Journal of Quantum Information Science, **2011, 1, 43-49**
doi:10.4236/jqis.2011.12006 Published Online September 2011 (http://www.SciRP.org/journal/jqis)

Adaptive Phase Matching in Grover's Algorithm

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Received June 28, 2011; *revised August* 16, 2011; *accepted August* 26, 2011

Abstract

When the Grover's algorithm is applied to search an unordered database, the successful probability usually decreases with the increase of marked items. In order to solve this problem, an adaptive phase matching is proposed. With application of the new phase matching, when the fraction of marked items is greater $\left(3-\sqrt{5}\right)/8$, the successful probability is equal to 1 with at most two Grover iterations. The validity of the new phase matching is verified by a search example.

Keywords: Quantum Computing, Quantum Searching, Grover's Algorithm, Phase Matching, Adaptive Phase Shifting

1. Introduction

Shor's prime factoring algorithm and Grover's quantum search algorithm are two of the great quantum algorithms [1,2]. Grover's search algorithm provides a dramatic example of the potential speedup offered by quantum computers [2,3]. The problem addressed by Grover's algorithm can be viewed as trying to find a marked element in an unsorted database of size *N*. To solve this problem, a classical computer would need, on average, $N/2$ database queries and *N* queries in the worst case. Using Grover's algorithm, a quantum computer can accomplish the same task using merely $\mathbf{O}\left(\sqrt{N}\right)$ queries. The importance of Grover's result stems from the fact that it proves the enhanced power of quantum computers compared to classical ones for a whole class of oracle-based problems, for which the bound on the efficiency of classical algorithms is known. At present, Grover's quantum search algorithm has been greatly noticed and has become a challenging research field. However, the Grover's algorithm also has some limitations. When the fraction of marked items is greater than a quarter of the total items in the database, the success probability will rapidly decrease, and when the fraction of marked items is greater than half of the total items in the database, the algorithm will be disabled.

Up to now, many efforts in improving Grover's original algorithm have been done. Boyer et al. have given analytical expressions for the amplitude of the states in Grover's search algorithm and tight bounds on the algorithm [4]. Zalka has improved these tight bounds and showed that Grover's algorithm is optimal [5]. Biron et al. generalized Grover's algorithm to an arbitrarily distributed initial state [6]. Pati recast the algorithm in geometric language and studied the bounds on the algorithm [7]. Ozhigov showed that quantum search can be further speeded up by a factor of $\sqrt{2}$ by parallelism [8]. Gingrich *et al.* also generalized Grover's algorithm with parallelism with improvement [9]. The Grover's original algorithm consists of inversion of the amplitude in the desired state and inversion-about-average operation [2]. In [10], Grover presented a general algorithm: $Q = -I_\gamma U^+ I_\tau U$, where *U* is any unitary operation, U^+ is the adjoint of *U*, $I_\gamma = I - 2|\gamma\rangle\langle\gamma|$, $I_\tau = I - 2|\tau\rangle\langle\tau|$, $|\gamma\rangle$ is an initial state and $|\tau\rangle$ is a desired state. When $U^+ = U = H$, where *H* is the Walsh-Hadamard transformation, and $|\gamma\rangle = |0\rangle$, the general algorithm becomes the original algorithm. Long extended Grover's algorithm [11]. In Long's algorithm, I_γ and I_τ are expressed as $I_\gamma = I - \left(1-\mathrm{e}^{i\theta}\right)|\gamma\rangle\langle\gamma|$ and $I_\tau = I - \left(1-e^{i\varphi}\right) \times|\tau\rangle\langle\tau|$, respectively. When $\theta = \varphi = \pi$, Long's algorithm becomes Grover's original algorithm. Li et al. proposed that *U* in Long's algorithm can be replaced by any unitary operation *V* [12,13]. Biham generalized the Grover's algorithm to deal with an arbitrary pure initial state and an arbitrary mixed initial state [14,15]. In [16], Grover presented the new algorithm by replacing the selective inversions by selective phase shifts of $\pi/3$.

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Just like classical search algorithms the algorithm has a fixed point in state-space toward which it preferentially converges. Li *et al.* studied the changes of the approximation error in the fixed-point search algorithm obtained by replacing equal phase shifts of $\pi/3$ by different phase shifts [17].

The methods mentioned above cannot solve the problem that the algorithm efficiencies decrease as the marked items increase. In this paper, we study the phase matching in Grover's algorithm, and propose an adaptive matching, namely, $\theta = -\varphi = f(\lambda)$, where λ is the fraction of marked items, and $f(\lambda)$ is the polynomial for λ. With application of the new phase matching, when λ is a rational number in range $(3-\sqrt{5})/8$ to 1/4, the probability of getting correct results is equal to 1 with two Grover iterations, and when λ is greater than 1/4, the probability of getting correct results is equal to 1 with only one Grover iteration.

This paper is organized as follows: In Section 2, we introduce Grover's algorithm and its drawbacks. Section 3 is used to propose an adaptive phase matching with the higher success probability. Section 4 gives an example to verify the validity of new phase matching. Section 5 summarizes the whole paper.

2. Grover's Algorithm and Its Problem

2.1. Grover's Algorithm Summary

Suppose we wish to search through a search space of N elements. Rather than search the elements directly, we concentrate on the index to those elements, which is just a number in range 0 to $N-1$. For convenience we assume $N = 2^n$, so the index can be stored in n qubits, and that the search problem has exactly M solutions, with $1 \le M \le N$. The algorithm begins with the state $|0\rangle^{\otimes n}$. The Walsh-hadamard transform is used to put the state $|0\rangle^{\otimes n}$ in the equal superposition state,

$$|\phi\rangle = \frac{1}{N}\sum_{x=0}^{N-1}|x\rangle \tag{1}$$

The Grover quantum search algorithm then consists of repeated application of a quantum subroutine, know as the Grover iteration or Grover operator, which we denote G. The Grover iteration may be broken up into four steps.

1) Apply the oracle O. The oracle is a unitary operator defined by its action on the computational basis

$$\mathrm{O}(|x\rangle|q\rangle) \to |x\rangle|q \oplus f(x)\rangle \tag{2}$$

where $|x\rangle$ is the index register, $\oplus$ denotes addition modulo 2, and the oracle qubit $|q\rangle$ is a single qubit which is flipped if $f(x)=1$, and is unchanged otherwise.

2) Applying the Walsh-Hadamard transform $H^{\otimes n}$.

3) Perform a conditional phase shift, with every computational basis state except $|0\rangle$ receiving a phase shift of -1, $|x\rangle \to -(-1)^{\delta_{x0}}|x\rangle$.

4) Applying the Walsh-Hadamard transform $H^{\otimes n}$.

It is useful to note that the combined effect of steps 2, 3, and 4 is

$$H^{\otimes n}\left(2|0\rangle\langle 0| - I\right)H^{\otimes n} = 2|\phi\rangle\langle\phi| - I \tag{3}$$

where $|\phi\rangle$ is the equally weighted superposition of states, (1). Thus the Grover iteration, G, may be written as $G = \left(2|\phi\rangle\langle\phi| - I\right)O$.

Let $\lambda = M/N$, and CI(x) denote the integer closest to the real number x, where by convention we round halves down. Then repeating the Grover iteration

$$R = \mathrm{CI}\left(\frac{\arccos\left(\sqrt{\lambda}\right)}{2\arcsin\left(\sqrt{\lambda}\right)}\right) \tag{4}$$

times rotates $|\phi\rangle$ to within an angle $\arcsin\sqrt{\lambda} \le \pi/4$ of a superposition of marked states [18]. Observation of the state in the computational basis then yields a solution to the search problem with probability at least one-half.

2.2. Grover's Algorithm Success Probability

In fact, the Grover iteration can be regarded as a rotation in the two-dimensional space spanned by the starting vector $|\phi\rangle$ and the state consisting of a uniform superposition of solutions to the search problem. Let $|\alpha\rangle$ represent a normalized states of a sum over all which are not solutions to the search problem, and $|\beta\rangle$ represent a normalized states of a sum over all which are solutions to the search problem. Simple algebra shows that the initial state $|\phi\rangle$ may be re-expresses as

$$|\phi\rangle = \cos(t)|\alpha\rangle + \sin(t)|\beta\rangle \tag{5}$$

where $t = \arcsin\sqrt{\lambda}$. After R Grover iterations, the initial state is taken to

$$\begin{aligned} G^R|\phi\rangle &= \cos\left((2R+1)\arcsin\sqrt{\lambda}\right)|\alpha\rangle \\ &\quad + \sin\left((2R+1)\arcsin\sqrt{\lambda}\right)|\beta\rangle \end{aligned} \tag{6}$$

Hence, the success probability is

$$P = \sin 2\left((2R+1)\arcsin\sqrt{\lambda}\right) \tag{7}$$

The curve of P is shown in **Figure 1**.

2.3. The Drawback of Grover's Algorithm

It is easy to deduce from Equation (4) and Equation (7)

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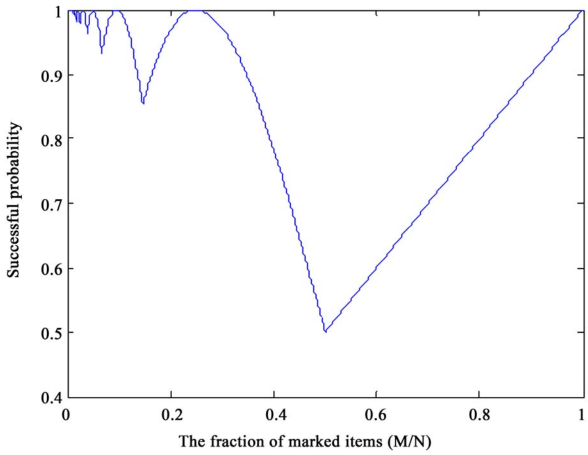

Figure 1. The success probability of Grover's algorithm.

that when $\left(3-\sqrt{5}\right)/8 < \lambda < 0.14645$, P_λ decreases rapidly. When $0.14645 \le \lambda < 1/4$, P_λ increases rapidly. When $1/4 \le \lambda < 1/2$, P_λ decreases rapidly. When $\lambda \ge 1/2$, there is $R = 0$, $P_\lambda = \lambda$, and the algorithm is disabled. Hence, the Grover's algorithm is no longer useful when $\lambda > 1/4$.

The reason for the problem is that the two phase rotations in Grover iteration are fully equivalent in both amplitude and direction, namely π. According to Ref. [18], the result of such phase rotations is that, for the one Grover iteration, the phase of the $|\phi\rangle$ increases $2\arcsin\sqrt{\lambda}$ radians, and that rotating through at least $\arccos\sqrt{\lambda}$ radians takes the $|\phi\rangle$ to $|\beta\rangle$. When $\lambda \in \left[\frac{3-\sqrt{5}}{8}, 1\right)$, there are only $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{3-\sqrt{5}}{8}$ make R an integer, namely, $R_1 = \frac{\arccos\sqrt{\lambda_1}}{2\arcsin\sqrt{\lambda_1}} = 1$, $R_2 = \frac{\arccos\sqrt{\lambda_2}}{2\arcsin\sqrt{\lambda_2}} = 2$.

3. The Adaptive Phase Matching for Grover's Algorithm

3.1. The Adaptive Phase Matching

The two phase shift operators in the Grover's algorithm may be generally expressed as follows

$$U = I - \left(1 - e^{i\alpha}\right)\sum_{m=1}^{M} |\tau_m\rangle\langle\tau_m| \tag{8}$$

$$V = \left(1 - e^{i\beta}\right)|\phi\rangle\langle\phi| + e^{i\beta} I \tag{9}$$

In the Grover's original algorithm, there is $\alpha = \beta = \pi$. According to the postulates of quantum mechanics [18], the evolution of a closed quantum system is described by a unitary transformation. So as far as the unitarity of operators described by Equation (8) and Equation (9) is concerned, we have the following conclusions.

Theorem 1 The operators described by Equation (8) and Equation (9) are unitary.

Proof $U^{+} = I - \left(1 - e^{i\alpha}\right)\sum_{m=1}^{M} |\tau\rangle\langle\tau|$

$$U^{+}U = I - \left(2 - e^{i\alpha} - e^{-i\alpha}\right)\sum_{m=1}^{M}\sum_{m=1}^{M} |\tau\rangle\langle\tau| + \left(1 - e^{i\alpha}\right)\left(1 - e^{-i\alpha}\right)\left(\sum_{m=1}^{M} |\tau\rangle\langle\tau|\right)^2 = I$$

Hence, the operator described by Eqation (8) is unitary.

$$V^{+} = \left(1 - e^{-i\beta}\right)\sum_{m=1}^{M} |\tau\rangle\langle\tau| + e^{-i\beta} I$$

$$V^{+}V = \left(1 - e^{i\beta}\right)\left(1 - e^{-i\beta}\right)|\phi\rangle\langle\phi|\phi\rangle\langle\phi| + \left(e^{i\beta} + e^{-i\beta} - 2\right)|\phi\rangle\langle\phi| + I = I$$

Hence, the operator described by Equation (9) is unitary. □

For the matching of α and β, we propose an adaptive phase matching described by Theorem 2.

Theorem 2

1) When $1/4 \le \lambda < 1$ and $\alpha = -\beta = \arccos\left(\frac{2\lambda - 1}{2\lambda}\right)$, the success probability $P = 1$ can be obtained after only one iteration.

2) When $\frac{3-\sqrt{5}}{8} < \lambda < \frac{1}{4}$ and

$$\alpha = -\beta = \arccos\left(1 - \frac{3-\sqrt{5}}{4\lambda}\right),$$

the success probability $P = 1$ can be obtained after only two iterations.

Proof 1) For $|\phi\rangle$ described by Equation (1), applying one Grover iteration gives

$$|\phi_1\rangle = UV|\phi\rangle = \frac{1}{\sqrt{N^3}}\left(A|x_j\rangle + B|x_k\rangle\right)$$

where

$$A = \sum_{j=0}^{N-M-1}\left[M\left(e^{i\alpha} + e^{i\beta} - e^{i(\alpha+\beta)}\right) + N - M\right]$$

$$B = \sum_{k=0}^{M-1}\left[\left(N-M\right)\left(e^{i(\alpha+\beta)} - e^{i\beta} + 1\right) + Me^{i\alpha}\right]$$

Let $p = \left(N-M\right)\left(e^{i(\alpha+\beta)} - e^{i\beta} + 1\right) + Me^{i\alpha}$, the success probability P is equal to $M\left(|p|/\sqrt{N^3}\right)^2$. When $\alpha = -\beta$, using some simple algebra gives $8\lambda^3-12\lambda^2+4\lambda-\left(8\lambda^3-12\lambda^2+4\lambda\right)\cos\alpha+4\lambda^3-8\lambda^2+5\lambda$. When $\cos\alpha=\left(2\lambda-1\right)/2\lambda$, namely, $\alpha=-\beta=\arccos\left(\left(2\lambda-1\right)/2\lambda\right)$,

$$P=P_{\max}=4\lambda^3-8\lambda^2+5\lambda-\frac{\left(8\lambda^3-12\lambda^2+4\lambda\right)^2}{4\left(3\lambda^3-4\lambda^2\right)}=1.$$

From $\left|\cos\alpha\right|\le 1$, the range $1/4\le\lambda<1$ may be obtained.

2) Reapplying one Grover iteration to $\left|\phi_1\right\rangle$ gives

$$\left|\phi_2\right\rangle=UV\left|\phi_1\right\rangle=\frac{1}{\sqrt{N^5}}\left(A\left|x_j\right\rangle+B\left|x_k\right\rangle\right)$$

where

$$A=\sum_{j=0}^{N-M-1}\Big[\left(1-e^{i\beta}+Ne^{i\beta}\right)C+\left(N-M-1\right)\left(1-e^{i\beta}\right)C+M\left(1-e^{i\beta}\right)D\Big]$$

$$B=\sum_{k=0}^{M-1}\Big[\left(1-e^{i\beta}+Ne^{i\beta}\right)D+\left(M-1\right)\left(1-e^{i\beta}\right)D+\left(N-M\right)\left(1-e^{i\beta}\right)C\Big]$$

$$C=M\left(e^{i\alpha}+e^{i\beta}-e^{i(\alpha+\beta)}\right)+N-M$$

$$D=\left(N-M\right)\left(e^{i(2\alpha+\beta)}-e^{i(\alpha+\beta)}+e^{i\alpha}\right)+Me^{i2\alpha}$$

Let

$$p=\left(1-e^{i\beta}+Ne^{i\beta}\right)D+\left(M-1\right)\left(1-e^{i\beta}\right)D+\left(N-M\right)\left(1-e^{i\beta}\right)C$$

the success probability P is equal to $M\left(\left|p/\sqrt{N^5}\right|\right)^2$. Where $\alpha=-\beta$, use some simple algebra gives

$$\begin{aligned}P=&\left(16\lambda^5-16\lambda^4\right)\cos^4\alpha-\left(64\lambda^5-112\lambda^4+48\lambda^3\right)\cos^3\alpha\\&+\left(96\lambda^5-240\lambda^4+188\lambda^3-44\lambda^2\right)\cos^2\alpha\\&-\left(64\lambda^5-208\lambda^4+232\lambda^3-100\lambda^2+12\lambda\right)\cos\alpha\\&+\left(16\lambda^5-64\lambda^4+92\lambda^3-56\lambda^2+13\lambda\right).\end{aligned}$$

where $\cos\alpha=1-\frac{3-\sqrt{5}}{4\lambda}$, $P=P_{\max}=1$.

From $\left|\cos\alpha<1\right|$, the range $\left(3-\sqrt{5}\right)/8<\lambda<1$ may be obtained. Taking into account the only iteration is needed when $1/4\le\lambda<1$, hence, the range for λ takes the form $\left(3-\sqrt{5}\right)/8<\lambda<1/4$, and the adaptive phase matching takes the form $\alpha=-\beta=\arccos\left(1-\left(3-\sqrt{5}\right)/4\lambda\right)$. □

According to thorem 2, applying the adaptive phase matching, the Equation (8) and Equation (9) can be re-expressed as follows

$$U=I-\left(1-e^{i\alpha}\right)\sum_{m=1}^{M}\left|\tau\right\rangle\left\langle\tau\right| \quad (10)$$

$$V=\left(1-e^{-i\alpha}\right)\left|\phi\right\rangle\left\langle\phi\right|+e^{-i\alpha}I \quad (11)$$

On the basis of the adaptive phase matching, the success probability curve is shown in **Figure 2**.

3.2. The Algorithm Description Based on the Adaptive Phase Matching

According to λ, we divide the search process into three cases.

1) When $0<\lambda<\left(3-\sqrt{5}\right)/8$, the original phase matching is applied.

2) When $\left(3-\sqrt{5}\right)/8<\lambda<1/4$, the adaptive phase matching is applied. The search process can be described as follows:

Step 1 Applying Equation (10) to create the phase of the marked states rotate $\alpha=\arccos\left(\left(4\lambda-3+\sqrt{5}\right)/\left(4\lambda\right)\right)$ radians, namely, $\left|\phi_1\right\rangle=U\left|\phi\right\rangle$.

Step 2 Applying Equation (11) to rotate the system superposition state $\left|\phi_1\right\rangle$ to $\left|\bar{\phi}_1\right\rangle$, namely, $\left|\bar{\phi}_1\right\rangle=V\left|\phi_1\right\rangle=V\left(U\left|\phi_1\right\rangle\right)$.

Step 3 Reapplying Equation (10) to $\left|\bar{\phi}_1\right\rangle$, namely, $\left|\phi_2\right\rangle=U\left|\bar{\phi}_1\right\rangle$.

Step 4 Reapplying Equation (11) to $\left|\phi_2\right\rangle$, namely, $\left|\bar{\phi}_2\right\rangle=V\left|\phi_2\right\rangle=V\left(U\left|\bar{\phi}_1\right\rangle\right)$.

Step 5 Measuring $\left|\bar{\phi}_2\right\rangle$.

3) When $1/4\le\lambda<1$, the adaptive phase matching is applied. The search process can be described as follows:

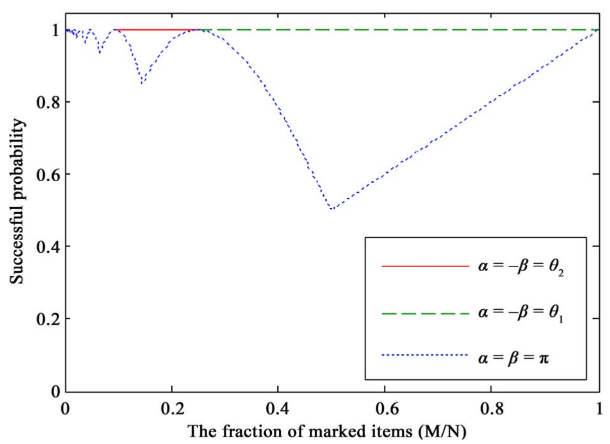

Figure 2. Comparison of success probability curves between Grover's original algorithm and improved ones, where $\theta_1=\arccos\left(\left(2\lambda-1\right)/2\lambda\right)$, $\theta_2=\arccos\left(1-\left(\left(3-\sqrt{5}\right)/4\lambda\right)\right)$.

Step 1 Applying Equation (10) to create the phase of the marked states rotate $\alpha = \text{arc}\cos\left((2\lambda - 1)/(2\lambda)\right)$ radians, namely, $|\phi_1\rangle = U|\phi\rangle$.

Step 2 Applying Equation (11) to rotate the system superposition state $|\phi_1\rangle$ to $|\bar{\phi}_1\rangle$, namely, $|\bar{\phi}_1\rangle = V|\phi_1\rangle = V\left(U|\phi\rangle\right)$.

Step 3 Measuring $|\bar{\phi}_1\rangle$.

4. Searching Example

There are 32 students in a class whose serial numbers are in range 0 to 31. 1) The search targets are the students whose serial number satisfies $n = \text{CI}\left((5k+3)/3\right)$, where $k = 0, 1, \cdots, 18$. The target serial numbers and marked states are shown in **Table 1**. 2) The search targets are the students whose serial number satisfies $n = 9k + 2$, where $k = 0, 1, 2, 3$. The target serial numbers and marked states are shown in **Table 2**.

In these two searches, $n = 32$, using 5 qubits can store all serial numbers. The initial state of the $|\phi\rangle$ is expressed as follows

Table 1. The target serial numbers and marked states.

k	Serial numbers	Marked states
0	1	$\|00001\rangle$
1	3	$\|00011\rangle$
2	4	$\|00100\rangle$
3	6	$\|00110\rangle$
4	8	$\|01000\rangle$
5	9	$\|01001\rangle$
6	11	$\|01011\rangle$
7	13	$\|01101\rangle$
8	14	$\|01110\rangle$
9	16	$\|10000\rangle$
10	18	$\|10010\rangle$
11	19	$\|10011\rangle$
12	21	$\|10101\rangle$
13	23	$\|10111\rangle$
14	24	$\|11000\rangle$
15	26	$\|11010\rangle$
16	28	$\|11100\rangle$
17	29	$\|11101\rangle$
18	31	$\|11111\rangle$

Table 2. The target serial numbers and marked states.

k	Serial numbers	Marked states
0	2	$\|00010\rangle$
1	11	$\|01011\rangle$
2	20	$\|10100\rangle$
3	29	$\|11101\rangle$

$$|\phi\rangle = \frac{1}{4\sqrt{2}}\left(|0\rangle + |1\rangle + \cdots + |31\rangle\right)$$

1) In this search, $\lambda = M/N = 19/32$. According to theorem 2, applying $\alpha = -\beta = \text{arc}\cos\left((2\lambda-1)/(2\lambda)\right) = \text{arc}\cos(3/19)$ to this search, the probability of getting correct results is equal to 1 only one Grover iteration. The search process can be described as follows

$$|\phi_1\rangle = \left(I - \left(1 - e^{i\alpha}\right)\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{32}}\begin{pmatrix} 1,\ e^{i\alpha},\ 1,\ e^{i\alpha},\ e^{i\alpha},\ 1,\ e^{i\alpha},\ 1,\ e^{i\alpha},\ e^{i\alpha},\ 1, \\ e^{i\alpha},\ 1,\ e^{i\alpha},\ e^{i\alpha},\ 1,\ e^{i\alpha},\ 1,\ e^{i\alpha},\ e^{i\alpha},\ 1,\ e^{i\alpha}, \\ 1,\ e^{i\alpha},\ e^{i\alpha},\ 1,\ e^{i\alpha},\ 1,\ e^{i\alpha},\ e^{i\alpha},\ 1,\ e^{i\alpha} \end{pmatrix}$$

$$|\bar{\phi}_1\rangle = \left(\left(1 - e^{-i\alpha}\right)|\phi\rangle\langle\phi| + e^{-i\alpha}I\right)|\phi_1\rangle$$

$$= \frac{1}{32\sqrt{32}}\begin{pmatrix} a, b, a, b, b, a, b, a, b, b, a, b, a, b, b, a, \\ b, a, b, b, a, b, a, b, b, a, b, a, b, b, a, b \end{pmatrix}$$

where $a = 19\left(e^{i\alpha} + e^{-i\alpha}\right) - 6 = 0$, $b = 19e^{i\alpha} - 13e^{-i\alpha} + 26 = \frac{512}{19} + \frac{32\sqrt{32}}{19}i$. The probability of finding the marked states is given as follows

$$P = 19\left(\frac{1}{32\sqrt{32}}\right)^2\left(\left(\frac{512}{19}\right)^2 + \left(\frac{32\sqrt{32}}{19}\right)^2\right) = 1.$$

For the original phase matching, as $\lambda = 19/32 > 0.5$, the success probability, according to **Figure 1** is $P_\lambda = \lambda$. In fact, here, the success probability descends rapidly after applying a Grover iteration, which is described as follows

$$|\phi_1\rangle = \left(I - 2\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{32}}\begin{pmatrix} 1,\ -1,\ 1,\ -1,\ -1,\ 1,\ -1,\ 1,\ -1,\ -1,\ 1, \\ -1,\ 1,\ -1,\ -1,\ 1,\ -1,\ 1,\ -1,\ -1,\ 1,\ -1, \\ 1,\ -1,\ -1,\ 1,\ -1,\ 1,\ -1,\ -1,\ 1,\ -1 \end{pmatrix}$$

$$\left|\bar{\phi}_1\right\rangle = \left(2|\phi\rangle\langle\phi| - I\right)|\phi_1\rangle$$

$$= \frac{1}{32\sqrt{32}}\begin{pmatrix} a, b, a, b, b, a, b, a, b, b, a, b, a, b, b, a, \\ b, a, b, b, a, b, a, b, b, a, b, a, b, b, a, b \end{pmatrix},$$

where $a = -11$, $b = 5$.

2) In this search, $\lambda = M/N = 1/8$. According to theorem 2, applying $= \arccos\left(2\sqrt{5}-5\right) = \arccos\left(2\sqrt{5}-5\right)$ to this search, the probability of getting correct results is equal to 1 after only two Grover iterations. The search process can be described as follows

$$|\phi_1\rangle = \left(I - \left(1 - e^{-i\alpha}\right)\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{32}}\begin{pmatrix} 1,\ 1,\ e^{i\alpha},\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ e^{i\alpha},\ 1,\ 1,\ 1,\ 1, \\ 1,\ 1,\ 1,\ 1,\ e^{i\alpha},\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ e^{i\alpha},\ 1,\ 1, \end{pmatrix}$$

$$\left|\bar{\phi}_1\right\rangle = \left(\left(1 - e^{-i\alpha}\right)|\phi\rangle\langle\phi| + e^{-i\alpha}I\right)|\phi_1\rangle$$

$$= \frac{1}{32\sqrt{32}}\begin{pmatrix} a, a, b, a, a, a, a, a, a, a, a, b, a, a, a, a, \\ a, a, a, a, b, a, a, a, a, a, a, a, a, b, a, a \end{pmatrix}$$

where $a = 4\left(e^{i\alpha} + e^{-i\alpha}\right) + 24$, $b = 28\left(2 - e^{-i\alpha}\right) + 4e^{i\alpha}$.

$$|\phi_2\rangle = \left(I - \left(1 - e^{i\alpha}\right)\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|\right)\left|\bar{\phi}_1\right\rangle$$

$$= \frac{1}{32\sqrt{32}}\begin{pmatrix} a, a, c, a, a, a, a, a, a, a, a, c, a, a, a, a, \\ a, a, a, a, c, a, a, a, a, a, a, a, a, c, a, a \end{pmatrix}$$

where $a = 4\left(e^{i\alpha} + e^{-i\alpha}\right) + 24$, $c = 28\left(2e^{i\alpha} - 1\right) + 4e^{i2\alpha}$.

$$\left|\bar{\phi}_2\right\rangle = \left(\left(1 - e^{-i\alpha}\right)|\phi\rangle\langle\phi| + e^{-i\alpha}\mathbf{I}\right)|\phi_2\rangle$$

$$= \frac{1}{32^2\sqrt{320}}\begin{pmatrix} d, d, f, d, d, d, d, d, d, d, d, f, d, d, d, d, \\ d, d, d, d, f, d, d, d, d, d, d, d, d, f, d, d \end{pmatrix}$$

where $d = 16\left(e^{i2\alpha} + e^{-i2\alpha}\right) + 320\left(e^{i\alpha} + e^{-i\alpha}\right) + 352 = 0$,

$$f = 16e^{i2\alpha} + 448e^{i\alpha} - 1344e^{-i\alpha} - 112e^{-i2\alpha} + 2016$$

$$= 512\left(4\sqrt{5} - 4 + i\left(\sqrt{5}+1\right)\sqrt{20\sqrt{5} - 44}\right).$$

The probability of finding the marked states is given as follows

$$P = 4\left(\frac{512}{32^2\sqrt{32}}\right)^2\left(\left(4\sqrt{5}-4\right)^2 + \left(\sqrt{5}+1\right)^2\left(\sqrt{20\sqrt{5}-44}\right)^2\right) = 1$$

For the original phase matching, $\alpha = \beta = \pi$, the number of iterations is

$$R = \left[\frac{\arccos\left(\sqrt{\lambda}\right)}{2\arcsin\left(\sqrt{\lambda}\right)}\right] = \left[\frac{\arccos\left(\sqrt{1/8}\right)}{2\arcsin(\sqrt{1/8})}\right] = 2$$

The search process can be described as follows

$$|\phi_1\rangle = \left(I - 2\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|\right)|\phi\rangle$$

$$= \frac{1}{\sqrt{32}}\begin{pmatrix} 1,\ 1,\ -1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -1,\ 1,\ 1,\ 1,\ 1, \\ 1,\ 1,\ 1,\ 1,\ -1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -1,\ 1,\ 1 \end{pmatrix}$$

$$\left|\bar{\phi}_1\right\rangle = \left(2|\phi\rangle\langle\phi| - I\right)|\phi_1\rangle$$

$$= \frac{1}{2\sqrt{32}}\begin{pmatrix} 1,\ 1,\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 5,\ 1,\ 1,\ 1,\ 1, \\ 1,\ 1,\ 1,\ 1,\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 5,\ 1,\ 1 \end{pmatrix}$$

$$|\phi_2\rangle = \left(I - 2\sum_{m=1}^{M}|\tau_m\rangle\langle\tau_m|\right)\left|\bar{\phi}_1\right\rangle$$

$$= \frac{1}{2\sqrt{32}}\begin{pmatrix} 1,\ 1,\ -5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -5,\ 1,\ 1,\ 1,\ 1, \\ 1,\ 1,\ 1,\ 1,\ -5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -5,\ 1,\ 1 \end{pmatrix}$$

$$\left|\bar{\phi}_2\right\rangle = \left(2|\phi\rangle\langle\phi| - I\right)|\phi_2\rangle$$

$$= \frac{1}{4\sqrt{32}}\begin{pmatrix} a, a, b, a, a, a, a, a, a, a, a, b, a, a, a, a, \\ a, a, a, a, b, a, a, a, a, a, a, a, a, b, a, a \end{pmatrix}$$

where $a = -1$, $b = 11$.

$$P = 4\left(\frac{11}{4\sqrt{32}}\right)^2 = 0.9453125.$$

5. Conclusions

An adaptive phase matching in Grover's algorithm is proposed. With application of the new phase matching, when the fraction of marked items is greater than $\left(3-\sqrt{5}\right)/8$, the probability of getting correct results is equal to 1 after at most two Grover iterations. The validity of the new phase matching is verified by two search examples.

6. Acknowledgements

This paper was supported by National Natural Science Foundation of China (Grant No. 61170132), Chinese Postdoctoral Science Foundation (Grant Nos.20090460864 and 201003405), Heilongjiang Province Postdoctoral Science Foundation of China (Grant No. LBH-Z09289), Scientific Research Foundations of Heilongjiang Provincial Education Department (Grant No. 11551015).

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