Scientific Research Publishing

Journal of Modern Physics, 2026, 17(2), 148-158
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Relativistic Quantum Dirac-Like Equation for Arbitrary Spin in General Global Curved Spacetime

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How to cite this paper: Moalem, A. (2026) Relativistic Quantum Dirac-Like Equation for Arbitrary Spin in General Global Curved Spacetime. *Journal of Modern Physics*, **17**, 148-158.
https://doi.org/10.4236/jmp.2026.172010

Received: November 13, 2025
Accepted: February 10, 2026
Published: February 13, 2026

Abstract

A relativistic quantum equation for free particles of any spin in a global curved spacetime is derived. Using this equation, it is demonstrated that in axially symmetric spacetimes, the wave function factorizes into a reduced wave function and an analytically determined normalization function. In spherical coordinates, the reduced wave function splits into a reduced angular wave function identically the same as in a flat Minkowski spacetime and a radial wave function which satisfies a second order non-homogeneous differential equation with nonhomogeneous terms depending on the ratio of time to space curvatures.

Keywords

Relativistic Quantum Equation, Curved Spacetime, Vierbein Fields

1. Introduction

Based on Spacetime Algebra (STA), we have demonstrated in a previous contribution [1] that, in flat Minkowski spacetime (ST), a Dirac like equation [2] with $2n \times 2n$ gamma matrices, $n = 2s + 1$ accounts for free particles of any spin. STA incorporates scalars, vectors and higher-grade multi-vectors into a single consistent and compact approach to quantum mechanics. Particularly, a Dirac like equation can be presented in a form which can be analysed and solved without requiring the construction of an explicit matrix representation. Yet, in a formalism where the particle wave function is written as a spinor, the order of a matrix representation determines the spinor order and elucidates clearly what a specific spin eigenstate it presents. Our main interest in the present note is to reformulate the equation reported in [1] in a global curved ST. Our approach will be to define connections and covariant derivatives which allow the notion of parallel transport for multi-component spinor wave functions and provide relationships between Dirac physical quantities in Minkowski ST across various curved ST's. General Relativity (GR) provides a unified description of gravity as a geometric property of four dimensional spacetime (ST) (see for example Refs. [3]-[5]). The Minkowski ST is flat, unchanging and uniform throughout, providing a framework for particles and interaction other than gravitation. It serves merely as a static and inactive background for whatever physical phenomena present. Worth mentioning though that Minkowski ST provides all essential concepts that are adapted and generalized in GR to describe gravity in global curved ST. A global ST is dynamic, non-uniform four-dimensional geometrical manifold which accounts for gravity. In the presence of matter and energy a global ST is curved and interacts actively with physical systems. Rather than a real force, Gravity is due to ST curvatures. Thus, the impact of gravity on a given physical system depends on location and ST geometry. To illustrate how ST curvatures influence particle dynamics, we consider free particles in axially symmetric ST's, namely, the Schwarzschild, Freedman-Le Maître-Robertson-Walker (FLRW) and the stationary rotating Kerr ST. All these ST's are known to be exact axisymmetric solutions of the Einstein's Field Equations (EFE's) [5] and serve as models of black holes. We demonstrate that in these ST's, the wave function separates into an analytically calculable normalization function, a common angular wave function identically the same as for a Minkowski ST, and radial wave functions which satisfy nonhomogeneous second order differential equations with nonhomogeneous terms, all depending on the ratio of time to space curvatures. The article is organized as follows. In Section 2 we consider a Dirac Like equation in flat Minkowski ST. In Section 3, this equation is generalized to account for particles in a global ST. In Section 4 we analyse free particles of any spin in stationary axially symmetric ST's. We conclude in Section 5. Throughout, we use natural units : $\hbar = c = 1$, $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ for the Minkowski metric tensor, and $g_{\mu\nu}(x^0,x^1,x^2,x^3) = e^a_\mu e^b_\nu \eta_{ab}$ for a global metric with e^a_μ denoting a vierbein field and E^μ_a its inverse [6].

2. Relativistic Quantum Equation for any Spin in Minkowski ST

A Dirac like equation for any spin [1] [2] in stationary flat Minkowski ST is,

$$\left(i\gamma^\mu \eta_{\mu\nu}\partial^\nu - \beta M\right)\Phi^{(2n)} = 0, \tag{1}$$

where M is the particle rest mass, γ^μ are $2n \times 2n$ diagonal block matrices defined as,

$$\gamma_0^{(2n)} = I^{2n}; \gamma_\mu^{(2n)} \equiv \mathrm{diag}\left(i\sigma_\mu, \cdots, i\sigma_\mu\right); \mu = x, y, z, \tag{2}$$

σ_μ are Pauli matrices, $\beta = \mathrm{diag}\left(I^n, -I^n\right)$, and $I^{(n)}$ denotes $n \times n$ unit matrix, $n = 2s+1$. The matrices above are Hermitian anti-commuting and square to unity, *i.e.*,

$$\left(\gamma^\mu\right)^\dagger = \gamma^\mu; \left(\gamma^\mu\right)^2 = -1; \gamma^0 = I^{(2n)}; \mu = 1,2,3, \tag{3a}$$

$$\left\{\gamma^a,\gamma^b\right\} = 2\gamma^0\delta^{ab}; a,b = x,y,z\,, \tag{3b}$$

$$\left[\gamma^a,\gamma^b\right] = i\epsilon^{abc}\gamma_c; a,b,c = x,y,z\,. \tag{3c}$$

In a Clifford $Cl_3(1,3)$ algebra, the matrices γ^0, γ^1, γ^2, γ^3 represent 4-unit vectors parallel to the coordinate axes, with γ^0 which squares to 1 being time-like, and γ^μ ; $\mu = 1,2,3$ which square to −1 are space-like. Like the Dirac gamma matrices, the above matrices are isomorphic to the Cliford $Cl_3(1,3)$ algebra [7]. It is to be noted that for $s = 1/2$ fermions the matrices (2) differ from those known to be the Dirac matrices [2]. In fact, there are several choices of Dirac matrices in use. Among these we name, the so-called standard Dirac matrices [2],

$$\gamma^0 = \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(2)} \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}; i = 1,2,3\,, \tag{4}$$

the Chiral basis,

$$\gamma^0 = \begin{pmatrix} 0 & I^{(2)} \\ I^{(2)} & -0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; i = 1,2,3\,, \tag{5}$$

the Chiral Weyl basis,

$$\gamma^0 = \begin{pmatrix} 0 & I^{(2)} \\ -I^{(2)} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}; i = 1,2,3\,, \tag{6}$$

and the Majorana basis,

$$\gamma^0 = \begin{pmatrix} \sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix}, \gamma^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^1\sigma^3 \\ \sigma^1\sigma^3 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \tag{7}$$

All above bases satisfy the rules (3). Based on the Pauli Fundamental Theorem, any two sets of 4 × 4 anti-commuting matrices which square to ±1 are related by a similarity transformation and therefore, any of the above sets, forms 4 × 4 representation of a Clifford algebra. This is correctly true for other representations of order higher than 4. The matrices (2) form $2n \times 2n$ representation of a Clifford $Cl_3(1,3)$ algebra. In fact, the Dirac equation can be presented in a form in which it can be analysed and solved without requiring an explicit matrix representation [7]. Formally, apart from the gamma matrices being different, Equation (2) has the form of the Dirac equation and subjected to the same $Cl_3(1,3)$ algebra rules. The minimal order of matrix representation that satisfies the rules Equation (2) is four [8]. Taking $n = 2s+1$ is just right to satisfy this restriction and account for the allowed spin eigenstates $\mathcal{X}_s^{m_s}$ with components $m_s = s, s-1, s-2, \cdots, -s+2, -s+1, -s$. For $s = 1/2$ fermions, Equation (1) with the gamma matrices of Equation (2) yields two positive and two negative energy states with spin up and spin down which are, as expected, the four familiar Dirac Spinors. Likewise, for $s = 3/2$ fermions one obtains four spin eigenstates which correspond to $m_s = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ without any redundant states. Below we identify the gamma matrices with the spin components. Indeed, the commutative relations of the gamma matrices (3c) are reminiscent of the spin commutation relations,

$$\left[S_x, S_y\right] = iS_z, \left[S_z, S_x\right] = iS_y, \left[S_y, S_z\right] = iS_x . \tag{8}$$

Furthermore, neither the orbital angular momentum components $L_i \equiv i\epsilon^{ijk} x_j \partial_k$, nor the spin components $s_i \equiv \epsilon^{ijk} \gamma_j \partial_k$ commute with the free particle Hamiltonian H_0 of Equation (1),

$$\left[L_i, H_0\right] = +\epsilon^{ijk} \gamma_j \partial_k , \tag{9}$$

and,

$$\left[s_i, H_0\right] = -\epsilon^{ijk} \gamma_j \partial_k . \tag{10}$$

Yet as should be, the components of total angular momentum, $j_i \equiv L_i + \gamma_i$, commutes with H_0 so that the total angular momentum taken to be $J \equiv L + S \equiv L + \gamma$ is conserved. Furthermore, since H_0 is $2n \times 2n$ matrix, the wave functions are two-component spinors $\Phi^{(2n)} = \left(\Phi_1^{(n)} / \Phi_2^{(n)}\right)$ where either $\Phi_1^{(n)}$ or $\Phi_2^{(n)}$ designates a spin eigenstates. This is just right to allow for a complete set of $2(2s+1)$ orthonormal solutions, a set of $(2s+1)$ positive energy and a set of $(2s+1)$ negative energy solutions, each with helicity Eigenvalues, $h = s, s-1, \cdots, -s+1, -s$. We prefer here to use the presentation (2) mostly because it allows a clear relation to spin eigenstates. Again, for spin $s = 1/2$ fermions, one obtains four component Dirac spinors, corresponding to 2 positive energy solutions, one with spin up and one with spin down and 2 negative energy solutions one with spin up and one with spin down, identically the same as the four familiar Dirac spinors. All be it; we may conclude that Equation (1) provides a unified framework to deal with the dynamics of particles of any spin in Minkowski ST.

3. Relativistic Particle Equation in Global Spacetimes

Based on the Principle of Equivalence (POE), a ST is locally Minkowskian [9]. This means that the mathematical and physical rules which are valid in Minkowski ST are applicable in infinitesimal region around any point of a global ST. Thus, in a sufficiently small region around any point of a global ST Equation (1) is valid and may well serve as a starting point to reformulate Equation (1) in a global ST. To this aim we apply the tetrad formalism [6]. Tetrads (or vierbein fields) e_μ^a generalize a flat Minkowski ST to a curved ST by relating the curved space metric $g_{\mu\nu}$ to the flat metric through $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$. This allows defining local Minkowski coordinates system and using SR locally, while the vierbein fields taking care for the curved geometry. Saying this, Equation (1) may serve as a good starting point to extend its validity to global curved ST. To this aim, three modifications are to be introduced [10]: 1) Replace the Minkowski metric by a global tensor metric, $\eta_{\mu\nu} \to g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$; 2) Transform the local matrices to global matrices, $\gamma^\mu \to E_a^\mu \gamma^a$; 3) Replace partial derivatives by covariant derivatives, $\nabla_\nu \to \partial_\nu + \Omega_\nu$. Here Ω_ν stands for the connection coefficient for a multi component spinor wave function Φ [9] to be determined below. Inserting these modifications in

Equation (1) yields,

$$\left[E_a^{\mu}\gamma^a g_{\mu\nu}\left(\partial_\nu+\Omega_\nu\right)-E_a^0\gamma^a\beta M\right]\Phi=0\,. \tag{10}$$

To determine the connection Ω_ν , we recall that the rules for parallel transport of a spinor wave function and its Hermitian conjugate are,

$$\Phi\left(x\to x+dx\right)=\Phi\left(x\right)-\Phi\left(x\right)\Omega_\nu dx^\nu\,, \tag{11a}$$

$$\Phi^H\left(x\to x+dx\right)=\Phi^H\left(x\right)-\Phi^H\left(x\right)\Omega_\nu^H\left(x\right)dx^\nu\,. \tag{11b}$$

Above, Φ^H is the Hermitian conjugate of Φ defined as, the transposed of the complex conjugate of Φ, $\Phi^H\equiv\left[\left(\Phi^*\right)\right]^{\mathrm{T}}$. To determine the connections Ω_ν and $\Omega_\nu^H\left(x\right)$ we may apply two conditions. First, the quantity $S\left(x\right)\equiv\Phi^H\left(x\right)\gamma^0\Phi\left(x\right)$ is a scalar and must transform as a scalar. To first order, using (11),

$$S\left(x+dx\right)-S\left(x\right)=-\Phi^H\left(x\right)\left[\gamma^0\Omega_\nu^H+\Omega_\nu\gamma^0\right]\Phi\left(x\right)=0\,, \tag{12}$$

thus,

$$\Omega_\nu=-\Omega_\nu^H\,. \tag{13}$$

Secondly, $V^a\left(x\right)\equiv\Phi^H\left(x\right)\gamma^a\Phi\left(x\right)$ is a vector and must transform as a vector, *i.e.*,

$$V^a\left(x\to x+dx\right)-V^a\left(x\right)=-\omega_{\mu b}^a V^b\left(x\right)dx^\mu\,, \tag{14}$$

where $\omega_{\mu b}^a$ stands for the spin connection [6]. Again using (11) one finds,

$$\Phi^H\left(x\right)\left[\gamma^a,\Omega_\mu\right]\Phi\left(x\right)dx^\mu=\omega_{\mu b}^a V^b\left(x\right)dx^\mu\,. \tag{15}$$

Thus, the commutator above must be proportional to γ^b and the connection is related to a product of two gammas via,

$$\Omega_\mu=C\omega_{\mu bc}\gamma^b\gamma^c\,. \tag{16}$$

Above C is a constant which can be determined algebraically by using the gamma matrices (2). One finds,

$$\left[\gamma^a,\Omega_\mu\right]=4C\omega_{\mu bc}\gamma^b\gamma^c\,. \tag{17}$$

Comparing the above with (15) gives $C=1/4$, so that,

$$\Omega_\mu=\omega_{\mu bc}\gamma^b\gamma^c\,. \tag{18}$$

Now, combining the above with the well-known expression for the spin connection [6] [9], $\omega_{\mu b}^a=e_\mu^a E_b^\sigma\Gamma_{\sigma\mu}^\nu+e_\nu^a\partial_\mu E_b^\nu$, one obtains,

$$E_a^\mu\gamma^a g_{\mu\nu}\Omega_\nu=\frac{1}{2}\gamma^b\left[E_b^\sigma\partial_\sigma\ln e+\partial_\sigma E_b^\sigma\right], \tag{19}$$

where, $e=\sqrt{-g}$ and $g=\det\left(g_{\mu\nu}\right)$. Substituting the above in (1) gives,

$$\left[E_a^\mu\gamma^a\frac{1}{\sqrt{e}}\partial_\mu\left(\sqrt{e}\Phi\right)-E_a^0\gamma^a\beta M\Phi\right]=-\frac{1}{2}\gamma^b\partial_\sigma E_b^\sigma\Phi\,. \tag{20}$$

Several comments are to be made concerning the above expression. First, for the Minkowski ST all terms on the *r.h.s* vanish, and the equation above reduces to (1). Secondly, in the limit $M \to 0$ Equation (20) converges to the equation reported in Ref. [11] for massless particles of any spin. Thirdly, Equation (20) can be derived from the Lagrangian,

$$L = \Phi^H \left[E_a^\mu \gamma^a \frac{1}{\sqrt{e}} \partial_\mu \left(\sqrt{e} \Phi \right) - E_a^0 \gamma^a \beta M \Phi = -\frac{1}{2} \gamma^b \partial_\sigma E_b^\sigma \Phi \right]. \tag{21}$$

Clearly, a variation with respect to Φ^H yields Equation (20), while a variation with respect to Φ yields the Hermitian conjugate of (20), *i.e.*,

$$E_a^\mu \gamma^a \frac{1}{\sqrt{e}} \partial_\mu \left(\sqrt{e} \Phi^* \right) - E_a^0 \gamma^a \beta M \Phi^* = -\frac{1}{2} \gamma^b \partial_\sigma E_b^\sigma \Phi^*. \tag{22}$$

Furthermore, using the Noether's theorem, the conserved current density satisfies the continuity equation,

$$j^\mu = \frac{\delta L}{\delta \left(\partial_\mu \Phi_b \right)} \frac{\delta \Phi_b}{\delta \alpha} - j_0^\mu = \Phi_b^H E_a^\mu \gamma^a \Phi^b, \tag{23}$$

with $j^0 = \Phi_b^H \gamma^0 \Phi^b = \Phi_b^* \Phi^b$ being the probability density. Note that j^μ does not depend on the spin connection. Following Ref. [11], we quote without details that field quantization can be accomplished by using a complete set of $2n$ orthonormal positive energy solutions, n particle and n anti-particle wave functions. Saying this we conclude that, Equation (20) is a generic relativistic quantum equation for particles of any spin in a ST with a global tensor metric $g_{\mu\nu}$.

4. Free Particles in Axially Symmetric Spacetimes

Consider now Equation (20) for axially symmetric STS. Symmetry is encoded in the vierbeins and spin connection. Specifically, the vierbeins are functions of x^1, x^2 only and the axially symmetric solutions of the Einstein's Fields Equations restrict the form of the vierbeins to be [3],

$$E_\mu^a \left(x^1, x^2 \right) = \begin{pmatrix} E_0^0 & 0 & 0 & E_0^3 \\ 0 & E_1^1 & 0 & 0 \\ 0 & 0 & E_2^2 & 0 \\ E_3^0 & 0 & 0 & E_3^3 \end{pmatrix}. \tag{24}$$

The fields in (24) are listed below in **Table 1** for the ST's to be considered. In what follows we assume that the wave function factorizes as,

$$\Phi \left(t^0, r, \vartheta, \varphi \right) = \psi \left(t^0, r, \vartheta, \varphi \right) f \left(r, \vartheta \right), \tag{25}$$

and ψ satisfies the equation,

$$\left(E_a^\mu \gamma^a \partial_\mu - E_a^0 \gamma^a \beta M \right) \psi = 0. \tag{26}$$

We refer to $\psi \left(t^0, r, \vartheta, \varphi \right)$ as the reduced wave function and to $f \left(r, \vartheta \right)$ as the normalization function. Substituting Equation (25) in Equation (20) gives,

$$E_a^\mu \gamma^a \partial_\mu \ln \left(\sqrt{e} f \right) = -\frac{1}{2} \gamma^b \partial_\sigma e_b^\sigma. \tag{27}$$

Note that due to the wave function factorization, all complexities due to the spin connection are restricted in the expression above. The reduced wave function and likewise its solutions do not depend on the spin connection. For stationary axially symmetric spacetimes the normalization function $f(r,\vartheta)$ does not depend either on time or on the spherical angles, $\partial_\vartheta \ln\left[\sqrt{e} f(r,\vartheta)\right] = \partial_\varphi \ln\left[\sqrt{e} f(r,\vartheta)\right] = 0$ so that (27) simplifies to,

$$\partial_r \ln\left[\sqrt{e} f(r,\vartheta)\right] = -\frac{1}{2} e_1^1 \partial_r E_1^1 . \quad (28)$$

The expression above is numerically integrable, and the function f resumes simple analytic form as listed in **Table 1**.

Table 1. The normalization function.

Spacetime	E_a^μ	E_0^0/E_1^1	$N(r,\vartheta)$
Minkowski	$\mathrm{diag}\left(1,1,\frac{1}{r},\frac{1}{r\sin\vartheta}\right)$	1	$r\sin^{1/2}\vartheta$
FLRW[2]	$\mathrm{diag}\left(1,\frac{F}{a},\frac{1}{ar},\frac{1}{ar\sin\vartheta}\right)$	$\frac{a}{F}$	$a^3 r^{1/2}\sin^{1/2}\vartheta$
Schwartzchild[3]	$\mathrm{diag}\left(\sqrt{F},\frac{1}{\sqrt{F}},\frac{1}{r},\frac{1}{r\sin\vartheta}\right)$	$\frac{1}{F}$	$F^{1/4} r\sin^{1/2}\vartheta$
Kerr[4]	$\mathrm{diag}\left(\frac{a^2+r^2}{\rho\sqrt{\Delta}},\frac{\sqrt{\Delta}}{\rho},\frac{1}{\rho},\frac{1}{\varrho r\sin\vartheta}\right)$ $E_0^3=\frac{a}{\rho\sqrt{\Delta}}, E_3^0=\frac{a}{\rho}r\sin\vartheta$	$\frac{a^2+r^2}{\Delta}$	$\Delta^{1/4}\rho^{1/2}\sin^{1/2}\vartheta$ $g(r,\vartheta)$ [5]

[1] $N(r,\vartheta) = f(r,\vartheta)$ is the normalization function for the Minkowski, Schwarzschild, FLRW ST's. For the Kerr ST the normalization function is $N(r,\vartheta) = f(r,\vartheta) g(r,\vartheta)$.

[2] $F=\sqrt{1-kr^2}$, $k=\pm 1$, a universe size, r dimensionless parameter.

[3] $F=(1-2GM/r)$, G gravitation constant, M mass.

[4] $\Delta = r^2+a^2-2M_{bh}r$, $\rho^2 = r^2+a^2\cos^2\vartheta$; $a=J/M_{bh}$; J —angular momentum, M_{bh} —BH mass.

[5] $g(r,\vartheta)=\exp\left[ia\left(\sqrt{(\omega^2+M^2)}\cos\vartheta+\frac{m_j}{\sqrt{a^2-M_{bh}}}\tan^{-1}\frac{(r-M_{bh})}{\sqrt{a^2-M_{bh}}}\right)\right]$.

Following the factorization (25) the reduced wave equation, for the Kerr ST (unlike for the other spacetimes considered), involves two non-diagonal vierbeins,

$$\left(E_0^0\gamma^0\partial_t + E_1^1\gamma^1\partial_r + E_2^2\gamma^2\partial_\vartheta + E_3^3\gamma^3\partial_\varphi + E_0^3\gamma^0\partial_\varphi + E_3^0\gamma^0\partial_t\right)\phi - E_0^0\gamma^0\beta M\phi = 0 . \quad (29)$$

Note that the diagonal vierbeins E_2^2 and E_3^3 for all cases in Table 1 are mathematically the same as for the Minkowski ST (the ρ variable for the Kerr ST cancels out). This ascertains that by eliminating the non-diagonal terms in Equation (29) the reduced angular wave functions for all ST's considered are commonly the same as that of the Minkowski ST. Below we demonstrate that E_0^3 and

E_3^0 can be eliminated by a second factorization of the reduced function ϕ as,

$$\phi(t,r,\vartheta,\varphi)=\psi(t,r,\vartheta,\varphi)g(r,\vartheta), \tag{30}$$

where ψ satisfies an equation which involves diagonal fields only, *i.e.*,

$$\left[E_0^0\gamma^0\partial_t+E_1^1\gamma^1\partial_r+E_2^2\gamma^2\partial_\vartheta+E_3^3\gamma^3\partial_\varphi-E_0^0\gamma^0\beta M\right]\psi=0. \tag{31}$$

Assume $\phi\sim \mathrm{e}^{-i\omega t+im_j\varphi}$ and substitute Equation (30) in Equation (29) one obtains,

$$E_1^1\gamma^1\partial_r\ln g+E_2^2\gamma^2\partial_\vartheta\ln g-E_0^0\gamma^0\beta M=-im_jE_0^3\gamma^0+i\omega E_3^0\gamma^3. \tag{32}$$

With the vierbeins listed in **Table 1** for the Kerr ST and, by taking the square of both sides of the expression above and then their traces one obtains,

$$\left[\sqrt{\Delta}\partial_r(\ln g)\right]^2+\left[\partial_\vartheta(\ln g)\right]^2=-\left[m_j\frac{a}{\sqrt{\Delta}}\right]^2-\left[a\sin\vartheta\right]^2\left(\omega^2+M^2\right). \tag{33}$$

The above is analytically integrable. One obtains (For more details see Ref. [11]),

$$g(r,\vartheta)=\exp\left[ia\left(\sqrt{\left(\omega^2+M^2\right)}\cos\vartheta+\frac{m_j}{\sqrt{a^2-M_{bh}}}\tan^{-1}\frac{\left(r-M_{bh}\right)}{\sqrt{a^2-M_{bh}}}\right)\right]. \tag{34}$$

With this accomplished, the reduced Equation (31) for the stationary rotating Kerr ST involves diagonal vierbein fields only, essentially, with the same E_2^2 and E_3^3 vierbeins as for the other spacetimes. Consequently, the reduced angular wave functions are all the same as for the Minkowski ST. The E_0^0 and E_1^1 vierbeins are characteristics of ST but these affect the reduced radial wave functions and normalization functions only. Note though that in the limit $J=0$ (no rotation) $g(r,\vartheta)$ goes to 1 and, the normalization function and the ratio E_0^0/E_1^1 reduce to those of the Schwarzschild ST.

Following Ref. [1], the reduced wave function and free particle Hamiltonian in spherical coordinates are set to be,

$$\psi(t,r,\vartheta,\varphi)=R(t,r)\Phi(\vartheta,\varphi), \tag{35}$$

and,

$$H_0=iE_0^0\gamma^0\beta M=iE_1^1\,\mathrm{diag}\left(\sigma^1,\cdots,\sigma^1\right)\left[\partial_r+\frac{s}{r}-K\right], \tag{36}$$

Above, $R(t,r)$ and $\Phi(\vartheta,\varphi)$ are the reduced radial and reduced angular wave functions and, K is an angular operator defined as,

$$K\equiv\frac{1}{r}(2S\cdot L+s), \tag{37}$$

with S and L denoting the particle spin and angular momentum. It is straight forward to show that the Hamiltonian H_0, total angular momentum squared J^2, z component of the total angular momentum J_z, parity P and the angular operator K form a complete set of commuting variables. The reduced angular wave function is readily written as spherical harmonic spinors,

$$\mathcal{Y}_{lm_j}^j(\theta,\varphi)=\sum_{-s}^{+s}C\left(l,s,j;m_j-m_s,m_s,m_j\right)Y_l^{m_l}(\vartheta,\varphi)\mathcal{X}_s^{m_s}. \tag{38}$$

Above C is a Clebsch-Gordon coefficient for combining orbital angular momentum l and spin s to a total angular momentum j and magnetic numbers $m_j - m_s$, m_s, m_j, respectively. The $Y_l^{m_l}(\vartheta,\varphi)$ are spherical harmonics eigenfunctions of L^2, L_3, $\mathcal{X}_s^{m_s}$ stands for Eigen functions of S^2 and S_3. Note that the spherical harmonics are orthonormal,

$$\int \mathrm{d}\Omega \left(\mathcal{Y}_{l'm_{j'}}^{j'}(\Omega)\right)^H \mathcal{Y}_{lm_j}^{j}(\Omega) = \delta_{jj'}\delta_{ll'}\delta_{mm'}. \tag{39}$$

Also note that the reduced angular wave function is an eigen function of J^2, J_3, K and parity satisfying,

$$J^2 \Phi_{m_j,\kappa_j}^{j} = j(j+1)\phi_{m_j,\kappa_j}^{j}, \tag{40}$$

$$J_3 \Phi_{m_j,\kappa_j}^{j} = m_j \phi_{m_j,\kappa_j}^{j}, \tag{41}$$

$$K^2 \phi_{m_j,\kappa_j}^{j} = \left(\kappa_j\right)^2 \phi_{m_j,\kappa_j}^{j}. \tag{42}$$

The eigenvalues of K^2 are $\left(\kappa_j\right)^2 = \left[s(2l+1)\right]^2$ and $\kappa_j = \pm s(2l+1)$. Then all values $\kappa_j = -s(2l+1),\cdots,+s(2l+1)$ are allowed for both $j = l+s$ and $j = l-s$. Thus, there exist two solutions $R_1(r)\phi^+(\vartheta,\varphi)$ and $R_2(r)\phi^-(\vartheta,\varphi)$, corresponding to positive and negative κ_j values,

$$R_1(r)K\phi^+(\theta,\varphi) = R_1(r)\kappa_j\phi^+(\vartheta,\varphi), \tag{43}$$

$$R_2(r)K\phi^-(\theta,\varphi) = -R_2(r)\kappa_j\phi^-(\vartheta,\varphi), \tag{44}$$

With the free particle Hamiltonian (36) one obtains $n/2$ identical pairs of non-autonomous equations,

$$iE_0^0(\omega - M)R_1(r)\phi^+ = E_1^1\left[\frac{\mathrm{d}}{\mathrm{d}r} + \frac{s+\kappa_j}{r}\right]R_2(r)\phi^-(\vartheta,\varphi), \tag{45}$$

$$iE_0^0(\omega + M)R_2(r)\phi^- = E_1^1\left[\frac{\mathrm{d}}{\mathrm{d}r} + \frac{s-\kappa_j}{r}\right]R_1(r)\phi^+(\vartheta,\varphi). \tag{46}$$

These we may rearrange as a pair of autonomous equations, namely,

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2s}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{s-\kappa_j}{r^2}\left[1-s+\kappa_j\right] + \left(\omega^2 - M^2\right)\right]R_1 \\ &= \left(\omega^2 - M^2\right)\left[1-C^2\right]R_1 + \frac{\mathrm{d}\ln C}{\mathrm{d}r}\left[\frac{s+\kappa_j}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right]R_1,\end{aligned} \tag{47}$$

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2s}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{s-\kappa_j}{r^2}\left[1-s+\kappa_j\right] + \left(\omega^2 - M^2\right)\right]R_2 \\ &= \left(\omega^2 - M^2\right)\left[1-C^2\right]R_2 + \frac{\mathrm{d}\ln C}{\mathrm{d}r}\left[\frac{s-\kappa_j}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right]R_2.\end{aligned} \tag{48}$$

In the above $C = \left(E_0^0/E_1^1\right)$, is the ratio of time to space curvatures. All nonhomogeneous terms depend on C, representing the impact of gravity and coupling of gravity to all other interactions. The first of these accounts for gravity effects on mass and energy. As expected, gravity affects both mass and energy the same way. The second term represents a spin dependent potential, with strength proportional to the logarithmic derivative of C,

$$\frac{\mathrm{d}\ln C}{\mathrm{d}r}\begin{pmatrix}\dfrac{s-\kappa_j}{r}\\ \dfrac{s+\kappa_j}{r}\end{pmatrix}=\frac{\mathrm{d}\ln C}{\mathrm{d}r}\frac{1}{r}\begin{pmatrix}2s(l+1)\\ 2sl\end{pmatrix}. \tag{49}$$

The fourth term refers to a particular Eigenstate with a strength proportional to the wave function gradient at a point. For a flat Minkowski ST all these terms vanish, $C=1$, leading to the homogeneous radial equations as reported in Ref. [1]. In the limit of $M\to 0$ Equations (47) and Equation (48) reduce to those reported for massless particles [11].

To summarize, Equation (20) promises a unified relativistic quantum dynamic formalism for particles of any spin in global spacetime. It is fully consistent with quantum mechanics and general relativity. In the private case of axially symmetric spacetimes the particle wave function factorizes into analytically determined normalization function $N(r,\vartheta)$, a common angular wave function identically the same as in Minkowski ST and radial wave functions which satisfy nonhomogeneous second order differential equations with nonhomogeneous terms all depending on the ratio of time to space curvatures.

5. Concluding Remarks

A Dirac like equation for any spin is reformulated to account for free particles in a global ST. The generalization from stationary flat Minkowski ST to dynamic and curved ST is accomplished by using the tetrad formalism, where the flat ST derivative is replaced with covariant derivatives and particles living on a curved ST manifold interact with gravity via the metric. Such a procedure is well justified by the claim that in an infinitesimal region around any point, ST is local.

The equation derived is fully consistent with quantum mechanics and general relativity with a conserved current satisfying the density equation. Though formidable, the equation is soluble in axially symmetric spacetimes, candidates of black hole models and may serve to study various cosmological physical phenomena in the vicinity of black holes. It is demonstrated that the wavefunction factorizes into a normalization function depending on the spin connection and a reduced wave function which splits into a common angular wave function identically the same as for a Minkowski ST and radial wave functions which are solutions of second order non-homogeneous differential equations. These non-homogeneous terms are explicit functions of the ratio of time to space curvatures and represent the impact of gravity. Asymptotically, in the limit of vanishing black hole parameters, all these terms vanish, and the radial equations converge to the homogeneous equations of a Minkowski ST as reported in Ref [1]. In the limit $M\to 0$ the equation converges to those reported for massless particles [11].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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