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Physical Constraints Simplify 5 Dimensions in Relativistic Waves

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Abstract

The classical mechanics of Newton—where momentum energy is less than rest mass energy, $pc < m_oc^2 = E_o$—transitions, with increasing momentum, to the relativistic mechanics of Einstein and Klein-Gordon—where $pc > m_oc^2$. The transition replaces the classical rest mass energy constant E_o by Einstein's varying relativistic energy $E = m'c^2 = \sqrt{\boldsymbol{p}^2c^2 + m_o^2c^4}$. We proceed to apply this 5-dimensional relativistic equation to dispersion dynamics in electron microscope probes[1]. The two scalers, E and m_o are initially presented as concentric, spherically-symmetric, spheres that are constrained by the Pythagorean triangle that is implicit in the relativistic expression $E^2 = \boldsymbol{p}^2c^2 + m_o^2c^4$, where the speed of light c is constant. Then $\boldsymbol{p}$, $\boldsymbol{E}$ and $\boldsymbol{m_o}$ are co-planar, and all act like vectors in 5 independent dimensions. The fact is significant because E is conjugate to time t in the free particle wave equation. Band structures and band gaps are derived. Remarkable physical effects are explained with mathematics hardly more complex than Pythagoras's theorem in a realistic framework of conventional and verified hypotheses. Consistent with Occham's razor, a proper economy in dimensionality is retained. On this firm footing, other dimensions extend to 2-dimensional displays of physical properties including Minkowski space and phase velocity in internal motion.

Keywords

5-Fold Dimensionality, Relativity, Wave Packet, Phase Velocity, Group Velocity, Dispersion Dynamics, Quanta

1. Introduction

Einstein's equation for a free particle in special relativity $E = m'c^2$ is 5-dimen-

[1]Typically 10 keV - 2 MeV.

sional[2]. Displays in 3-dimensions are easy enough in Cartesian space; and the mathematical tesseract is sometimes used [1] to simulate 4-dimensional structures. The last is generally unenlightening because unintuitive, and the tesseract is also short of one variable. The only easy solution is found by combining physical constraints within Cartesian geometry. This solution is here developed to display not only the conjugate variables of space-time, x, y, z, t, p_x, p_x, p_x and E *and* ω, (as in the free particle wave function below), but also m_o, phase and group velocities v_p, v_g, plus other variables that describe the dynamics of internal motion [2]-[5].

2. Five Dimensions

We begin by expanding the two scalers for rest mass m_o and energy E into two concentric, symmetric, 3-dimensional spheres, before cutting them through to a single plane that contains the two concentric radii in common with the 3-dimensional momentum $\boldsymbol{p}$ in the free particle (Figure 1). A legend for the symbols used is given in **Appendix 1**.

The two scalers are thus represented by concentric circles whose surfaces contact the vector $\boldsymbol{p}$, at both ends. The radii at the points of contact form a Pythagorean triangle with right angle at the point of contact between tangential $\boldsymbol{p}$ and $m_o c$. It is convenient to choose the orientation $\boldsymbol{p} = p_x$ as indicated in the figure. The triangle is a 2-dimensional cut from the spheres into the triangular plane bounded by the lines $p_x - E/c - m_o c$. The construction is cylindrically symmetric as indicated, and the rest mass and energy can be treated as vectors $\boldsymbol{m_o c}$ and $\boldsymbol{E}$ within the planar physical constraint.

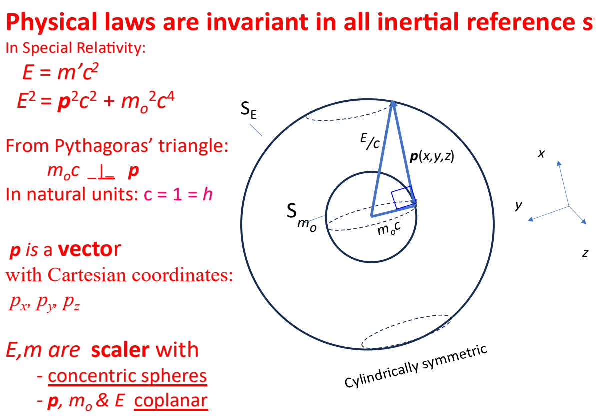

Figure 1. The momentum vector $\boldsymbol{p}$ is shown displaced from its Cartesian coordinate axes (right). The vector is tangential to a sphere S_{m_o} that represents the scalar mass $m_o c$, which touches the vector at its base. At its point of contact, $\boldsymbol{p}$ is perpendicular to the radius $m_o c$. Similarly, a concentric spherical surface S_E represents the scalar energy E/c. This contacts the other end of $\boldsymbol{p}$. Physics constrains the three sides of the right-angled triangle to coplanarity.

[2]Rest mass m_o is constant in Special Relativity but not necessarily in General Relativity. m_o is normal to the momentum $\boldsymbol{p}$ in the Pythagorean triangle. In General Relativity, space is locally Euclidean as it is generally in Special Relativity.

3. Band Structures

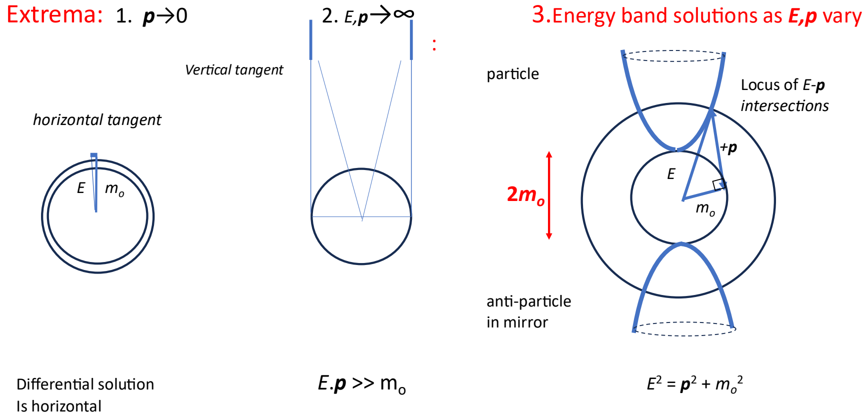

Figure 2. Band structures are constructed by physical constraints: *firstly* as $\boldsymbol{p} \to 0$, a tangential and horizontal p_y band at the intersection of a vertical $\boldsymbol{m_o}$ and nearly parallel $\boldsymbol{E}$; *secondly* as $\boldsymbol{p}, \boldsymbol{E} \gg \boldsymbol{m_o}$, almost vertical tangential bands; and *thirdly* intermediate solutions derived as in Figure 3.

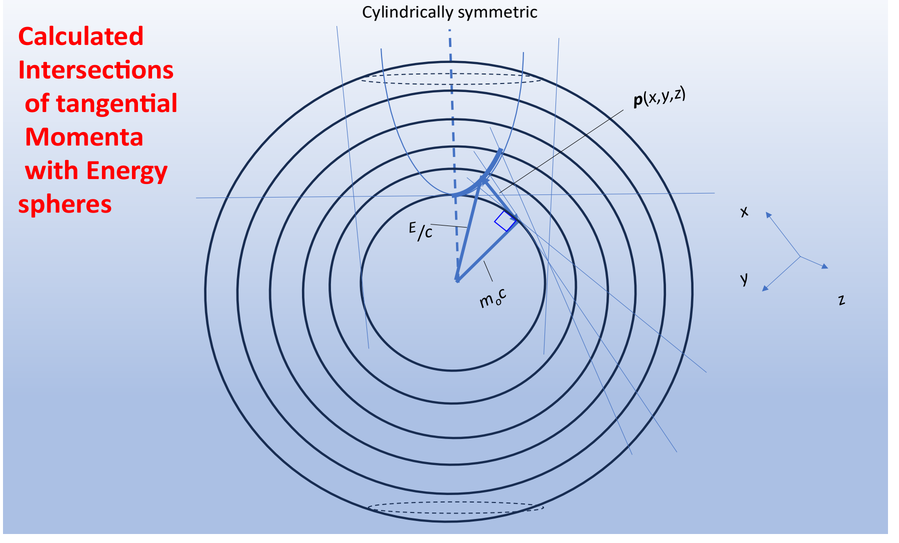

Figure 3. Construction of bands at various tangents on $\boldsymbol{m_o c}$ that plot the constrained intersections of $\boldsymbol{p}$ with $\boldsymbol{E/c}$ as the latter changes with increasing concentric radius. Notice that the 3-D construction (including spheres expanded from the m_o scaler and various E scalers) is reduced (by physical constraint due to the relativity equation) to a plane dictated by the vector $\boldsymbol{p}$. This is the planar section displayed by broad lines in the figure.

Energy bands occur on the locus of intersections of $\boldsymbol{p}$ with $\boldsymbol{E}$, consistent with varying tangents at $\boldsymbol{m_o c}$ (Figure 2, Figure 3). The bands are cylindrically symmetric, with particle bands mirrored by anti-particle bands where $m_o c$, E/c and p are all negative in dispersion dynamics [4]. The particle to anti-particle band gap is, obviously, $2m_o c$. These figures describe, simply and intuitively, the display of 5-D structures onto 2-D displays. Next, we extend their application by simplifying to variables in dispersion dynamics with natural units.

4. Extension to further Dimensions

To define the point of our discussion, it is necessary to outline the perspective of dispersion dynamics. This transforms the variables we have been using to their equivalent wave properties in order to associate them with their origins: in emission, in their resonant interactions, in their measurement, and in their quantum values. The free-particle wave-function in **Figure 4** serves not only the classical optics of Huygens, Fresnel and Fraunhoffer; and furthermore the modern, quantized waves of Schrödinger and de Broglie; but the wave also explains Heisenberg's uncertainty with advanced precision.

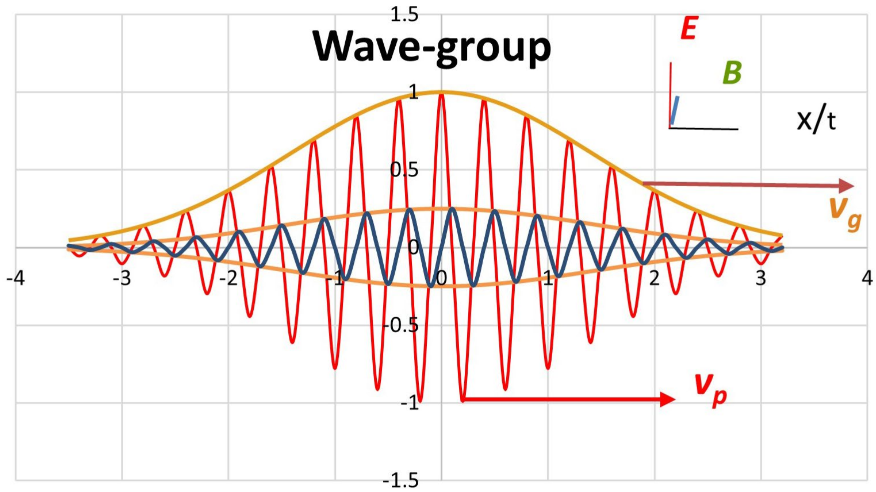

Figure 4. The electromagnetic quantum due to dipole emission is derived from Maxwell's equations. The electric vector is shown in the vertical plane with the magnetic vector horizontal out of phase π/2. The phase velocity of a free photon is equal to its group velocity.

The wave group is the product of the normal distribution with Euler's wave:

$$\phi = A \cdot \exp\left(\frac{X^2}{2\sigma^2}\right) \cdot \exp\left(i\omega t - ikx\right) \tag{1}$$

where $X = i\left(\bar{\omega}t - \bar{k}x\right)$ when $\bar{\omega}$ coincides with peak ω at $x = 0$;
while $\bar{k}$ coincides with peak k at $t = 0$.

Then referring to **Figure 4**, it is obvious that when $x = 0$:

$$\Delta t = \frac{2\sigma}{\bar{\omega}} \tag{2}$$

i.e. the group width at height ϕ_{max}/e. The transform of a Gaussian is Gaussian. The Fourier transform of Equation (1) is written:

$$\phi'(\omega)_{x=0} = F\left(\phi(t)^*_{x=0} \cdot \phi(t)_{x=0}\right) = \frac{\sigma}{\bar{\omega}\sqrt{2}} \exp\left(-\frac{\sigma^2\omega^2}{4\bar{\omega}^2}\right) \tag{3}$$

$$\text{Then:}\quad \Delta\omega = \frac{4\bar{\omega}}{\sigma} \tag{4}$$

$$\Delta t \Delta\omega = 8, \tag{5}$$

$$\Delta x_i \cdot \Delta k_i = 8,\ \ i = x, y, z \tag{6}$$

This clear, wave result is 16x greater than Heisenberg's 'limits', whatever they may be.

In optics, the wave is real for both the electric field and magnetic field, while their phase relationship is expressed by making the magnetic field imaginary in Euler's wave formulation. The wave also applies to electron optics when allowance is made for mass, charge and internal motion that replace polarization in massless light optics. The finite mass of the electron results in complicated internal motion [2]. By differentiating Einstein's formula in special relativity, the phase velocity in a massive free particle is shown and measured to be faster than the speed of light within the wave group. In free space typically, $v_g < c$ and $v_p > c$ (**Figure 5**) [4]: This is demonstrated in 6 quick steps:

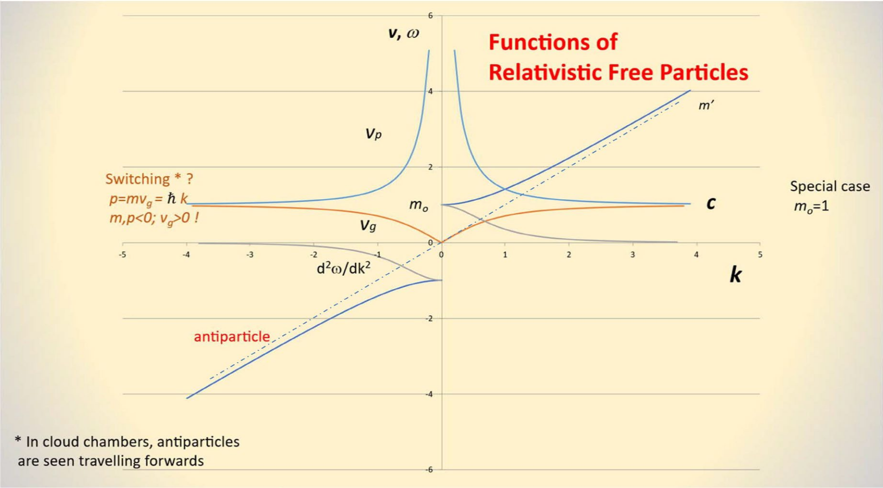

Figure 5. Functions of relativistic free particles. The velocities in massless light is represented by the dashed diagonal through the origin; in free electrons the internal phase velocity is faster than light and $v_p \cdot v_g = c^2$.

1) In wave mechanics, phase velocity is the product of wavelength and frequency:

$$v_p = \lambda n = \frac{\omega}{k} \tag{7}$$

2) *The phase velocity is proportional to* λ. Proceeding from Newton to relativity to quanta as below:

$$p = m' v_g = \frac{h}{\lambda} = \hbar k = \frac{m_o v_g}{\sqrt{1 - v_g^2 / c^2}} = \frac{m_o v_g c}{\sqrt{c^2 - v_g^2}} \tag{8}$$

3) Differentiate relativity: $\omega^2 = k^2 + m_o^2$ in natural units to find $v_g \cdot v_p = 1$:

$$\frac{\mathrm{d}\left(E^2 = p^2 c^2 + m_o^2 c^4\right)}{\mathrm{d}k} \supset \frac{\omega}{k} \cdot \frac{\mathrm{d}\omega}{\mathrm{d}k} = c^2 = v_p \cdot v_g \tag{9}$$

where v_g is the beat velocity in the normal distribution, Equation (3).

4) Semi-Classically, $p \to 0$:

$$E_o = \hbar\omega_o = m_o c^2 \tag{10}$$

5) *Substituting the variable E in relativity for the classical constant E_o:*

$$E = \hbar\omega = \frac{m_o c^2}{\sqrt{1 - v_g^2/c^2}} = \frac{m_o c^3}{\sqrt{c^2 - v_g^2}}$$

6) The group velocity v_g is proportional to λ^{-1} cf. Equation (7)!

$$\frac{E}{p} = \frac{m_o c^3}{\sqrt{c^2 - v_g^2}} \cdot \frac{\sqrt{c^2 - v_g^2}}{m_o v_g c} = \frac{c^2}{v_g} = v_p = \frac{\omega}{k}; \quad \frac{pc}{E} = \frac{v_g}{c} \tag{11}$$

4.1. Co-planarity of $v_{g,x}$-t-x and $v_{p,x}$-t-x

In dispersion dynamics we need to add to Figure 2, dimensions of time and phase velocity v_p. We already include group velocity in momentum $\boldsymbol{p} = m_o \boldsymbol{v_g}$. So if we consider a free electron that is physically constrained by relativity and directed to react to some experimental arrangement, its velocity follows its momentum: if we are considering the p_x-component as in Figure 2, then $\boldsymbol{v} = v_x$ and its change of position Δx after time Δt is $v_x \cdot \Delta t$. The definition of velocity, in this case v_x, is the ratio of its components $\Delta x/\Delta t$. We make the velocity v_g bisect the angle between the axes for t and x as a modification of Minkowsksi's diagram for the velocity of light that has a different bisecting angle of π/2.

Our construction is confirmed by Planck's law in quantum mechanics, where t and E are conjugate variables as in Euler's wave. Both are scalers and are therefore drawn on a uni-directional axis in Figure 6. In this particular configuration the dot product $\boldsymbol{t} \cdot \boldsymbol{v_x}/(|t| \cdot |v_x|)$ is the same angle as $\cos^{-1}(\boldsymbol{E} \cdot \boldsymbol{p}/(|E| \cdot |p|))$. This adaptation of Minkowski space [6] therefore has the divergent angle between the axes for t and v_x, not Minkowski's right angle, but the configured angle $\cos^{-1}(\boldsymbol{E} \cdot \boldsymbol{p}/(|E| \cdot |p|))$. It is important to make v_x bisect the angle between t and the x-axis so that x/t is the constant v_x in a free particle.

Typically the Cartesian axial scales are also different. As in Figure 5, logarithmic plots are better adapted to compare group and phase velocities than are linear plots. Since $v_p/v_g = c^2$, the speed of light (dotted circle in Figure 6) limits both group velocities and phase velocities, for the former velocity c is maximal; for the the latter c is minimal.

In Figure 6, the main components of the internal velocity $v_x^p = x/t$ are shown as an extensions of $\boldsymbol{p}$. Time t that is conjugate to energy E and angular momentum ω, aligns with the latter two variables, and the major components of the velocities align with the vector $\boldsymbol{p}$. Time is common also to corresponding transverse components v_y and v_z if they occur as products of whatever experimental arrangement exists. The vector axes (green in Figure 6), describing internal motion (red), are shifted and rotated from the axes applied in Figure 1 (blue).

In the configuration shown, while the directions of $\boldsymbol{p}$, t and x are fixed for the free particle, there remains some freedom to optimize the y and z spatial axes. The y- and z-axes are both normal to the Cartesian x-axis; and are mutually normal to each other. It is assumed that these will be assigned to suit details of any scattering that will be described.

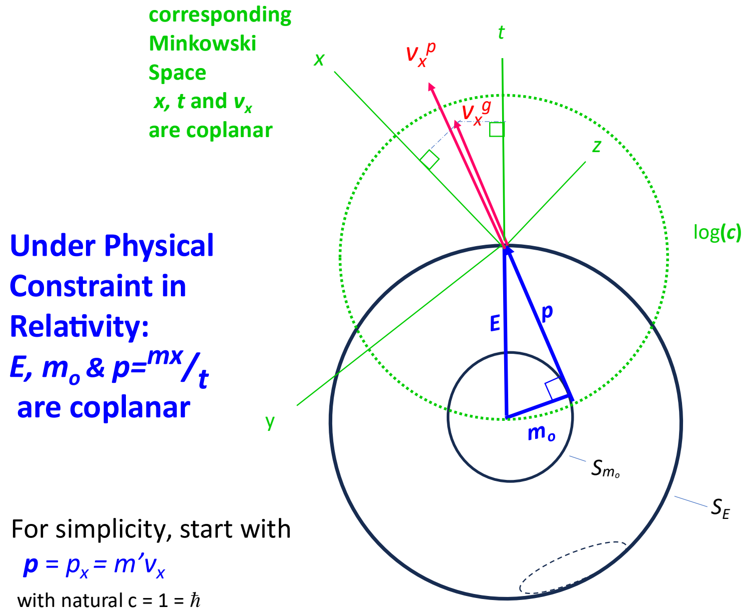

Figure 6. Extended dimensional analysis of a massive particle scattering diffractively. The products include the initial vector $\boldsymbol{p}$, related to 2 concentric relativity spheres, S_E *and* S_{mo}, on scaler radii E and m_o respectively, both restricted to vectorial operation by co-planarship with $\boldsymbol{p}$. After diffractive scattering, further products include vector phase and group velocities that are limited by c, and that depend on scattering details in 4 space-time coordinates which are analyzed in (green) t-v_x-x space.

4.2. Co-Planarity of v_y-t-y and of v_z-t-z

Consider the motion of an electron, with initial momentum $\boldsymbol{p}$, that is elastically scattered into 3-dimensional space. Typically, as in Bragg diffraction, the motion depends on internal properties while the group properties such as momentum is projected inherently as in Equation (1). Resulting waves have superposed components such as $v_{g,x}$, $v_{g,y}$, $v_{g,z}$. It is obvious that co-planarity that was described in Section 4.1 and in Figure 1 and Figure 6, applies with appropriate changes to the v_y-t-y and v_z-t-z planes. These changes include bisections of the scattering angles between relevant spatial axes and t.

The examples show how complicated axial dimensional problems can be solved by simplifying physical constraints. For the free electron $\boldsymbol{p} = m\boldsymbol{v}$ we have included axes for p_i, $v_{g,i}$, $v_{p,i}$, k_i, x_i, *i.e.* 15 dimensions. These double with the positron where E and m_o have negative values; though t, v_g[3] and v_p are always positive.

5. Conclusion

The five variables in special relativity, one vector and two scalers, are recognized

[3]As in chamber photographs of particle-antiparticle creation.

as co-planar in a Pythagorean triangle. This fact transforms the scalers to vector look-alikes in many cases that are represented in 2-dimensional drawings. These are easily recognized by perspectives on Cartesian axes. The variables are conjugated, as in wave equations and in quantum mechanics: energy with time, and momentum with space, velocity and mass. Further co-planarities are discovered in time-velocity-space properties, and in dispersion dynamics with internal motion. At least 26 dimensions are translated to 2-dimensional drawings by these procedures.

Generally, multi-dimensional structures in Hilbert space can often be represented in 2-dimensional space when they are subject to physical constraints. The structures and dynamics are readily understood by a common sense of Cartesian space with time as a static variable. The drawings provide not only proper understandings of temporal change and energy constraints, but also representations of the dynamics of internal motion and of the de Broglie hypothesis in modern physics [3]: Planck's energy depends on phase velocity; de Broglie's momentum on group velocity. The number of possible applications, for 2-dimensional reductions of other physical constraints onto multi-variate dimensions, is unlimited. The mathematics of dimensionality is more informative when combined with physical constraints.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix 1. Legend for Symbols Used in Natural Units $c = \hbar = 1$

p:	momentum
m_o:	rest mass
E:	energy
E_0:	rest mass energy = m_oc^2 *in SI units*
c:	speed of light = 1 or c *in SI units*
m':	relativistic mass = $m_o / \sqrt{1 + v_g^2 / c^2}$
v_g:	group velocity (Beat of the normal distribution, $\mathrm{d}w/\mathrm{d}k$)
h:	Planck's constan $t = 1$ in natural units
	or in dispersion dynamics $h/2\pi$:
ω:	angular frequency
k:	wave vector
n:	frequency
λ:	wavelength
v_p:	phase velocity (frequency x wavelength and w/k)