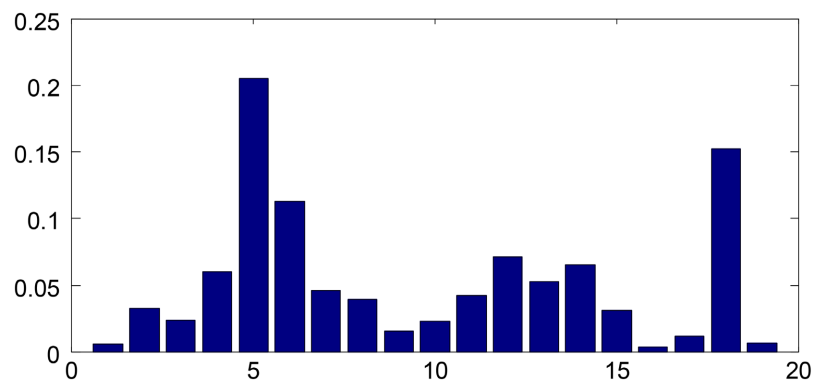
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.001, 2. Australia: 0.020, 3. Brazil: 0.016, 4. Canada: 0.053, 5. China: 0.153, 6. Germany: 0.073, 7. France: 0.046, 8. The United Kingdom: 0.049, 9. Indonesia: 0.015, 10. India: 0.041, 11. Italy: 0.034, 12. Japan: 0.067, 13. Republic of Korea: 0.048, 14. Mexico: 0.053, 15. Russia: 0.019, 16. Saudi Arabia: 0.011, 17. Turkey: 0.017, 18. The United States: 0.262, 19. South Africa: 0.007.

Figure 3. The part of demand of the k-th country for goods of G20 countries, 2018.



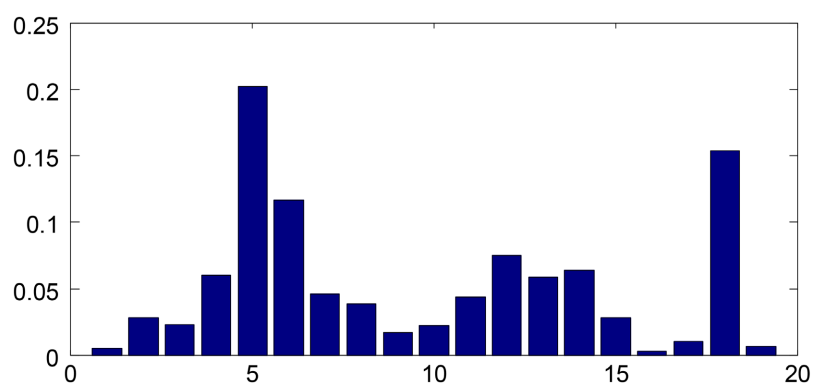
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.005, 2. Australia: 0.020, 3. Brazil: 0.017, 4. Canada: 0.054, 5. China: 0.153, 6. Germany: 0.073, 7. France: 0.047, 8. The United Kingdom: 0.053, 9. Indonesia: 0.015, 10. India: 0.033, 11. Italy: 0.033, 12. Japan: 0.067, 13. Republic of Korea: 0.048, 14. Mexico: 0.053, 15. Russia: 0.021, 16. Saudi Arabia: 0.013, 17. Turkey: 0.017, 18. The United States: 0.263, 19. South Africa: 0.007.

Figure 4. The part of demand of the k-th country for goods of G20 countries, 2019.



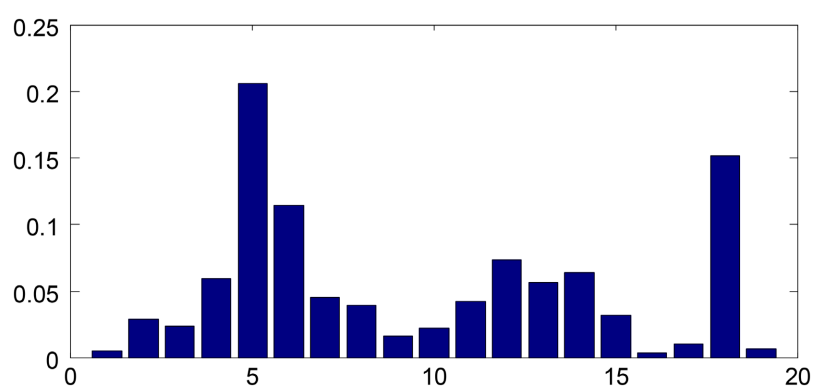
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.005, 2. Australia: 0.022, 3. Brazil: 0.021, 4. Canada: 0.061, 5. China: 0.20, 6. Germany: 0.120, 7. France: 0.048, 8. the United Kingdom: 0.040, 9. Indonesia: 0.015, 10. India: 0.020, 11. Italy: 0.043, 12. Japan: 0.075, 13. Republic of Korea: 0.055, 14. Mexico: 0.064, 15. Russia: 0.024, 16. Saudi Arabia: 0.002, 17. Turkey: 0.010, 18. The United States: 0.157, 19. South Africa: 0.006.

Figure 5. The part of goods supply by the k-th country in general supply of G20 countries, 2016.



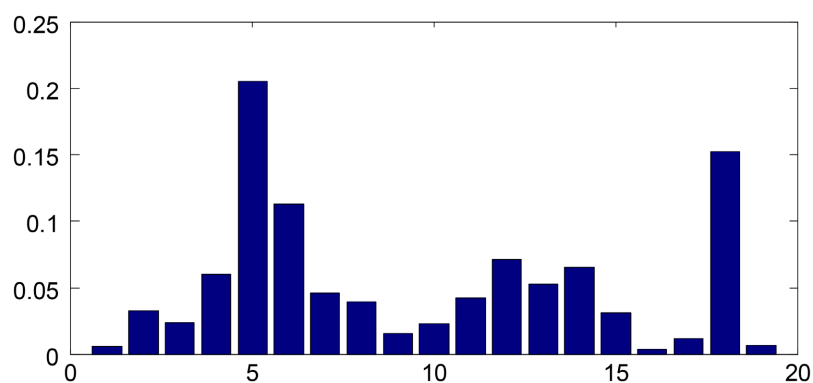
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.005, 2. Australia: 0.027, 3. Brazil: 0.020, 4. Canada: 0.060, 5. China: 0.200, 6. Germany: 0.116, 7. France: 0.045, 8. The United Kingdom: 0.038, 9. Indonesia: 0.016, 10. India: 0.021, 11. Italy: 0.043, 12. Japan: 0.074, 13. Republic of Korea: 0.058, 14. Mexico: 0.063, 15. Russia: 0.027, 16. Saudi Arabia: 0.002, 17. Turkey: 0.010, 18. The United States: 0.153, 19. South Africa: 0.006.

Figure 6. The part of goods supply by the k-th country in general supply of G20 countries, 2017.



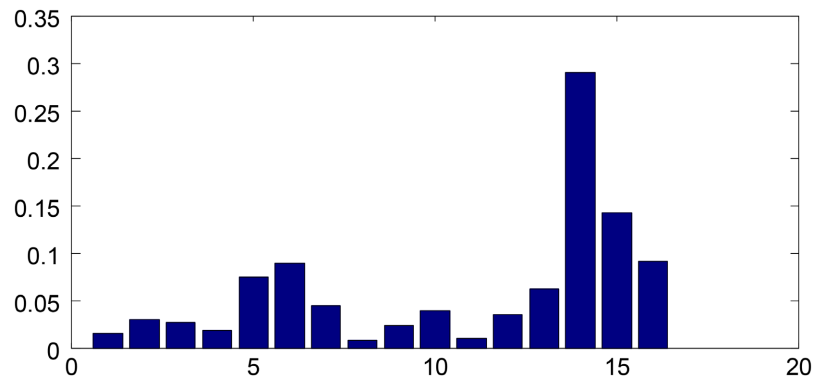
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.004, 2. Australia: 0.028, 3. Brazil: 0.023, 4. Canada: 0.059, 5. China: 0.205, 6. Germany: 0.114, 7. France: 0.045, 8. the United Kingdom: 0.039, 9. Indonesia: 0.016, 10. India: 0.022, 11. Italy: 0.041, 12. Japan: 0.073, 13. Republic of Korea: 0.056, 14. Mexico: 0.063, 15. Russia: 0.031, 16. Saudi Arabia: 0.003, 17. Turkey: 0.010, 18. The United States: 0.151, 19. South Africa: 0.006.

Figure 7. The part of goods supply by the k-th country in general supply of G20 countries, 2018.



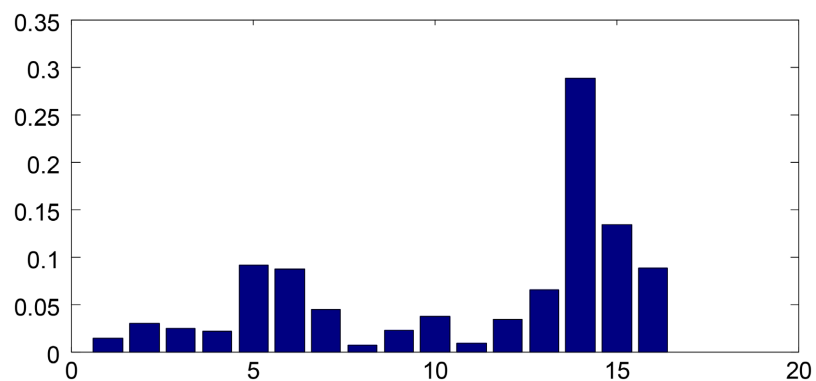
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Argentina: 0.005, 2. Australia: 0.032, 3. Brazil: 0.023, 4. Canada: 0.060, 5. China: 0.200, 6. Germany: 0.110, 7. France: 0.045, 8. The United Kingdom: 0.039, 9. Indonesia: 0.015, 10. India: 0.023, 11. Italy: 0.042, 12. Japan: 0.071, 13. Republic of Korea: 0.052, 14. Mexico: 0.065, 15. Russia: 0.031, 16. Saudi Arabia: 0.003, 17. Turkey: 0.011, 18. The United States: 0.150, 19. South Africa: 0.006.

Figure 8. The part of goods supply by the k-th country in general supply of G20 countries, 2019.



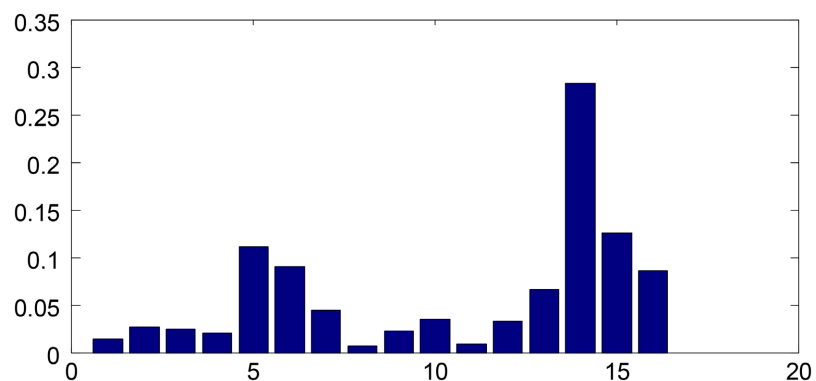
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.014, 2. Vegetable: 0.029, 3. FoodProd: 0.026, 4. Minerals: 0.018, 5. Fuels: 0.074, 6. Chemicals: 0.089, 7. PlastiRub: 0.044, 8. HidesSkin: 0.007, 9. Wood: 0.023, 10. TextCloth: 0.039, 11. Footwear: 0.009, 12. StoneGlas: 0.035, 13. Metals: 0.062, 14. MachElec: 0.290, 15. Transport: 0.142, 16. Miscellan: 0.091.

Figure 9. The part of demand for the k-th type of goods by all G20 countries, 2016.



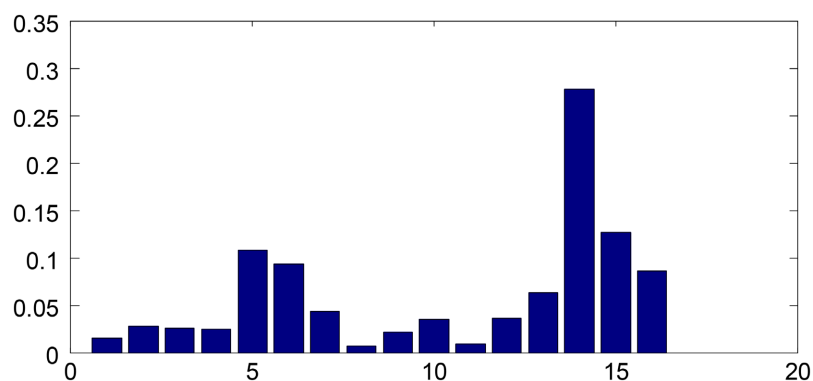
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.014, 2. Vegetable: 0.029, 3. FoodProd: 0.025, 4. Minerals: 0.021, 5. Fuels: 0.091, 6. Chemicals: 0.087, 7. PlastiRub: 0.044, 8. HidesSkin: 0.007, 9. Wood: 0.022, 10. TextCloth: 0.036, 11. Footwear: 0.009, 12. StoneGlas: 0.033, 13. Metals: 0.065, 14. MachElec: 0.288, 15. Transport: 0.134, 16. Miscellan: 0.088.

Figure 10. The part of demand for the k-th type of goods by all G20 countries, 2017.



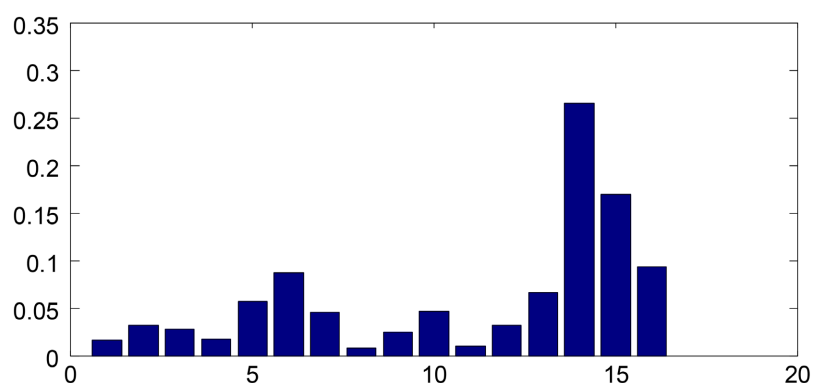
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.014, 2. Vegetable: 0.027, 3. FoodProd: 0.024, 4. Minerals: 0.021, 5. Fuels: 0.111, 6. Chemicals: 0.090, 7. PlastiRub: 0.044, 8. HidesSkin: 0.006, 9. Wood: 0.022, 10. TextCloth: 0.035, 11. Footwear: 0.008, 12. StoneGlas: 0.032, 13. Metals: 0.066, 14. MachElec: 0.282, 15. Transport: 0.125, 16. Miscellan: 0.086.

Figure 11. The part of demand for the k-th type of goods by all G20 countries, 2018.



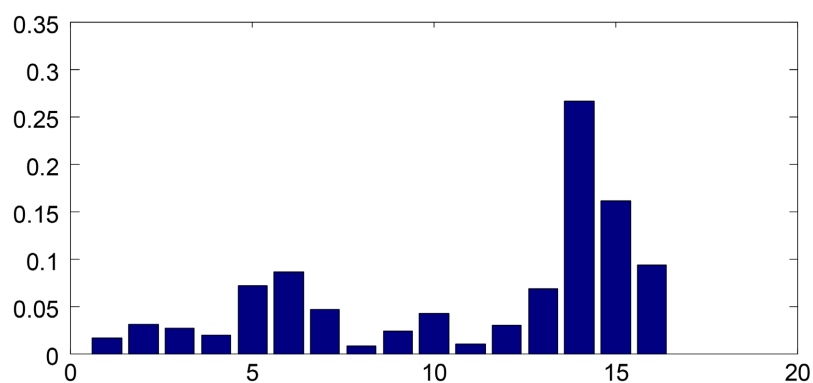
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.015, 2. Vegetable: 0.027, 3. FoodProd: 0.025, 4. Minerals: 0.024, 5. Fuels: 0.108, 6. Chemicals: 0.093, 7. PlastiRub: 0.043, 8. HidesSkin: 0.006, 9. Wood: 0.021, 10. TextCloth: 0.035, 11. Footwear: 0.009, 12. StoneGlas: 0.036, 13. Metals: 0.062, 14. MachElec: 0.278, 15. Transport: 0.126, 16. Miscellan: 0.086.

Figure 12. The part of demand for the k-th type of goods by all G20 countries, 2019.



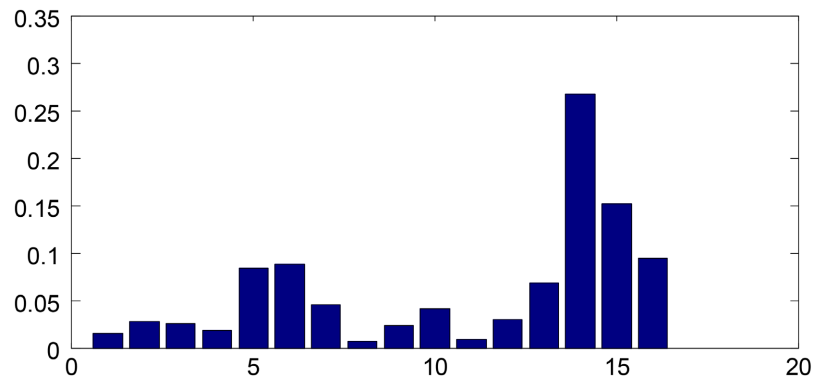
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.016, 2. Vegetable: 0.031, 3. FoodProd: 0.028, 4. Minerals: 0.017, 5. Fuels: 0.057, 6. Chemicals: 0.087, 7. PlastiRub: 0.046, 8. HidesSkin: 0.007, 9. Wood: 0.024, 10. TextCloth: 0.046, 11. Footwear: 0.010, 12. StoneGlas: 0.031, 13. Metals: 0.066, 14. MachElec: 0.265, 15. Transport: 0.169, 16. Miscellan: 0.093.

Figure 13. The part of supply of the k-th goods type by all G20 countries, 2016.



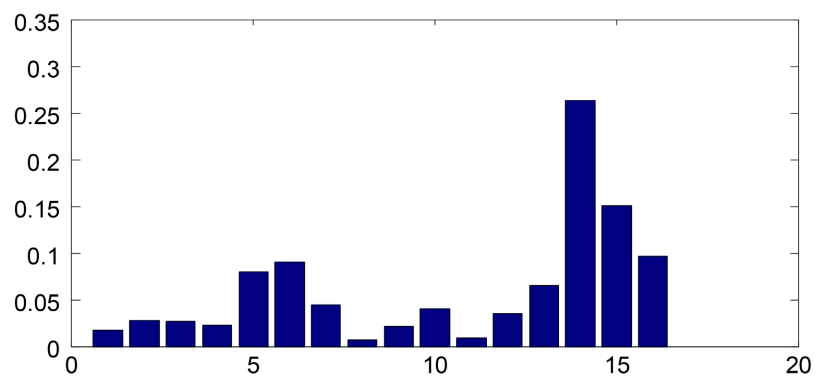
(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.015, 2. Vegetable: 0.030, 3. FoodProd: 0.026, 4. Minerals: 0.019, 5. Fuels: 0.071, 6. Chemicals: 0.086, 7. PlastiRub: 0.046, 8. HidesSkin: 0.007, 9. Wood: 0.023, 10. TextCloth: 0.042, 11. Footwear: 0.009, 12. StoneGlas: 0.029, 13. Metals: 0.068, 14. MachElec: 0.266, 15. Transport: 0.161, 16. Miscellan: 0.093.

Figure 14. The part of supply of the k-th goods type by all G20 countries, 2017.



(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.015, 2. Vegetable: 0.027, 3. FoodProd: 0.026, 4. Minerals: 0.018, 5. Fuels: 0.084, 6. Chemicals: 0.088, 7. PlastiRub: 0.045, 8. HidesSkin: 0.007, 9. Wood: 0.023, 10. TextCloth: 0.040, 11. Footwear: 0.009, 12. StoneGlas: 0.030, 13. Metals: 0.068, 14. MachElec: 0.267, 15. Transport: 0.151, 16. Miscellan: 0.094.

Figure 15. The part of supply of the k-th goods type by all G20 countries, 2018.



(<http://wits.worldbank.org>, <https://data.oecd.org>): 1. Animal: 0.017, 2. Vegetable: 0.028, 3. FoodProd: 0.026, 4. Minerals: 0.022, 5. Fuels: 0.080, 6. Chemicals: 0.090, 7. PlastiRub: 0.044, 8. HidesSkin: 0.007, 9. Wood: 0.022, 10. TextCloth: 0.040, 11. Footwear: 0.009, 12. StoneGlas: 0.035, 13. Metals: 0.064, 14. MachElec: 0.262, 15. Transport: 0.150, 16. Miscellan: 0.096.

Figure 16. The part of supply of the k-th goods type by all G20 countries, 2019.

a significant part of the supply of goods, in descending order: China, USA, Germany, Japan, Mexico, Canada, Republic of Korea. The following goods are in great demand in G20 countries: MachElec, Transport, Chemicals, Miscellan, Fuels. The growth in demand for fuels was characteristic. The following goods had the largest shares of supply: MachElec, Transport, Miscellan, Chemicals, Fuels, Metals. There has been an increase in fuels supply.

8. Conclusion

We have elaborated a new method to investigate international trade. It is based on the theory of economy equilibrium. We formulated a model of international trade containing an ideal equilibrium state of the exchange of goods when the balance of every country is zero. The deviation from this ideal state of equilibrium characterizes the real equilibrium states. In Theorems proved, we give the algorithms to construct equilibrium states. For this purpose, we use the notion

of consistency for the structure of supply with the structure of demand early introduced in a more general case (Gonchar, 2008). To construct equilibrium price vectors, we created a new method based on the representation of the supply matrix after the demand matrix. The conditions for a recession state to occur are formulated, under which in the economy system the equilibrium price vector has a high degree of degeneracy. In such a case, the destabilization of the monetary system occurs. This is a state in which money is not able to activate the economy without appropriate changes in the structure of supply and demand. We introduced an important concept of a generalized equilibrium price vector defined as a solution to a degenerate system of equations with real consumption. Using the concept of a generalized equilibrium vector, we defined a recession level parameter. It is a characteristic of the stability for the international exchange currency. As we have shown, the international currency became more stable during 2016-2019.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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