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General Relativity and Gauge Theory: Beyond the Mirror

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Abstract

Lie pseudogroups are groups of transformations solutions of systems of ordinary or partial differential equations. The purpose of this paper is to present an elementary summary of a few recent results obtained through the application of the formal theory of systems of OD or PD equations and Lie pseudogroups to engineering (elasticity, electromagnetism) or mathematical physics (general relativity, gauge theory) and their couplings (piezoelectricity, photoelasticity). The work of Cartan is superseded by the use of the canonical Spencer sequence while the work of Vessiot is superseded by the use of the canonical Janet sequence but the link between these two sequences and thus these two works is still not known today. Using differential duality in the linear framework, the adjoint of the Spencer operator for the group of conformal transformations provides the Cauchy, Cosserat, Clausius, Maxwell and Weyl equations on equal footing. However, such a unifying result also leads to deep contradictions in the case of gravitational waves and black holes. Indeed, the Beltrami operator (1892) which is parametrizing the Cauchy operator of elasticity by means of 6 stress functions is nothing else than the self-adjoint Einstein operator (1915) in dimension 3 for the deformation of the metric which is parametrizing the *div* operator induced from the Bianchi identities. The same confusion between the Cauchy and *div* operators is existing on space-time as the Cauchy operator can be parametrized by the adjoint of the Ricci operator. Accordingly, the foundations of engineering and mathematical physics must be revisited within this new framework, though striking it may sometimes look like.

Keywords

Lie Groups, Lie Pseudogroups, Differential Sequences, Riemann Tensor, Weyl Tensor, Ricci Tensor, Differential Duality, Maxwell Equations, Einstein Equations, Gravitational Waves

1. Introduction

The purpose of this self-contained but difficult paper is to revisit *general relativity* (GR) and *gauge theory* (GT), but also Elasticity (EL) and electromagnetism (EM) in view of the latest mathematical developments existing today in *group theory*, *system theory* and *module theory*, namely:

- *Systems*: The orders of the successive operators appearing in the conformal Killing resolution highly depend on the dimension n, a result confirmed in 2016 by A. Quadrat (INRIA) while using computer algebra [1]. They are respectively $3 \underset{1}{\to} 5 \underset{3}{\to} 5 \underset{1}{\to} 3 \to 0$ when $n=3$, $4 \underset{1}{\to} 9 \underset{2}{\to} 10 \underset{2}{\to} 9 \underset{1}{\to} 4 \to 0$ when $n=4$ and $5 \underset{1}{\to} 14 \underset{2}{\to} 35 \underset{1}{\to} 35 \underset{2}{\to} 14 \underset{1}{\to} 5 \to 0$ when $n=5$. This result leads to revisit conformal geometry.
- *Groups*: The Ricci and the Maxwell tensors have only to do with the second order jets of the conformal group, called *elations* by Cartan (1922) [2]-[5]. This result questions the mathematical foundations of both general relativity (GR) and gauge theory (GT).
- *Modules*: Contrary to the Maxwell equations, the Einstein equations cannot be parametrized by a potential. Such a result is not coherent with the use of "*extension modules*" in homological algebra and the Cauchy stress equations must not be thus confused with the divergence-type condition for the Einstein tensor obtained by contracting the Bianchi identities [6]-[12]. This result questions the origin and existence of both gravitational waves and black holes [12] [13].

We briefly recall the historical framework leading to these new results.

The concept of "*group*" has been introduced in mathematics for the first time by E. Galois (1830) and slowly passed from algebra to geometry with the work of S. Lie on *Lie groups* (1880) and *Lie pseudogroups* (1890) of transformations. The concept of a finite length *differential sequence*, now called *Janet sequence*, has been described for the first time as a footnote by M. Janet (1920). Then, the work of D. C. Spencer (1970) has been the first attempt to use the formal theory of systems of partial differential equations in order to study the formal theory of Lie pseudogroups. However, the linear and nonlinear *Spencer sequences* for Lie pseudogroups, though never used in physics, largely supersede the "*Cartan structure equations*" (1905) and are quite different from the "*Vessiot structure equations*" (1903), introduced for the same purpose but still totally unknown today because they have never been acknowledged by E. Cartan or successors [14]-[16]:

		CARTAN	$\to$	*SPENCER*	*GT*
	$\nearrow$		*CAUCHY*		
LIE		$\updownarrow$	*COSSERAT, CLAUSIUS*	$\updownarrow$	$\updownarrow$
	$\searrow$		*MAXWELL*		
		VESSIOT	$\to$	*JANET*	*GR*

Meanwhile, mixing differential geometry with homological algebra, M. Kashiwara (1970) [[6] has created "*differential homological algebra*", in order to study *dif-*

ferential modules by means of *double duality* and the corresponding *extension modules* but his work has only been accessible in 1995 and a similar work of U. Oberst in 1990 [9] is only restricted to systems with constant coefficients (See [17] for more references and Zbl 1079.93001).

By chance, unexpected arguments have been introduced by the brothers E. and F. Cosserat (1909) in order to revisit elasticity and by H. Weyl (1918) in order to revisit electromagnetism through a *unique differential sequence* only depending on the structure of the *conformal group*. However, while the Cosserat brothers were only using (*translations* + *rotations*), Weyl has only been dealing with (*dilatation* + *elations*) as we shall explain [18]-[20].

The initial motivation for studying the methods used in this paper has been a 1000$ challenge proposed in 1970 by J. Wheeler in the physics department of Princeton University while the author of this paper was a visiting student of D.C. Spencer in the close-by mathematics department:

Is it possible to express the generic solutions of Einstein equations in vacuum by means of the derivatives of a certain number of arbitrary functions, like the potentials for Maxwell equations?

After recalling the negative answer we already provided in 1995 [7], the main purpose of this paper is to use the new techniques of *differential double duality* in order to revisit the mathematical foundations of general relativity and gauge theory that are leading to gravitational waves [12] [13]. We point out the fact that all the formulas presented could be obtained by computer algebra while using the packages developed by my former PhD student A. Quadrat and collaborators.

The origin of this striking but difficult paper is a series of lectures given at the Albert Einstein Institute (AEI, Potsdam, October, 23-27, 2017) "General Relativity and Gauge Theory: Beyond the Mirror" (hal-01632085, 03-11-2017) and a more applied presentation under the title "From Elasticity to Electromagnetism: Beyond the Mirror" (arXiv:1802.02430) published in [3]. In this approach, we advise the reader to have a look to the photo-elastic beam experiment (photo taken by the author) showing the link that may exist between Hooke constitutive laws in elasticity (EL) and Minkowski constitutive laws in electromagnetism (EM). We also point out the way to use high level mathematical tools and their phenomenological byproducts done by J.C. Maxwell, explaining why the interference pattern is made by hyperbolas, a result highly not evident at first sight!

We finally say that this paper is a kind of wink to the English writer Lewis Carroll who first wrote his famous book "*Alice in Wonderland*" in 1865 but added a second part six years later under the title "*Beyond the Mirror*" in which Alice, sitting in front of a mirror, is falling asleep and dreams that she is passing through this mirror, discovering the strange things that happen on the other side. After spending a life-time on general relativity (GR), it has been quite a surprise to discover the close link existing between differential homological algebra and the mathematical foundations of GR, *not found during one century* (up to our knowledge) that we now point out.

2. Parametrization

Starting with the well known linear map

$C : S_2T^* \to S_2T^* : R_{ij} \to E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij} tr(R)$ between symmetric covariant tensors, where ω is a metric with $det(\omega) \neq 0$ and $tr(R) = \omega^{rs} R_{rs}$, we may introduce the linear second order operators $Ricci : \Omega \to R$ and $Einstein : \Omega \to E$ obtained by linearization over ω and we have the relation $Einstein = C \circ Ricci$ where C does not depend on any conformal factor and is only invertible when $n \geq 3$ because $tr(E) = \frac{(2-n)}{2} tr(R)$ [21]. We recall the method used in any textbook for studying gravitational waves, which " *surprisingly* " brings the *same map* $C : \Omega \to \bar{\Omega} = \Omega - \frac{1}{2}\omega tr(\Omega)$ in order to introduce the key composite operator $\mathcal{X} : \bar{\Omega} \to \Omega \to E$ which is such that $Einstein = \mathcal{X} \circ C$.

The first goal will be to prove that only the use of *differential homological algebra*, a mixture of differential geometry (differential sequences, formal adjoint) and homological algebra (module theory, double duality, extension modules) *totally unknown by physicists*, is able to explain why the Einstein operator (with 6 terms) defined above is useless as it can be replaced by the Ricci operator (with 4 terms only) in the search for gravitational waves equations. Indeed, the *Einstein* operator is self-adjoint as we shall see later on when ω is the Minkowski metric, contrary to the *Ricci* operator [6] [8] [9] [12] [13]. Taking the respective (formal) adjoint operators (that is multiplying by convenient test functions and integrating by parts), we get:

$$\boxed{\begin{array}{l} ad(Einstein) = ad(C) \circ ad(\mathcal{X}) \Rightarrow Einstein = C \circ ad(\mathcal{X}) \\ \Rightarrow ad(\mathcal{X}) = Ricci \Rightarrow \mathcal{X} = ad(Ricci) \end{array}}$$

Meanwhile, the *Riemann* operator can be considered as a second order operator describing the *compatibility conditions* (CC) for the *Killing* operator

$\xi \in T \to \mathcal{L}(\xi)\omega = \Omega \in S_2T^*$ with standard notations where $\mathcal{L}$ is the Lie derivative [16] [22]. In this new framework, we shall prove that we no longer need to use the *Bianchi* operator as the first order CC for the *Riemann* operator. Also, we shall prove that the *relative parametrization* with *div*-type *differential constraints* needed in order to keep only the *Dalembert* operator in the wave equations has *nothing to do* with any gauge transformation in the corresponding adjoint differential sequence, but has *only to do* with the search for a *minimal parametrization*, exactly like Maxwell in 1870 for elasticity [13] [23].

Example 2.1: When $n = 2$, the stress equations become

$d_1\sigma^{11} + d_2\sigma^{12} = 0, d_1\sigma^{21} + d_2\sigma^{22} = 0$. Their second order parametrization $\sigma^{11} = d_{22}\phi$, $\sigma^{12} = \sigma^{21} = -d_{12}\phi$, $\sigma^{22} = d_{11}\phi$ has been provided by George Biddell Airy (1801-1892) in 1863. It can be simply recovered as follows:

$$\begin{array}{lll} d_1\sigma^{11} - d_2\left(-\sigma^{12}\right) = 0 & \Rightarrow & \exists \varphi, \sigma^{11} = d_2\varphi, \sigma^{12} = -d_1\varphi \\ d_2\sigma^{22} - d_1\left(-\sigma^{21}\right) = 0 & \Rightarrow & \exists \psi, \sigma^{22} = d_1\psi, \sigma^{21} = -d_2\psi \\ \sigma^{12} = \sigma^{21} \Rightarrow d_1\varphi - d_2\psi = 0 & \Rightarrow & \exists \phi, \varphi = d_2\phi, \psi = d_1\phi \end{array}$$

When constructing a long prismatic dam with concrete, we may transform a problem of 3-dimensional elasticity into a problem of 2-dimensional elasticity by supposing that the axis x^3 of the dam is perpendicular to the river with $\Omega_{ij}\left(x^1,x^2\right)$, $\forall i,j=1,2$ and $\Omega_{33}=0$ because the rocky banks of the river are supposed to be fixed and we have $d_{22}\Omega_{11}+d_{11}\Omega_{22}-2d_{12}\Omega_{12}=0$. Introducing the two *Lamé constants* (λ,μ) in order to describe the constitutive relations of an homogeneous isotropic medium, we may restrict them from the standard case $n=3$ to the case $n=2$ by setting:

$$\sigma=\frac{1}{2}\lambda\, tr(\Omega)\omega+\mu\Omega,\ tr(\Omega)=\Omega_{11}+\Omega_{22}$$

$$\Leftrightarrow\ \mu\Omega=\sigma-\frac{\lambda}{2(\lambda+\mu)}tr(\sigma)\omega,\ tr(\sigma)=\sigma^{11}+\sigma^{22}$$

even though $\sigma^{33}=\frac{1}{2}\lambda\left(\Omega_{11}+\Omega_{22}\right)=\frac{1}{2}\lambda tr(\Omega)$. Let us consider the *right square* of the diagram below with locally exact rows, where any vector bundle is simply denoted by its fiber dimension:

$$\begin{array}{ccccccc} & 2 & \xrightarrow{Killing} & 3 & \xrightarrow{Riemann} & 1 & \rightarrow 0 \\ & & & \downarrow\uparrow & & \vdots & \\ 0\leftarrow & 2 & \xleftarrow{Cauchy} & 3 & \xleftarrow{Airy} & 1 & \end{array}$$

Taking into account the linearization of the only component of the Riemann tensor over the Euclidean metric ω when $n=2$ and substituting the Airy parametrization, we obtain as in [13]:

$$tr(R)\equiv d_{11}\Omega_{22}+d_{22}\Omega_{11}-2d_{12}\Omega_{12}=0$$

$$\Rightarrow\ \mu tr(R)\equiv\frac{\lambda+2\mu}{2(\lambda+\mu)}\Delta\Delta\phi=0\ \Rightarrow\ \Delta\Delta\phi=0$$

where the linearized *scalar curvature* $tr(R)$ is allowing to define the *Riemann operator* in the previous diagram, namely the only *compatibility condition* (CC) of the Killing operator. It remains to exhibit an arbitrary homogeneous polynomial solution of degree 3 and to determine its 4 coefficients by the boundary pressure conditions on the upstream and downstream walls of the dam. The Airy potential ϕ has *nothing to do* with the perturbation Ω of the metric ω and the *Airy* operator is nothing else but the adjoint of the *Riemann* operator, that is $Airy=ad(Riemann)$.

Example 2.2: When $n=3$, we may now use the *left square* of the following diagram with locally exact rows:

$$\begin{array}{cccccc} & 3 & \xrightarrow{Killing} & 6 & \xrightarrow{Riemann} & 6 \\ & \vdots & & \downarrow\uparrow & & \vdots \\ 0\leftarrow & 3 & \xleftarrow{Cauchy} & 6 & \xleftarrow{Beltrami} & 6 \end{array}$$

where the self-adjoint operator $Beltrami=ad(Riemann)$ has been introduced by E. Beltrami in 1892. We may substitute the 3-dimensional constitutive relations with Lamé constants (λ,μ) in the Cauchy stress equations and get, when $\vec{f}\sim\vec{g}$ (*gravity*) is now the right member:

$$(\lambda+\mu)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\xi}\right)+\mu\Delta\vec{\xi}=\vec{f}\overset{\vec{\nabla}}{\Rightarrow}(\lambda+2\mu)\Delta tr(\Omega)=0\Rightarrow\Delta tr(\Omega)=0\Rightarrow\Delta tr(\sigma)=0$$

We discover at once that the origin of elastic waves is shifted by *one step backwards, from the right square to the left square* of the diagram. Indeed, using inertial forces $\vec{f}=\rho\partial^2\vec{\xi}/\partial t^2$ for a medium with mass ρ per unit volume in the right member of Cauchy stress equations because of Newton law and the vector identity $\vec{\nabla}\wedge\left(\vec{\nabla}\wedge\vec{\xi}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\xi}\right)-\Delta\vec{\xi}$, we discover the existence of two types of *elastic waves* $\vec{A}\exp i\left(\vec{k}\cdot\vec{x}-\omega t\right)$ with *wave vector* $\vec{k}$, *period* T, *pulsation* $\omega=2\pi/T$ along standard notations. We obtain thus the *longitudinal* and *transversal* waves with different speeds $v_T<v_L$, which are really existing because they are responsible for earthquakes:

$$\begin{cases}\vec{\nabla}.\vec{\xi}=0 & \Rightarrow \vec{k}\cdot\vec{A}=0 & \Rightarrow \mu\Delta\vec{\xi}=\vec{f} & \Rightarrow v_T=\sqrt{\dfrac{\mu}{\rho}}\\ \vec{\nabla}\wedge\vec{\xi}=0 & \Rightarrow \vec{k}\wedge\vec{A}=0 & \Rightarrow (\lambda+2\mu)\Delta\vec{\xi}=\vec{f} & \Rightarrow v_L=\sqrt{\dfrac{\lambda+2\mu}{\rho}}\end{cases}$$

These comments pushed the author to a systematic use of the *formal adjoint* of an operator as explained and illustrated in the platform "ideXlab" on the net.

Let us explain the origin of the definition of *extension modules* in homological algebra by means of an elementary example. With $d_{22}\xi=\eta^2, d_{12}\xi=\eta^1$ for $\mathcal{D}$, we get $d_1\eta^2-d_2\eta^1=\zeta$ for the CC $\mathcal{D}_1$. Then $ad(\mathcal{D}_1)$ is defined by $\mu^2=-d_1\lambda, \mu^1=d_2\lambda$ while $ad(\mathcal{D})$ is defined by $\nu=d_{12}\mu^1+d_{22}\mu^2$ but the CC of $ad(\mathcal{D}_1)$ are simply generated by $\nu'=d_1\mu^1+d_2\mu^2$ with $\nu=d_1\nu'$. Using operators, we have the *three* differential sequences:

$$\begin{array}{ccccc}\xi & \overset{\mathcal{D}}{\rightarrow} & \eta & \overset{\mathcal{D}_1}{\rightarrow} & \zeta\\ \nu & \overset{ad(\mathcal{D})}{\leftarrow} & \mu & \overset{ad(\mathcal{D}_1)}{\leftarrow} & \lambda\\ \uparrow & \swarrow & & & \\ \nu' & & & & \end{array}$$

where $\mathcal{D}_1$ generates the CC of $\mathcal{D}$ in the upper sequence but $ad(\mathcal{D})$ does not generate the CC of $ad(\mathcal{D}_1)$ in the lower sequence, even though $\mathcal{D}_1\circ\mathcal{D}=0\Rightarrow ad(\mathcal{D})\circ ad(\mathcal{D}_1)=0$, contrary to what happens in the Poincaré sequence for the exterior derivative used in electromagnetism for exhibiting the Maxwell equations when $n=4$. We shall see that this "*gap*" brings the need to introduce the *first extension module* $ext^1(M)$ of the differential module M determined by $\mathcal{D}$.

More generally, using the same notation for a vector bundle and its set of local sections, when $E\overset{\mathcal{D}}{\rightarrow}F$ is a given operator, its formal adjoint is $\wedge^n T^*\otimes E^*\overset{ad(\mathcal{D})}{\leftarrow}\wedge^n T^*\otimes F^*$ where E^* and F^* are respectively obtained from

E and F by inverting the transition matrices, like T and T^*. The stress is thus *not* a tensor but a 2-contravariant tensor density, that is $\sigma \in \wedge^n T^* \otimes S_2 T$ [12] [13] [17].

Before going ahead, let us prove that there may be mainly two types of differential sequences, the *Janet sequence* introduced by M. Janet in 1920 [24], having to do with the tools we have studied, and a different sequence called *Spencer sequence* introduced by D. C. Spencer in 1970 with totally different operators ([14] does not contain explicit examples while the examples in the Introduction of [15] have no relation with the core of the book). For this, if E is a vector bundle over the base X, we introduce the q-jet bundle $J_q(E)$ with sections $\xi_q:(x) \to (\xi^k(x), \xi_i^k(x), \xi_{ij}^k(x), \cdots)$ transforming like the sections $j_q(\xi):(x) \to (\xi^k(x), \partial_i \xi^k(x), \partial_{ij}\xi^k(x), \cdots)$ up to order q. The *Spencer operator* $d: J_{q+1}(E) \to T^* \otimes J_q(T)$ allows to compare these sections by considering the differences $(\partial_i \xi^k(x) - \xi_i^k(x), \partial_i \xi_j^k(x) - \xi_{ij}^k(x), \cdots)$ and so on. It can be extended to an operator:

$$\boxed{d: \wedge^s T^* \otimes R_{q+1} \to \wedge^{s+1} T^* \otimes R_q : (\xi_{\mu,I}^k(x) dx^I) \to ((\partial_i \xi_{\mu,I}^k(x) - \xi_{\mu+1_i,I}^k(x)) dx^i \wedge dx^I)}$$

with standard multi-index notation for exterior forms and one can easily check that $d \circ d = 0$. This is the reason for which we have used the same notation for the Spencer operator and the exterior derivative that are thus "*interlaced*" in the above formula. Such a notation also avoids any confusion with the ring D of differential operators that will be introduced in the next section. Its restriction to the symbol $g_{q+1} = R_{q+1} \cap S_{q+1} T^* \otimes E \subset J_{q+1}(E)$ is *minus* the Spencer map:

$$\boxed{\delta: \wedge^r T^* \otimes g_{q+1} \to \wedge^{r+1} T^* \otimes g_q : \xi_{\nu,I}^k dx^I \to \xi_{\mu+1_i,I}^k dx^i \wedge dx^I}$$

with $\delta^2 = \delta \circ \delta = 0$ because $\xi_{\mu+1_i+1_j,I}^k dx^i \wedge dx^j = 0$ with lengths $|\nu| = q+1$, $|\mu| = q$. A symbol g_q is said to be involutive if all the δ-sequences are exact at all the $\wedge^s T^* \otimes g_{q+r}$ and *finite type* if there exists r large enough such that $g_{q+r} = 0$. It is known that if g_q is involutive *and* finite type, then necessarily $g_q = 0$ [8] [22] [25]. A vector field $\xi^i(x)\partial_i$ may be written as $\xi \in T$ but, in the module framework, we must choose $\xi^i \in K$ when $\xi^i d_i \in D$, using the formal notation $d_i \xi^k$ for operators but $(\partial_i \xi^k(x) - f_i^k(x))$ for the Spencer operator when dealing with sections of $J_1(T)$.

When ω is a non-degenerate metric with Christoffel symbols γ and Levi-Civita isomorphism $j_1(\omega) \simeq (\omega, \gamma)$ while $T = T(X)$ is the tangent bundle to X, we consider the second order involutive system $R_2 \subset J_2(T)$ defined by considering the first order Killing system $\mathcal{L}(\xi)\omega = 0$, adding its first prolongation $\mathcal{L}(\xi)\gamma = 0$ and using ξ_2 instead of $j_2(\xi)$. Looking for the first order generating *compatibility conditions* (CC) $\mathcal{D}_1$ of the corresponding second order operator $\mathcal{D}$ just described, we may then look for the generating CC $\mathcal{D}_2$ of $\mathcal{D}_1$

and so on. We may proceed similarly for the injective operator $T \xrightarrow{j_2} C_0(T) = J_2(T)$, finding successively $C_0(T) \xrightarrow{D_1} C_1(T)$ and $C_1(T) \xrightarrow{D_2} C_2(T)$ induced by d. When $n = 2$ and ω is the *Euclidean* metric, we have a Lie group of isometries with the 3 infinitesimal generators $\{\partial_1, \partial_2, x^1\partial_2 - x^2\partial_1\}$. If we now consider the Weyl group defined by $\mathcal{L}(\xi)\omega = 2A\omega$ with $A = cst$ and $\mathcal{L}(\xi)\gamma = 0$, we have to add the only dilatation $x^1\partial_1 + x^2\partial_2$ and get the strict inclusions $R_2 \subset \tilde{R}_2 \subset J_2(T)$. As for the conformal system $\hat{R}_3 \subset J_3(T)$, according to [4], we have to add the two elations $\theta^1 = \frac{1}{2}\left(\left(x^1\right)^2 + \left(x^2\right)^2\right)\partial_1 + x^1x^2\partial_2$ and θ^2 obtained by exchanging x^1 with x^2. Both systems have vanishing third symbol and we have the strict inclusions $R_3 \subset \tilde{R}_3 \subset \hat{R}_3 \subset J_3(T)$ with respective dimensions $3 < 4 < 6$.

For a later use, it is important to notice that, if we define the infinitesimal conformal transformations by $\mathcal{L}(\xi)\omega = 2A(x)\omega$, $\left(\mathcal{L}(\xi)\gamma\right)_{ij}^k = \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{kr}A_r(x)$ while considering the couple (ω, γ) as a geometric object, we let the reader check that $\partial_i A - A_i = 0 \Rightarrow \partial_i A_j - \partial_j A_i = 0$ a reason for using the factor 2 and explaining the link existing between the work of Weyl in [20] and the Spencer operator.

Collecting the results and exhibiting the induced kernel upper differential sequence, we get the following commutative *fundamental diagram I* where the upper down arrows are monomorphisms while the lower down arrows are epimorphisms Φ_0, Φ_1, Φ_2 [3] [22] [25]:

$$
\begin{array}{cccccccccccc}
 & & & & & 0 & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & \hat{\Theta} & \xrightarrow{j_3} & 6 & \xrightarrow{D_1} & 12 & \xrightarrow{D_2} & 6 & \rightarrow 0 & \\
 & 0 & \rightarrow & \tilde{\Theta} & \xrightarrow{j_3} & 4 & \xrightarrow{D_1} & 8 & \xrightarrow{D_2} & 4 & \rightarrow 0 & \\
 & 0 & \rightarrow & \Theta & \xrightarrow{j_3} & 3 & \xrightarrow{D_1} & 6 & \xrightarrow{D_2} & 3 & \rightarrow 0 & \textit{Spencer} \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & 2 & \xrightarrow{j_3} & 20 & \xrightarrow{D_1} & 30 & \xrightarrow{D_2} & 12 & \rightarrow 0 & \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \\
0 \rightarrow & \Theta & \rightarrow & 2 & \xrightarrow{\mathcal{D}} & 17 & \xrightarrow{\mathcal{D}_1} & 24 & \xrightarrow{\mathcal{D}_2} & 9 & \rightarrow 0 & \textit{Janet} \\
0 \rightarrow & \tilde{\Theta} & \rightarrow & 2 & \xrightarrow{\mathcal{D}} & 16 & \xrightarrow{\mathcal{D}_1} & 22 & \xrightarrow{\mathcal{D}_2} & 8 & \rightarrow 0 & \\
0 \rightarrow & \hat{\Theta} & \rightarrow & 2 & \xrightarrow{\mathcal{D}} & 14 & \xrightarrow{\mathcal{D}_1} & 18 & \xrightarrow{\mathcal{D}_2} & 6 & \rightarrow 0 & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & & 0 & &
\end{array}
$$

It follows that "*Spencer and Janet play at see-saw*", the dimension of each *Janet bundle* being decreased by the same amount as the dimension of the corresponding *Spencer bundle* is increased. The Poincaré sequence for the exterior derivative d is $\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \rightarrow 0$ but it is only at the end of the paper that we shall understand the link with Maxwell equations when $n=4$.

3. Differential Modules

Let A be a *unitary ring*, that is $1,a,b \in A \Rightarrow a+b, ab \in A, 1a = a1 = a$ and even an *integral domain* ($ab=0 \Rightarrow a=0$ or $b=0$) with *field of fractions* $K=Q(A)$. However, we shall not always assume that A is commutative, that is ab may be different from ba in general for $a,b \in A$. We say that $M={}_A M$ is a *left module* over A if $x,y \in M \Rightarrow ax, x+y \in M, \forall a \in A$ or a *right module* M_B over B if the operation of B on M is $(x,b) \rightarrow xb, \forall b \in B$. If M is a left module over A and a right module over B with $(ax)b = a(xb), \forall a \in A, \forall b \in B, \forall x \in M$, then we shall say that $M={}_A M_B$ is a *bimodule*. Of course, $A={}_A A_A$ is a bimodule over itself. We define the *torsion submodule* $t(M)=\{x \in M \mid \exists 0 \neq a \in A, ax=0\} \subseteq M$ and M is a *torsion module* if $t(M)=M$ or a *torsion-free module* if $t(M)=0$. We denote by $hom_A(M,N)$ the set of morphisms $f: M \rightarrow N$ such that $f(ax)=af(x)$. We finally recall that a sequence of modules and maps is exact if the kernel of any map is equal to the image of the map preceding it. When A is commutative, $hom(M,N)$ is again an A-module for the law $(bf)(x)=f(bx)$ as we have $(bf)(ax)=f(bax)=f(abx)=af(bx)=a(bf)(x)$. In the non-commutative case, things are more complicate and, given ${}_A M$ and ${}_A N_B$, then $hom_A(M,N)$ becomes a right module over B for the law $(fb)(x)=f(x)b$ (See [2] [8] [17] for more details or [12] [26] [27] for homological algebra).

Definition 3.1: A module F is said to be *free* if it is isomorphic to a (finite) power of A called the *rank* of F over A and denoted by $rk_A(F)$ while the rank $rk_A(M)$ of a module M is the rank of a maximum free submodule $F \subset M$. It follows from this definition that M/F is a torsion module. In the sequel we shall only consider *finitely presented* modules, namely *finitely generated* modules defined by exact sequences of the type $F_1 \xrightarrow{d_1} F_0 \xrightarrow{p} M \rightarrow 0$ where F_0 and F_1 are free modules of finite ranks m_0 and m_1 often denoted by m and p in examples. A module P is called *projective* if there exists a free module F and another (projective) module Q such that $P \oplus Q \simeq F$.

Proposition 3.2: For any short exact sequence $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$, we have the important relation $rk_A(M)=rk_A(M')+rk_A(M'')$, even in the non-commutative case. As a byproduct, if M admits a finite length free *resolution* $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{p} M \rightarrow 0$, we may define the *Euler-Poincaré characteristic* $\chi_A(M)=\sum_r (-1)^r rk_A(F_r)=rk_A(M)$.

The following classical proposition is quite useful:

Proposition 3.3: A short exact sequence *splits* if one of the following three conditions holds:

$$0 \rightarrow M' \underset{f}{\overset{\overset{u}{\leftarrow}}{\rightarrow}} M \underset{g}{\overset{\overset{v}{\leftarrow}}{\rightarrow}} M'' \rightarrow 0$$

- There exists a monomorphism $v: M'' \to M$ called *lift* of g and such that $g \circ v = id_{M''}$.
- There exists an epimorphism $u: M \to M'$ called *lift* of f and such that $u \circ f = id_{M'}$.
- There exist isomorphisms $\varphi = (u, g): M \to M' \oplus M''$ and $\psi = f + v: M' \oplus M'' \to M$ that provide an isomorphism $M \simeq M' \oplus M''$ with $f \circ u + v \circ g = id_M$ and thus $ker(u) = im(v)$.

These conditions are automatically satisfied if M'' is free or projective.

Using the notation $M^* = hom_A(M, A)$, for any morphism $f: M \to N$, we shall denote by $f^*: N^* \to M^*$ the morphism which is defined by $f^*(h) = h \circ f$, $\forall h \in hom_A(N, A)$ and satisfies $rk_A(f) = rk_A(im(f)) = rk_A(f^*)$, $\forall f \in hom_A(M, N)$. We may take out M in order to obtain the *deleted sequence* $\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$ and apply $hom_A(\bullet, A)$ in order to get the sequence $\cdots \xleftarrow{d_2^*} F_1^* \xleftarrow{d_1^*} F_0^* \leftarrow 0$ [8] [12] [17] [26] [27].

Proposition 3.4: If we define the *extension modules* $ext_A^0(M) = ker(d_1^*) = hom_A(M, A) = M^*$ and $ext^i(M) = ext_A^i(M) = ker(d_{i+1}^*)/im(d_i^*)$, $\forall i \geq 1$, they do not depend on the resolution chosen and are torsion modules for $i \geq 1$.

We now study modules over the ring $D = K[d_1, \cdots, d_n] = K[d]$ of differential operators with coefficients in a differential field K with n commuting derivations $(\partial_1, \cdots, \partial_n)$, also called D-modules. Any operator is of the form $P = \sum a^\mu d_\mu$ and its (formal) adjoint is $ad(P) = \sum (-1)^{|\mu|} d_\mu a^\mu$ but we may also use the rule $ad(a) = a$, $ad(d_i) = -d_i$, $ad(PQ) = ad(Q)ad(P)$ with $d_i a = a d_i + \partial_i a$.

In a more general setting, if a differential operator $\xi \xrightarrow{\mathcal{D}} \eta$ is given, a *direct problem* is to find generating *compatibility conditions* (CC) as an operator $\eta \xrightarrow{\mathcal{D}_1} \zeta$ such that $\mathcal{D}\xi = \eta \Rightarrow \mathcal{D}_1 \eta = 0$. Conversely, given $\eta \xrightarrow{\mathcal{D}_1} \zeta$, the *inverse problem* will be to look for $\xi \xrightarrow{\mathcal{D}} \eta$ such that $\mathcal{D}_1$ generates the CC of $\mathcal{D}$ and we shall say that $\mathcal{D}_1$ *is parametrized by* $\mathcal{D}$ *if such an operator* $\mathcal{D}$ *is existing*. Of course the main problem will be to solve these two problems for *all* the physical explicit or physical operators that we shall meet.

Theorem 3.5: If M is the differential module defined by a differential operator $\mathcal{D}$ and N is the differential operator similarly defined by $ad(\mathcal{D})$, we have the crucial formula $t(M) = ext^1(N)$ and $\mathcal{D}$ can be parametrized if and only if $t(M) = 0$. As $ad(ad(\mathcal{D})) = \mathcal{D}$, we have also $t(N) = ext^1(M)$.

Introducing the morphism $\epsilon: M \to M^{**}$ such that $\epsilon(m)(f) = f(m)$, $\forall m \in M$, $\forall f \in M^*$ and defining the differential modules M and N from $\mathcal{D}_1$ instead of $\mathcal{D}$ (*care*), we obtain:

Corollary 3.6: (*Reflexivity test*) Checking whether M is *reflexive* or not, that is to find out a parametrization if $ext^1(N)=t(M)=0$ which *can be again parametrized* amounts to prove that ϵ is an isomorphism of D-modules in the following long exact sequence:

$$\boxed{0 \to ext^1(N) \to M \xrightarrow{\epsilon} M^{**} \to ext^2(N) \to 0}$$

$$\mathcal{D}_1 \text{ parametrized by } \mathcal{D} \Leftrightarrow \mathcal{D}_1 = \mathcal{D}_1' \Leftrightarrow \epsilon \text{ injective} \Leftrightarrow ext^1(N)=0 \Leftrightarrow t(M)=0$$

$$\mathcal{D} \text{ parametrized by } \mathcal{D}_{-1} \Leftrightarrow \mathcal{D} = \mathcal{D}' \Leftrightarrow \epsilon \text{ surjective} \Leftrightarrow ext^2(N)=0$$

The 5 steps are described as follows, where $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$, $\mathcal{D}_1'$ generates the CC of $\mathcal{D}=ad(ad(\mathcal{D}))$, $ad(\mathcal{D}_{-1})$ generates the CC of $ad(\mathcal{D})$ and $\mathcal{D}'$ generates the CC of $\mathcal{D}_{-1}$:

$$\begin{array}{|ccccccccc|}
 & & & & & \eta' & & \zeta' & \boxed{5} \\
 & & & & \nearrow^{\mathcal{D}'} & & \nearrow^{\mathcal{D}_1'} & & \\
\boxed{4} & \phi & \xrightarrow{\mathcal{D}_{-1}} & \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta & \boxed{1} \\
 & & & & & & & & \\
\boxed{3} & \theta & \xleftarrow{ad(\mathcal{D}_{-1})} & \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda & \boxed{2}
\end{array}$$

Corollary 3.7: In the differential module framework, if $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{p} M \to 0$ is a finite free presentation of $M = coker(\mathcal{D}_1)$ with $t(M)=0$, then we may obtain an exact sequence $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{\mathcal{D}} E$ of free differential modules where $\mathcal{D}$ is the parametrizing operator. However, there may exist other parametrizations $F_1 \xrightarrow{\mathcal{D}_1} F_0 \xrightarrow{\mathcal{D}'} E'$ called *minimal parametrizations* such that $coker(\mathcal{D}')$ is a torsion module and we have thus $rk_D(M) = rk_D(E')$.

Example 3.8: When $n=3$, the *div* operator can be parametrized by the *curl* operator which can be itself parametrized by the *grad* operator. However, the new minimal parametrization $-d_3\xi^2=\eta^1$, $d_3\xi^1=\eta^2$, $d_1\xi^2-d_2\xi^1=\eta^3$ $\Rightarrow d_1\eta^1+d_2\eta^2+d_3\eta^3=0$ cannot be again parametrized.

The differential module M defined by the first set of Maxwell equations $\wedge^2 T^* \xrightarrow{d} \wedge^3 T^*$ is reflexive because we have the adjoint differential sequence $0 \leftarrow \wedge^4 T^* \xleftarrow{d} \wedge^3 T^* \xleftarrow{d} \wedge^2 T^* \xleftarrow{d} \wedge^1 T^*$. Similarly, the differential module defined by the second set of Maxwell equations is also reflexive because it is defined by the adjoint operator $\wedge^3 T^* \xleftarrow{d} \wedge^2 T^*$ which is the adjoint of the parametrization $\wedge^1 T^* \xrightarrow{d} \wedge^2 T^*$ defined by $A \to dA = F$.

Similarly, it is even less evident that the differential module defined by the Cauchy operator is reflexive as we shall see when $n=3$ but only torsion-free as we saw when $n=2$.

Theorem 3.9: The Einstein operator, namely the linearization of the Einstein tensor over the locally constant Minkowski metric ω, is self-adjoint and *Cauchy* = *ad*(*Killing*) is parametrized by *ad*(*Ricci*).

Proof: First of all, the linearizations of the Christoffel symbols γ_{ij}^k and the Riemann tensor $\rho_{l,ij}^k$ are:

$$\Gamma_{ij}^k = \frac{1}{2}\omega^{kr}\left(d_i\Omega_{rj} + d_j\Omega_{ir} - d_r\Omega_{ij}\right) \Rightarrow R_{l,ij}^k = d_i\Gamma_{lj}^k - d_j\Gamma_{li}^k.$$

Setting $tr(\Omega) = \omega^{rs}\Omega_{rs}$, we deduce $\Gamma_{ri}^r = \frac{1}{2}d_i tr(\Omega)$ and get:

$$\boxed{\begin{array}{l} 2R_{ij} = \omega^{rs}d_{rs}\Omega_{ij} + d_{ij}tr(\Omega) - \omega^{rs}\left(d_{ri}\Omega_{sj} + d_{rj}\Omega_{si}\right) \\ \Rightarrow tr(R) = \omega^{rs}d_{rs}tr(\Omega) - \omega^{rs}\omega^{ij}d_{ri}\Omega_{sj} \end{array}}$$

Setting $E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij}tr(R)$ with $tr(R) = \omega^{rs}R_{rs}$, we obtain the linear Einstein operator (6 terms):

$$\boxed{2E_{ij} = \omega^{rs}d_{rs}\Omega_{ij} + d_{ij}tr(\Omega) - \omega^{rs}\left(d_{ri}\Omega_{sj} + d_{rj}\Omega_{si}\right) - \omega_{ij}\left(\omega^{rs}d_{rs}tr(\Omega) - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv}\right)}$$

It is essential to notice that the *Ricci operator is not self-adjoint* because we have for example:

$$\lambda^{ij}\left(\omega^{rs}d_{ij}\Omega_{rs}\right) \xrightarrow{ad} \left(\omega^{rs}d_{ij}\lambda^{ij}\right)\Omega_{rs} = \left(\omega^{ij}d_{rs}\lambda^{rs}\right)\Omega_{ij}$$

and ad provides a term appearing in $-\omega_{ij}tr(R)$ but *not* in $2R_{ij}$.

After two integrations by parts, we obtain successively for *ad*(*Ricci*):

$$\Omega_{ij}\omega^{rs}d_{rs}\lambda^{ij} + tr(\Omega)d_{ij}\lambda^{ij} - \omega^{rs}\left(\Omega_{sj}d_{ri}\lambda^{ij} + \Omega_{si}d_{rj}\lambda^{ij}\right)$$

$$\boxed{d_{ij}(\omega^{ij}\lambda^{rs} + \omega^{rs}\lambda^{ij} - \omega^{sj}\lambda^{ri} - \omega^{ri}\lambda^{sj}) = \sigma^{rs} \Rightarrow d_r\sigma^{rs} = 0}$$

Setting $tr(\lambda) = \omega_{ij}\lambda^{ij}$, we may change the indices in order to factor out Ω_{ij} and finally get the adjoint of the Einstein operator:

$$\omega^{rs}d_{rs}\lambda^{ij} + \omega^{ij}d_{rs}\lambda^{rs} - \left(\omega^{ri}d_{rs}\lambda^{sj} + \omega^{rj}d_{rs}\lambda^{si}\right) - \omega^{ij}\omega^{rs}d_{rs}tr(\lambda) + \omega^{ri}\omega^{sj}d_{rs}tr(\lambda)$$

the 6 terms being exchanged between themselves with $(1,2,3,4,5,6) \to (1,6,3,4,5,2)$.

Example 3.10: When $n=3$ and the Euclidean metric, we obtain:

$$\begin{aligned} tr(R) = {} & \left(d_{11}\Omega_{22} + d_{11}\Omega_{33} + d_{22}\Omega_{11} + d_{22}\Omega_{33} + d_{33}\Omega_{11} + d_{33}\Omega_{22}\right) \\ & - 2\left(d_{12}\Omega_{12} + d_{13}\Omega_{13} + d_{23}\Omega_{23}\right) \end{aligned}$$

$$2R_{12} = 2E_{12} = d_{33}\Omega_{12} + d_{12}\Omega_{33} - d_{13}\Omega_{23} - d_{23}\Omega_{13}$$

$$2R_{11} = \left(d_{22} + d_{33}\right)\Omega_{11} + d_{11}\left(\Omega_{22} + \Omega_{33}\right) - 2\left(d_{12}\Omega_{12} + d_{13}\Omega_{13}\right)$$

$$-2E_{11} = d_{22}\Omega_{33} + d_{33}\Omega_{22} - 2d_{23}\Omega_{23}$$

We let the reader check that eight to twelve terms are disappearing each time, a reason for which nobody saw that the Einstein equations had been written *exactly* (up to sign) by E. Beltrami in 1892 in order to parametrize the Cauchy stress equations while using 6 stress functions $\phi_{ij} = \phi_{ji}$ in place of $\Omega_{ij} = \Omega_{ji}$, 25 years before Einstein [3] [12] [13]. *The comparison needs no comment!*

Remark 3.11: When $n=3$, we have already noticed in the first page of [2] that one has to take into account the factors "2" in the duality summation with Lagrange multipliers $\left(\lambda^{ij} = \lambda^{ji}\right)$ which is used in order to exhibit the operator

$ad(Einstein)$, namely (Compare to [11]):

$$\lambda^{ij}E_{ij}=\lambda^{11}E_{11}+2\lambda^{12}E_{12}+2\lambda^{13}E_{13}+\lambda^{22}E_{22}+2\lambda^{23}E_{23}+\lambda^{33}E_{33}$$

In the present situation, this is sufficient in order to obtain a self-adjoint operator as follows:

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -d_{33} & 2d_{23} & -d_{22} \\ 0 & 2d_{33} & -2d_{23} & 0 & -2d_{13} & 2d_{12} \\ 0 & -2d_{23} & 2d_{22} & 2d_{13} & -2d_{12} & 0 \\ -d_{33} & 0 & 2d_{13} & 0 & 0 & -d_{11} \\ 2d_{23} & -2d_{13} & -2d_{12} & 0 & 2d_{11} & 0 \\ -d_{22} & 2d_{12} & 0 & -d_{11} & 0 & 0 \end{pmatrix}\begin{pmatrix}\Omega_{11}\\ \Omega_{12}\\ \Omega_{13}\\ \Omega_{22}\\ \Omega_{23}\\ \Omega_{33}\end{pmatrix}=\begin{pmatrix}E_{11}\\ 2E_{12}\\ 2E_{13}\\ E_{22}\\ 2E_{23}\\ E_{33}\end{pmatrix}$$

We shall finally prove below that the *Einstein parametrization* of the stress equations is neither canonical nor minimal in the following diagrams ([23]):

$$\begin{array}{ccccccccccccc} & & 4 & \xrightarrow{Killing} & 10 & \xrightarrow{Riemann} & 20 & \xrightarrow{Bianchi} & 20 & \rightarrow & 6 & \rightarrow & 0 \\ & & & & \| & & \downarrow & & \downarrow & & & & \\ & & & & 10 & \xrightarrow{Einstein} & 10 & \xrightarrow{div} & 4 & \rightarrow & 0 & & \\ & & & & & & & & & & & & \\ 0 & \leftarrow & 4 & \xleftarrow{Cauchy} & 10 & \xleftarrow{Beltrami} & 20 & \xleftarrow{Lanczos} & 20 & & & & \\ & & & & \| & & \uparrow & & & & & & \\ & & & & 10 & \xleftarrow{Einstein} & 10 & & & & & & \end{array}$$

The upper *div* induced by *Bianchi* has *strictly nothing to do* with the lower *Cauchy* operator, contrary to what is still believed today while the 10 *on the right* of the lower diagram has *strictly nothing to do* with the perturbation of a metric which is the 10 *on the left* in the upper diagram. It also follows that the Einstein equations in vacuum cannot be parametrized as we have the following diagram of operators recapitulating the five steps of the parametrizability criterion (A computer agebra exhibition of this result can be easily provided):

$$\begin{array}{ccccccc} & & & & & & 20 \quad \boxed{5} \\ & & & & & \nearrow^{Riemann} & \\ \boxed{4} & 4 & \xrightarrow{Killing} & 10 & \xrightarrow{Einstein} & 10 & \boxed{1} \\ & & & & & & \\ \boxed{3} & 4 & \xleftarrow{Cauchy} & 10 & \xleftarrow{Einstein} & 10 & \boxed{2} \end{array}$$

As a byproduct, we are facing *only two* possibilities, both leading to a contradiction [16]:

1) If we use the operator $S_2T^* \xrightarrow{Einstein} S_2T^*$ in the “geometrical” setting of H. Poincaré, the S_2T^* on the left has indeed *someting to do* with the perturbation Ω of the metric ω *by definition* but the S_2T^* on the right has *strictly nothing to do* with the stress which is a tensor density.

2) If we use the adjoint operator $\wedge^n T^* \otimes S_2 T \overset{Einstein}{\leftarrow} \wedge^n T^* \otimes S_2 T$ in the "physical " setting of H. Poincaré, then $\wedge^n T^* \otimes S_2 T$ on the left has of course *something to do* with the stress σ but the $\wedge^n T^* \otimes S_2 T$ on the right has *strictly nothing to do* with a perturbation of the metric as it is just the definition of the stress functions ϕ allowing to parametrize the Cauchy operator for example.

These purely mathematical results question the origin and existence of gravitational waves. From what has been proved at the beginning of the paper, we now prove that the parametrization of the *Cauchy* operator by $ad(Ricci)$ is neither canonical nor minimal in the following diagrams:

$$\begin{array}{cccccccc}
 & 4 & \overset{Killing}{\rightarrow} & 10 & \overset{Riemann}{\rightarrow} & 20 \\
 & & & \parallel & & \downarrow \\
 & & & 10 & \overset{Ricci}{\rightarrow} & 10 \\
 & & & & & \\
0 \leftarrow & 4 & \overset{Cauchy}{\leftarrow} & 10 & \overset{Beltrami}{\leftarrow} & 20 \\
 & & & \parallel & & \uparrow \\
 & & & 10 & \overset{ad(Ricci)}{\leftarrow} & 10
\end{array}$$

Corollary 3.12: When $n=2$, we already met the differential module M defined by the second order system $d_{22}\xi = 0$, $d_{12}\xi = 0$). In this case, we discover easily that $t(M) = M$ is indeed a torsion module. However, the differential module N is defined by the single PD equation $d_{12}\mu^1 + d_{22}\mu^2 = 0$ and $T(N) \subset N$ with a strict inclusion and $t(N)$ is generated by $\nu' = d_1\mu^1 + d_2\mu^2$.

When $n \geq 4$, as the extension modules are torsion modules, each component of the Weyl tensor is a torsion element killed by the *Dalembert* operator whenever the Einstein equations in vacuum are satisfied by the metric. Differentiating the Bianchi identities, there exists a second order operator Q such that we have an identity describing the so-called *Lichnerowicz wave equations*:

$$\boxed{\Box\, R_{kl,ij} = d_i(d_k R_{lj} - d_l R_{kj}) - d_j(d_k R_{li} - d_l R_{ki}) \Rightarrow \Box \circ Weyl = Q \circ Ricci}$$

Remark 3.13: When $n=3$, using the *Lamé constants* (λ, μ) for homogeneous isotropic medium like iron or concrete, it can be found in any textbook of elasticity that the *constitutive relations* are described by the six linear relations $2\sigma_{ij} = \lambda tr(\Omega)\omega_{ij} + 2\mu\Omega_{ij}$ by introducing the *trace* $\omega^{ij}\Omega_{ij}$ of Ω. The corresponding invertible 6 × 6 matrix $Cst(\lambda, \mu)$ is even symmetric because the underlying *energy of deformation* can be described by a quadratic Lagrangian. Contracting by ω^{ij} we obtain $2tr(\sigma) = (3\lambda + 2\mu)tr(\Omega)$ and obtain the inverse relations $\mu\Omega_{ij} = \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} tr(\sigma)\omega_{ij}$.

When $n=2$, similar relations do not exist for plane elasticity unless we are in a particular situation. Indeed, when constructing a dam with concrete between two fixed rocky banks and over a fixed rocky bottom, we may suppose that, *in the* *main central part of the dam*, vertical slices remain vertical slices that obey to 2-dimensional elasticity (See the picture in the Introduction of [17], p 42). Using local coordinates x^3 along the horizontal axis joining the banks, it is important to notice that we may have $\Omega_{33}=0$ even though $2\sigma_{33}=\lambda(\Omega_{11}+\Omega_{22})\neq 0$ because the dam has a tendency to expand horizontally and push against the banks. We obtain therefore:

$$2\sigma_{11}=(\lambda+2\mu)\Omega_{11}+\lambda\Omega_{22},\ 2\sigma_{12}=2\sigma_{21}=2\mu\Omega_{12},\ 2\sigma_{22}=\lambda\Omega_{11}+(\lambda+2\mu)\Omega_{22}$$

$$Cst=\begin{Bmatrix}(\lambda+2\mu) & 0 & \lambda\\ 0 & 2\mu & 0\\ \lambda & 0 & (\lambda+2\mu)\end{Bmatrix}$$

Introducing now the second order operator $Airy=ad(Riemann)$ in the commutative diagram:

$$\begin{array}{ccccccc} & 2 & \xrightarrow{Killing} & 3 & \xrightarrow{Riemann} & 1\\ & & & \uparrow Cst^{-1} & & \uparrow\\ 0\leftarrow & 2 & \xleftarrow{Cauchy} & 3 & \xleftarrow{Airy} & 1\end{array}$$

we obtain the Airy parametrization of the $Cauchy=ad(Killing)$ operator in a quite unusual way:

$$\phi\to\left(\sigma^{11}=d_{22}\phi,\sigma^{12}=-d_{12}\phi,\sigma^{22}=d_{11}\phi\right)$$

as in Example 2.1. With $\Delta=d_{11}+d_{22}$, the idea is now to invert the constitutive relation and to consider the composite operator $Riemann\circ Const^{-1}\circ Airy$ which is of order 4 as follows:

$$\boxed{d_{22}\Omega_{11}+d_{11}\Omega_{22}-2d_{12}\Omega_{12}=\frac{2(\lambda+\mu)}{3\lambda+2\mu}\Delta^2\phi=0\Rightarrow\Delta^2\phi=0\Rightarrow\Delta^2\sigma_{ij}=0,\forall i,j\leq 2}$$

Surprisingly, the vanishing kernel does not depend anymore on the Lamé constants. It follows that the Airy function parametrizing any elastic stress σ killed by the *Cauchy* operator must be biharmonic with harmonic trace $\Delta tr(\sigma)=0$.

When $n=3$, our purpose is finally to prove that such a technical result can be extended to 3-dimensional elasticity along methods first discovered by Beltrami in 1892 [13], setting simply Ω in place of $tr(\Omega)$ and Σ in place of $tr(\sigma)$. Using the formulas of Theorem 3.9, Example 3.10 and Remark 3.11, let us suppose that $\Omega\in ker(Killing)$ as follows with the previous constitutive relation and its inverse:

$$2R_{ij}\equiv\Delta\Omega_{ij}+d_{ij}\Omega-\omega^{rs}\left(d_{ri}\Omega_{sj}+d_{rj}\Omega_{si}\right)=0$$

Substituting $\mu\Omega_{ij}=\sigma_{ij}-\frac{\lambda}{3\lambda+2\mu}\Sigma\omega_{ij}$ and introducing the Poisson coefficient $\nu=\frac{\lambda}{2(\lambda+\mu)}$, we obtain:

$$\Delta\sigma_{ij} + d_{ij}\Sigma - \omega^{rs}\left(d_{ri}\sigma_{sj} + d_{rj}\sigma_{si}\right) = \frac{\nu}{1+\nu}\left(\omega_{ij}\Delta\Sigma + d_{ij}\Sigma\right)$$

However, we have $d_r\sigma_s^r = 0$ because $\sigma \in ker(Cauchy)$ by assumption and we obtain the so-called Beltrami equations (1892) that can be found in any textbook on elasticity theory or even on the net:

$$\boxed{\Delta\sigma_{ij} + \frac{1}{1+\nu}d_{ij}\Sigma = 0 \Rightarrow \Delta^2\sigma^{ij} = 0, \forall i, j \le 3}$$

We thus obtain when $n = 3$ the same diagram as the one we already found when $n = 2$:

$$\begin{array}{cccccc} & 3 & \xrightarrow{Killing} & 6 & \xrightarrow{Riemann} & 6 \\ & & & \uparrow Cst^{-1} & & \uparrow \\ 0 \leftarrow & 3 & \xleftarrow{Cauchy} & 6 & \xleftarrow{Beltrami} & 6 \end{array}$$

However, such a result cannot exist for $n = 4$ because we have $n^2\left(n^2-1\right)/12 \ge n(n+1)/2$, $\forall n \ge 3$. It must also be noticed that the Riemann operator is also self-adjoint when $n = 3$ as we proved in [13] and that, *in fact*, Beltrami had been using the Einstein operator for parametrizing the Cauchy operator "*backwards*". The only possibility to extend these results for $n \ge 3$ is to use the fact that extension modules do not depend on the resolution used in order to define a differential module *namely the Killing module* and that the corresponding Spencer sequence is isomorphic to the tensor product of the Poincaré exterior differential sequence by a finite dimensional Lie algebra... but this is another story [12].

We end this section by applying these results in order to study gravitational waves as in [21]. For this, let us consider again the linear (symmetric and thus self-adjoint) map $C : S_2T^* \to S_2T^*$ already defined in the beginning of section 2. and exhibit the following commutative diagram:

$$\begin{array}{ccc} \Omega & \xrightarrow{Ricci} & R \\ \downarrow C & \searrow Einstein & \downarrow C \\ \bar{\Omega} & \xrightarrow{\mathcal{X}} & E \end{array} \quad \xrightarrow{ad} \quad \begin{array}{ccc} \sigma & \xleftarrow{ad(Ricci)} & \lambda \\ \uparrow C & \nwarrow ad(Einstein) & \uparrow C \\ \bar{\sigma} & \xleftarrow{ad(\mathcal{X})} & \bar{\lambda} \end{array}$$

in which we have now used C instead of Cst. We have indeed $Einstein = C \circ Ricci = \mathcal{X} \circ C$ by definition as in [21]. We have proved in Section 2 that the *only* coherent way to compare these two diagrams is to set $\mathcal{X} = ad(Ricci) \Leftrightarrow ad(\mathcal{X}) = Ricci$ in order to discover that the diagram on the right is just the "*turnover*" of the diagram on the left but this "*game with mirror*" is simply proving that gravitational waves cannot exist because, *again by a kind of "mirror effect"*, the *Cauchy* operator on the left of the lower sequence has *strictly nothing to do* with the *div* operator induced by the Bianchi identities not even appearing in the induced upper sequence! It is also coming from the fact that the adjoint functor is reversing the composition because $ad(P \circ Q) = ad(Q) \circ ad(P)$ for any two operators by construction [12].

4. Gauge Theory

Gauging procedure: If $y=a(t)x+b(t)$ with $a(t)$ a time depending orthogonal matrix (*rotation*) and $b(t)$ a time depending vector (*translation*) describes the movement of a rigid body in $\mathbb{R}^3$, then the projection of the speed $v=\dot{a}(t)x+\dot{b}(t)$ in an orthogonal frame fixed in the body is $a^{-1}v=\left(a^{-1}\dot{a}\right)x+a^{-1}\dot{b}$ and the kinetic energy is a quadratic function of the 1-forms $a^{-1}\dot{a}$ and $a^{-1}\dot{b}$.

More generally, we may consider a map $a:X\rightarrow G:x\rightarrow a(x)$, introduce the tangent mapping $T(a):T=T(X)\rightarrow T(G):dx\rightarrow da=\frac{\partial a}{\partial x}dx$ and consider the family of left invariant 1-forms $a^{-1}da=A=\left(A_i^\tau(x)dx^i\right)$ with value in the *Lie algebra* $\mathcal{G}=T_e(G)$, the tangent space of G at the identity $e\in G$ with *structure constants* $c=\left(c_{\rho\sigma}^\tau\right)$ for $1\le\rho,\sigma,\tau\le p$. We may introduce the 2-forms $\partial_iA_j^\tau-\partial_jA_i^\tau-c_{\rho\sigma}^\tau A_i^\rho A_j^\sigma=F_{ij}^\tau$ with value in $\mathcal{G}$, simply denoted by $dA-[A,A]=F$, and we have $A=a^{-1}da\Leftrightarrow F=0$ by pulling back on X the *Maurer-Cartan equations* (MC) on G [19].

In 1956, at the birth of GT, the above notations were coming from the EM potential A and EM field $dA=F$ of relativistic Maxwell theory. Indeed, $G=U(1)$ (unit circle in the complex plane) $\rightarrow dim(\mathcal{G})=1$ was the *only possibility* to get a 1-form A and a 2-form F when $c=0$. Such a choice convinced people that the EM field F should be related to the curvature introduced by Cartan, as a 2-form with value in a Lie algebra, exactly like the Riemann tensor is considered as a 2-form with value in rotation and that *curvature + torsion* should be therefore a generalization of curvature alone, roughly that the "*field*" should be a section of the Spencer bundle C_2.

On the contrary, in the conformal framework, that is when G *is acting on* X, the second order jets (*elations*) $\xi_{ij}^k=\delta_i^kA_j+\delta_j^kA_i-\omega_{ij}\omega^{kr}A_r\Rightarrow\xi_{ri}^r=nA_i$ behave like the 1-form $A_i(x)dx^i$ and the corresponding part of the Spencer operator d is a 1-form with value in 1-form, that is a $(1,1)$-covariant tensor providing the EM field as a 2-form by skewsymmetrization. This result, namely to construct lagrangians on the image of the induced Spencer operator D_1, is thus *perfectly coherent* with rigid body dynamics, Cosserat elasticity and Maxwell theory but in *total contradiction* with GT because $U(1)$ is not acting on space-time and there is a *shift by one step* in the interpretation of the Poincaré sequence involved because the fields are now described by 1-forms [2] [11].

Gauging procedure revisited: Finally, we may extend the action $y=f(x,a)$ to $y_q=j_q(f)(x,a)$ in order to eliminate the parameters when q is large enough. In this case, we may set $f(x)=f(x,a(x))$ and $f_q(x)=j_q(f)(x,a(x))$ in order to obtain $a(x)=a=cst\Leftrightarrow f_q=j_q(f)$ because $df_{q+1} = j_1(f_q) - f_{q+1} = \left(\frac{\partial f_q(x, a(x))}{\partial a^\tau} \partial_i a^\tau(x) \right)$ and the matrix involved has maximum rank p.

5. Cosserat Versus Weyl

Computing the formal adjoint $ad(D_1)$ of the first Spencer operator D_1 induced by d for the group of rigid motions when $n = 2$, we get [28]:

$$\sigma^{11}(\partial_1 \xi_1 - \xi_{1,1}) + \sigma^{12}(\partial_2 \xi_1 - \xi_{1,2}) + \sigma^{21}(\partial_1 \xi_2 - \xi_{2,1}) + \sigma^{22}(\partial_2 \xi_2 - \xi_{2,2}) + \mu^{12,r}(\partial_r \xi_{1,2} - \xi_{1,2r})$$

Integrating by parts with $\xi_{1,1} = 0, \xi_{1,2} + \xi_{2,1} = 0, \xi_{2,2} = 0 \Rightarrow \xi_{1,2r} = 0$, we obtain *at once* and *exactly* the *Cosserat couple-stress equations*:

$$\boxed{\partial_1 \sigma^{11} + \partial_2 \sigma^{12} = f^1,\ \partial_1 \sigma^{21} + \partial_2 \sigma^{22} = f^2,\ \partial_r \mu^{12,r} + \sigma^{12} - \sigma^{21} = m^{12}}$$

allowing to have now a non-symmetrical stress and a new *first order parametrization*:

$$\boxed{\begin{array}{l} \sigma^{11} = \partial_2 \phi^1,\ \sigma^{12} = -\partial_1 \phi^1,\ \sigma^{21} = -\partial_2 \phi^2,\ \sigma^{22} = \partial_1 \phi^2, \\ \mu^{12,1} = \partial_2 \phi^3 + \phi^1,\ \mu^{12,2} = -\partial_1 \phi^3 - \phi^2 \end{array}}$$

with a potential $(\phi^1, \phi^2, \phi^3) \in \wedge^2 T^* \otimes \wedge^2 T \otimes R_2^* \simeq R_2^*$ with $dim(R_2) = 3$. These equations can be extended by adding the only dilatation with infinitesimal generator $x^i \partial_i$ in order to provide the *virial equations* (See [19] for more details).

Similarly, going along the idea pioneered by Weyl in 1918 [20], we obtain with $i < j$:

$$\boxed{(\partial_i \xi^r_{rj} - \xi^r_{rji}) - (\partial_j \xi^r_{ri} - \xi^r_{rij}) = \partial_i \xi^r_{rj} - \partial_j \xi^r_{ri} \ \Rightarrow\ \mathcal{J}^i(\partial_i \xi^r_r - \xi^r_{ri}) + \mathcal{F}^{ij}(\partial_i \xi^r_{rj} - \partial_j \xi^r_{ri})}$$

An integration by parts brings the equations $\boxed{\partial_r \mathcal{F}^{ir} - \mathcal{J}^i = 0}$ and $\boxed{\partial_i \mathcal{J}^i = 0}$, leading thus to:

There is no conceptual difference at all between the Cosserat couple-stress equations and the second set of Maxwell equations. Only the groups are different.

As a next crucial step, let us consider the Lie group of transformations of X described by the action of a Lie group G with local coordinates (a^τ), identity $e \in G$ and Lie algebra $\mathcal{G} = T_e(G)$, on X with infinitesimal generators $\theta_\tau = \theta^k_\tau(x)\partial_k$ and introduce the section $\xi_q = \lambda^\tau(x) j_q(\theta_\tau)$ with $\lambda \in \wedge^0 T^* \otimes \mathcal{G}$. We have thus $(\xi^k(x) = \lambda^\tau(x)\theta^k_\tau(x), \xi^k_i(x) = \lambda^\tau(x)\partial_i \theta^k_\tau(x), \cdots)$ and get at once $D_1 \xi_q = d\xi_{q+1} = (d\lambda^\tau) j_q(\theta_\tau)$ where $d : \wedge^0 T^* \otimes \mathcal{G} \to \wedge^1 T^* \otimes \mathcal{G}$ is the exterior derivative, a result proving that the Spencer sequence is (locally) isomorphic to the tensor product by $\mathcal{G}$ of the Poincaré sequence for d, in a coherent way with the second Example of the Introduction. As the extension modules of a module M do not depend on the resolution of M, it follows that, if $\mathcal{D}_{r+1}$ generates the CC of $\mathcal{D}_r$ in a Janet sequence like in the Introduction, then

$ad(\mathcal{D}_r)$ generates the CC of $ad(\mathcal{D}_{r+1})$ while, if D_{r+1} generates the CC of D_r in the corresponding Spencer sequence, then $ad(D_r)$ generates the CC of $ad(D_{r+1})$, *though all these operators are quite different*, a result not evident at all that Lanczos and followers could have not even been able to imagine [29].

It remains to prove that, in this new framework, the Ricci tensor only depends on the symbol $\hat{g}_2 \simeq T^* \subset S_2T^* \otimes T$ of the first prolongation $\hat{R}_2 \subset J_2(T)$ of the conformal Killing system $\hat{R}_1 \subset J_1(T)$ with symbol $\hat{g}_1 \subset T^* \otimes T$ defined by the equations $\omega_{rj}\xi^r_i + \omega_{ir}\xi^r_j - \frac{2}{n}\omega_{ij}\xi^r_r = 0$ *not depending on any conformal factor*. The next commutative diagram covers both situations, taking into account that the equations of both the classical and conformal Killing operator are homogeneous. The purely algebraic *Spencer map* $\delta : g_{q+1} \to T^* \otimes g_q$ with symbol $g_q = R_q \cap S_qT^* \otimes E \subset J_q(E)$ is induced by $-d$ and all the sequences are exact by definition but perhaps the left column:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & g_3 & \to & S_3T^* \otimes T & \to & S_2T^* \otimes F_0 & \to & F_1 \to 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 \to & & T^* \otimes g_2 & \to & T^* \otimes S_2T^* \otimes T & \to & T^* \otimes T^* \otimes F_0 & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 \to & & \boxed{\wedge^2T^* \otimes g_1} & \to & \underline{\wedge^2T^* \otimes T^* \otimes T} & \to & \wedge^2T^* \otimes F_0 & \to & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow & & \\
0 \to & & \wedge^3T^* \otimes T & = & \wedge^3T^* \otimes T & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Theorem 5.1: Introducing the δ-cohomology bundles $Riemann = H^2_1(g_1)$ at $\wedge^2T^* \otimes g_1$ and $Weyl = H^2_1(\hat{g}_1)$ at $\wedge^2T^* \otimes \hat{g}_1$ while taking into account that $g_1 \subset \hat{g}_1$, $g_2 = 0$ $\hat{g}_2 \simeq T^*$ and $\hat{g}^3 = 0$, we have the commutative and exact "*fundamental diagram II*" [7]:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & 0 & & Ricci & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & Z^2_1(g_1) & \to & Riemann & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \to & T^* \otimes \hat{g}_2 & \xrightarrow{\delta} & Z^2_1(\hat{g}_1) & \to & Weyl & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to S_2T^* & \xrightarrow{\delta} & T^* \otimes T^* & \xrightarrow{\delta} & \wedge^2T^* & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Needless to say that no one of these results could be obtained by classical methods (!).

The splitting sequence $0 \to Ricci \to Riemann \to Weyl \to 0$ of vector bundles provides a quite unusual interpretation of the successive Ricci, Riemann and Weyl tensors. Similarly, the well known splitting sequence

$0 \to S_2T^* \stackrel{\delta}{\to} T^* \otimes T^* \stackrel{\delta}{\to} \wedge^2 T^* \to 0$, namely

$A_{ij} \to \left(R_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right), F_{ij} = A_{ij} - A_{ji} \right)$, provides an unusual conformal interpretation of the EM field $F = \left(F_{ij} = -F_{ji}\right) \in \wedge^2 T^*$ in a coherent way with the dream of H. Weyl in 1916 [20]. It follows that:

$$\boxed{\begin{aligned} &dim(Weyl) = dim(Riemann) - dim(Ricci) \\ &\qquad = \frac{n^2\left(n^2-1\right)}{12} - \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)(n-3)}{12} \end{aligned}}$$

though the Weyl operator is of order 3 when $n=3$ but of order 2 when $n \geq 4$ because $\hat{g}_2$ is 2-acyclic only when $n \geq 4$, a result still not known and not even acknowledged today (See arXiv: 1603.05030 for a computer algebra checking by our former PhD student A. Quadrat of INRIA). We finally point out that the *Bianchi*-type operator is of order 2 when $n=4$ but of order 1 when $n=5$ as $\hat{g}_2$ becomes 3-acyclic (See[1] [3] [10] [29] for more details).

6. Janet versus Spencer

We shall say that $R_q \subseteq J_q(E)$ is an involutive system of order q on E if its symbol g_q is involutive, that is all the δ-sequences are exact, and $\pi_q^{q+1} : R_{q+1} \to R_q$ is an epimorphism, a result leading to the fact that R_q is *formally integrable* (FI), that is all the projections $\pi_{q+r}^{q+r+1} : R_{q+r+1} \to R_{q+r}$ are epimorphisms. In actual practice it means that all the generating equations of order $q+r$ can be simply obtained by differentiating only r times the equations of R_q. In this case, we may define the *Janet bundles* F_r for $r = 0,1,\cdots,n$ by the short exact sequences [19] [22] [25]:

$$0 \to \wedge^r T^* \otimes R_q + \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E\right) \to \wedge^r T^* \otimes J_q(E) \to F_r \to 0$$

We may pick up a section of F_r, lift it up to a section of $\wedge^r T^* \otimes J_q(E)$ that we may lift up to a section of $\wedge^r T^* \otimes J_{q+1}(E)$ and apply d in order to get a section of $\wedge^{r+1} T^* \otimes J_q(E)$ that we may project onto a section of F_{r+1} in order to construct a *first order operator* $\mathcal{D}_{r+1} : F_r \to F_{r+1}$ generating the CC of $\mathcal{D}_r$ in the canonical *linear Janet sequence*:

$$\boxed{0 \to \Theta \to E \underset{q}{\stackrel{\mathcal{D}}{\to}} F_0 \underset{1}{\stackrel{\mathcal{D}_1}{\to}} F_1 \underset{1}{\stackrel{\mathcal{D}_2}{\to}} \cdots \underset{1}{\stackrel{\mathcal{D}_n}{\to}} F_n \to 0}$$

If we have two involutive systems $R_q \subset \hat{R}_q \subset J_q(E)$, *the Janet sequence for* R_q *projects onto the Janet sequence for* $\hat{R}_q$ and we may define inductively *canonical epimorphisms* $F_r \to \hat{F}_r \to 0$ for $r = 0,1,\cdots,n$ by comparing the previous sequences for R_q and $\hat{R}_q$.

A similar procedure can also be obtained if we define the Spencer bundles $C_r \subset C_r(E)$ for $r = 0,1,\cdots,n$ by the short exact sequences [22] [25]:

$$0 \to \delta\left(\wedge^{r-1}T^* \otimes g_{q+1}\right) \to \wedge^r T^* \otimes R_q \to C_r \to 0$$

$$0 \to \delta\left(\wedge^{r-1}T^* \otimes S_{q+1} \otimes E\right) \to \wedge^r T^* \otimes J_q(E) \to C_r(E) \to 0$$

We may pick up a section of C_r, lift it to a section of $\wedge^r T^* \otimes R_q$, lift it up to a section of $\wedge^r T^* \otimes R_{q+1}$ and apply d in order to construct a section of $\wedge^{r+1} \otimes R_q$ that we may project to C_{r+1} in order to construct an operator $D_{r+1}: C_r \to C_{r+1}$ generating the CC of D_r in the canonical *linear Spencer sequence* which is *another completely different resolution* of the set Θ of (formal) solutions of R_q:

$$\boxed{0 \to \Theta \xrightarrow{j_q} C_0 \underset{1}{\xrightarrow{D_1}} C_1 \underset{1}{\xrightarrow{D_2}} \cdots \underset{1}{\xrightarrow{D_n}} C_n \to 0}$$

However, if we have two systems as above, *the Spencer sequence for* R_q *is now contained into the Spencer sequence for* $\hat{R}_q$ and we may construct inductively *canonical monomorphisms* $0 \to C_r \to \hat{C}_r$ for $r = 0,1,\cdots,n$ by comparing the previous sequences for R_q and $\hat{R}_q$.

In actual practice, it is important to notice that *the Spencer sequence for* $R_q \subset J_q(E)$ *is nothing else than the Janet sequence for the first order involutive system* $R_{q+1} \subset J_1(R_q)$. Also, and though it has never been acknowledged by Spencer and coworkers, the Janet and the Spencer sequences are related from a purely mathematical point of view by the following *fundamental diagram I*. In this diagram, $\Theta \subset T$ is the set of solutions of $\mathcal{D} = \Phi \circ j_q$ and $\Phi = \Phi_0$ is a given epimorphism while $\Phi_1,\cdots,\Phi_n$ are inductively induced from $\Phi = \Phi_0$ while $C_0 = R_q$ and $C_0(E) = J_q(E)$:

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2} & \cdots & \xrightarrow{D_n} & C_n & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & \to & E & \xrightarrow{j_q} & C_0(E) & \xrightarrow{D_1} & C_1(E) & \xrightarrow{D_2} & \cdots & \xrightarrow{D_n} & C_n(E) & \to & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & & & \downarrow \Phi_n & & \\
0 & \to & \Theta & \to & E & \underset{q}{\xrightarrow{\mathcal{D}}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2} & \cdots & \xrightarrow{\mathcal{D}_n} & F_n & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & & & 0 & & 0 & & & & 0 & &
\end{array}$$

In this diagram, the central sequence is *at the same time* a Janet sequence for the injective operator $j_q : E \to J_q(E)$ of order q and a Spencer sequence for the trivially involutive first order system $J_{q+1}(E) \subset J_1\left(J_q(E)\right)$ on $J_q(E)$ thanks to the commutative and exact diagram:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & R_{q+1} & \rightarrow & J_1(R_q) & \rightarrow & C_1 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & J_{q+1}(E) & \rightarrow & J_1(J_q(E)) & \rightarrow & C_1(E) & \rightarrow & 0 \\
 & & & & \| & & \downarrow J_1(\Phi_0) & & \downarrow \Phi_1 & & \\
0 & \rightarrow & R_{q+1} & \rightarrow & J_{q+1}(E) & \rightarrow & J_1(F_0) & \rightarrow & F_1 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

The main "*trick*" is to cut vertically this diagram in two parts and use the right one on the symbol level in order to obtain the strikingly simple commutative and exact diagram allowing to introduce the involutive symbol $h_1 \subset T^* \otimes F_0$ of the system defining the first order operator $\mathcal{D}_1$.

$$\boxed{\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_{q+1} & \xrightarrow{\delta} & T^* \otimes R_q & \rightarrow & C_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S_{q+1}T^* \otimes E & \xrightarrow{\delta} & T^* \otimes J_q(E) & \rightarrow & C_1(E) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & h_1 & \xrightarrow{\delta} & T^* \otimes F_0 & \rightarrow & F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}}$$

As a byproduct, continuing inductively, we obtain the short exact sequences:

$$\boxed{0 \rightarrow \wedge^{r-1} T^* \otimes h_1 \xrightarrow{\delta} \wedge^r T^* \otimes F_0 \rightarrow F_r \rightarrow 0}$$

When dealing with applications, we have set $E = T$ and considered systems of finite type Lie equations determined by Lie groups of transformations and $ad(\mathcal{D}_r)$ generates the CC of $ad(\mathcal{D}_{r+1})$ while $ad(D_r)$ generates the CC of $ad(D_{r+1})$. We have obtained in particular $C_r = \wedge^r T^* \otimes R_q \subset \wedge^r T^* \otimes \hat{R}_q = \hat{C}_r$ when comparing the classical and conformal Killing systems, but *these bundles have never been used in physics*. Therefore, instead of the classical Killing system $R_2 \subset J_2(T)$ defined by $\Omega \equiv \mathcal{L}(\xi)\omega = 0$ *and* $\Gamma \equiv \mathcal{L}(\xi)\gamma = 0$ or the conformal Killing system $\hat{R}_2 \subset J_2(T)$ defined by $\Omega \equiv \mathcal{L}(\xi)\omega = 2A(x)\omega$ and $\Gamma \equiv \mathcal{L}(\xi)\gamma = \left(\delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{ks} A_s(x)\right) \in S_2T^* \otimes T$, we may introduce the *intermediate differential system* $\tilde{R}_2 \subset J_2(T)$ defined by $\mathcal{L}(\xi)\omega = 2A\omega$ with $A = cst$ and $\Gamma \equiv \mathcal{L}(\xi)\gamma = 0$, for the *Weyl group* obtained by adding the only dilatation with infinitesimal generator $x^i\partial_i$ to the Poincaré group. We have now $R_1 \subset \tilde{R}_1 = \hat{R}_1$ with the strict inclusions $R_2 \subset \tilde{R}_2 \subset \hat{R}_2$ and we discover *exactly* the group scheme used through this paper, both with the need to *shift by one* *step to the left* the physical interpretation of the various differential sequences used. Indeed, the symbol $\hat{g}_2 \simeq T^*$ is defined by the $n(n-1)(n+2)/2$ linear equations:

$$\boxed{\xi_{ij}^k - \frac{1}{n}\left(\delta_i^k \xi_{rj}^r + \delta_j^k \xi_{ri}^r - \omega_{ij}\omega^{ks}\xi_{rs}^r\right) = 0}$$

that do not depend on any conformal factor. The first Spencer operator $\hat{R}_2 \xrightarrow{D_1} T^* \otimes \hat{R}_2$ is induced by the usual Spencer operator $\hat{R}_3 \xrightarrow{d} T^* \otimes \hat{R}_2 : \left(0, 0, \xi_{rj}^r, \xi_{rij}^r = 0\right) \rightarrow \left(0, \partial_i 0 - \xi_{ri}^r, \partial_i \xi_{rj}^r - 0\right)$ and thus projects by cokernel onto the induced operator $T^* \rightarrow T^* \otimes T^*$. Composing with δ, it projects therefore onto $T^* \xrightarrow{d} \wedge^2 T^* : (A_i) \rightarrow \left(\partial_i A_j - \partial_j A_i = F_{ij}\right)$ as in EM with parametrization described by the Spencer operator $(A, A_i) \rightarrow (\partial_i A - A_i)$ because of the factor 2 and so on by using the fact that D_1 *and* d *are both involutive*, or the composition of epimorphisms:

$$\boxed{\hat{C}_r \rightarrow \hat{C}_r / \tilde{C}_r \simeq \wedge^r T^* \otimes \left(\hat{R}_2 / \tilde{R}_2\right) \simeq \wedge^r T^* \otimes \hat{g}_2 \simeq \wedge^r T^* \otimes T^* \xrightarrow{\delta} \wedge^{r+1} T^*}$$

The main result we have obtained is thus to be able to increase the order and dimension of the underlying jet bundles and groups, proving therefore that any 1-form with value in the second order jets $\hat{g}_2$ (*elations*) of the conformal Killing system (conformal group) can be decomposed uniquely into the direct sum (R, F) where R is a section of the *Ricci bundle* $S_2 T^*$ and the EM field F is a section of $\wedge^2 T^*$, thanks to the fundamental diagram II.

This was exactly the dream of Weyl in [20].

7. Applications

MOTIVATING EXAMPLE 7.1: With $m = 1, n = 2, q = 2$, let us consider the second order system $R_2 \subset J_2(E)$ written $d_{22}\xi - x^2 d_1\xi = \eta^2$, $d_{12}\xi - d_{11}\xi = \eta^1$ or $\mathcal{D}\xi = \eta$ with ground differential field $K = \mathbb{Q}(x)$. Differentiating once, we obtain the third order system $R_3 \subset J_3(E)$ with corresponding Janet tabular:

$$\left\{\begin{array}{lcl} d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\ d_{12}\xi - d_{11}\xi & = & \eta^1 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} d_{222}\xi - x^2 d_{12}\xi - d_1\xi & = & d_2\eta^2 \\ d_{122}\xi - x^2 d_{11}\xi & = & d_1\eta^2 \\ d_{112}\xi - x^2 d_{11}\xi & = & d_1\eta^2 - d_2\eta^1 \\ d_{111}\xi - x^2 d_{11}\xi & = & d_1\eta^2 - d_2\eta^1 - d_1\eta^1 \\ d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\ d_{12}\xi - d_{11}\xi & = & \eta^1 \end{array}\right. \begin{array}{|cc|} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}$$

Though the symbol $g_3 = 0$ is trivially involutive, this system is *not* even formally integrable (FI) because, trying all the dots, we discover that we have the strict inclusions $R_2^{(2)} \subset R_2^{(1)} = R_2 \subset J_2(E)$ with respective dimension $3 < 4 = 4 < 6$. Indeed, after a few tricky substitutions and eliminations, we obtain the new second order PD equation:

$$\boxed{A \equiv d_{11}\xi = d_{22}\eta^1 - d_{12}\eta^2 + d_{11}\eta^2 - x^2 d_1\eta^1 \in j_2(\eta)}$$

The hard step is to look for generating CC in the form of an operator $\mathcal{D}_1\eta = \zeta$. We obtain the commutative and exact diagram with $dim(R_3) = 4$, $dim(R_4) = 3$:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \rightarrow & S_4T^*\otimes E & \rightarrow & S_2T^*\otimes F_0 & \rightarrow & h_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_4 & \rightarrow & J_4(E) & \rightarrow & J_2(F_0) & \rightarrow & Q_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_3 & \rightarrow & J_3(E) & \rightarrow & J_1(F_0) & \rightarrow & Q_1 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

and the long exact connecting sequence $0 \rightarrow R_4 \rightarrow R_3 \rightarrow h_2 \rightarrow Q_2 \rightarrow Q_1 \rightarrow 0$. It follows that $Q_1 \simeq Q_2 = 0$ because $dim(h_2) = 1$ and there cannot exist *any* first or second order CC. We may start afresh with the new system $R'_2 = R_2^{(2)} \subset R_2 \subset J_2(E$ which is involutive with symbol $g'_2 = 0$:

$$
\left\{
\begin{array}{lcl}
d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\
d_{12}\xi & = & A + \eta^1 \\
d_{11}\xi & = & A
\end{array}
\right.
\quad
\begin{array}{|cc|}
\hline
1 & 2 \\
1 & \bullet \\
1 & \bullet \\
\hline
\end{array}
$$

We obtain therefore the *Fundamental Diagram I* for R'_2 while introducing the 3 new jet coordinates ($z^1 = \xi$, $z^2 = \xi_1$, $z^3 = \xi_2$) as R'_2 may be equivalently defined by the first order involutive system ($d_1z^1 - z^2 = 0$, $d_2z^1 - z^3 = 0$, $d_1z^2 = 0$, $d_2z^2 = 0$, $d_1z^3 = 0$, $d_2z^3 - x^2z^2 = 0$) with 6 equations.

$$
\begin{array}{|cccccccccccc|}
\hline
 & & & & & 0 & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & \Theta & \xrightarrow{j_2} & 3 & \xrightarrow[1]{D_1} & 6 & \xrightarrow[1]{D_2} & 3 & \rightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \| & & \\
 & 0 & \rightarrow & 1 & \xrightarrow[2]{j_2} & 6 & \xrightarrow[1]{D_1} & 8 & \xrightarrow[1]{D_2} & 3 & \rightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow \Theta & & \rightarrow & 1 & \xrightarrow[2]{\mathcal{D}'} & 3 & \xrightarrow[1]{\mathcal{D}'_1} & 2 & \rightarrow & 0 & & \\
 & & & & & \downarrow & & \downarrow & & & & \\
 & & & & & 0 & & 0 & & & & \\
\hline
\end{array}
$$

It finally remains to find out the generating CC for the initial second order operator $\mathcal{D}\xi = \eta$ which is neither FI nor involutive. Checking the two dots separately we have the two *third order* (!) CC:

$$
\boxed{B_1 \equiv d_1A - x^2A - d_1\eta^2 + d_2\eta^1 + d_1\eta^1 = 0, \quad B_2 \equiv d_2A - x^2A - d_1\eta^2 + d_2\eta^1 = 0}
$$

$$
d_2A - d_1A - d_1\eta^1 \equiv d_{222}\eta^1 - d_{122}\eta^2 - d_{122}\eta^1 + 2d_{112}\eta^2 - d_{111}\eta^2 - x^2d_{12}\eta^1 - x^2d_{11}\eta^1 = 0
$$

Exactly like in [13], we now provide the link existing between these two third order CC and the Spencer operator. Indeed, using R_3 we obtain at once:

$$\boxed{\begin{array}{lclcl} d_1\xi_{11}-\xi_{111} & = & d_1A-x^2A-d_1\eta^2+d_2\eta^1+d_1\eta^2 & = & B_1 \\ d_2\xi_{11}-\xi_{112} & = & d_2A-x^2A-d_1\eta^2+d_2\eta^1 & = & B_2 \end{array}}$$

a result that we shall obtain after one more prolongation by the long exact sequence:

$$0\rightarrow R_5\rightarrow J_5(E)\rightarrow J_3(F_0)\rightarrow Q_3\rightarrow 0 \qquad 0\rightarrow 3\rightarrow 21\rightarrow 20\rightarrow 2\rightarrow 0$$

We may thus define $F_1=Q_3$ with $dim(F_1)=2$ and proceed similarly in order to define F_2 with $dim(F_2)=1$ by the long exact sequence:

$$0\rightarrow R_6\rightarrow J_6(E)\rightarrow J_4(F_0)\rightarrow J_1(F_1)\rightarrow F_2\rightarrow 0 \qquad 0\rightarrow 3\rightarrow 28\rightarrow 30\rightarrow 6\rightarrow 1\rightarrow 0$$

We have indeed $d_1B_2-d_2B_1=0$ and the exact differential sequence which is *not* a Janet sequence:

$$\boxed{0\rightarrow\Theta\rightarrow 1\underset{2}{\overset{\mathcal{D}}{\rightarrow}}2\underset{3}{\overset{\mathcal{D}_1}{\rightarrow}}2\underset{1}{\overset{\mathcal{D}_2}{\rightarrow}}1\rightarrow 0}$$

More generally, we have the long exact sequences $\forall r\geq 0$:

$$0\rightarrow 3\rightarrow J_{r+6}(E)\rightarrow J_{r+4}(F_0)\rightarrow J_{r+1}(F_1)\rightarrow J_r(F_2)\rightarrow 0$$

Using finally the basis $\{\theta_\tau(x)\,|\,1\leq\tau\leq 3\}=\left\{1,x^2,x^1+\frac{1}{6}(x^2)^3\right\}$ for the vector space $\mathcal{V}$ over the constants, the Spencer sequence of the *Fundamental Diagram I* is the tensor product by $\mathcal{V}$ of the Poincaré sequence

$\wedge^0T^*\overset{d}{\rightarrow}\wedge^1T^*\overset{d}{\rightarrow}\wedge^2T^*\rightarrow 0$ for the exterior derivative when $n=2$.

SCHWARZSCHILD VERSUS KERR 7.2:

We now write the Kerr metric in Boyer-Lindquist coordinates (t,r,θ,ϕ) as in [30]:

$$\boxed{\begin{array}{l}\omega=\dfrac{\rho^2-mr}{\rho^2}dt^2-\dfrac{\rho^2}{\Delta}dr^2-\rho^2d\theta^2-\dfrac{2amr\sin^2(\theta)}{\rho^2}dtd\phi \\ \qquad -\left(r^2+a^2+\dfrac{mra^2\sin^2(\theta)}{\rho^2}\right)\sin^2(\theta)d\phi^2\end{array}}$$

where we have set $\Delta=r^2-mr+a^2$, $\rho^2=r^2+a^2\cos^2(\theta)$ as usual and we check that we recover the Schwarzschild metric when $a=0$ as follows with

$A(r)=1-\dfrac{m}{r}$:

$$\boxed{\omega=A(r)dt^2-(1/A(r))dr^2-r^2d\theta^2-r^2\sin^2(\theta)d\phi^2}$$

We notice that t or ϕ do not appear in the coefficients of the metric. We shall change the coordinate system in order to confirm theses results by using computer algebra. The idea is to use the so-called "*rational polynomial*" coefficients as follows with $c=\cos(\theta)\Rightarrow dc=-\sin(\theta)d\theta\Rightarrow dc^2=(1-c^2)d\theta^2$ and set ($x^0=t$, $x^1=r$, $x^2=c=\cos(\theta)$, $x^3=\phi$).

We obtain over the differential field $K=\mathbb{Q}(a,m)(t,r,c,\phi)$:

$$\omega = \frac{\rho^2 - mr}{\rho^2} dt^2 - \frac{\rho^2}{\Delta} dr^2 - \frac{\rho^2}{1-c^2} dc^2 - \frac{2amr\left(1-c^2\right)}{\rho^2} dt d\phi$$
$$-\left(1-c^2\right)\left(r^2 + a^2 + \frac{ma^2 r\left(1-c^2\right)}{\rho^2}\right) d\phi^2$$

with now $\Delta = r^2 - mr + a^2$ and $\rho^2 = r^2 + a^2c^2$ and we have $det(\omega) = -\left(r^2 + a^2c^2\right)^2$.

Looking at the symbol $g_1 \subset T^* \otimes T$, elementary linear combinatorics allow to prove [18] [22]:

$$\omega_{33}\boxed{\xi_3^3} + \omega_{03}\xi_3^0 = 0 \; mod(\xi),$$
$$\omega_{33}\boxed{\xi_0^3} + \omega_{00}\xi_3^0 = 0 \; mod(\xi),$$
$$\omega_{33}\boxed{\xi_0^0} - \omega_{03}\xi_3^0 = 0 \; mod(\xi)$$

Then, multiplying Ω_{22} by ω_{11}, Ω_{11} by ω_{22} and adding, we finally obtain:

$$2\left(\omega_{11}\omega_{22}\right)\left(\xi_1^1 + \xi_2^2\right) + \xi\partial\left(\omega_{11}\omega_{22}\right) = 0$$

However, we have also successively:

$$\begin{cases} R_{03,03} \equiv 2\rho_{03,03}\left(\xi_0^0 + \xi_3^3\right) + \xi\partial\rho_{03,03} = 0 \\ R_{12,12} \equiv 2\rho_{12,12}\left(\xi_1^1 + \xi_2^2\right) + \xi\partial\rho_{12,12} = 0 \\ R_{01,23} \equiv \rho_{01,23}\left(\xi_0^0 + \xi_1^1 + \xi_2^2 + \xi_3^3\right) + \xi\partial\rho_{01,23} = 0 \end{cases}$$

Now, the coefficients of the metric are rational functions in K and the various geometric objects appearing in $\omega \to \gamma \to \rho$ can be obtained through the rules of differential algebra. It is thus possible to obtain the 13 non-zero components of the Riemann tensor for the Kerr metric according to K. R. Koehler in (http://kias.dyndns.org/crg/blackhole.html) by adding factorizations as follows:

$$\begin{cases} \rho_{01,01} = -\dfrac{mr\left(2\left(r^2 - mr + a^2\right) + a^2\left(1-c^2\right)\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3\left(r^2 - mr + a^2\right)} \\ \rho_{02,02} = \dfrac{mr\left(r^2 - mr + a^2 + 2a^2\left(1-c^2\right)\right)\left(r^2 - 3a^2c^2\right)}{2\left(1-c^2\right)\left(r^2 + a^2c^2\right)^3} \\ \rho_{03,03} = \dfrac{mr\left(1-c^2\right)\left(r^2 - mr + a^2\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \\ \rho_{12,12} = -\dfrac{mr\left(r^2 - 3a^2c^2\right)}{2\left(1-c^2\right)\left(r^2 + a^2c^2\right)\left(r^2 - mr + a^2\right)} \\ \rho_{13,13} = \dfrac{-\left(1-c^2\right)mr\left(r^4 - 2a^2c^2r^2 + 4a^2r^2 - 2a^4c^2 + 3a^4 - 2a^2mr\left(1-c^2\right)\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3\left(r^2 - mr + a^2\right)} \\ \rho_{23,23} = \dfrac{mr\left(2r^4 - a^2c^2r^2 + 5a^2r^2 - a^4c^2 + 3a^4 - a^2mr\left(1-c^2\right)\right)\left(r^2 - 3a^2c^2\right)}{2\left(r^2 + a^2c^2\right)^3} \end{cases}$$

$$\left\{\begin{array}{l}
\rho_{01,23}=\dfrac{amc\left(2r^2-a^2c^2+3a^2\right)\left(3r^2-a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3}\\
\rho_{02,31}=-\dfrac{amc\left(r^2-2a^2c^2+3a^2\right)\left(3r^2-a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3}\\
\rho_{03,12}=-\dfrac{amc\left(3r^2-a^2c^2\right)}{2\left(r^2+a^2c^2\right)^2}\\
\rho_{02,10}=\dfrac{3a^2mc\left(3r^2-a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3}\\
\rho_{02,32}=\dfrac{amr\left(3r^2-mr+3a^2\right)\left(r^2-3a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3}\\
\rho_{13,23}=-\dfrac{3a^2mc\left(1-c^2\right)\left(r^2+a^2\right)\left(3r^2-a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3}\\
\rho_{01,13}=\dfrac{amr\left(1-c^2\right)\left(3r^2+3a^2-2mr\right)\left(r^2-3a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3\left(r^2-mr+a^2\right)}
\end{array}\right.$$

among which the 7 last ones are vanishing when $a=0$ in the case of the Schwarzschild metric.

We have to add the 8 vanishing components:

$$\boxed{\rho_{01,03}=0,\rho_{01,12}=0,\rho_{02,03}=0,\rho_{02,12}=0,\rho_{03,13}=0,\rho_{03,23}=0,\rho_{12,13}=0,\rho_{12,23}=0}$$

In fact, as the Riemann tensor has $n^2\left(n^2-1\right)/12$ components, that is 20 when $n=4$, we have to take into account the only identity:

$$\boxed{\rho_{01,23}+\rho_{02,31}+\rho_{03,12}=0\Rightarrow R_{01,23}+R_{02,31}+R_{03,12}=0}$$

We obtain therefore $\xi\partial\left(\rho_{12,12}/\left(\omega_{11}\omega_{22}\right)\right)=0$ but we have also $\xi\partial\left(\rho_{03,03}\rho_{12,12}/det\left(\omega\right)\right)=0$.

The following invariants are obtained successively in a coherent way:

$$\omega_{11}\omega_{22}=\frac{\left(r^2+a^2c^2\right)^2}{\left(1-c^2\right)\left(r^2-mr+a^2\right)}\Rightarrow\left|\rho_{12,12}\right|/\left(\omega_{11}\omega_{22}\right)=\frac{mr\left(r^2-3a^2c^2\right)}{2\left(r^2+a^2c^2\right)^3}$$

Also, as $a\in K$, then $\rho_{01,23}$ and $\rho_{02,13}$ can be both divided by a and we get the new invariant:

$$\boxed{\rho_{01,23}/\rho_{03,12}=\frac{2r^2-a^2c^2+3a^2}{r^2+a^2c^2}}$$

These results are leading to $\boxed{\xi^1=0}$, $\boxed{\xi^2=0}$, thus to $\boxed{\xi_1^1=0}$, $\boxed{\xi_2^2=0}$ and $\boxed{\xi_0^0+\xi_3^3=0}$ after substitution in the equations defining the first order symbol g_1 of R_1.

In the case of the S-metric with $a=0$, the previous division has no meaning and we have only $\boxed{\xi_1=0}$ as the only equation of zero order.

Let us now introduce the new equation:

$$R_{01,13}\equiv\rho_{01,13}\left(\xi_0^0+2\xi_1^1+\xi_3^3\right)-\rho_{13,13}\xi_0^3-\rho_{01,01}\xi_3^0+\left(\rho_{01,23}+\rho_{02,13}\right)\xi_1^2=0$$

As we have $\xi_0^0+\xi_3^3=0$ and $\xi_1^1=0$, we obtain therefore a linear equation of the form:

$$\boxed{\rho_{13,13}\xi_0^3+\rho_{01,01}\xi_3^0-\left(\rho_{01,23}+\rho_{02,13}\right)\xi_1^2=0}$$

Similarly, we have also:

$$R_{01,02}\equiv\rho_{01,02}\left(2\xi_0^0+\xi_1^1+\xi_2^2\right)-\left(\rho_{01,23}+\rho_{02,13}\right)\xi_0^3+\rho_{01,01}\xi_2^1+\rho_{02,02}\xi_1^2=0$$

and we obtain therefore a linear equation of the form:

$$\boxed{2\rho_{01,02}\xi_0^0-\left(\rho_{01,23}+\rho_{02,13}\right)\xi_0^3+\rho_{01,01}\xi_2^1+\rho_{02,02}\xi_1^2=0}$$

In the case of the S-metric, that is when $a=0$, we obtain respectively $\xi_3^0=0$ and $\xi_2^1=0$ as in [18] because $\xi_0^0\simeq\xi_3^0$. The previous linear system has thus a rank equal to 2 and we obtain therefore because $\xi_0^3\simeq\xi_3^0$, $\xi_1^2\simeq\xi_2^1$:

$$\boxed{\xi_3^0=0},\boxed{\xi_2^1=0}\Leftrightarrow\boxed{\xi_0^3=0},\boxed{\xi_1^2=0},\boxed{\xi_0^0=0},\boxed{\xi_3^3=0}$$

It remains to study the following 4 linear equations, namely:

$$\boxed{R_{01,03}=0,\ R_{03,23}=0,\ R_{03,13}=0,\ R_{02,03}=0}$$

The rank of the previous system with respect to the 4 jet coordinates $\left(\xi_0^1,\xi_0^2,\xi_3^1,\xi_3^2\right)$ is equal to 2, for both the S and K-metrics thanks to the two striking identities [30]:

$$\boxed{R_{03,13}+a\left(1-c^2\right)R_{01,03}=0,\ R_{02,03}+\frac{a}{\left(r^2+a^2\right)}R_{03,23}=0}$$

Two prolongations only provide 6 additional equations of order one that we provide in the following list which is obtained $mod\left(j_2\left(\Omega\right)\right)$, namely:

$$\boxed{\xi^1=0,\ \xi^2=0,\ \xi_2^1=0,\ \xi_3^0=0,\ \xi_3^1+lin\left(\xi_0^1,\xi_0^2\right)=0,\ \xi_3^2+lin\left(\xi_0^1,\xi_0^2\right)}$$

We have therefore obtained the inclusion of Lie algebroids $R_1^{(2)}\subset R_1^{(1)}=R_1\subset J_1\left(T\right)$ with respective dimensions $4<10=10<20$. Using the standard *ker-coker* long exact sequence of prolongations:

$$\boxed{0\to R_3\to J_3\left(T\right)\to J_3\left(S_2T^*\right)\to Q_2\to 0,\ \ 0\to 4\to 140\to 150\to 14\to 0}$$

we discover that the initial Killing system for the Kerr metric has 14 compatibility conditions of second order contrary to the 20 existing for the Minkowski metric. Such a result has been obtained *totally independently of any specific GR technical object* like the *Teukolski scalars* or the *Killing-Yano tensors* introduced in [31]. However, *this system is not involutive* because its symbol is finite type but non-zero [22] [25].

Using one more prolongation, all the *sections* (*care again*) vanish but ξ^0 and ξ^3, a result leading to $dim\left(R_1^{(3)}\right)=2$ in a coherent way with the only nonzero Killing vectors $\left\{\partial_t,\partial_\phi\right\}$. We have indeed:

$$\boxed{\xi_0^1=0\ \ \xi_0^2=0}\Leftrightarrow \xi_3^1=0,\ \xi_3^2=0\Rightarrow \xi_1^0=0,\ \xi_1^0=0,\ \xi_2^0=0,\ \xi_2^3=0$$

Taking therefore into account that the metric only depends on $\left(x^1=r,x^2=cos(\theta)\right)$ we obtain *after three prolongations* the inclusions of first order systems:

$$\boxed{R_1^{(3)}\subset R_1^{(2)}\subset R_1^{(1)}=R_1\subset J_1(T)\Leftrightarrow 2<4<10=10<20}$$

Surprisingly and contrary to the situation that will be found for the S metric, we have now an involutive first order system with only solutions ($\xi^0=cst$, $\xi^1=0$, $\xi^2=0$, $\xi^3=cst$) and notice that $R_1^{(3)}$ does not depend any longer on the parameters $(m,a)\in K$. The difficulty is to know what second members must be used along the procedure met for all the motivating examples, in particular we have again identities to zero like $d_0\xi^1-\xi_0^1=0$, $d_0\xi^2-\xi_0^2=0$.

We finally obtain 14 second order generating CC and their prolongations as we already said but also 6 third order CC coming from the 6 following components of the Spencer operator, namely:

$$\boxed{\begin{array}{l} d_1\xi^1-\xi_1^1=0,\ d_2\xi^1-\xi_2^1=0,\ d_3\xi^1-\xi_3^1=0,\\ d_1\xi^2-\xi_1^2=0,\ d_2\xi^2-\xi_2^2=0,\ d_3\xi^2-\xi_3^2=0\end{array}}$$

a result that cannot be even imagined from [31]. Of course, proceeding like in the motivating examples, we must substitute in the right members the values obtained from $j_2(\Omega)$ and set for example $\xi_1^1=-\frac{1}{2\omega_{11}}\xi^r\partial_r\omega_{11}$ while replacing ξ^1 and ξ^2 by the corresponding linear combinations of the Riemann tensor already obtained for the right members of the two zero order equations. The corresponding *Fundamental Diagram I* is no longer depending on (m,a) as follows:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & \Theta & \overset{j_1}{\to} & & 2 & \overset{D_1}{\to} & 8 & \overset{D_2}{\to} & 12 & \overset{D_3}{\to} & 8 & \overset{D_4}{\to} & 2 & \to & 0\\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & 4 & \overset{j_1}{\to} & & 20 & \overset{D_1}{\to} & 40 & \overset{D_2}{\to} & 40 & \overset{D_3}{\to} & 20 & \overset{D_4}{\to} & 4 & \to & 0\\
 & & & \| & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0\to\Theta & & \to & 4 & \overset{\mathcal{D}}{\to} & & 18 & \overset{\mathcal{D}_1}{\to} & 32 & \overset{\mathcal{D}_2}{\to} & 28 & \overset{\mathcal{D}_3}{\to} & 12 & \overset{\mathcal{D}_4}{\to} & 2 & \to & 0\\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with the Euler-Poincaré characteristic $4-18+32-28+12-2=0$. However, the only intrinsic concepts associated with a differential sequence are the "*extension modules*" that only depend on the Kerr differential module but *not* on the differential sequence and we repeat once more that:

THE ONLY IMPORTANT CONCEPT IS THE GROUP INVOLVED, NOT THE METRIC.

Needless to say that the group involved in this case has no physical usefulness.

The study of the S metric is much simpler when $a=0$ and we have only:

$$\omega = A(r)dt^2 - \frac{1}{A(r)}dr^2 + \frac{r^2}{(1-c^2)}dc^2 - r^2(1-c^2)d\phi^2$$

As we already said, we have only six non-zero components for the Riemann tensor also provided in the coordinates (t,r,θ,ϕ) while setting $A = 1 - \frac{m}{r}$:

$$\left\{\begin{array}{lcll} \rho_{01,01} & = & -\frac{m}{r^3} & \rightarrow -\frac{m}{r^3} \\ \rho_{02,02} & = & \frac{m(r-m)}{2(1-c^2)r^2} & \rightarrow \frac{mA}{2r} \\ \rho_{03,03} & = & \frac{m(1-c^2)(r-m)}{2r^2} & \rightarrow \frac{mA\sin^2(\theta)}{2r} \\ \rho_{12,12} & = & -\frac{m}{2(1-c^2)(r-m)} & \rightarrow -\frac{m}{2rA} \\ \rho_{13,13} & = & -\frac{(1-c^2)m}{2(r-m)} & \rightarrow -\frac{m\sin^2(\theta)}{2rA} \\ \rho_{23,23} & = & mr & \rightarrow mr\sin^2(\theta) \end{array}\right.$$

Such a result leads to the inclusions of algebroids:

$$\boxed{R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T) \leftrightarrow 4 < 5 < 10 = 10 < 20}$$

The subsystem $R_1^{(2)}$ is defined by adding the 5 new first order equations:

$$\xi^1 = 0,\ \xi_2^1 = 0,\ \xi_3^1 = 0,\ \xi_2^0 = 0,\ \xi_3^0 = 0$$

while $R_1^{(3)}$ is obtained by adding again $\boxed{\xi_0^1 = 0}$.

We have now 15 generating second order CC and 3 new third order CC determined by the Spencer operator, namely:

$$\boxed{d_1\xi^1 - \xi_1^1 = 0,\ d_2\xi^1 - \xi_2^1 = 0,\ d_3\xi^1 - \xi_3^1 = 0}$$

in which we substitute $\xi^1 \in j_2(\Omega)$, $\xi_1^1 = \frac{A'}{2A}\xi^1 \in j_2(\Omega)$, $\xi_2^1 \in j_2(\Omega)$, $\xi_3^1 \in j_2(\Omega)$ with $A' = \frac{m}{r^2}$.

However, *these results do not agree at all* with [31] that must be compared to [30].

8. Variational Calculus

Adapting variational calculus to Lie pseudogroups and geometric objects is based on *two* ideas:

1) One must vary *sections* but not *points*. Hence we may consider formulas like

$\delta f^k(x) = \eta^k(f(x))$ and set $\eta^k(f(x)) = \xi^i(x)\partial_i f^k(x)$ *or* $\eta^k(f(x)) = f_i^k(x)\xi^i(x)$ as we shall see.

2) The Lie pseudogroup and thus the underlying geometric object used *must* not be changed. It thus follows that *the Einstein-Hilbert variational calculus has no meaning in this framework*.

If X is an n-dimensional manifold with local coordinates (x^i) and Y is a copy of X with local coordinates (y^k), then $X \times Y$ is a fibered manifold over X and we shall denote by Π_q the open sub-fibered manifold of the q-jet bundle $J_q(X \times Y)$ defined independently of the coordinate system by $det(y_i^k) \neq 0$ with *source projection* $\alpha_q : \Pi_q \to X : (x, y_q) \to (x)$ and *target projection* $\beta_q : \Pi_q \to Y : (x, y_q) \to (y)$. We denote by $id : X \to Y : x \to y = x$ the *identity* map and we have the identification $T = id^{-1}(V(\mathcal{E}))$. In order to construct the nonlinear Spencer sequence, we need a few basic definitions on *Lie groupoids* and *Lie algebroids* that will become substitutes for Lie groups and Lie algebras. Introducing the operator $j_q : X \times Y \to J_q(X \times Y) = f(x) \to (f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), \cdots)$, the first idea is to use the chain rule for derivatives $j_q(g \circ f) = j_q(g) \circ j_q(f)$ whenever $f, g \in aut(X)$ can be composed and to replace both $j_q(f)$ and $j_q(g)$ respectively by f_q and g_q in order to obtain the new section $g_q \circ f_q$. This kind of "composition" law can be written in a pointwise symbolic way by introducing another copy Z of X with local coordinates (z) as follows:

$$\gamma_q : \Pi_q(Y, Z) \times_Y \Pi_q(X, Y)$$
$$\to \Pi_q(X, Z) : \left(y, z, \frac{\partial z}{\partial y}, \cdots\right), \left(x, y, \frac{\partial y}{\partial x}, \cdots\right) \to \left(x, z, \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}, \cdots\right)$$

We may also define $j_q(f)^{-1} = j_q(f^{-1})$ and obtain similarly an "inversion" law $\iota_q : \Pi_q \to \Pi_q$. As usual we may introduce the Jacobian matrix $\Delta = det\partial_i f^k(x)$ and the transformation $y = f(x)$ is invertible iff $\Delta \neq 0$ with inverse $x = g(y)$ and we have the useful n technical *identities* [13]:

$$\boxed{\frac{\partial}{\partial y^k}\left(\frac{1}{\Delta}\partial_i f^k(g(y))\right) = 0}$$

A fibered submanifold $\mathcal{R}_q \subset \Pi_q$ is called a *system of finite Lie equations* or a *Lie groupoid* of order q if we have an induced *source projection* $\alpha_q : \mathcal{R}_q \to X$, *target projection* $\beta_q : \mathcal{R}_q \to Y$, *composition* $\gamma_q : \mathcal{R}_q \times_Y \mathcal{R}_q \to \mathcal{R}_q$, *inversion* $\iota_q : \mathcal{R}_q \to \mathcal{R}_q$ and *identity* $j_q(id) = id_q : X \to \mathcal{R}_q$. In the sequel we shall only consider *transitive* Lie groupoids such that the map $(\alpha_q, \beta_q) : \mathcal{R}_q \to X \times Y$ is an epimorphism and we shall denote by $\mathcal{R}_q^0 = id^{-1}(\mathcal{R}_q)$ the *isotropy Lie group bundle* of $\mathcal{R}_q$. One can prove that the system $\rho_r(\mathcal{R}_q) = \mathcal{R}_{q+r}$ obtained by differentiating r times all the defining equations of $\mathcal{R}_q$ is a Lie groupoid of order $q+r$. The vector sub-bundle $R_q = id_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$ is called a *system of infinitesimal Lie equations* or a *Lie algebroid* of order q.

Using the canonical inclusion $\Pi_{q+1} \subset J_1(\Pi_q)$ defined by $y_{\mu,i}^k = y_{\mu+1_i}^k$ and the composition $f_{q+1}^{-1}\circ j_1(f_q)$ is a well defined section of $J_1(\Pi_q)$ over the section $f_q^{-1}\circ f_q = id_q$ of Π_q like id_{q+1}. The difference $\chi_q = f_{q+1}^{-1}\circ j_1(f_q) - id_{q+1}$ is thus a section of $T^*\otimes J_q(T)$. We get with $g_1 = f_1^{-1}$:

$$\boxed{\chi_{,i}^k = g_l^k \partial_i f^l - \delta_i^k = A_i^k - \delta_i^k,\ \chi_{j,i}^k = g_l^k\left(\partial_i f_j^l - A_i^r f_{rj}^l\right)}$$

By restriction, we may define similarly the non-linear operator:

$$\boxed{\bar{D}: \mathcal{R}_{q+1}\to T^*\otimes R_q : f_{q+1}\to f_{q+1}^{-1}\circ j_1(f_q) - id_{q+1} = \chi_q}$$

For any composition $f'_{q+1} = g_{q+1}\circ f_{q+1}$ we get:

$$\bar{D}f'_{q+1} = f_{q+1}^{-1}\circ g_{q+1}^{-1}\circ j_1(g_q)\circ j_1(f_q) - id_{q+1} = f_{q+1}^{-1}\circ \bar{D}g_{q+1}\circ j_1(f_q) + \bar{D}f_{q+1}$$

Definition 8.1: For any section $f_{q+1}\in\mathcal{R}_{q+1}$, we may define the *finite gauge transformation*:

$$\boxed{\chi_q \in T^*\otimes R_q \to \chi'_q = f_{q+1}^{-1}\circ \chi_q \circ j_1(f_q) + \bar{D}f_{q+1} \in T^*\otimes R_q}$$

Introducing the *bilinear algebraic bracket* $\{j_{q+1}(\xi), j_{q+1}(\eta)\} = j_q([\xi,\eta])$, we may then introduce both the *formal Lie derivative* and the *differential algebroid bracket* on $J_q(T)$ by the formulas:

$$\boxed{L(\xi_{q+1})\eta_q = \{\xi_{q+1},\eta_{q+1}\} + i(\xi)d\eta_{q+1} = [\xi_q,\eta_q] + i(\eta)d\xi_{q+1}}$$

in which $d\xi_{q+1} = j_1(\xi_q) - xi_{q+1}\subset T^*\otimes R_q$ in such a way that $[R_q,R_q]\subset R_q$.

We have proved in many books and papers [1] [2] [17] [22]:

Proposition 8.2: Setting:

$$\boxed{(d\chi_q)_{\mu,ij}^k = \partial_i\chi_{\mu,j}^k - \partial_j\chi_{\mu,i}^k - \chi_{\mu+1_i,j}^k + \chi_{\mu+1_j,i}^k}$$

there are first order nonlinear compatibility conditions:

$$\boxed{(\bar{D}'\chi_q)(\xi,\eta) = (d\chi_q)(\xi,\eta) - \{\chi_q(\xi),\chi_q(\eta)\} = 0}$$

For any composition $f'_{q+1} = f_{q+1}\circ h_{q+1}$ we obtain:

$$f'_{q+1} = h_{q+1}^{-1}\circ f_{q+1}^{-1}\circ j_1(f_q)\circ j_1(h_q) - id_{q+1} = h_{q+1}^{-1}\circ \bar{D}f_{q+1}\circ j_1(h_q) + \bar{D}h_{q+1}$$

Example 8.3: We obtain for $q=1$:

$$\boxed{\begin{array}{l}\partial_i A_j^k - \partial_j A_i^k - A_i^r\chi_{r,j}^k + A_j^r\chi_{r,i}^k = 0\\ \partial_i\chi_{,j}^k - \partial_j\chi_{,i}^k - \chi_{i,j}^k + \chi_{j,i}^k - \left(\chi_{,i}^r\chi_{r,j}^k - \chi_{,j}^r\chi_{r,i}^k\right) = 0\\ \partial_i\chi_{l,j}^k - \partial_j\chi_{l,i}^k - \chi_{li,j}^k + \chi_{lj,i}^k - \left(\chi_{,i}^r\chi_{lr,j}^k + \chi_{l,i}^r\chi_{r,j}^k - \chi_{l,j}^r\chi_{r,i}^k - \chi_{,j}^r\chi_{lr,i}^k\right) = 0\end{array}}$$

These are *exactly* the formulas of the first order nonlinear CC obtained by the Cosserat brothers in 1909 for the Lie pseudogroup of Euclidean rigid motions [18]. Contracting the lower formula in k/l, we obtain $\varphi_{ij} = A_i^r\chi_{sr,j}^s - A_j^r\chi_{sr,i}^s = \partial_i\chi_{r,j}^r - \partial_j\chi_{r,i}^r = \partial_i\alpha_j - \partial_j\alpha_i$ where $(\alpha_i = \chi_{r,i}^r)\in T^*$ when $\gamma=0$ ([25] p 436, 437). As $\partial_i\varphi_{jk} + \partial_j\varphi_{ki} + \partial_k\varphi_{ij} = 0$, this is a way to relate EM with the second order jets of the conformal group along the tentative of H. Weyl in [20].

Lemma 8.4: Passing to the limit *over the source* with $\chi_q = \bar{D}f_{q+1}$ and $h_{q+1} = id_{q+1} + t\xi_{q+1} + \cdots$ for $t \to 0$, we get an *infinitesimal gauge transformation* leading to the *infinitesimal variation*:

$$\boxed{\delta\chi_q = d\xi_{q+1} + L\left(j_1\left(\xi_{q+1}\right)\right)\chi_q}$$

which *only depends on* χ_q but *does not depend on* the parametrization of χ_q.

Lemma 8.5: Passing to the limit *over the target* with $\chi_q = \bar{D}f_{q+1}$ and $g_{q+1} = id_{q+1} + t\eta_{q+1} + \cdots$ for $t \to 0$, we get the other *infinitesimal variation*:

$$\boxed{\delta\chi_q = f_{q+1}^{-1} \circ d\eta_{q+1} \circ j_1\left(f_q\right)}$$

which depends on the parametrization of χ_q.

Example 8.6: We obtain for $q = 1$:

$$\begin{array}{|lcl|}\delta A_i^k & = & \xi^r \partial_r A_i^k + A_r^k \partial_i \xi^r - A_i^r \xi_r^k \\ \delta\xi_i^k & = & \left(\partial_i \xi^k - \xi_i^k\right) + \left(\xi^r \partial_r \chi_{,i}^k + \chi_{,r}^k \partial_i \xi^r - \chi_{,i}^r \xi_r^k\right) \\ \delta\chi_{j,i}^k & = & \left(\partial_i \xi_j^k - \xi_{ij}^k\right) + \left(\xi^r \partial_r \chi_{j,i}^k + \chi_{j,r}^k \partial_i \xi^r + \chi_{r,i}^k \xi_j^r - \chi_{j,i}^r \xi_r^k - \chi_{,i}^r \xi_{jr}^k\right)\end{array}$$

and in particular $\boxed{\delta det(A) = \xi^r \partial_r \left(det(A)\right) + det(A)\left(\partial_r \xi^r - \xi_r^r\right)}$.

For the Killing system $R_1 \subset J_1(T)$ with $g_2 = 0$, these variations are *exactly* the ones that had been found in 1909 by the Cosserat brothers([C], (50)+(49), p 124). *The two last unavoidable Lemmas are thus essential in order to bring back the nonlinear framework of finite elasticity to the linear framework of infinitesimal elasticity that only depends on the linear Spencer operator* d.

For the conformal Killing system $\hat{R}_1 \subset J_1(T)$, we obtain:

$$\begin{aligned}\alpha_i = \chi_{r,i}^r \Rightarrow \delta\alpha_i &= \left(\partial_i \xi_r^r - \xi_{ri}^r\right) + \left(\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r - \chi_{,i}^s \xi_{rs}^r\right) \\ &= \left(\partial_i \xi_r^r - A_i^s \xi_{rs}^r\right) + \left(\alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i\right)\end{aligned}$$

$$\varphi_{ij} = \partial_i \alpha_j - \partial_j \alpha_i \Rightarrow \delta\varphi_{ij} = \left(\partial_j\left(A_i^s \xi_{rs}^r\right) - \partial_i\left(A_j^s \xi_{rs}^r\right)\right) + \left(\varphi_{rj} \partial_i \xi^r + \varphi_{ir} \partial_j \xi^r + \xi^r \partial_r \varphi_{ij}\right)$$

These are *exactly* the variations obtained in 1918 by Weyl ([20], (76), p 289) who was assuming implicitly $A = 0$. Accordingly, ξ_{ri}^r is the variation of the EM potential itself, that is the δA_i of engineers used in order to exhibit the Maxwell equations from a variational principle but the introduction of the Spencer operator is new in this framework. Indeed, if $f_1 = id_1$, we have $\chi_0 = 0$ and thus $\delta\alpha_i = \left(\partial_i \xi_r^r - \xi_{ri}^r\right) + \left(\alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i\right)$.

The explicit general formulas of the two previous lemmas cannot be found somewhere else (The reader may compare them to the ones obtained in [15] by means of the so-called "*diagonal*" method that cannot be applied to the study of explicit examples). We provide a new elementary and constructive proof of the following difficult but crucial theorem:

Theorem 8.7: The same variation is obtained whenever $\eta_q = f_{q+1}\left(\xi_q + \chi_q(\xi)\right)$ with $\chi_q = \bar{D}f_{q+1}$, a transformation only depending on $j_1\left(f_q\right)$ and invertible if and only if $det(A) \neq 0$.

Proof. Using $\eta_q \in R_q(Y)$ *over the target* or $\xi_q \in R_q$ *over the source*, the formal derivative d_i for $i=1,\cdots,n$ and the composition of jets, we may define $\eta_q = f_{q+1}\left(\xi_q\right)$, we obtain:

$$\begin{array}{ccccccc} \delta f^k & = & \eta^k & = & \eta^k & = & f_r^k \xi^r \\ \delta f_i^k & = & d_i\eta^k & = & \eta_u^k f_i^u & = & f_r^k \xi_i^r + f_{ri}^k \xi^r \\ \delta f_{ij}^k & = & d_{ij}\eta^k & = & \eta_{uv}^k f_i^u f_j^v + \eta_u^k f_{ij}^u & = & f_r^k \xi_{ij}^r + f_{ri}^k \xi_j^r + f_{rj}^k \xi_i^r + f_{rij}^k \xi^r \\ \cdots & = & \cdots & = & \cdots & = & \cdots \\ \delta f_\mu^k & = & d_\mu \eta^k & = & \eta_u^k f_\mu^u + \cdots & = & f_r^k \xi_\mu^r + \cdots + f_{\mu+1_r}^k \xi^r \end{array}$$

Finally, we may write the symbolic formula $f_{q+1}\left(\chi_q\right) = j_1\left(f_q\right) - f_{q+1} = df_{q+1}$ defining the *Spencer operator* in the more explicit inductive form:

$$f_r^k \chi_{\mu,i}^r + \cdots + f_{\mu+1_r}^k \chi_{,i}^r = \partial_i f_\mu^k - f_{\mu+1_i}^k$$

Contracting by ξ^i and substituting in the previous formula provides $\eta_q = f_{q+1}\left(\xi_q + \chi_q\left(\xi\right)\right)$.

Checking directly the proposition is not evident when $q=0,1,2$ but cannot be done by hand when $q \geq 3$. Finally, setting $\bar{\xi}_q = \xi_q + \chi_q\left(\xi\right)$, we get $\bar{\xi} = A(\xi)$ for $q=0$, a transformation which is invertible if and only if $det(A) \neq 0$ or $det(\Delta) \neq 0$ because $det\left(f_i^k\right) \neq 0$ by assumption.

Example 8.8: Let us compute directly the variation of the 1-form α over the target and over the source, recalling that $\alpha = \alpha_i dx^i$ with $\alpha_i = \chi_{r,i}^r = g_k^r \partial_i f_r^k - A_i^r g_k^s f_{rs}^k = n\left(\partial_i a - A_i^r a_r\right)$ and $na_i = g_k^r f_{ri}^k$. We have successively for the conformal Killing system:

$$\delta f^k = \eta^k = \xi^r \partial_r f^k, \quad \delta f_i^k = \eta_u^k f_i^u = \xi^r \partial_r f_i^k + f_r^k \xi_i^r$$

$$\delta f_{ij}^k = \eta_{uv}^k f_i^u f_j^v + \eta_u^k f_{ij}^u = \xi^r \partial_r f_{ij}^k + f_{rj}^k \xi_i^r + f_{ri}^k \xi_j^r + f_r^k \xi_{ij}^r$$

$$n\delta a_i = g_k^r \delta f_{ri}^k + f_{ri}^k \delta g_k^r = g_k^r\left(\eta_{uv}^k f_i^u f_r^v + \eta_u^k f_{ir}^u\right) - f_{ri}^u g_k^r \eta_u^k = f_i^r \eta_{sr}^s$$

$$\begin{aligned} n\delta a_i &= g_k^s\left(\xi^r \partial_r f_{is}^k + f_{rs}^k \xi_i^r + f_{ri}^k \xi_s^r + f_r^k \xi_{si}^r\right) - f_{si}^u g_k^s\left(g_u^t\left(\xi^r \partial_r f_t^k + f_r^k \xi_t^r\right)\right) \\ &= n\left(\xi^r \partial_r a_i + a_r \xi_i^r\right) + \xi_{ri}^r \end{aligned}$$

Then, using the definition of $a = a(x)$, namely $det\left(f_i^k(x)\right) = e^{na(x)}$, we have:

$$\begin{aligned} n\delta a &= \left(1/det\left(f_i^k\right)\right)\delta det\left(f_i^k\right) = g_k^i \delta f_i^k = \eta_s^s \\ &= g_k^i\left(\xi^r \partial_r f_i^k + f_r^k \xi_i^r\right) = n\xi^r \partial_r a + \xi_r^r \end{aligned}$$

Using the variation Example 8.6, we finally get:

$$\begin{aligned} \delta\alpha_i &= n\delta\partial_i a - nA_i^r \delta a_r - na_r \delta A_i^r \\ &= \left(\partial_i \xi_r^r - \xi_{ri}^r\right) + \left(\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r\right) - \chi_{,i}^s \xi_{rs}^r \\ &= \left(\partial_i \xi_r^r - A_i^s \xi_{rs}^r\right) + \left(\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r\right) \end{aligned}$$

The terms $\partial_i \xi_r^r + \left(\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r\right)$ of the variation, including the variation of $\alpha = \alpha_i dx^i$ as a 1-form, are *exactly* the ones introduced by Weyl in ([20] formula (76), p 289). We also recognize the variation δA_i of the 4-potential ξ_{ri}^r used by engineers by using now second order jets.

We have over the target with $a(x)=b(f(x))$ and $a_i = f_i^k b_k$:

$$n\delta a = n\xi^r \partial_r a + \xi_r^r \Rightarrow n\delta a = \delta b + \eta^k \frac{\partial b}{\partial y^k} = \eta_s^s$$

$$f_r^k A_i^r = \partial_i f^k \Rightarrow f_r^k \delta A_i^r + A_i^r \eta_u^k f_r^u = \frac{\partial \eta^k}{\partial y^u}\partial_i f^u \Rightarrow \delta A_i^r = g_l^r\left(\frac{\partial \eta^l}{\partial y^k} - \eta_k^l\right)\partial_i f^k$$

$$\alpha_i = n\left(\frac{\partial b}{\partial y^k} - b_k\right)\partial_i f^k \Rightarrow \delta\alpha_i = \left(\left(\frac{\partial \eta_s^s}{\partial y^k} - \eta_{sk}^s\right) - nb_l\left(\frac{\partial \eta^l}{\partial y^k} - \eta_k^l\right)\right)\partial_i f^k$$

a result only depending on the components of the Spencer operator, in a coherent way with the general variational formulas that could have been used otherwise. We notice that these formulas show the importance and usefulness of the general formulas providing the Spencer non-linear operators for an arbitrary order, in particular for the study of the conformal group which is defined by second order lie equations with a 2-acyclic symbol when $n=4$.

The novelty brought by the fundamental diagram II is that we have now only n^2 components for $\left(\tau_{lj,i}^k\right)\in T^*\otimes \hat{g}_2$ and no longer the $n^2\left(n^2-1\right)/12$ components of the Riemann tensor. Hence, as we have already used the $n(n-1)/2$ components $\varphi_{ij} = \tau_{i,j} - \tau_{j,i} = -\varphi_{ji}$ with $\tau_{i,j} = \tau_{ri,j}^r$ for describing the EM field, we may choose the $n(n+1)/2$ components $\tau_{ij} = \frac{1}{2}\left(\tau_{i,j} + \tau_{j,i}\right) = \tau_{ji}$ while defining $\rho_{l,ij}^k = \tau_{li,j}^k - \tau_{lj,i}^k$ and setting $\rho_{i,j} = \rho_{i,rj}^r \neq \rho_{j,i}$ in such a way that $\rho_{r,ij}^r = \tau_{i,j} - \tau_{j,i} = \rho_{i,j} - \rho_{j,i}$. As $\hat{g}_2 \simeq T^*$ is 2-acyclic when $n\geq 4$ [5] [6], we may express $\rho_{l,ij}^k$ by means of ρ_{ij} or τ_{ij}. When there is no EM, that is to say when $\varphi=0$, then we can express $\rho_{l,ij}^k$ by means of $\rho_{ij} = \rho_{i,j} = \rho_{j,i} = \rho_{ji}$ or $\tau_{ij} = \tau_{ji}$ similarly. Setting $tr(\rho) = \omega^{ij}\rho_{ij}, tr(\tau) = \omega^{ij}\tau_{ij}$, we have $ntr(\rho) = 2(n-1)tr(\tau)$ [5]. Linearizing, we may set $nR_{l,ij}^k = \xi_{li,j}^k - \xi_{lj,i}^k$ with $\xi_{ij,s}^k = \delta_i^k A_{j,s} + \delta_j^k A_{i,s} - \omega^{kr}A_{r,s}$ but we must always remember that we are in the Spencer sequence at C_1 and *not* in the Janet sequence at F_1.

In the nonlinear framework [5], the variation of the action over the source is:

$$\Sigma \mathcal{X}_k^{\mu,i}\delta\chi_{\mu,i}^k = \Sigma \mathcal{X}_k^{\mu,i}\left(\partial_i \xi_\mu^k - \xi_{\mu+1_i}^k\right) + \cdots$$

However, we have only to look for the adjoint of the Spencer operator *over the target* while integrating by parts the summation up to sign [12] [25]:

$$\mathcal{Y}_k^u\left(\frac{\partial \eta^k}{\partial y^u} - \eta_u^k\right) + \mathcal{Y}_k^{v,u}\left(\frac{\partial \eta_v^k}{\partial y^u} - \eta_{uv}^k\right) + \cdots$$

$$\Rightarrow \eta^k \to \frac{\partial \mathcal{Y}_k^u}{\partial y^u}, \eta_u^k \to \frac{\partial \mathcal{Y}_k^{u,r}}{\partial y^r} + \mathcal{Y}_k^{,u}, \eta_{uv}^k \to \frac{\partial \mathcal{Y}_k^{uv,r}}{\partial y^r} + \mathcal{Y}_k^{v,u}, \cdots$$

Example 8.9: When $q=1$, we have the "*pure*" variations:

$$\mathcal{X}_k^i \delta A_i^k = \mathcal{X}_k^i \left(\xi^r \partial_r A_i^k + A_r^k \partial_i \xi^r - A_i^r \xi_r^k \right)$$
$$= \mathcal{X}_k^i g_l^k \left(\frac{\partial \eta^l}{\partial y^u} - \eta_u^l \right) \partial_i f^u = \Delta \mathcal{Y}_k^u \left(\frac{\partial \eta^k}{\partial y^u} - \eta_u^k \right)$$

Setting $\Delta \mathcal{Y}_k^u = g_k^r \mathcal{X}_r^i \partial_i f^u$ and $tr(\mathcal{Y}) = \mathcal{Y}_r^r$, we obtain (See [25], p 449 for details):

$$\boxed{\xi^r \to \partial_i \left(\mathcal{X}_k^i A_r^k \right) - \mathcal{X}_k^i \partial_r A_i^k = 0} \Rightarrow \boxed{\eta^k \to \frac{\partial Y_k^u}{\partial y^u} - tr(\mathcal{Y}) \frac{1}{\Theta} \frac{\partial \Theta}{\partial y^k} = 0}$$

after tricky computations with conformal factor $\Theta(y) = e^{-b(y)}$ over the target. Using the n identities provided in the beginning of this section, we may pass from target to source as follows:

$$\Delta \frac{\partial Y_k^u}{\partial y^u} = \Delta \frac{\partial}{\partial y^u} \left(\frac{1}{\Delta} \partial_i f^u g_k^r \mathcal{X}_r^i \right) = \partial_i f^u \frac{\partial}{\partial y^u} \left(g_k^r \mathcal{X}_r^i \right) = \partial_i \left(g_k^r \mathcal{X}_r^i \right)$$

However, we also obtain $\xi_i^k \to \mathcal{X}_k^i A_i^r \xi_r^k = 0$ and thus $\mathcal{X}_k^i A_i^k = 0 \Rightarrow tr(\mathcal{Y}) = 0$ because of the dilatation subgroup.

With $dy = \Delta dx$ and $det(A) = \Delta / det\left(f_i^k(x) \right) = e^{-na(x)} \Delta$, we obtain along [5]:

$$\int \tau det(A) dx = \int n \Theta^{(n-2)} \left(\omega^{kl}(y) \frac{\partial b_k}{\partial y^l} - \frac{(n-2)}{2} \omega^{kl}(y) b_k b_l \right) dy$$

Caring only about the second order jets, we may vary the b_k alone and get after integration by parts:

$$\delta b_l \to (n-2) \Theta^{(n-3)} \omega^{kl} \frac{\partial \Theta}{\partial y^k} + (n-2) \Theta^{(n-2)} \omega^{kl} b_k = 0 \Rightarrow \boxed{\frac{\partial b}{\partial y^k} - b_k = 0 \Rightarrow \alpha_i = 0}$$

a situation only existing when there is no EM, that is when
$\frac{\partial b_l}{\partial y^k} - \frac{\partial b_k}{\partial y^l} = 0 \Leftrightarrow \varphi_{ij} = 0$.

Substituting, we obtain:

$$\int \tau det(A) dx = -\int \left(n \Theta^{(n-3)} \omega^{kl}(y) \frac{\partial^2 \Theta}{\partial y^k \partial y^l} + \frac{n(n-4)}{2} \Theta^{(n-4)} \omega^{kl}(y) \frac{\partial \Theta}{\partial y^k} \frac{\partial \Theta}{\partial y^l} \right) dy$$

while, integrating by parts, we also get:

$$\int \tau det(A) dx = -\int \frac{n(n-2)}{2} \Theta^{(n-4)} \omega^{kl}(y) \frac{\partial \Theta}{\partial y^k} \frac{\partial \Theta}{\partial y^l} dy$$

If we only vary the section $y = f(x)$ of $X \times Y$ over X, we have $dy = \Delta dx$, $\delta \Delta = \Delta \frac{\partial \eta^u}{\partial y^u}$ and:

$$\Theta^n det\left(f_i^k(x) \right) = 1 \Rightarrow 0 = \delta(\partial_i \Theta) = \delta \left(\frac{\partial \Theta}{\partial y^k} \right) \partial_i f^k + \frac{\partial \Theta}{\partial y^u} \frac{\partial \eta^u}{\partial y^k} \partial_i f^k$$
$$\Rightarrow \delta \left(\frac{\partial \Theta}{\partial y^k} \right) = -\frac{\partial \Theta}{\partial y^u} \frac{\partial \eta^u}{\partial y^k}$$

It follows that the variation of the last integral is:

$$-\int n(n-2)\Theta^{(n-4)}\omega^{kl}(y)\left(\frac{\partial\Theta}{\partial y^l}\frac{\partial\Theta}{\partial y^u}\frac{\partial\eta^u}{\partial y^k}-\frac{1}{2}\frac{\partial\Theta}{\partial y^k}\frac{\partial\Theta}{\partial y^l}\frac{\partial\eta^u}{\partial y^u}\right)\mathrm{d}y$$

After integration by parts, we get, up to a divergence:

$$-n(n-2)\int\frac{\partial}{\partial y^u}\left(\Theta^{(n-4)}\left(\omega^{ru}(y)\frac{\partial\Theta}{\partial y^r}\frac{\partial\Theta}{\partial y^k}-\frac{1}{2}\delta^u_k\omega^{rs}(y)\frac{\partial\Theta}{\partial y^r}\frac{\partial\Theta}{\partial y^s}\right)\right)\eta^k\Bigg)\mathrm{d}y$$

Example 8.10: When $n=4$ *only*, the direct computation becomes simpler because a part of the integral disappears. We are left with $\tau det(A)=-n\Theta\Box\Theta$ and we recognize the well known *Abraham tensor* in the bracket, *without any other assumption*. Indeed, setting over the target:

$$\delta(\tau det(A))=-n\Theta^{n-2}\Delta\omega^{kl}(y)\frac{\partial b_k}{\partial y^r}\frac{\partial\eta^r}{\partial yl}+\tau det(A)\frac{\partial\eta^s}{\partial y^s}+\cdots$$

$$\Rightarrow\mathcal{Y}^u_{k,add}=\tau\Theta^4\delta^u_k-n\Theta^{n-2}\omega^{ru}(y)\frac{\partial b_r}{\partial y^k}$$

With $b_k=-\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}$, we finally obtain:

$$\frac{\partial\mathcal{Y}^u_{k,add}}{\partial y^u}=-2n\frac{\partial\Theta}{\partial y^k}\omega^{rs}(y)\frac{\partial^2\Theta}{\partial y^r\partial y^s}$$

$$=-2n\frac{\partial}{\partial y^u}\left(\omega^{ru}(y)\frac{\partial\Theta}{\partial y^r}\frac{\partial\Theta}{\partial y^k}-\frac{1}{2}\delta^u_k\omega^{rs}(y)\frac{\partial\Theta}{\partial y^r}\frac{\partial\Theta}{\partial y^s}\right)$$

Setting $\mathcal{Y}^u_k=\mathcal{Y}^u_{k,pure}+\mathcal{Y}^u_{k,add}$ in a coherent way with the condition $\frac{\partial\mathcal{Y}^u_k}{\partial y^u}=0$, it is thus *necessary* to have the so-called *Poisson equation*:

$$\boxed{2n\Theta\omega^{kl}(y)\frac{\partial^2\Theta}{\partial y^k\partial y^l}=tr(\mathcal{Y}_{pure})}\Rightarrow\frac{\partial\mathcal{Y}^u_{k,add}}{\partial y^u}=-tr(\mathcal{Y}_{pure})\frac{1}{\Theta}\frac{\partial\Theta}{\partial y^k}=tr(\mathcal{Y}_{pure})b_k$$

Such a result proves that *the gravitational force appearing in the right member of the Cauchy equation can be expressed as the divergence of the additional Abraham stress tensor density*, exactly like *the EM Lorentz force can be expressed as the divergence of the so-called Maxwell stress tensor density*. Accordingly, we may say, as in the previous section, that *the whole gravitational scheme only depends on the structure of the conformal group*. As a byproduct, we may say that there is no conceptual difference between the so-called "*virial*" theorem of Clausius (1870) and the Poisson equation of gravity (1823) as *both are involving the trace of the stress tensor density* [32].

9. Conclusion

These *new unavoidable methods* based on the formal theory of systems of partial differential equations and Lie pseudogroups provide the common secret of the three famous books [18] [20] [33] published about at the same time at the beginning of the last century. Indeed, the Spencer operator can always be exhibited even if there is no group background and, when only constant sections are considered, one recovers exactly (up to sign) the operator introduced by Macaulay for studying *inverse systems* [34]. As a main consequence, *the following results, totally new from both a historical and even a mathematical point of view, will explain the title of this paper*:

1) In General Relativity, the Einstein equations of 1915 for space-time had already been exhibited by E. Beltrami for space alone in 1892, that is almost 25 years before but this FACT is not even known though the comparison we made... needs no comment [12] [13].

2) As the corresponding *Einstein* operator is self-adjoint in any dimension $n \geq 3$, it follows from *diagram chasing* that the mathematical foundations of gravitational waves (GW) are not compatible with purely mathematical results to be found in homological algebra, namely differential double duality and extension modules, and cannot thus exist. As can be checked *at once* from any textbook of continuum mechanics, the confusion done between the *Cauchy* operator (adjoint of the *Killing* operator) and the *div* operator (induced from the *Bianchi* operator) cannot be accepted any longer [2] [7] [9] [12] [13] [35].

3) In Special Relativity, there is no reason for using only the Lorentz transformations. Indeed, the conformal group of spacetime preserving the Minkowski metric up to a multiplication by a non-vanishing function called *conformal factor*, is the *only* good candidate for the invariance of Maxwell equations in a coherent way with the second part of the 1905 paper of Einstein. It follows that conformal geometry must be entirely revisited by using Spencer δ -cohomology *in any dimension* n because the *Weyl* operator is indeed a self-adjoint third order operator when $n=3$ with first order generating CC while the well known second order *Weyl* operator has only second order generating CC when $n=4$, a result that can even be established by using computer algebra and is leading to the two following conformal differential sequences still totally unknown today in the literature:

$$\boxed{0 \rightarrow \hat{\Theta} \rightarrow 3 \underset{1}{\rightarrow} 5 \underset{3}{\rightarrow} 5 \underset{1}{\rightarrow} 3 \rightarrow 0} \qquad \boxed{0 \rightarrow \hat{\Theta} \rightarrow 4 \underset{1}{\rightarrow} 9 \underset{2}{\rightarrow} 10 \underset{2}{\rightarrow} 9 \underset{1}{\rightarrow} 4 \rightarrow 0}$$

4) I did provide (as early as in 1983!) the "*fundamental diagram II*" proving through a *diagonal diagram chasing*, that *electromagnetism and gravitation only depend on the structure of the conformal group of space-time through the second order jets of transformations* [6] [12] [13].

5) This paper can also be considered as an elementary summary of certain recent results presented in the references below. A much more difficult non-linear version of the preceding results can be found in [2] [5]. We hope to have convinced the reader that most of the applications presented are providing explicit examples that are tricky enough in order to justify the use of computer algebra in a near future for proving, as in [35] but contrary to [31], that *black holes cannot exist*. Indeed, we did prove that *the group of invariance of a metric is more important than the metric itself* by using the *fundamental diagram II* in which the Spencer sequences only depend on such a group in the spirit of GT, contrary to the Janet sequences that could be quite intricate and are only involved in the origin

of GR through the Riemann curvature without any torsion [Compare to [36]).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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