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A Note on the Speed of Light and Lorentz Transformation

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Abstract

In 1905, Einstein formulated special relativity. He derived the results of Lorentz and Poincaré by employing two fundamental principles: the postulate of relativity and the postulate of the invariance of the speed of light. This note presents an alternative derivation of the Lorentz transformation. This approach relies solely on the existence of a constant speed C, invariant across all inertial reference frames, and the assumption that coordinate transformations between such frames are given by real and linear functions of space and time. The analysis yields two classes of transformations; one encompasses the Lorentz transformation, while the other is physically untenable.

Keywords

Special Relativity, Classical Electromagnetism, Maxwell Equations, History of Science

1. Introduction

In 1905, Einstein formulated special relativity [1], obtaining the Lorentz transformation (LT) [2], under which Maxwell's equations in vacuum were invariant, and the results of Poincaré [3]. Einstein derived the LT and the consequences that followed using two postulates: the postulate of relativity, according to which Galileo's principle of relativity is valid for all the laws of physics, and the postulate of the invariance of the speed of light, according to which the speed of light has the same value in every inertial reference frame.

In subsequent years, several alternative derivations of the LT were published. V. Ignatowski was the first, in 1910, to try to derive the transformation by group theory, only using the relativity principle and without the postulate of the constancy of the speed of light [4]-[6]. He established a connection between the group structure implied by the relativity principle and the rules for the transformations of space-time coordinates. Under the hypotheses of locality, linearity, and isotropy, the relativity principle almost uniquely leads to either the LT or Galileo's transformation. This result makes no appeal to the constancy of the speed of light.

Other derivations of LT were published around the same time [7] [8]. Ignatowski's work has been repeatedly rediscovered and re-analyzed over the past century. See for instance, [9] and the references 5 - 25 therein. More recently, Richard Feynman provided a derivation of the LT from electrodynamics [10].

A fundamental difference from classical mechanics is the existence, in relativistic mechanics, of a maximum finite speed, C, which is constant in all inertial reference frames.

There is no trace, in classical mechanics, where the speed of a particle can theoretically range from zero to infinity, of a speed that has this property.

In this note, the LT is derived, starting from the hypothesis that there is such speed C and that the coordinate transformation between two inertial reference frames is given by real and linear functions of the space and the time. Isotropy and homogeneity of space are assunmed.

Neither the principle of relativity in Einstein's extension, nor the postulate that light propagates at the same speed in every reference frame are used to deduce the transformation. However, in retrospect, it is observed that there is a particle, the photon, whose speed precisely matches the hypothesized speed.

2. Existence of an Invariant Speed

Consider two inertial reference frames S and S'. Frame S' moves along the positive x-axis with velocity V relative to S. We assume the existence of an invariant, finite speed C that enables the synchronization of clocks. Let $x' = x = 0$ when $t' = t = 0$. The coordinate transformation law from S to S' is given by:

$$x' = f(x,t,V,C), \tag{1}$$

$$t' = g(x,t,V,C). \tag{2}$$

We then have, for the speed of a particle that in S' moves along the x'-axis:

$$v'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial t}}{\frac{\partial g}{\partial x}v_x + \frac{\partial g}{\partial t}}. \tag{3}$$

Now a particle, at rest in S', moves with speed V in S. From (3), this implies the relationship

$$\frac{\partial f}{\partial t} = -V\frac{\partial f}{\partial x}. \tag{4}$$

For the invariant speed C, we also have:

$$C = \frac{C\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}}{C\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t}}, \tag{5}$$

which implies:

$$C = \frac{-\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right) \pm \sqrt{\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right)^2 - 4V\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}}}{2\frac{\partial g}{\partial x}}. \tag{6}$$

In the case of Galilean transformations:

$$x' = x - Vt, \tag{7}$$

$$t' = t, \tag{8}$$

it follows that: $\partial f/\partial x = \partial g/\partial t = 1$, $\partial f/\partial t = -V$, $\partial g/\partial x = 0$ and C is given by the indeterminate form 0/0.

Assuming the argument of the square root in Equation (6) is positive, if $\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right) \neq 0$, there can be two solutions for C. Excluding cases of double solution, a unique value C is obtained from Equation (6) if:

$$\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x} = 0, \tag{9}$$

or:

$$\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right)^2 - 4V\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} = 0. \tag{10}$$

3. The First Case

Let's consider the first case. From (6) and (9), we have:

$$C = \sqrt{-V\frac{\partial f}{\partial x}\Big/\frac{\partial g}{\partial x}} \tag{11}$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}. \tag{12}$$

Now suppose that f and g are linear functions of the variables x and t taking the form:

$$x' = \eta x + \alpha t, \tag{13}$$

$$t' = \theta x + \xi t, \tag{14}$$

where η, α, θ and ξ depend on the parameters V and C.

It follows that:

$$C^2 = -\frac{V\eta}{\theta}. \tag{15}$$

Additionally, we have $\xi = \eta$, and from Equation (4), $\alpha = -V\eta$.

The transformation law of the coordinates from S to S' now takes the form:

$$x' = \eta(x - Vt), \tag{16}$$

$$t' = \eta\left(t - \frac{V}{C^2}x\right). \tag{17}$$

When transforming from S we pass to S' (where S' moves with relative speed V with respect to S), we can obtain the inverse transformation from S' to S by simply inverting V. If η, which is dimensionless, depended on even powers of V, its value would remain the same when going from S' back to S. Since a dependence on odd powers of V cannot be excluded a priori, we introduce a function η^* for the inverse transformation from S' to S. If we then consider the round-trip transformation $S \to S' \to S$, we obtain:

$$x = \eta^* \eta x(1 - \beta^2), \tag{18}$$

$$t = \eta^* \eta t(1 - \beta^2). \tag{19}$$

Thus:

$$\eta^* \eta = \frac{1}{1 - \beta^2}, \tag{20}$$

where $\beta = V/C$.

Since the product $\eta^* \eta$ must satisfy Equation (20) and η^* is obtained from η by substituting β for $-\beta$, the possible values of η are:

$$\eta_- = \frac{1}{1 - \beta}, \tag{21}$$

$$\eta_+ = \frac{1}{1 + \beta}, \tag{22}$$

$$\eta_\gamma = \eta_\gamma^* = \frac{1}{\sqrt{1 - \beta^2}} \equiv \gamma, \tag{23}$$

which lead to the three transformation laws:

$$x' = \eta_-(x - \beta C t), \tag{24}$$

$$t' = \eta_-\left(t - \frac{\beta}{C}x\right), \tag{25}$$

$$x' = \eta_+(x - \beta C t), \tag{26}$$

$$t' = \eta_+\left(t - \frac{\beta}{C}x\right), \tag{27}$$

$$x' = \gamma(x - \beta C t), \tag{28}$$

$$t' = \gamma\left(t - \frac{\beta}{C}x\right). \tag{29}$$

It's straightforward to verify that, for each of the three transformations, a velocity C, along the x-axis in frame S, corresponds to the same velocity C, along the x'-axis, in frame S'. This is evident because the factors η_-, η_+ and γ don't affect the calculation of $\mathrm{d}x'/\mathrm{d}t'$. What, then, differentiates the three transformations?

The Jacobian matrices for the three inverse transformations (from $x' \to x$) are obtained by substituting $-\beta$ for β in the matrices of transformations from $x \to x'$. The same applies to their respective determinants:

$$Det_{\eta_-} = \frac{1+\beta}{1-\beta}, \tag{30}$$

$$Det_{\eta_+} = \frac{1-\beta}{1+\beta}, \tag{31}$$

$$Det_{\gamma} = 1. \tag{32}$$

Let's now consider a clock at the origin O' of the reference frame S'. With this clock, an observer measures a time interval $T' = t_2' - t_1'$. In the reference frame S, the corresponding interval is $T = t_2 - t_1 = k\left(t_2' - t_1'\right) = kT'$, where $k \in \{\eta_-, \eta_+, \gamma\}$. Since the proper time τ of a body in motion is the time measured by a clock that moves together with the body itself, by setting $t_1' = t_1 = 0$, $t_2' = \tau$ e $t_2 = t$, we obtain $\tau = t/k$. However, since proper time is independent of the specific reference frame S, τ must be invariant under all three transformations.

In the case $k = \gamma$ we have:

$$\tau_{\gamma}^2 = t^2\left(1 - \frac{V^2}{C^2}\right) = \frac{1}{C^2}\left(C^2t^2 - x^2\right), \tag{33}$$

and it is straightforward to verify, using the Lorentz transformations directly, that the quantity $C^2t^2 - x^2$ is indeed invariant.

In the case $k = \eta_-$ and $k = \eta_+$, we have respectively:

$$\tau_{\eta_-} = (1-\beta)t = \frac{1}{C}(Ct - x) \tag{34}$$

and

$$\tau_{\eta_+} = (1+\beta)t = \frac{1}{C}(Ct + x), \tag{35}$$

so the quantities $Ct - x$ and $Ct + x$ should be invariant. Using the transformation laws directly, we obtain instead:

$$Ct' - x' = \frac{1+\beta}{1-\beta}(Ct - x) = Det_{\eta_-}(Ct - x), \tag{36}$$

and

$$Ct' + x' = \frac{1-\beta}{1+\beta}(Ct + x) = Det_{\eta_+}(Ct + x). \tag{37}$$

The physical requirement that the proper time must not depend on the reference frame S clearly indicates that the coordinate transformations with parameters η_- ed η_+ are not physically acceptable. This suggests that the LT is the only transformation arising from the fundamental principle that there exists an invariant speed.

Furthermore, the requirement for the transformation be real, implies that the relative velocity V cannot be greater than or equal to C, and this value can only be approached asymptotically.

If we consider a particle at rest in the frame S', we see that C is not only a speed that maintains the same value in every inertial reference frame, but it is also the maximum speed that a material point, stationary in S', can asymptotically reach in S.

However, our initial hypothesis was a particle that moves with the same speed C along the x'-axis in both S and S'. Electromagnetic waves are an example of propagation at a speed that does not depend on the reference frame. Therefore, the speed C must then be identified with the speed of light, or of the photon, which has no mass. The hypothesized particle, which moves at the same speed in S and S', must consequently be identified with the photon. The synchronization of clocks using the invariant velocity C, as initially hypothesized, then satisfies Einstein's synchronization convention.

If the speed of light were infinite, we would obtain Galileo's transformations, and Maxwell's equations, no longer invariant under these transformations, could not have their current form.

The hypothesis that there exists an invariant speed C and that for speeds much smaller than C, the classical Galilean transformation is recovered, then implies that C is also finite, if classical electromagnetism is described by Maxwell's equations.

4. The Second Case

Assume that in this case too, functions f and g are linear functions of the variables x and t. Taking Equation (4) into account, we get:

$$x' = \eta x - V\eta t, \tag{38}$$

$$t' = \theta x + \xi t, \tag{39}$$

where now η, θ and ξ depend on the relative velocity V and the new solution C^*.

From Equations (6) and (10) we have:

$$\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right)^2 = 4V\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}, \tag{40}$$

$$C^* = -\frac{\left(\frac{\partial g}{\partial t} - \frac{\partial f}{\partial x}\right)}{2\frac{\partial g}{\partial x}} = -\frac{\xi - \eta}{2\theta}, \tag{41}$$

from which:

$$\xi = \eta - 2C^*\theta, \tag{42}$$

and

$$\theta = \frac{V\eta}{C^{*2}}. \tag{43}$$

The transformation law now takes the form:

$$x' = \eta\left[x - \beta C^* t\right], \tag{44}$$

$$t' = \eta\left[\left(1 - 2\beta\right)t + \frac{\beta}{C^*}x\right], \tag{45}$$

where $\beta = V/C^*$.

The key difference between the first and second case lies in the presence of the $-2\beta t$ term and the positive sign in front of $\frac{\beta}{C^*}x$ in the expression for t'.

It's straightforward to verify that to the speed C^* in frame S corresponds the same speed C^* in frame S'. Additionally, Galileo's transformation is obtained in the limit $V/C^* \to 0$ with $\eta = 1$.

Now, as in the first case, let's consider the transformation $S \to S' \to S$. By introducing a function η^* in the passage from S' to S, we get:

$$x = \eta\eta^*\left[\left(1 + \beta^2\right)x - 2\beta^2 C^* t\right], \tag{46}$$

$$t = \eta\eta^*\left[\left(1 - 3\beta^2\right)t + 2\frac{\beta^2}{C^*}x\right], \tag{47}$$

and the identities $x = x$ and $t = t$ are only possible when $\eta\eta^* = 1$ and $\beta = 0$, meaning when $V = 0$ and $S' \equiv S$.

Therefore, in the second case, it's not possible to obtain a physically acceptable coordinate transformation law under the hypothesis that an invariant speed exists.

5. Conclusions

We found that the Lorentz transformation is the only physically acceptable transformation that arises from the hypothesis of the existence of an invariant speed.

Naturally, the arguments above don't explain why our universe has a finite, invariant speed limit associated with linear coordinate transformations.

Both electromagnetic and gravitational interactions are long-range, possessing an infinite interaction radius. The electromagnetic interaction is considerably more intense than the gravitational one.

Both electromagnetic and gravitational waves propagate at the same speed C. This is not possible for a massive particle, whose mass is partly due to its interaction with the Higgs field. This suggests that the existence of an invariant limiting speed may not depend on the type of interaction.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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