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Gravitational Electromagnetic Gauge Field

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Abstract

This paper proposes a gauge field theory of gravitation and electromagnetism based on principal and associated bundles, aiming to realize mutual transformations between electromagnetic and gravitational fields through Generalized Gauge Transformations (GGE) and explore its implications for unifying the four fundamental interactions. The theory is founded on the structural group $G = SO(1,3)\times U(1)\times SU(2)\times SU(3)$, where $SO(1,3)$ and $U(1)$ govern gravitational and electromagnetic interactions, respectively. We construct the gauge potentials A_μ^{ab} and gauge tensors $F_{\mu\nu}^{ab}$ for the gravitational-electromagnetic gauge field, verifying their compliance with the geralized gauge equations (GGEs) for connection and curvature. The study demonstrates that the "generalized" nature of GGE lies in its capacity to mediate cross-interaction transformations, enabling electromagnetic and gravitational fields to mutually convert in overlapping regions via principal bundle theory while preserving gauge invariance of connections and curvatures. A conversion model from optical solitons to gravitational solitons is established to validate GGE, accompanied by derivations of the relationship between the electromagnetic tensor and the Weyl tensor, revealing profound connections to solitonic transformations. Furthermore, we prove the gauge invariance of the gravitational-electromagnetic field and propose that the four fundamental interactions represent projected components of a unified cosmic gauge field across distinct domains, with their quantum or classical nature determined by gauge representations. This work establishes a theoretical framework for unifying fundamental interactions and highlights potential applications in high-energy physics, gravitational wave detection, and cosmology.

Keywords

Gravitational Gauge Field, Generalized Gauge Transformation, The Great Unification of Physics

1. Introduction

The unification of gravitational and electromagnetic interactions, as fundamental forces of nature, remains a central challenge in theoretical physics. Recent advances in quantum gravity and the unification of fundamental interactions have achieved notable progress, yet significant obstacles persist. In quantum gravity, superstring theory and loop quantum gravity (LQG) represent two primary directions. Superstring theory employs higher-dimensional spaces and supersymmetry to reconcile gravity with quantum mechanics, with recent breakthroughs in AdS/CFT correspondence and black hole information paradox studies [1] [2]. For instance, the work of Maldacena *et al.* has deepened our understanding of the intrinsic connection between gravity and quantum field theory [3]. However, the hypothesis of higher-dimensional spaces and the challenges in experimental verification continue to hinder its widespread acceptance. In contrast, LQG provides a non-perturbative quantum description of gravity by discretizing spacetime geometry, with recent advancements in black hole entropy calculations and cosmological applications [4]. Nevertheless, theoretical issues persist in reconciling LQG with the Standard Model and recovering classical gravity in the low-energy limit.

For the unification of the four fundamental interactions (gravity, electromagnetism, weak, and strong forces), Grand Unified Theories (GUTs) and supersymmetry (SUSY) remain prominent. GUTs aim to unify electromagnetic, weak, and strong interactions under a single gauge group (e.g., SU (5) or SO (10)), with recent studies focusing on proton decay experiments and neutrino mass constraints [5]. SUSY addresses Standard Model limitations by introducing supersymmetric particles and offers a potential bridge to gravitational unification. However, the absence of direct evidence for supersymmetric particles at the LHC has significantly constrained SUSY parameter spaces [6]. Additionally, extra-dimensional frameworks such as Kaluza-Klein theory and Randall-Sundrum models attempt geometric unification of gravity and electromagnetism, though their low-energy predictions require further experimental validation [7].

In the context of unifying electromagnetism and gravity, classical approaches like Kaluza-Klein theory unify electromagnetic and gravitational fields as geometric effects in five-dimensional spacetime [8]. However, challenges arise in quantization, particularly regarding gauge invariance and renormalizability. Recent developments in Double Field Theory and certain branches of M-theory explore electromagnetic-gravity duality through holographic principles [9]. Furthermore, gauge/gravity duality studies (e.g., Witten's proposal of gravity as the "square" of gauge theory) offer novel perspectives on unification [10]. Despite these efforts, mathematical complexity and physical observability remain critical barriers, especially in reconciling the non-perturbative nature of gravity with the Abelian properties of electromagnetism.

While existing frameworks provide diverse pathways toward unification, conventional quantum gravity approaches often assume that gravity must conform to quantum mechanical principles—a perspective that may overlook gravity's unique role as the geometric essence of spacetime. Motivated by this, we propose a gravitational-electromagnetic gauge field theory based on principal and associated bundles, aiming to realize mutual transformations between electromagnetic and gravitational fields via Generalized Gauge Equation (GGE) [11]-[22] and to explore their implications for a unified cosmic gauge field. We emphasize that gravity need not be "quantized" in the traditional sense but can instead be unified through gauge group representations on principal bundle sections. This perspective offers fresh philosophical and physical insights into the nature of fundamental interactions.

Within this framework, structural group $G = SO(1,3) \times U(1) \times SU(2) \times SU(3)$ governs the four fundamental interactions, with $SO(1,3)$ and $U(1)$ corresponding to gravity and electromagnetism, respectively. Our primary objective is to construct gauge potentials $A_{\mu\nu}^{ab}$ and gauge tensors $F_{\mu\nu}^{ab}$ for the gravitational-electromagnetic field, ensuring their compliance with GGEs for connections and curvatures. The "generalized" nature of GGEs manifests in their capacity to mediate cross-interaction transformations, enabling mutual conversion between electromagnetic and gravitational fields. Such transformations hold not only for gravitational-electromagnetic fields but also extend to a unified gauge field encompassing all fundamental interactions. In overlapping regions of different interactions, GGEs (rooted in principal bundle theory) facilitate conversions while preserving the gauge invariance of connections and curvatures.

To validate the theory's physical relevance, we demonstrate the conversion of electromagnetic optical solitons into gravitational solitons via GGEs. Additionally, we establish a profound relationship between the electromagnetic tensor and the Weyl tensor, linking their dynamics to solitonic transformations. This connection underscores the internal consistency of the gravitational-electromagnetic gauge field and supports the existence of a unified cosmic gauge field. Crucially, we prove the gauge invariance of the principal bundle connections and curvatures, a fundamental property of the unified cosmic gauge field. The relevant theories and explanations can be systematically seen in references [12] [13] [16], the principle theories of the two specific models can be found in references [19]-[21], and their applications in curvature engine spacecraft and time machines can be found in references [18] [20]-[22]. It is hoped that the introduction of these theoretical and application backgrounds can further dispel readers' doubts about the new theory.

This paper is organized into nine sections. Section 2 defines the mathematical and physical foundations of connection and curvature GGEs, elucidating the relationship between structural groups and gauge representations. Section 3 analyzes gravitational gauge field theory, extracting gauge potentials $A_{\mu\nu}^{ab}$ and tensors $F_{\mu\nu}^{ab}$, and contrasts them with connection (ω) and curvature (Ω) in GGEs. Section 4 verifies the compatibility of gravitational gauge fields with quantum gauge theories under GGE constraints. Section 5 extends the framework to gravitational-electromagnetic fields, proving their adherence to GGEs, with Appendix

A and B outlining a unified transformation framework for gravitational, electromagnetic, weak, and strong interactions. Section 6 applies the theory to solitonic conversion, validating GGE efficacy. Section 7 derives the electromagnetic-Weyl tensor relationship, revealing its connection to gravitational solitons. Section 8 further generalizes the gauge invariance of principal bundle connections and curvatures, unifying all four interactions. Section 9 concludes with future research directions.

Through this exploration, we aim to establish rigorous mathematical and physical foundations for gravitational-electromagnetic gauge field theory, paving the way for a unified framework of fundamental interactions.

2. Structure of Connection and Curvature GGEs

2.1. Connection (Gauge Potential) GGE

We first define the connection GGE as:

$$\omega_V(Y) = \mathcal{A}d_{g_{UV}(x)^{-1}}\omega_U(Y) + L^{-1}_{g_{UV}(x)*}g_{UV*}(Y), \forall x \in U \cap V, Y \in T_xM \tag{1}$$

where (ω_U , ω_V) represents the Lie algebra g-valued 1-form on the underlying base manifold region (U , V), representing a local connection; ($g_{UV} : U \cap V \to G$) is a transformation function, G is a structure group (e.g., $SO(1,3)\times U(1)\times SU(2)\times SU(3)$); $\mathcal{A}d_{g_{UV}^{-1}}\omega_U = g_{UV}^{-1}\omega_U g_{UV}$ is an adjoint action, a transformation connection; $L^{-1}_{g_{UV}(x)*}g_{UV*}(Y)$ is the inverse mapping of the left translation, involving the derivative of (g_{UV}), here $*$ means forward mapping [23]. In fact, from reference [23] we can get three clear equivalent definitions of the principal bundle connection, and on this basis we can prove very generally that the connection on the underlying manifold must satisfy relation (1). Here GGE represents the generalized gauge transformation equation, which is a conceptual generalization of the original formula by the author, indicating that the equation of this gauge transformation can be transformed across fundamental interactions. One of the most important manifestations is that the conversion function $g_{UV}(x)$ can belong to the product of two subgroups, for example: if it is electromagnetic-gravitational conversion, $g_{UV} \in SO(1,3)\otimes U(1)$; if it is strong-gravitational conversion, $g_{UV} \in SO(1,3)\otimes SU(3)$, and so on. For the derivation and discussion in this regard, see the author's published references [11]-[22]. On the other hand, from the perspective of quantum field theory (such as Yang-Mills field), the literature [23] also gives a very general proof that if the Lagrangian is to remain unchanged under gauge transformation, the gauge potential must satisfy relationship GGE. In this sense, since the transfer function in the GGE is the gauge transformation, it can be considered that the GGE is the transformation law that the connection (gauge potential) or curvature (gauge field strength) must satisfy. For example, this is the statement in Appendix 3 of reference [24].

Taking these background conditions into consideration, the author again constructs the cross-fundamental interaction gauge potential or gauge field strength transformation equation GGE in a more general way from the perspective of cross-fundamental interaction gauge theory in Section 8 and Appendix B, that is, GGE is also valid for weak force, strong force, electromagnetic and gravitational gauge fields.

Therefore, this article discusses the gauge field theory from the perspective of the GGE, not from the perspective of the gauge invariance of the Lagrangian, which may be a theoretical optimization approach, because the Lagrangian is not so easy to construct for general physical systems, especially for physical processes across fundamental interactions. In addition, even if the constructed Lagrangian is gauge invariant, it is not necessarily easy to see which elementary particle is transformed into which one. Even if there is certain symmetry, such as the local gauge invariance of the free complex scalar field, the conclusion that the rest mass of the photon is 0 can be "obtained", but it is still difficult to see which elementary particle is transformed into which one. However, the advantage of GGE across fundamental interactions is obvious, for example: $\omega_U \to \omega_V$, for $U \cap V \neq \varnothing$. Where ω_U represents the gauge potential of one fundamental physical field (on the bottom manifold) and is converted to ω_V represents the gauge potential of another fundamental physical field (on the bottom manifold). However, the gauge potential $\tilde{\boldsymbol{\omega}}$ of the principal bundle (representing the gauge potential of the cosmic gauge field) is invariant in this gauge transformation, *i.e.*, $\sigma_U^* \tilde{\omega} = \omega_U$; $\sigma_V^* \tilde{\omega} = \omega_V$, where σ_U^* and σ_V^* represents the pullback mapping of the principal bundle section on the bottom manifold U and V, respectively. This is the meaning of gauge symmetry and the meaning of the unification of fundamental interactions, *i.e.*, $\tilde{\omega}$ is the "subject quantity", ω_U and ω_V are the projections of $\tilde{\omega}$ on the bottom manifold through σ_U^* or σ_V^*, respectively, and ω_U and ω_V are converted into each other through the GGE. The meaning of gauge symmetry is very clear. We will discuss this issue again in Section 8.

Therefore, the physical meaning of the above formula: (ω_U, ω_V) represents the gauge potential of different regions (such as electromagnetic potential A_μ^{EM}, gravitational gauge potential A_μ^{ab}).

Formula (1) describes the generalized gauge transformation of the gauge potential in the region ($U \cap V$), which is applicable to cross-basic interactions (such as electromagnetic to gravitational) [11]-[15]. The above formula (1) can be simplified in matrix expression, that is, when G is a matrix Lie group, we can get

$$\omega_V = g_{UV}^{-1} \omega_U g_{UV} + g_{UV}^{-1} d g_{UV} \tag{2}$$

Here, the first term on the left side of formula (2) is the similarity transformation of the connection; the second term is the exterior differential of the transformation function, reflecting the change of the gauge selection between regions.

2.2. Curvature (Gauge Field Strength) GGE

We define the curvature by the Cartan second structural equation as:

$$\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \tag{3}$$

where $\tilde{\Omega}$ and $\tilde{\omega}$ are the principal bundle curvature and principal bundle connection respectively, projected to the (U, V) region as (Ω_U, Ω_V), then transformed to the bottom manifold as:

$$\Omega_V = \mathcal{A}d_{g_{UV}^{-1}}\Omega_U = g_{UV}^{-1}\Omega_U g_{UV} \tag{4}$$

or written as the transformation of the gauge tensor on the bottom manifold as:

$$F'_{\mu\nu} = g_{UV}^{-1}F_{\mu\nu}g_{UV} \tag{5}$$

where $F_{\mu\nu}$ is the gauge field strength in region U; $F'_{\mu\nu}$ is the gauge field strength in region V, $U \cap V \neq \varnothing$. The physical meaning of formula (5) is (Ω_U, Ω_V) corresponds to the gauge field strength (such as the electromagnetic field $F_{\mu\nu}^{EM}$, the gravitational field strength $F_{\mu\nu}^{ab}$); GGE ensures the consistency of the transformation of curvature under different gauges and supports the unification across interactions [11] [18] [19] [22].

2.3. Structure Group and Region

We construct the structure group as:

$$G = SO(1,3)\times U(1)\times SU(2)\times SU(3)$$

Here $SO(1,3)$ can represent the operating rules of gravity in region V; $U(1)$ expresses the operating rules of electromagnetic force in region U; and $SU(2)$ and $SU(3)$ express the operating rules of weak and strong interactions in regions W_2 and W_1 respectively. We can also call these group rules the gauge group representations of the corresponding forces. Obviously, they are related to the scale of the region. In this sense, the so-called quantization is nothing more than a gauge group representation naturally selected by the principal bundle in a sufficiently small regional scale, such as in W_2 and W_1. However, there are generalized gauge transformations across fundamental interactions. For example, if the generalized gauge transformation $g_{UV} \in SO(1,3)\times U(1)$, the gravitational-electromagnetic transformation can be described. Therefore, the four fundamental interactions can be converted into each other through generalized gauge transformations in the intersection region [11] [12]. Of course, choosing $GL(n)$ (general linear group, the group of $n\times n$ invertible matrices over the real or complex fields) as the structure group, including $SO(1,3), U(1), SU(2), SU(3)$ as subgroups, is also mathematical physics possible, because $GL(n)$ is the largest linear transformation group that can cover all possible linear symmetries. However, because in some sense, the basic models of weak and strong interactions with electromagnetical interaction have already been unified [5] [25] [26], and the key to the unification of the four basic interactions is now the unification of electromagnetism and gravity [4] [8] [27]-[29], this article will focus on the unified theory of gravity-electromagnetism under generalized gauge transformation, such as $G = SO(1,3)\times U(1)$.

3. Analysis of Gravitational Gauge Theory

Next, we start from the gravitational gauge theory [24] [30] [31] and expand it and establish a relation with GGEs [11]-[16] [19]. Here, the gravitational gauge field is based on the principal bundle structure, and the structure group is $SO(N)$ (usually $SO(1,3)$). Then we describe gravity in a gauge field manner and establish a relationship with the above connection GGE (1), (2) and curvature GGE (4), (5).

3.1. Gauge Potential and Connection

The principal bundle of the gravitational gauge field framework we express is: $P(M,SO(1,3))$, the underlying manifold M is 4-spacetime with Riemann metric $g_{\mu\nu}$; the gauge potential is:

$$A_\mu = A_\mu^{ab} J_{ab} \tag{7}$$

Here A_μ^{ab} is defined as the gauge potential component, which is related to the frame e_μ^a or the metric perturbation $h_{\mu\nu}$; J_{ab} is the $SO(1,3)$ Lie algebra generator, satisfying:

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc} \tag{8}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$, then the corresponding connection expression is:

$$\omega = A_\mu dx^\mu = A_\mu^{ab} J_{ab} dx^\mu \Rightarrow A_\mu = \frac{1}{2} A_\mu^{ab} J_{ab} = \omega_\mu \tag{9}$$

here ω is a Lie algebra so(1,3)-valued 1-form. Corresponding to the connection GGE, the above $A_\mu^{ab} J_{ab}$ corresponds to ω_U in formulas (1) and (2), which represents the local gauge potential in region U (gravitational region V). Thus, $\omega_U(Y)$ in formula (1) can be expressed on the bottom manifold as follows:

$$\omega_U(Y) = A_\mu^{ab} J_{ab} Y^\mu, Y = Y^\mu \partial_\mu \tag{10}$$

The GGE transformation can be expressed as follows:

$$\omega_V(Y) = Ad_{g_{UV}^{-1}} \omega_U(Y) + L_{g_{UV}(x)*}^{-1} g_{UV*}(Y) \tag{11}$$

The first term of Equation (11) above is:

$$Ad_{g_{UV}^{-1}} \omega_U(Y) = g_{UV}^{-1} \left(A_\mu^{ab} J_{ab} Y^\mu \right) g_{UV} \tag{12}$$

If $g_{UV} \in SO(1,3)$, then J_{ab} is transformed into:

$$g_{UV}^{-1} J_{ab} g_{UV} = \left(g^{-1}\right)_a^c \left(g^{-1}\right)_b^d J_{cd}, \quad g \in SO(1,3) \tag{13}$$

Here $\left(g^{-1}\right)_b^d$ is a scalar coefficient, J_{cd} is a generator, and the index d is bound to J_{cd} and can be summed. Specifically, $\left(g^{-1}\right)_b^d$ is the b-th row and d-th column element of the g^{-1} matrix; g^{-1} is the matrix representation abbreviation of g_{UV}^{-1}.

If g_{UV} is a rotation matrix, then

$$g_{UV}^{-1} = g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

$\left(g^{-1}\right)^c_a$ = matrix element, for example $\left(g^{-1}\right)^2_2 = \cos\theta$; J_{ab} is a $SO(1,3)$ generator, in matrix form: $\left(J_{ab}\right)^e_f = \eta_{af}\delta^e_b - \eta_{bf}\delta^e_a$, where $\delta^e_a = 1$, if $e = a$, $\delta^e_a = 0$, if $e \neq a$.

Next, we calculate the matrix elements:

$$\left(gJ_{ab}g^{-1}\right)^e_f = g^e_m\left(\eta_{an}\delta^m_b - \eta_{bn}\delta^m_a\right)\left(g^{-1}\right)^n_f = g^e_b\eta_{an}\left(g^{-1}\right)^n_f - g^e_a\eta_{bn}\left(g^{-1}\right)^n_f \tag{15}$$

See if we can achieve the goal:

$$\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b\left(\eta_{df}\delta^e_c - \eta_{cf}\delta^e_d\right) \tag{16}$$

In fact, if we use $SO(1,3)$ orthogonality: $\eta_{mn}g^m_e g^n_f = \eta_{ef}$; and adjust the index, we can get:

$$g^{-1}J_{ab}g = \left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b J_{cd} \tag{17}$$

The right side of the above formula (17) comes from: $\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b$ is the tensor product of g^{-1}, reflecting the transformation of J_{ab} index a, b in the adjoint representation, and J_{cd} is the transformed generator basis, and the index c, d is reallocated by g^{-1}. Therefore, from (12), A^{ab}_μ can be obtained:

$$A^{ab}_\mu \to A'^{ab}_\mu = g^{-1}_{UV}A^{ab}_\mu g_{UV} = \left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b A^{cd}_\mu \tag{18}$$

Note that it will be proved later (see formulas (26) - (28)):

$$A'^{ab}_\mu = A^{cd}_\mu\left(g^{-1}\right)^a_c\left(g^{-1}\right)^b_d = A'^{ab}_\mu = A^{cd}_\mu\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b \tag{19}$$

The second term of Equation (11) above by definition is:

$$L^{-1}_{g_{UV}(x)*}g_{UV*}(Y) = g^{-1}_{UV}\left(\partial_\mu g_{UV}\right)Y^\mu \tag{20}$$

If g_{UV} is a matrix, then there is

$$\omega_V = g^{-1}_{UV}A_\mu g_{UV}dx^\mu + g^{-1}_{UV}\partial_\mu g_{UV}dx^\mu \tag{21}$$

which is consistent with the connection GGE (2).

The following is the core consistency verification of the gauge potential transformation in the generalized gauge theory (GGE). We need to confirm whether the derived gauge potential transformation formula (19) $A'^{ab}_\mu = A^{cd}_\mu\left(g^{-1}\right)^a_c\left(g^{-1}\right)^b_d$, which can directly derive $\omega_V = g^{-1}_{UV}A_\mu g_{UV}dx^\mu + g^{-1}_{UV}\partial_\mu g_{UV}dx^\mu$ and verify its consistency with the GGE transformation $\omega_V = g^{-1}_{UV}\omega_U g_{UV} + g^{-1}_{UV}dg_{UV}$. This step is crucial because it connects the transformation form of the gravitational gauge field based on $SO(1,3)$ and the quantum gauge field, such as electromagnetic gauge field [11] [15] [19] [24] [30] [31], reflecting the consistency of the gauge fields theory.

3.2. Detailed Derivation and Analysis

1) Understand the matrix form of A_μ

In order to verify the target formula, we first need to clarify the meaning of A_μ in $\omega_V = g^{-1}_{UV}A_\mu g_{UV}dx^\mu + g^{-1}_{UV}\partial_\mu g_{UV}dx^\mu$, and then give the definition of gauge potential, that is, in region U, $\omega_U = A^{ab}_\mu J_{ab}dx^\mu$, A^{ab}_μ is the component (tensor) of gauge potential, J_{ab} is the generator (matrix) of $SO(1,3)$ Lie algebra, so

$A_\mu^{ab} J_{ab}$ can be regarded as the Lie algebra value matrix:

$$A_\mu = A_\mu^{ab} J_{ab} \tag{22}$$

where A_μ is a matrix, representing the connection component on the Lie algebra; A_μ^{ab} is the component of gauge potential, with indices a and b corresponding to the basis J_{ab} of $SO(1,3)$ Lie algebra. Mathematically, A_μ^{ab} is a tensor (double index, antisymmetric) whose values are scalars (or numerical coefficients) and does not carry a matrix structure. For fixed μ, the a, b are numbers (e.g., real or complex) and do not participate in matrix multiplication. J_{ab} is a Lie algebra generator of the $SO(1,3)$, represented as a matrix (in some representations, such as the adjoint representation or the fundamental representation). J_{ab} satisfies the Lie algebra commutation relation: $[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}$; J_{ab} is a matrix and is therefore subject to matrix multiplication by g_{UV}^{-1}. And $A_\mu = A_\mu^{ab} J_{ab}$ is a linear combination of Lie algebra elements: $A_\mu = \sum_{a,b} A_\mu^{ab} J_{ab}$; (Einstein's summation convention, $a, b = 0,1,2,3$, but $J_{ab} = -J_{ba}$, only sums over independent combinations). Here, A_μ^{ab} is the coefficient, J_{ab} is the basis, and A_μ is a matrix (Lie algebra value). Because the target formula $g_{UV}^{-1} A_\mu g_{UV}$ implies that A_μ is in matrix form, so it is: $A_\mu = A_\mu^{ab} J_{ab}$, where $g_{UV} = g \in SO(1,3)$, is a group element (here in matrix form), acting on the Lie algebra space.

2) Verify the target formula

We start from the contact GGE transformation formula to derive ω_V and check whether we get the target form. From GGE transformation (2), we get the first term:

$$\omega_U = A_\mu^{ab} J_{ab} dx^\mu \tag{23}$$

Substituting, we get:

$$g_{UV}^{-1} \omega_U g_{UV} = g_{UV}^{-1} \left(A_\mu^{ab} J_{ab} dx^\mu \right) g_{UV} \tag{24}$$

Since dx^μ is a scalar, we have:

$$g_{UV}^{-1} \omega_U g_{UV} = g_{UV}^{-1} \left(A_\mu^{ab} J_{ab} g_{UV} \right) dx^\mu \tag{25}$$

Using the known formula (17):

$$g_{UV}^{-1} J_{ab} g_{UV} = \left(g^{-1} \right)_a^c \left(g^{-1} \right)_b^d J_{cd}$$

Then we can get:

$$g_{UV}^{-1} A_\mu^{ab} J_{ab} g_{UV} = A_\mu^{ab} \left(g_{UV}^{-1} J_{ab} g_{UV} \right) = A_\mu^{ab} \left(R^{-1} \right)_a^c \left(R^{-1} \right)_b^d J_{cd} \tag{26}$$

Rename indicator $c, d \to a, b$:

$$A_\mu^{ab} \left(R^{-1} \right)_a^c \left(R^{-1} \right)_b^d J_{cd} = A_\mu^{cd} \left(R^{-1} \right)_c^a \left(R^{-1} \right)_d^b J_{ab} \tag{27}$$

defining:

$$A_\mu'^{ab} = A_\mu^{cd} \left(R^{-1} \right)_c^a \left(R^{-1} \right)_d^b \tag{28}$$

hence we obtain:

$$g_{UV}^{-1} A_\mu^{ab} J_{ab} g_{UV} = A_\mu'^{ab} J_{ab} \tag{29}$$

So the first term becomes:

$$g_{UV}^{-1}\omega_U g_{UV} = g_{UV}^{-1}\left(A_\mu^{ab}J_{ab}dx^\mu\right)g_{UV} = \left(A_\mu'^{ab}J_{ab}\right)dx^\mu \tag{30}$$

where note that the matrix form $A_\mu = A_\mu^{ab}J_{ab}$ then having $\omega_U = A_\mu dx^\mu$; so the first term is:

$$g_{UV}^{-1}\omega_U g_{UV} = g_{UV}^{-1}\left(A_\mu dx^\mu\right)g_{UV} = \left(g_{UV}^{-1}A_\mu g_{UV}\right)dx^\mu$$

here we have verified: $g_{UV}^{-1}A_\mu g_{UV} = g_{UV}^{-1}\left(A_\mu^{ab}J_{ab}\right)g_{UV} = A_\mu^{ab}\left(g_{UV}^{-1}J_{ab}g_{UV}\right)$.

Why can A_μ^{ab} be extracted? Because A_μ^{ab} is a scalar coefficient and does not participate in matrix operations, and again consider $a,b \to c,d$, the above formula is

$$g_{UV}^{-1}A_\mu g_{UV} = A_\mu^{ab}\left(R^{-1}\right)_a^c\left(R^{-1}\right)_b^d J_{cd} = A_\mu^{cd}\left(R^{-1}\right)_c^a\left(R^{-1}\right)_d^b J_{ab} = A_\mu'^{ab}J_{ab} \tag{31}$$

Therefore we arrive at

$$g_{UV}^{-1}A_\mu g_{UV} = A_\mu'^{ab}J_{ab} \tag{32}$$

The first term matches:

$$g_{UV}^{-1}A_\mu g_{UV}dx^\mu = A_\mu'^{ab}J_{ab}dx^\mu \tag{33}$$

The second item is:

$$g_{UV}^{-1}dg_{UV} = g_{UV}^{-1}\left(\partial_\mu g_{UV}\right)dx^\mu \tag{34}$$

This term is in Lie algebra value 1-form: $g_{UV}^{-1}\partial_\mu g_{UV} \in Lie\left(SO(1,3)\right)$, which can be expanded as:

$$g_{UV}^{-1}\partial_\mu g_{UV} = B_\mu^{ab}J_{ab} \tag{35}$$

where B_μ^{ab} is the coefficient that depend on the specific form of g_{UV}.

Combining (33) and (35) we get:

$$\begin{aligned}\omega_V &= \left(g_{UV}^{-1}A_\mu g_{UV}\right)dx^\mu + g_{UV}^{-1}\left(\partial_\mu g_{UV}\right)dx^\mu \\ &= \left(A_\mu'^{ab}J_{ab} + B_\mu^{ab}J_{ab}\right)dx^\mu \\ &= \left(A_\mu'^{ab} + B_\mu^{ab}\right)J_{ab}dx^\mu\end{aligned} \tag{36}$$

and

$$A_\mu'^{ab} = A_\mu^{ab}\left(R^{-1}\right)_a^c\left(R^{-1}\right)_b^d + B_\mu^{ab} \tag{37}$$

also can directly obtain:

$$\omega_V = A_\mu'^{ab}J_{ab}dx^\mu + g_{UV}^{-1}\left(\partial_\mu g_{UV}\right)dx^\mu \tag{38}$$

where $A_\mu'^{ab} = A_\mu^{ab}\left(R^{-1}\right)_a^c\left(R^{-1}\right)_b^d$, and $g_{UV}^{-1}\partial_\mu g_{UV} = B_\mu^{ab}J_{ab}$, namely:

$$\omega_V = \left(A_\mu'^{ab}J_{ab} + B_\mu^{ab}J_{ab}\right)dx^\mu = \left(A_\mu'^{ab} + B_\mu^{ab}\right)J_{ab}dx^\mu \tag{39}$$

This is consistent with the target formula (2):

$$\omega_V = g_{UV}^{-1}A_\mu g_{UV}dx^\mu + g_{UV}^{-1}\left(\partial_\mu g_{UV}\right)dx^\mu \tag{40}$$

Conclusion: Equation (40) accurately reflects the GGE transformation. $A_\mu'^{ab} = A_\mu^{cd}\left(R^{-1}\right)_c^a\left(R^{-1}\right)_d^b$ is the coefficient transformation of the first term $g_{UV}^{-1}A_\mu g_{UV}$.

4. The Consistency between Gravitational Gauge Field and Quantum Gauge Field

Our goal is to show that $A_\mu dx^\mu$ is consistent with ω_U as a gauge potential, and to verify that the transformed forms of the gravitational gauge field (based on $SO(1,3)$) and the quantum gauge field (based on the general gauge group) are consistent.

4.1. The Structure of the Gauge Transformation

In Yang-Mills gauge theory or gravitational gauge theory, the transformation of the gauge potential always consists of two parts:

- Adjoint transformation: $g^{-1}A_\mu g$
- Inhomogeneous term: $g^{-1}\partial_\mu g$

For $A_\mu = A_\mu^{ab}J_{ab}$, we have $g^{-1}A_\mu g = A_\mu'^{ab}$, $A_\mu'^{ab} = A_\mu^{cd}\left(R^{-1}\right)_c^a\left(R^{-1}\right)_d^b$, and inhomogeneous term $g_{UV}^{-1}\partial_\mu g_{UV} = B_\mu^{ab}J_{ab}$, so the complete transformation is:

$$A_\mu \to A_\mu' = g_{UV}^{-1}A_\mu g_{UV} + g_{UV}^{-1}\partial_\mu g_{UV}$$

which corresponding to

$$A_\mu'^{ab}J_{ab} = \left(A_\mu^{cd}\left(g^{-1}\right)_c^a\left(g^{-1}\right)_d^b\right)J_{ab} + \left(B_\mu^{ab}J_{ab}\right)$$

where $A_\mu'^{ab}$ is the transformation result of the first term, while the second term retains its Lie algebraic form. So the connection $\omega_U = A_\mu dx^\mu = A_\mu^{ab}J_{ab}dx^\mu$ of the gravitational gauge field (based on $SO(1,3)$) follows the same gauge transformation rules as the quantum gauge field, *i.e.* the connection GGE, and the same transformation form as the quantum gauge field (based on general Lie groups, such as $SU(N)$, except that the Lie algebra is different:

- Quantum gauge fields: $A_\mu = A_\mu^a T_a$, T_a is the Lie algebra generator;
- Gravitational gauge field: $A_\mu = A_\mu^{ab}J_{ab}$, J_{ab} is the generator of $SO(1,3)$;
- So the generalized transformation formula is still:

$$g_{UV}^{-1}\omega_U g_{UV} = A_\mu^{cd}\left(g^{-1}\right)_c^a\left(g^{-1}\right)_d^b$$

The adjoint representation transformation corresponding to the quantum gauge field is still:

$$A_\mu^a \to \left(g^{-1}\right)_b^a A_\mu^b$$

The second term $g_{UV}^{-1}\partial_\mu g_{UV}$ in the connection GGE (40) above ensures the non-homogeneous nature of the gauge transformation, which is consistent with the quantum gauge field. From this we can conclude that: $A_\mu dx^\mu = A_\mu^{ab}J_{ab}dx^\mu = \omega_U$, which is completely consistent with the connection GGE transformation as a gauge potential.

In fact, the transformation form of the gravitational gauge field is consistent with the quantum gauge field, only the Lie algebra and group ($SO(1,3)$ vs. $SU(N)$) are different. So it can maintain

$$\omega_V = g_{UV}^{-1}A_\mu g_{UV}dx^\mu + g_{UV}^{-1}\partial_\mu g_{UV}dx^\mu$$

The above formula can be directly derived from the GGE $\omega_V = g_{UV}^{-1}\omega_U g_{UV} + g_{UV}^{-1} dg_{UV}$, based on the idea that connection is gauge potential and gauge potential is connection [16] [23], where $g_{UV}^{-1}A_\mu g_{UV} = A_\mu^{cd}\left(g^{-1}\right)_c^a\left(g^{-1}\right)_d^b$.

4.2. Gauge Tensor and Curvature

The gauge tensor (field strength) can be defined as:

$$F_{\mu\nu} = F_{\mu\nu}^{ab} J_{ab} \tag{41}$$

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + \left[A_\mu, A_\nu\right]^{ab} \tag{42}$$

and

$$\left[A_\mu, A_\nu\right]^{ab} = A_\mu^{ac} A_{\nu c}{}^{b} - A_\nu^{ac} A_{\mu c}{}^{b} \tag{43}$$

then we have the curvature Cartan second structural equation:

$$F = dA + A \wedge A \tag{44}$$

In the coordinate base:

$$F = \frac{1}{2} F_{\mu\nu}^{ab} J_{ab} dx^\mu \wedge dx^\nu \tag{45}$$

In this way, $F_{\mu\nu}^{ab}$ is usually the associated frame curvature or gravitational field strength, but it is not directly equal to the Ricci curvature $R_{\mu\nu}$ or the Weyl tensor $C_{\mu\nu\rho\sigma}$, but needs to be mapped to the metric through the frame $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$.

Therefore, the Cartan transform (46) corresponding to the curvature GGE [23] [24] is:

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{46}$$

While the curvature component on the bottom manifold is:

$$\Omega_U = \frac{1}{2} F_{\mu\nu}^r e_r dx^\mu \wedge dx^\nu \tag{47}$$

$$F_{\mu\nu}^r = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r + k C_{st}^r A_\mu^s A_\nu^t \tag{48}$$

where C_{st}^r is the Lie algebra structure constant.

The curvature GGE (4) or (5) is:

$$\Omega_V = g_{UV}^{-1}\Omega_U g_{UV}$$

where, the gravitational gauge field is introduced

$$F_{\mu\nu}'^r = \left(g^{-1}\right)_s^r F_{\mu\nu}^s,\ g \in G \tag{49}$$

here the Lie algebra basis J_{ab} corresponds to e_r, the structure constant: $\left[J_{ab}, J_{cd}\right] = C_{ab,cd}^{ef} J_{ef}$ is equivalent to C_{st}^r; the transformation corresponding to $F_{\mu\nu}^{ab}$ is:

$$F_{\mu\nu}'^{ab} = \left(g^{-1}\right)_a^c \left(g^{-1}\right)_b^d F_{\mu\nu}^{cd}$$

which is consistent with $\Omega_V = g_{UV}^{-1}\Omega_U g_{UV}$, and the detailed derivation can be found in Appendix A and B. Therefore, the gauge tensor $F_{\mu\nu}^{ab}$ has the same form as the curvature Ω_U on the manifold of the principal bundle, both of which are Lie algebraic valued 2-forms, and $F_{\mu\nu}^{ab}J_{ab}$ and $\Omega_U = \frac{1}{2}F_{\mu\nu}^{ab}J_{ab}dx^\mu \wedge dx^\nu$ are also Lie algebraic valued 2-forms, and their forms are consistent (the factor 1/2 is the exterior differential convention). Here, the definition of $F_{\mu\nu}^{ab}$ includes $\partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + \left[A_\mu, A_\nu\right]^{ab}$, see formula (42) above, which is also consistent with GGE's

$$F_{\mu\nu}^r = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r + kC_{st}^r A_\mu^s A_\nu^t$$

and its structural constant C_{st}^r corresponds to $\left[J_{ab}, J_{cd}\right]$.

4.3. Differences and Problems

Although the above $A_\mu^{ab}J_{ab}$ and $F_{\mu\nu}^{ab}J_{ab}$ directly correspond to ω_U and Ω_U, and the transformation formula is compatible with the connection GGE and the curvature GGE, the above framework still needs to be extended for the following reasons:

- The above $F_{\mu\nu}^{ab}$ usually uses the Vielbein correlation metric and does not directly reflect the gravitational field strength (such as $h_{\mu\nu}$ or Weyl tensor $C_{\mu\nu\rho\sigma}$).
- The above curvature is expanded in the coordinate basis and lacks intuitive physical meaning.
- Extension across interactions: The usual gravitational gauge theory focuses on $SO(1,3)$ and is not explicitly extended to $SO(1,3)\times U(1)$. It needs to be adjusted to the generalized gauge transformation to support the gravitational-electromagnetic transformation.

5. Extension to General Gauge Fields

So to ensure that the gravitational electromagnetic gauge field generally satisfies the formula GGE, we propose the following adjustments.

5.1. Gauge Potential and Connection

We expand the structure group:

$$G = SO(1,3)\times U(1)\times SU(2)\times SU(3) \tag{50}$$

and define the gauge potential that unifies the four fundamental interactions:

$$A_\mu = A_\mu^{EM}T_{EM} + A_\mu^W T_W + A_\mu^s T_s + A_\mu^{ab}J_{ab} \tag{51}$$

Here T_{EM}, T_W, T_s are the generators of $U(1)$, $SU(2)$, $SU(3)$ respectively; $A_\mu^{ab}J_{ab}$ is the gravitational gauge potential, then the general connection (gauge potential) can be defined as:

$$\omega_U = A_\mu dx^\mu = \left(A_\mu^{EM}T_{EM} + A_\mu^W T_W + A_\mu^s T_s + A_\mu^{ab}J_{ab}\right)dx^\mu \tag{52}$$

According to the principal bundle theory of generalized gauge transformations, the general connection (gauge potential) GGE holds:

$$\omega_V = g_{UV}^{-1}\omega_U g_{UV} + g_{UV}^{-1}dg_{UV} \tag{53}$$

where $\omega_U \in$ any fundamental gauge potential (e.g. $\omega_U = A_\mu^W T_W$) in the region $U \cap V \neq \varnothing$, $\omega_V \in$ another fundamental gauge potential (e.g. $\omega_V = A_\mu^{EM} T_{EM}$); $g_{UV} \in$ the product of the corresponding two or four subpopulations (e.g. $g_{UV} = g_{U(1)} \times g_{SU(2)}$), or

$$g_{UV} = g_{SO(1,3)} \times g_{U(1)} \times g_{SU(2)} \times g_{SU(3)} \tag{54}$$

Although according to the generalized gauge transformation theory, we believe that the GGE of connection and curvature is generally valid for the mutual conversion between the four basic interactions at the intersection of the two, and the certain framework will be shown in section 8, but considering the unification of weak, strong and electromagnetic interactions, there is already a model that has laid the foundation. The problem of the unification of the four basic interactions in modern times is mainly the unification of electromagnetism and gravity, so this article still focuses on how the electromagnetic force is converted into gravity through the generalized gauge transformation. In fact, for the gravitational-electromagnetic transformation, $g_{UV} \in SO(1,3) \times U(1)$, there is

$$\omega_V = g_{UV}^{-1}\left(A_\mu^{EM} T_{EM} + A_\mu^{ab} J_{ab}\right) g_{UV} + g_{UV}^{-1} \partial_\mu g_{UV} dx^\mu \tag{55}$$

We can verify that the connection GGE (2) holds, that is, according to the previous definition, if we set:

$$\omega_U = A_\mu^{ab} J_{ab} dx^\mu, \omega_V = A_\mu'^{ab} J_{ab} dx^\mu \tag{56}$$

then the connection GGE is transformed into (gravity is transformed into gravity):

$$A_\mu'^{ab} J_{ab} = g_{UV}^{-1}\left(A_\mu^{cd} J_{cd}\right) g_{UV} + g_{UV}^{-1} \partial_\mu g_{UV} \tag{57}$$

If $g_{UV} \in SO(1,3)$, then we can obtain:

$$A_\mu'^{ab} = \left(g^{-1}\right)_a^c \left(g^{-1}\right)_b^d A_\mu^{cd}, J'_{ab} = g_{UV}^{-1} J_{ab} g_{UV} \tag{58}$$

For interactions across electromagnetic and gravitational ties:

$$A_\mu'^{ab} J_{ab} = g_{UV}^{-1}\left(A_\mu^{EM} T_{EM}\right) g_{UV} + g_{UV}^{-1} \partial_\mu g_{UV} \tag{59}$$

From the above analysis, we can find that it is necessary to define a mapping from T_{EM} to J_{ab}, that is, if $J_{ab} = g_{UV}^{-1} T_{EM} g_{UV}$, A_μ^{ab} can indeed be embedded in the unified gauge potential, connection GGE (2).

5.2. Gauge Tensor and Curvature

Further adjustment formally unifies the curvature of the four basic interactions into:

$$F_{\mu\nu} = F_{\mu\nu}^{EM} T_{EM} + F_{\mu\nu}^{W} T_W + F_{\mu\nu}^{s} T_s + F_{\mu\nu}^{ab} J_{ab} \tag{60}$$

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + A_\mu^{ac} A_{\nu c}{}^{b} - A_\nu^{ac} A_{\mu c}{}^{b} \tag{61}$$

Then the curvature GGE is transformed into:

$$F_{\mu\nu}'^{ab} = \left(g^{-1}\right)_a^c \left(g^{-1}\right)_b^d F_{\mu\nu}^{cd} \tag{62}$$

or written as:

$$F'_{\mu\nu} = g_{UV}^{-1} F_{\mu\nu} g_{UV} \tag{63}$$

For gravity-electromagnetism the formula (63) becomes:

$$F'^{ab}_{\mu\nu} J_{ab} = g_{UV}^{-1} \left(F_{\mu\nu}^{EM} T_{EM} \right) g_{UV} \tag{64}$$

In the physical mapping, it is also necessary to make $F_{\mu\nu}^{ab}$ associated with the metric, namely

$$F_{\mu\nu}^{ab} \sim e_\rho^a e_\sigma^b R_{\sigma\mu\nu}^\rho \tag{65}$$

where $R_{\sigma\mu\nu}^\rho$ is the Riemann curvature, *i.e.* the adjustment:

$$F_{\mu\nu}^{ab} = e_\rho^a e_\sigma^b \left(\partial_\mu \Gamma_{\sigma\nu}^\rho - \partial_\nu \Gamma_{\sigma\mu}^\rho + \Gamma_{\lambda\mu}^\rho \Gamma_{\sigma\nu}^\lambda - \Gamma_{\lambda\nu}^\rho \Gamma_{\sigma\mu}^\lambda \right) \tag{66}$$

here $\Gamma_{\sigma\mu}^\rho$ is the Christoffel symbol, and the associated Vielbein:

$$A_\mu^{ab} \sim e_\rho^a \partial_\mu e^{b\rho} \tag{67}$$

In short, for curvature GGE, the curvature transformation is $\Omega_V = g_{UV}^{-1} \Omega_U g_{UV}$; for gravity only, Equation (62) and Equation (63) hold; for the curvature interaction across electromagnetic gravity, Equation (64) holds, and the expression of $T_{EM} \to J_{ab}$ defined must satisfy

$$J_{ab} = g_{UV}^{-1} T_{EM} g_{UV} \tag{68}$$

5.3. Further Adjustments across Interactions

The goal of this adjustment is to support the gravitational-electromagnetic transformation by defining the hybrid representation as:

$$g_{UV} = \begin{pmatrix} R_{SO(1,3)} & 0 \\ 0 & e^{i\theta(x)} \end{pmatrix} \tag{69}$$

where $R_{SO(1,3)} \in SO(1,3)$, $e^{i\theta(x)} \in U(1)$; then we have:

- Gauge potential transformation, $\omega_U \to \omega_V$, see formula (53):

$$\omega_V = \begin{pmatrix} A'^{ab}_\mu J_{ab} & 0 \\ 0 & A'^{EM}_\mu T_{EM} \end{pmatrix} \tag{70}$$

where $A'^{ab}_\mu = \left(g^{-1}\right)_a^c \left(g^{-1}\right)_b^d A_\mu^{cd}$, $A'^{EM}_\mu = A_\mu^{EM} - \partial_\mu \theta(x)$.

- Curvature transformation $F_{\mu\nu}^{EM} \to F'^{ab}_{\mu\nu}$, formula (66):

$$F'^{ab}_{\mu\nu} = \left(g^{-1}\right)_a^c \left(g^{-1}\right)_b^d F_{\mu\nu}^{cd}, F'^{EM}_{\mu\nu} = F_{\mu\nu}^{EM} \tag{71}$$

The adjusted (69), (70) and (71) support the above Equation (53) and Equation (64), and it can be determined that the adjusted gravitational electromagnetic gauge field framework supports $SO(1,3) \times U(1)$, satisfying the connection and curvature GGE [32]-[35], the relevant details can be seen in Appendix A and B.

6. Gravitational Solitons

After confirming that the gravitational electromagnetic gauge field generally satisfies the connection and curvature GGE, I use gravitational solitons as a specific model to verify the generalized gauge transformation from electromagnetic field to gravity.

6.1. Gravitational Soliton Model

First, we define the polarization matrix of the gauge potential of the gravitational soliton is expressed as:

$$\omega_V = h_{ij} = \operatorname{sech}^2(ku)\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \tag{72}$$

where (u, v, x, y) are light cone coordinates. It can be proved that h_{ij} satisfies the vacuum Einstein Equation [19] Equation [22], that is, $R_{\mu\nu} = 0$, indicating a local solution of gravitational waves, and the polarization tensor is:

$$\epsilon_{ij} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \tag{73}$$

Moreover, we are very happy to find that this gravitational soliton can be connected with two polarization states of optical solitons through rotation transformation. Further, in the weak field approximation, these two optical solitons can be transformed into two photons, and the corresponding gravitational soliton can be transformed into graviton. They can still be transformed through the original rotational gauge transformation through the GGE equation. For detailed derivation and application on curvature engine spacecraft, please refer to references [19] [22]. Therefore, this should be regarded as an example of the existence of the generalized gauge transformation across fundamental interactions that we proposed. Further analysis is given below.

6.2. Electromagnetic to Gravitational Transformation

We define the electromagnetic gauge potential as the polarization state of two optical solitons:

$$\omega_U \sim A_\mu^{EM} T_{EM} dx^\mu \tag{74}$$

here the light cone coordinates are used here, and $A_\mu^{EM} \sim e_\mu \operatorname{sech}^2(ku)$ is taken to represent the polarization representation of the electromagnetic wave corresponding to the two optical solitons:

$$\omega_U = \operatorname{sech}^2(ku)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A_\mu^{EM} T_{EM} \tag{75}$$

where the polarization vector $e_\mu, \mu = x, y$, then the connection GGE (2) is:

$$\omega_V = g_{UV}^{-1}\omega_U g_{UV} + g_{UV}^{-1} dg_{UV}$$

the gauge transformation matrix is selected as the rotation transformation:

$$g_{UV} = \begin{pmatrix} \cos\theta(u) & -\sin\theta(u) \\ \sin\theta(u) & \cos\theta(u) \end{pmatrix} \tag{76}$$

then following calculation can be divided into two parts:

1) Prove that it satisfies the connection GGE, that is, starting from the connection GGE

$$\omega_V = g_{UV}^{-1}\left(A_\mu^{EM} T_{EM}\right) g_{UV} + g_{UV}^{-1}\partial_\mu g_{UV} dx^\mu$$

Assuming the generator $T_{EM} \to J_{ab}$ through the spinor mapping: $A_\mu^{EM} e_\mu \to h_{\mu\nu}\epsilon_\sigma^\nu$, the proof result can be obtained

$$\omega_V \sim h_{\mu\nu} J_\sigma^\nu dx^\mu \sim \operatorname{sech}^2(ku)\begin{pmatrix} A & B \\ B & -A \end{pmatrix} J_{ij} dx^i \tag{77}$$

2) Curvature verification, that is, through the gravitational tensor $F_{\mu\nu}^{ab}$ of the soliton and the electromagnetic tensor $F_{\mu\nu}^{EM}$ of the optical soliton constructed above, it is finally proved that the curvature GGE holds.

The relevant detailed derivations are given as follows:

1) Proof of contact with GGE

(1) Electromagnetic gauge potential and input

Assume the electromagnetic gauge potential is

$$\omega_U = A_\mu^{EM} T_{EM} dx^\mu$$

$$A_\mu^{EM} \sim e_\mu \operatorname{sech}^2(ku) \tag{78}$$

where T_{EM} is the generator of the $U(1)$ Lie algebra, usually in scalar or matrix form; e_μ is the polarization vector of the electromagnetic wave, satisfying the transverse condition (for example, $k^\mu e_\mu = 0$); $\operatorname{sech}^2(ku)$ is the soliton waveform, k^μ is the wave vector, and $u = k_\mu x^\mu$.

(2) The transformation group element g_{UV} is (76), which is a rotation matrix of $SO(2) \subset SO(1,3)$, and its inverse matrix is:

$$g_{UV}^{-1} = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix} \tag{79}$$

where, because $\det g_{UV} = 1$, Equation (79) can be obtained by transposing. Then, taking the derivative of g_{UV}, we can obtain

$$\partial_\mu g_{UV} = \partial_\mu\theta(x)\begin{pmatrix} -\sin\theta(x) & -\cos\theta(x) \\ \cos\theta(x) & -\sin\theta(x) \end{pmatrix} \tag{80}$$

(3) Calculate the GGE transform

By connection GGE transformation:

$$\omega_V = g_{UV}^{-1}\omega_U g_{UV} + g_{UV}^{-1}\partial_\mu g_{UV} dx^\mu$$

the first term is:

$$\omega_U = A_\mu^{EM} T_{EM} dx^\mu = e_\mu \operatorname{sech}^2(ku) T_{EM} dx^\mu \tag{81}$$

$$g_{UV}^{-1}\omega_U g_{UV} = g_{UV}^{-1}\left(e_\mu \operatorname{sech}^2(ku) T_{EM}\right) g_{UV} dx^\mu \tag{82}$$

Assume T_{EM} is the $U(1)$ generator, that is scalar (e.g., $T_{EM} = 1$) or matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents polarization $e_x = 1$, $e_y = -1$; $k_\mu = (1,0,0,1)$, $u = k_\mu x^\mu = t - z \approx t$ $(z \approx 0, x - y \text{ plane})$, then since $U(1)$ is an Abelian group, the transformation does not change the generator, so we have

$$g_{UV}^{-1} T_{EM} g_{UV} = T_{EM} \tag{83}$$

which allows us to get

$$g_{UV}^{-1}\left(e_{\mu}\operatorname{sech}^{2}(ku)T_{EM}\right)g_{UV}=e_{\mu}\operatorname{sech}^{2}(ku)T_{EM} \tag{84}$$

That is, the first term can be written as $g_{UV}^{-1}\omega_{U}g_{UV}=e_{\mu}\operatorname{sech}^{2}(ku)T_{EM}dx^{\mu}$, while the second term is

$$g_{UV}^{-1}\partial_{\mu}g_{UV}dx^{\mu}=g_{UV}^{-1}\left[\partial_{\mu}\theta(x)\begin{pmatrix}-\sin\theta(x) & -\cos\theta(x)\\ \cos\theta(x) & -\sin\theta(x)\end{pmatrix}\right] \tag{85}$$

Substituting the rotation transformation (79) into the above Equation (85) and calculating, we finally get:

$$\begin{pmatrix}\cos\theta(x) & \sin\theta(x)\\ -\sin\theta(x) & \cos\theta(x)\end{pmatrix}\begin{pmatrix}-\sin\theta(x) & -\cos\theta(x)\\ \cos\theta(x) & -\sin\theta(x)\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{86}$$

$$g_{UV}^{-1}\partial_{\mu}g_{UV}=\partial_{\mu}\theta(x)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{87}$$

Hence the above equation $\omega_{V}=g_{UV}^{-1}\omega_{U}g_{UV}+g_{UV}^{-1}\partial_{\mu}g_{UV}$ can be changed to the matrix representation of GGE as

$$\begin{aligned}&\operatorname{sech}^{2}(ku)\begin{pmatrix}A & B\\ B & -A\end{pmatrix}J_{12}\\ &=\operatorname{sech}^{2}(ku)\begin{pmatrix}\cos(2\theta(u)) & -\sin(2\theta(u))\\ -\sin(2\theta(u)) & -\cos(2\theta(u))\end{pmatrix}T_{EM}+\partial_{\mu}\theta(u)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{aligned} \tag{88}$$

- Match the diagonal terms of Equation (88):

$$\cos(2\theta(u))\operatorname{sech}^{2}(ku)T_{EM}=A\operatorname{sech}^{2}(ku)J_{12}\Rightarrow AJ_{12}=\cos(2\theta(u))T_{EM} \tag{89}$$

- Match the off-diagonal terms of Equation (88) ($u=t$):

Top right corner:

$$-\sin 2\omega(t)\operatorname{sech}^{2}(ku)T_{EM}-\frac{\mathrm{d}\theta(u)}{\mathrm{d}u}=B\operatorname{sech}^{2}(ku)J_{12} \tag{90}$$

Lower left corner:

$$-\sin 2\omega(t)\operatorname{sech}^{2}(ku)T_{EM}+\frac{\mathrm{d}\theta(u)}{\mathrm{d}u}=B\operatorname{sech}^{2}(ku)J_{12} \tag{91}$$

Solving the formulas (90) and (91) together, we get:

$$\frac{\mathrm{d}\theta(u)}{\mathrm{d}u}\cosh^{2}(ku)T_{EM}=0\Rightarrow\frac{\mathrm{d}\theta(u)}{\mathrm{d}u}=0 \tag{92}$$

Therefore, from the above solution, we can find that $\theta(u)$ is a constant, that is,

$$\theta(u)=\frac{1}{2}\arccos(A)=-\frac{1}{2}\arcsin(B),\quad A^{2}+B^{2}=1 \tag{93}$$

Substituting into Equation (88), the second term is 0, so we get

$$\omega_{V}=g_{UV}^{-1}\omega_{U}g_{UV}=\operatorname{sech}^{2}(ku)\begin{pmatrix}A & B\\ B & -A\end{pmatrix}J_{12} \tag{94}$$

$$\omega_V = g_{UV}^{-1} \omega_U g_{UV} = e_\mu \operatorname{sech}^2(ku) T_{EM} dx^\mu \sim h_{\mu\nu} J_\sigma^\nu dx^\mu \tag{95}$$

That is, the electromagnetic gauge potential $A_\mu^{EM} e_\mu$ is converted into the gravitational gauge potential through spinor mapping: $A_\mu^{EM} e_\mu \to h_{\mu\nu} \epsilon_\sigma^\nu$; gravitational gauge potential $A_\mu^{ab} = e_\rho^a e_\sigma^b h_{\mu\nu} \partial^\nu \left(\operatorname{sech}^2(ku)\right)$; polarization tensor: $\epsilon_{ij} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$; gravitational field $h_{ij} = \operatorname{sech}^2(ku) \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$; the key here is the relationship between T_{EM} and J_{12}. We can find later that $T_{EM} = J_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so Equation (94) and Equation (95) confirm that the constructed gravitational electromagnetic gauge field has no problem satisfying the connection GGE in the model of two optical solitons conversion into one gravitational soliton.

2) Verification of curvature GGE

In order to calculate $F_{\mu\nu}^{ab} = g_{UV}^{-1} F_{\mu\nu}^{EM} g_{UV}$, where $F_{\mu\nu}^{EM}$ is the electromagnetic field intensity tensor, and convert the electromagnetic curvature $F_{\mu\nu}^{EM} T_{EM}$ to the gravitational curvature $F_{\mu\nu}^{ab} J_{ab}$ through the spinor mapping, we need a detailed derivation. We use the given gauge group element g_{UV} and the generator relation $J_{ab} = g_{UV}^{-1} T_{EM} g_{UV}$, where $T_{EM} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $J_{ab} = J_{12}$, and we can give the following specific derivation.

(1) Calculation of electromagnetic gauge field strength $F_{\mu\nu}^{EM}$

- Gauge potential

$$A_\mu^{EM} = e_\mu \operatorname{sech}^2(ku), \quad T_{EM} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{96}$$

- Definition of electromagnetic gauge field strength (Abelian group Lie bracket term is 0):

$$F_{\mu\nu}^{EM} = \partial_\mu A_\nu^{EM} - \partial_\nu A_\mu^{EM} \tag{97}$$

- Derivative calculation:

$$\partial_\mu A_\nu^{EM} = e_\nu \partial_\mu \left(\operatorname{sech}^2(ku)\right) = -2ke_\nu \operatorname{sech}^2(ku) \tanh(ku) \cdot k_\mu \tag{98}$$

Same reason

$$\partial_\nu A_\mu^{EM} = -2ke_\mu \operatorname{sech}^2(ku) \tanh(ku) \cdot k_\nu \tag{99}$$

- Field strength expression:

$$F_{\mu\nu}^{EM} = -2k \operatorname{sech}^2(ku) \tanh(ku) \left(e_\nu k_\mu - e_\mu k_\nu\right) T_{EM} \tag{100}$$

(2) Calculation of gravitational gauge field strength $F_{\mu\nu}^{ab}$

- Gauge potential (via spinor mapping):

$$A_\mu^{ab} = e_\rho^a e_\sigma^b \operatorname{sech}^2(ku) \left(e_\mu k^\sigma - e^\sigma k_\mu\right) J_{12}^{\rho\sigma} \tag{101}$$

where $k_\mu x^\mu \approx t$,wave vector $k_\mu = (1,0,0,1)$, polarization vector e_μ satisfies $k_\mu e^\mu = 0$, and the generator of Li algebra is $J_{12}^{\rho\sigma} = \delta_1^\rho \delta_2^\sigma - \delta_2^\rho \delta_1^\sigma$.

- The field strength is defined as (non-Abelian group):

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + \left[A_\mu, A_\nu\right]^{ab}$$

- Derivative term: $\partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab}$
- The derivative of the scalar part $f(u) = \operatorname{sech}^2(ku)$ is:

$$\frac{\mathrm{d}f}{\mathrm{d}u} = -2k\operatorname{sech}^2(ku)\tanh(ku) \tag{102}$$

hence,

$$\partial_\mu A_\nu^{ab} = -2k\operatorname{sech}^2(ku)\tanh(ku)k_\mu e_\rho^a e_\sigma^b\left(e_\nu k^\sigma - e^\sigma k_\nu\right)J_{12}^{\rho\sigma} \tag{103}$$

After swapping $\mu \leftrightarrow \nu$, the difference in the derivative terms is:

$$\partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} = -2k\operatorname{sech}^2(ku)\tanh(ku)e_\rho^a e_\sigma^b J_{12}^{\rho\sigma}\left(e_\nu k^\sigma k_\mu - e_\mu k^\sigma k_\nu\right) \tag{104}$$

(3) Expand bracketed items $\left[A_\mu, A_\nu\right]^{ab}$

$$\left[A_\mu, A_\nu\right]^{ab} = A_\mu^{ac}A_\nu^{cb} - A_\nu^{ac}A_\mu^{cb} \tag{105}$$

After substituting into the gauge potential, we find that due to the antisymmetry of the generator $J_{12}^{\rho\sigma}$ and the transverse condition $k_\mu e^\mu = 0$, all non-zero terms cancel each other out and the final contribution is zero:

$$\left[A_\mu, A_\nu\right]^{ab} = 0 \tag{106}$$

(4) Result organization

Combining the derivative term and the Lie bracket term, the field strength is:

$$F_{\mu\nu}^{ab} = -2k\operatorname{sech}^2(ku)\tanh(ku)e_\rho^a e_\sigma^b J_{12}^{\rho\sigma}\left(e_\nu k_\mu - e_\mu k_\nu\right)k^\sigma \tag{107}$$

Considering the wave vector $k_\mu = (1,0,0,1)$ and the polarization condition $e_\mu k^\mu = 0$, the above formula is finally simplified to:

$$F_{\mu\nu}^{ab} = -2\operatorname{sech}^2(ku)\tanh(ku)\left(e_\nu k_\mu - e_\mu k_\nu\right)e_\rho^a e_\sigma^b J_{12}^{\rho\sigma} \tag{108}$$

therefore we obtain

$$F_{\mu\nu}^{ab} = -2\operatorname{sech}^2(ku)\tanh(ku)\left(e_\nu k_\mu - e_\mu k_\nu\right)J_{12}^{ab} \tag{109}$$

where J_{12}^{ab} represents the rotation generator of the $x-y$ plane, and the matrix form can be proven lately as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(5) Rigorous proof of Lie bracket term $\left[A_\mu, A_\nu\right]^{ab} = 0$

From the standard potential form (99),

$$A_\mu^{ab} = \operatorname{sech}^2(ku)e_\rho^a e_\sigma^b\left(e_\mu k^\sigma - e^\sigma k_\mu\right)J_{12}^{\rho\sigma}$$

consider the definition of Lie brackets as

$$\left[A_\mu, A_\nu\right]^{ab} = A_\mu^{ac}A_\nu^{cb} - A_\nu^{ac}A_\mu^{cb}$$

and substituting the gauge potential, then the each term takes the form:

$$A_\mu^{ac} = \operatorname{sech}^2(ku)e_\rho^a e_\sigma^c\left(e_\mu k^\sigma - e^\sigma k_\mu\right)J_{12}^{\rho\sigma}$$

$$A_{\nu}^{cb} = \operatorname{sech}^2\left(ku\right) e_{\tau}^{c} e_{\omega}^{b}\left(e_{\nu} k^{\omega} - e^{\omega} k_{\nu}\right) J_{12}^{\tau\omega}$$

the product term is:

$$A_{\mu}^{ac} A_{\nu}^{cb} = \operatorname{sech}^4\left(ku\right)\cdot e_{\rho}^{a} e_{\sigma}^{c} e_{\tau}^{c} e_{\omega}^{b}\left(e_{\mu} k^{\sigma} - e^{\sigma} k_{\mu}\right)\left(e_{\nu} k^{\omega} - e^{\omega} k_{\nu}\right) J_{12}^{\rho\sigma} J_{12}^{\tau\omega} \tag{110}$$

When summing over the index c, $e_{\sigma}^{c} e_{\tau}^{c} = \eta_{\sigma\tau}$ (orthogonal basis condition), so we have:

$$A_{\mu}^{ac} A_{\nu}^{cb} = \operatorname{sech}^4\left(ku\right)\cdot e_{\rho}^{a} e_{\omega}^{b}\left(e_{\mu} k^{\sigma} - e^{\sigma} k_{\mu}\right)\left(e_{\nu} k^{\omega} - e^{\omega} k_{\nu}\right)\eta_{\sigma\tau} J_{12}^{\rho\sigma} J_{12}^{\tau\omega} \tag{111}$$

Next we compute the product of the generators:

$$\eta_{\sigma\tau} J_{12}^{\rho\sigma} J_{12}^{\tau\omega} = \eta_{\sigma\tau}\left(\delta_1^{\rho}\delta_2^{\sigma} - \delta_2^{\rho}\delta_1^{\sigma}\right)\left(\delta_1^{\tau}\delta_2^{\omega} - \delta_2^{\tau}\delta_1^{\omega}\right) \tag{112}$$

After expansion, there are four terms, but all non-zero terms involve the lateral condition $k_{\mu} e^{\mu} = 0$. For example, when $\rho = 1$, $\sigma = 2$, $J_{12}^{12} J_{12}^{21} = -1 \Rightarrow \eta_{21} = 0$ (because $\eta_{\mu\nu}$ is diagonal), resulting in all cross terms being zero. Similarly, other non-diagonal terms also disappear due to the diagonal nature of $\eta_{\sigma\tau}$, and finally we have:

$$\left[A_{\mu}, A_{\nu}\right]^{ab} = 0 \tag{113}$$

So, combined with the wave vector $k_{\mu} = \left(1,0,0,1\right)$ and the polarization condition $k_{\mu} e^{\mu} = 0$, in the tensor terms, the role of $k^{\sigma} = 1$, when $\sigma = 0$ or 3. But since $k_{\mu} e^{\mu} = 0$ (transverse condition), if e^{μ} is in the spatial direction (such as x, y), then $e_0 = e_3 = 0$, thus $e_{\nu} k_{\mu} k^{\sigma} = e_{\nu} k_{\mu}$ (only when $\sigma = 0$ or 3 is non-zero, but e_{ν} is in the x, y direction). The final tensor term is simplified to $e_{\nu} k_{\mu} - e_{\mu} k_{\nu}$, therefore, the field strength is:

$$\begin{aligned} F_{\mu\nu}^{ab} &= -2\operatorname{sech}^2\left(ku\right)\tanh\left(ku\right)\left(e_{\nu} k_{\mu} - e_{\mu} k_{\nu}\right) e_{\rho}^{a} e_{\sigma}^{b} J_{12}^{\rho\sigma} \\ &= -2\operatorname{sech}^2\left(ku\right)\tanh\left(ku\right)\left(e_{\nu} k_{\mu} - e_{\mu} k_{\nu}\right) J_{12}^{ab} \end{aligned}$$

(6) Curvature GGE transformation verification $F_{\mu\nu}^{ab} = g_{UV}^{-1} F_{\mu\nu}^{EM} g_{UV}$

Transformation Matrix:

$$g_{UV} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\quad g_{UV}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Mode of action:

- The generator of electromagnetic field strength $F_{\mu\nu}^{EM}$ is T_{EM};
- The generator of the gravitational field strength $F_{\mu\nu}^{ab}$ is $J_{12}^{ab} = e_{\rho}^{a} e_{\sigma}^{b} J_{12}^{\rho\sigma}$.

Transformation Relationship:

$$g_{UV}^{-1} F_{\mu\nu}^{EM} g_{UV} = g_{UV}^{-1}\left[-2k\operatorname{sech}^2\tanh\cdot\left(e_{\nu} k_{\mu} - e_{\mu} k_{\nu}\right) T_{EM}\right] g_{UV} \tag{114}$$

The scalar part of the above formula $-2\operatorname{sech}^2\left(ku\right)\tanh\left(ku\right)\left(e_{\nu} k_{\mu} - e_{\mu} k_{\nu}\right)$ is not affected by matrix transformation, so we focus on the Lie algebra part and find that

$$g_{UV}^{-1} T_{EM} g_{UV} = g_{UV}^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{UV} = J_{12} \tag{115}$$

Then we get:

$$g_{UV}^{-1} T_{EM} g_{UV} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (116)$$

therefore we arrive at

$$g_{UV}^{-1} T_{EM} g_{UV} = g_{UV}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{UV} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J_{12}$$

This is consistent with the form of J_{12}, indicating that:

$$T_{EM} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow J_{ab} = J_{12} \quad (117)$$

The above result is reasonableness, namely T_{EM} stands for $U(1)$ electromagnetic polarization degree of freedom, and J_{12} represents the gravitational degree of freedom in the $x-y$ plane in $SO(1,3)$. Therefore from formula (109) we can obtain

$$F_{\mu\nu}^{ab} = -2\,\mathrm{sech}^2(ku)\tanh(ku)\left(e_\nu k_\mu - e_\mu k_\nu\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (118)$$

The spinor map is:

$$F_{\mu\nu}^{EM} T_{EM} \to F_{\mu\nu}^{ab} J_{ab} \to T_{EM} = J_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence we get:

$$F_{\mu\nu}^{ab} = g_{UV}^{-1} F_{\mu\nu}^{EM} g_{UV} \quad (119)$$

The verified gravitational field strength is

$$F_{\mu\nu}^{ab} = -2\,\mathrm{sech}^2(ku)\tanh(ku)\left(e_\nu k_\mu - e_\mu k_\nu\right) J_{12}^{ab} \quad (120)$$

where $J_{12}^{ab} = e_\rho^a e_\sigma^b J_{12}^{\rho\sigma}$ is the matrix representation of the rotation generator, satisfying $J_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

7. Conversion of Electromagnetic Tensor to Weyl Tensor

Furthermore, we start from the curvature GGE transformation framework, combine the spinor representation and Cartan form, and clarify how to construct the relationship between the electromagnetic tensor $F_{\mu\nu}$ and the Weyl tensor $C_{\mu\nu\rho\sigma}$ through the gauge transformation. The derivation will include the following steps: review the GGE transformation and the definition of gauge potential and curvature; introduce the spinor mapping to solve the difference in tensor order; apply the Cartan second structural equation to construct the gravitational curvature; derive $C_{\mu\nu\rho\sigma}$ through the GGE transformation; and verify the properties of the Weyl tensor. This should be the second example of the existence of the generalized gauge transformation phenomenon we proposed, that is, it can be strictly proved that the electromagnetic tensor and the Weyl tensor satisfy the transformation relationship: $C_{\mu\nu\rho\sigma} = \kappa\left(F_{\mu\rho}F_{\nu\sigma} - F_{\mu\sigma}F_{\nu\rho}\right)$, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, $F_{\mu\rho}$ is the electromagnetic tensor, and κ is the conversion coefficient. For detailed derivation and application in curvature engine spacecraft, see references [18] [20] [21]. Next, we will analyze it from the perspective of gravitational electromagnetic gauge field.

7.1. Background and Definition

- The parameters and GGE transformation we use are:
    - Electromagnetic Gauge Potential: $\omega_U = \operatorname{sech}^2(ku)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_{EM}$, where, as defined above, $T_{EM} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the generator of the $U(1)$ Lie algebra.
    - Gravitational gauge potential: $\omega_V = \operatorname{sech}^2(ku)\begin{pmatrix} A & B \\ B & -A \end{pmatrix} J_{12}$, where we take $A = -\frac{3}{4}$, $B = \frac{\sqrt{7}}{4}$, and $J_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the generator of $SO(1,3)$ as above.
    - GGE Transformation: $\omega_V = g_{UV}^{-1}\omega_U g_{UV}$, where we take $g_{UV} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta \approx 1.2094$ (*radian*).
    - Curvature formula: $F_U = d\omega_U + \omega_U \wedge \omega_U$; $F_V = d\omega_V + \omega_V \wedge \omega_V$; $F_V = g_{UV}^{-1} F_U g_{UV}$.
    - **Target formula:** $C_{\mu\nu\rho\sigma} = \kappa\left(F_{\mu\rho}F_{\nu\sigma} - F_{\mu\sigma}F_{\nu\rho}\right)$, where $F_{\mu\nu}$ is the electromagnetic field tensor, second-order antisymmetric, dimension L^{-2}; $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, fourth-order, dimension L^{-2}, and κ is the conversion coefficient, dimension L^{2}, whose value is determined by experiments.
- Spinor representation:
    - Electromagnetic polarization:

$$e_{\mu}^{(k)} \sim \phi_A^{(k)}\bar{\phi}_B^{(k)}\sigma_{AB}^{\mu} \tag{121}$$

where, $e_{\mu}^{(k)}$ is the polarization vector of the kth soliton in the μ direction; $\phi_A^{(k)}$ is a binary spinor, $A = 0,1$; $\bar{\phi}_B^{(k)}$ is the Hermitian conjugate of the spinor $\phi_A^{(k)}$ and its core function is to combine with the original spinor $\phi_A^{(k)}$ to generate the physical real four-vector polarization field $e_{\mu}^{(k)}$ through the Pauli matrix σ_{AB}^{μ} while maintaining the Lorentz covariance. The $e_{\mu}^{(k)}$ relationship with T_{EM} is that the T_{EM} acts on the polarization matrix to adjust the phase or direction of $\phi_A^{(k)}$, for example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_{EM}$ represents $e_x^{(1)} = 1$, $e_y^{(2)} = -1$.

- Gravitational polarization:

$$\epsilon_{\mu\nu} \sim \psi_{ABCD}\sigma_{AB}^{\mu}\sigma_{CD}^{\nu} \tag{122}$$

where $\epsilon_{\mu\nu}$ is the gravitational polarization tensor, describing the polarization state of gravitons, satisfying $k^{\mu}\epsilon_{\mu\nu} = 0$. The fourth order symmetric spinor, denoted as ψ_{ABCD}, encodes the five independent components of gravitational waves; σ_{AB}^{μ}, σ_{CD}^{ν} are the Pauli matrix, mapping spinors to second-order tensors. The $\epsilon_{\mu\nu}$ relationship with J_{12} is that J_{12} represents the rotational degrees of freedom in the $x-y$ plane, and can apply to $\epsilon_{ij}=\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ to generate gravitational wave polarization modes as well connect electromagnetic polarization to gravitational polarization through $J_{12}=g_{UV}^{-1}T_{EM}g_{UV}$.

  - Two photons conversion to one graviton:

$$\psi_{ABCD}\sim\phi_A^{(1)}\overline{\phi}_B^{(1)}\phi_C^{(2)}\overline{\phi}_D^{(2)} \tag{123}$$

- Cartan form:
  - Gravitational curvature: $R_b^a=d\omega_b^a+\omega_c^a\wedge\omega_b^c$
  - $R_{b\mu\nu}^a\sim C_{\mu\nu ab}$ (Under vacuum $R_{\mu\nu}=0$)

7.2. Rigorous Proof

Step 1: Electromagnetic curvature and F_U

- Electromagnetic gauge potential: $\omega_U=\operatorname{sech}^2(ku)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}T_{EM}$
  - $T_{EM}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
  - Corresponding to the electromagnetic four-vector potential: $\omega_U\sim A_\mu dx^\mu$
- For example, the polarization matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents: $e_x^{(1)}=1$, $e_y^{(2)}=-1$
  - Curvature: $F_U=d\omega_U+\omega_U\wedge\omega_U$
  - For the $U(1)$ gauge field, $\omega_U\wedge\omega_U=0$ (because $U(1)$ is an Abelian group): $F_U=d\omega_U$

$$F_U\sim F_{\mu\nu}dx^\mu\wedge dx^\nu$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

  - Spiner representation: $F_{\mu\nu}\sim\phi_A\overline{\phi}_B\sigma_{\mu\nu}^{AB}$
  - $\sigma_{\mu\nu}^{AB}=\sigma_{[\mu}^{AC}\overline{\sigma}_{\nu]CB}=\frac{1}{2}\left(\sigma_\mu^{AC}\overline{\sigma}_{\nu CB}-\sigma_\nu^{AC}\overline{\sigma}_{\mu CB}\right)$, where $\overline{\sigma}_{0CB}=\sigma_{0CB}$, $\overline{\sigma}_{iCB}=-\sigma_{iCB}$, $i=2,3,4$.

Step 2: Gravitational curvature and F_V

1) Gravitational gauge potential: $\omega_V=\operatorname{sech}^2(ku)\begin{pmatrix} A & B \\ B & -A \end{pmatrix}J_{12}$

- J_{12} is a $SO(1,3)$ generator, corresponding to the spinor connection: $\omega_V\sim\omega_{b\mu}^a dx^\mu$
- Curvature: $F_V=d\omega_V+\omega_V\wedge\omega_V$

$$F_V=\frac{1}{2}R_{b\mu\nu}^a dx^\mu\wedge dx^\nu$$

  - Under vacuum: $R_{b\mu\nu}^a=C_{b\mu\nu}^a$
  - $F_V\sim C_{\mu\nu ab}dx^\mu\wedge dx^\nu$
- Spiner representation: $C_{\mu\nu\rho\sigma}\sim\psi_{ABCD}\sigma_{\mu\nu}^{AB}\sigma_{\rho\sigma}^{CD}$

Step 3: GGE transformation and spinor synthesis

- **GGE** transformation: $F_V = g_{UV}^{-1} F_U g_{UV}$
  - Problem: $F_U \sim F_{\mu\nu}$ is 2-form, $F_V \sim C_{\mu\nu\rho\sigma}$ is a fourth-order tensor.
    - Solution: **Assume** that F_U represents the combined field intensity of two optical solitons: $F_U \sim F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)}$
    - Corresponding to the spinors of two photons: $F_{\mu\nu}^{(1)} \sim \phi_A^{(1)} \overline{\phi}_B^{(1)} \sigma_{\mu\nu}^{AB}$; $F_{\rho\sigma}^{(2)} \sim \phi_C^{(2)} \overline{\phi}_D^{(2)} \sigma_{\rho\sigma}^{CD}$
- Spiner mapping:
  - Gravitational spinor: $\psi_{ABCD} \sim \phi_A^{(1)} \overline{\phi}_B^{(1)} \phi_C^{(2)} \overline{\phi}_D^{(2)}$

Therefore we can obtain:

$$\begin{aligned} C_{\mu\nu\rho\sigma} &\sim \psi_{ABCD} \left(\sigma_{\mu\rho}^{AB} \sigma_{\nu\sigma}^{CD} - \sigma_{\mu\sigma}^{AB} \sigma_{\nu\rho}^{CD} \right) \\ &\sim \left(\phi_A^{(1)} \overline{\phi}_B^{(1)} \sigma_{\mu\rho}^{AB} \right) \left(\phi_C^{(2)} \overline{\phi}_D^{(2)} \sigma_{\nu\sigma}^{CD} \right) - \left(\phi_A^{(1)} \overline{\phi}_B^{(1)} \sigma_{\mu\sigma}^{AB} \right) \left(\phi_C^{(2)} \overline{\phi}_D^{(2)} \sigma_{\nu\rho}^{CD} \right) \\ &= F_{\mu\rho}^{(1)} F_{\nu\sigma}^{(2)} - F_{\mu\sigma}^{(1)} F_{\nu\rho}^{(2)} \end{aligned} \tag{124}$$

- Curvature GGE transformation effect:
  - Definition: $F_U = F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)}$
  - Transformation: $F_V = g_{UV}^{-1} \left(F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)} \right) g_{UV}$, here g_{UV} acts on the Lie algebraic index to adjust the spinor component.
  - Result:

$$\begin{aligned} &F_V \sim \kappa \left(F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)} - F_{\mu\sigma}^{(1)} F_{\nu\rho}^{(2)} \right) \\ &\Rightarrow \left(g_{UV}^{-1} \left(F_{\mu\nu}^{(1)} F_{\rho\sigma}^{(2)} \right) g_{UV} - g_{UV}^{-1} F_{\mu\sigma}^{(1)} F_{\nu\rho}^{(2)} g_{UV} \right) = F_V^1 + F_V^2 = F_V \sim C_{\mu\nu\rho\sigma} \end{aligned} \tag{125}$$

Finally, the target formula is obtained:

$$C_{\mu\nu\rho\sigma} = \kappa \left(F_{\mu\rho} F_{\nu\sigma} - F_{\mu\sigma} F_{\nu\rho} \right) \tag{126}$$

Step 4: Dimensions and κ

2) Dimensionality: $\left[F_{\mu\nu} \right] = L^{-2}$, $\left[F_{\mu\rho} F_{\nu\sigma} \right] = L^{-4}$

- $\left[C_{\mu\nu\rho\sigma} \right] = L^{-2}$
- $[\kappa] \cdot L^{-4} = L^{-2} \Rightarrow [\kappa] = L^2$

Because it is a generalized gauge transformation without going through Einstein's equations, κ is basically the conversion efficiency, which is determined by experiments.

Step 5: Verify the properties of the Weyl tensor

3) Antisymmetry:

$C_{\nu\mu\rho\sigma} = \kappa \left(F_{\nu\rho} F_{\mu\sigma} - F_{\nu\sigma} F_{\mu\rho} \right) = -\kappa \left(F_{\mu\rho} F_{\nu\sigma} - F_{\mu\sigma} F_{\nu\rho} \right) = -C_{\mu\nu\rho\sigma}$;

$C_{\mu\nu\sigma\rho} = \kappa \left(F_{\mu\sigma} F_{\nu\rho} - F_{\mu\rho} F_{\nu\sigma} \right) = -C_{\mu\nu\rho\sigma}$

4) Exchange symmetry:

$C_{\rho\sigma\mu\nu} = \kappa \left(F_{\rho\mu} F_{\sigma\nu} - F_{\rho\nu} F_{\sigma\mu} \right) = \kappa \left(F_{\mu\rho} F_{\nu\sigma} - F_{\mu\sigma} F_{\nu\rho} \right) = C_{\mu\nu\rho\sigma}$

5) Traceability: $C_{\nu\mu\sigma}^{\mu} = \kappa \left(F_\mu^\mu F_{\nu\sigma} - F_\sigma^\mu F_{\nu\mu} \right) = 0$, Because $F_{\mu\nu}$ is antisymmetric and untraceable, therefore $F_\mu^\mu = 0$, $F_\sigma^\mu F_{\nu\mu} = 0$.

- Conformal invariance:
  - $F_{\mu\nu}$ is invariant under the conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$.
  - κ is a constant, the formula remains invariant under conformal transformation.

So through the GGE transformation $F_V = g_{UV}^{-1} F_U g_{UV}$, combined with the spinor mapping $\psi_{ABCD} \sim \phi_A^{(1)} \bar{\phi}_B^{(1)} \phi_C^{(2)} \bar{\phi}_D^{(2)}$, we successfully deduce the formula (126),where κ has the dimension of meter squared and is the conversion coefficient determined by experiment; the tensor order is solved by combining the field strengths of two optical solitons; and the properties of the Weyl tensor (antisymmetry, exchange symmetry, and tracelessness) are verified [16]-[18] [22] [36].

8. Unified Gauge Field of the Universe

The important topic involved in the analysis and deduction in the above sections is: From the perspective of principal bundle theory, there exists a "universal unified gauge field" whose connection or curvature on the principal bundle is invariant to gauge transformations.

In fact, if we assume that the structure group G of the principal bundle (frame bundle) is the general linear transformation group $GL(n)$, which contains $SO(1,3)$, $U(1)$, $SU(2)$, $SU(3)$ as its subgroups, its representation group is $\hat{G}$, that is, there exists a homomorphism $\rho : G \to \hat{G}$, and the representation space is the N-dimensional (complex) inner product vector space V. Here, the inner product for real V or $\dim V$ greater than 1 refers to (ϕ, ψ), $\forall \psi \in V$, $\phi \in V^*$, where V^* is the dual space of V. This makes sense for the inner product of the expression space of the structure group $SO(1,3)$ corresponding to general relativity. Another issue is the choice of the bottom manifold M. We choose the bottom manifold corresponding to the frame bundle FM because the bottom manifold derivative operator corresponding to the given connection $\tilde{\omega}$ of the frame bundle allows the physical spacetime of general relativity to be expressed on the bottom manifold [23], but this is not necessarily the case for other principal bundles.

In this way, there is a nontrivial principal bundle (frame bundle) $P = M \times G$, where the free right action of G on P is $R : (M \times G) \times G \to M \times G$, then we can define: $\forall g_1 \in G$, $R_{g_1} : M \times G \to M \times G$ is

$$R_{g_1}(x, g_2) := (x, g_2 g_1), \forall (x, g_1) \in M \times G$$

If $\sigma_U : U \to P$ and $\sigma_V : V \to P$ are two local sections of P, then $\forall x \in U \cap V \neq \varnothing$, there exists a unique transformation function $g_{UV}(x) \in G$ such that

$$\sigma_V(x) = \sigma_U(x) g_{UV}(x) \tag{127}$$

Note that for the transformation across fundamental interactions, such as the transformation from electromagnetic to gravity, here $g_{UV}(x) \in SO(1,3) \times U(1)$; electromagnetic to strong force: $g_{UV}(x) \in SU(3) \times U(1)$, etc. The above formula (127) shows that the cross-sectional transformation of the principal bundle can give a group element field g_{UV} on $U \cap V \subset M$, $g_{UV} : U \cap V \to G$, which naturally allows the group element $g_{UV}(x)$ to construct a local gauge transformation, establishing a connection between the above principal bundle theory and the gauge field theory, that is, if $U(x) = \rho(g(x)) \in \hat{G}$ is the local gauge transformation of the gauge field (particle field) $\psi(x) \in V$ (*i.e.* $\psi(x)$ is actually the adjoint bundle cross section of the principal bundle on bottom manifold $x \in U \cap V$), then (square matrix multiplied by column matrix)

$$\psi'(x) = U(x)\psi(x) = \rho(g(x))\psi(x) \tag{128}$$

Here, $U(x) = \rho(g(x))$ comes from the following: If $\hat{\mathcal{G}}$ represents the Lie algebra of $\hat{G}$, then the forward mapping $\rho_* : \mathcal{G} \to \hat{\mathcal{G}}$ of the homomorphism $\rho : G \to \hat{G}$ on the identity element $e \in G$ is a Lie algebra homomorphism. If we let $U(\vec{\theta}) = \rho(g) \in \hat{G}$, then

$$\rho(\exp(A)) = U(\vec{\theta}) = \exp\left[\rho_*\left(\theta^r e_r\right)\right] = \exp\left(\theta^r \rho_* e_r\right) = \exp\left(-iL_r\theta^r\right) = \mathrm{e}^{-i\vec{L}\cdot\vec{\theta}} \tag{129}$$

Here we introduce the symbol: $-iL_r \equiv \rho_* e_r \in \hat{\mathcal{G}}$, so $-iL_r$ can be expressed as a $N \times N$ matrix, and thus $\rho(g) = U(\vec{\theta}) = \mathrm{e}^{-i\vec{L}\cdot\vec{\theta}}$ is also an $N \times N$ matrix equation. In this way, a cross-section transformation (127) of the principal bundle gives a local gauge transformation (128) of the gauge field (particle field) $\psi(x)$ on M, and vice versa. So the principal bundle cross section $\sigma : U \to P$ represents a choice of gauge.

So, the gauge transformations simply convert its projection component on the underlying manifold from one component to another. That is, if we 1) let the connection (gauge potential) on the principal bundle $P(M,G)$ be: $\tilde{\boldsymbol{\omega}}$ is a C^∞ (Lie algebra of G) valued 1 formal field on P; 2) let $\boldsymbol{\omega}_U$ be a connection $\boldsymbol{\omega}_V$ on the principal bundle $P(M,G)$ that specifies $U \subset M$ for every local trivial $T_U : \pi^{-1}[U] \to U \times G$, *i.e.*, a Lie algebra valued 1 formal field of G; if $T_V : \pi^{-1}[V] \to V \times G$ is another local trivial, $U \cap V \neq \varnothing$, the conversion function g_{UV} from T_U to T_V; 3) then the GGE equation holds, and $\boldsymbol{\omega}_U$ can be transformed from the GGE equation to $\boldsymbol{\omega}_V$ (just like the two optical solitons mentioned above are transformed into a gravitational soliton through GGE); 4) it can also be expressed as $\boldsymbol{\omega}_U = \sigma_U^* \tilde{\boldsymbol{\omega}}$; $\boldsymbol{\omega}_V = \sigma_V^* \tilde{\boldsymbol{\omega}}$, Here σ_U^* and σ_V^* represent the pullback mapping of σ_U and σ_V respectively; $\sigma_U : U \to P$ is a local section corresponding to T_U, $\sigma_V : V \to P$ is a local section corresponding to T_V; g_{UV} in the GGE equation reflects the gauge transformation, which is related to the sections σ_U and σ_V, so we can say that under the gauge transformation $\boldsymbol{\omega}_U = \sigma_U^* \tilde{\boldsymbol{\omega}}$; $\boldsymbol{\omega}_V = \sigma_V^* \tilde{\boldsymbol{\omega}}$, the connection $\tilde{\boldsymbol{\omega}}$ of the principal bundle is "gauge transformation" invariant! What changes is only the projection of the principal bundle section on the bottom manifold $\boldsymbol{\omega}_U \to \boldsymbol{\omega}_V$ or $\boldsymbol{\omega}_V \to \boldsymbol{\omega}_U$ through GGE, which is converted from one (basic) interaction to another (basic) interaction, but the gauge potential $\tilde{\boldsymbol{\omega}}$ or gauge field intensity $\tilde{\boldsymbol{\Omega}}$ representing the "unified gauge field" of the universe is gauge invariant. This is the meaning of the generalized gauge transformation invariance. Electromagnetism and gravity (including weak and strong interactions) are ultimately unified in this gauge transformation invariant cosmic unified field. Now we can infer from the extended gravitational electromagnetic gauge field GGE that the connection $\tilde{\boldsymbol{\omega}}$ and curvature $\tilde{\boldsymbol{\Omega}}$ of the principal bundle are "gauge" invariant. The generalized gauge transformation across fundamental interactions is derived as follows:

1) Gauge group structure and generator mapping

Let the unified standard group be:

$$G = SO(1,3) \times U(1) \times SU(2) \times SU(3)$$

Its Lie algebra is:

$$\mathrm{g} = so(1,3) \oplus u(1) \oplus su(2) \oplus su(3)$$

By embedding the generators of $SU(2)$ and $SU(3)$ into the generators of $SO(1,3)$ through spinor representation, we can obtain:

- Weak force generator mapping, namely the Pauli matrix $T_i = \frac{i}{2}\sigma_i$ of $su(2)$ maps to the spatial rotation generator J_{jk} of $so(3) \subset so(1,3)$:

$$T_1 \mapsto J_{23}, T_2 \mapsto J_{31}, T_3 \mapsto J_{12}$$

- Strong interaction force generator mapping, namely certain subsets of the Gell-Mann matrices λ_a of $su(3)$ (such as λ_1, λ_2, λ_3) can also be mapped to J_{jk}, forming rotational symmetries **similar** to $su(2)$.

2) Decomposition of the initial gauge potential

Geometric definition of gauge potential: Assume that the connection form ω of the principal bundle $P(M,G)$ is a Lie algebra-valued 1-form, $\omega \in \Omega^1(P,\mathrm{g})$. In the local coordinate system, the gauge potential A can be expressed as $A = A_\mu^i T_i dx^\mu$, where A_μ^i is a scalar function on the underlying manifold M as a component of the Lie algebra-valued 1-form, and T_i is a generator satisfying the Lie algebraic structure $\left[T_i, T_j\right] = f_{ij}^k T_k$.

The unified gauge potential includes the gauge potentials of all interactions:

$$\omega_U = \underbrace{A_\mu^{ab} J_{ab} dx^\mu}_{\text{Gravity}} + \underbrace{A_\mu^{EM} T_{EM} dx^\mu}_{\text{Electromagnetic force}} + \underbrace{A_\mu^W T_W dx^\mu}_{\text{Weak interaction}} + \underbrace{A_\mu^s T_s dx^\mu}_{\text{Strong interaction}} \tag{127}$$

3) Definition of generalized gauge transformation

Select group elements $g_{UV} \in SO(1,3) \times SU(2) \times SU(3)$, its role is as connection GGE:

$$\omega_V = g_{UV}^{-1} \omega_U g_{UV} + g_{UV}^{-1} d g_{UV}$$

Due to the direct product structure of the gauge group, the transformations of each subgroup act independently on the corresponding gauge potential.

4) Gravitationalization of the weak force gauge potential

- Transformation rules for non-Abelian gauge potentials: For a gauge transformation $g \in SU(2)$, the components of the gauge potential transform to $A_\mu^W T_W \to g^{-1} A_\mu^W T_W g + g^{-1} \partial_\mu g$, where the adjoint action of the generator $g^{-1} T_W g = Ad_g(T_W)$ is mapped to

$$Ad_g(T_W) \mapsto J_{jk} \tag{128}$$

through the double covering isomorphism $\phi : su(2) \to so(3)$.

- Transformation decomposition of gauge potential:
  - The role of the $SU(2)$ part:

$$g^{-1}\left(A_\mu^W T_W\right) g = A_\mu^W \cdot Ad_g(T_W) = A_\mu^W \cdot J_{jk} \tag{129}$$

where A_μ^W is still a scalar function, and the generators are transformed through isomorphic mapping.

- Lorentz transformation of $SO(1,3)$: The Lorentz transformation matrix Λ corresponding to the gravitational group element $g_{grav} \in SO(1,3)$, and its adjoint mixed generator index:

$$g_{grav}^{-1}\left(A_\mu^i J_{jk}\right)g_{grav} = A_\mu^i \cdot \Lambda_j^{j'}\Lambda_k^{k'}J_{j'k'} \tag{130}$$

where Λ is the Lorentz transformation matrix, satisfying $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$.

- Contribution of the Maurer-Cartan term: The transformation parameter $\theta^W(x)$ generates a gravitational correction by embedding:

$$g_{grav}^{-1}dg_{grav} \supset \theta_\mu^W J_{ab}dx^\mu \tag{131}$$

- After the transformation, the weak force part is the weak force gauge potential and is absorbed into the gravitational term:

$$\omega_V^{weak} = \left(A_\mu^i\Lambda_j^{j'}\Lambda_k^{k'} + \theta_\mu^W\right)J_{j'k'}dx^\mu \tag{132}$$

5) Gravitationalization of the strong gauge potential

- Mapping of strong generators: Similarly, the subset of generators T_s of $SU(3)$ maps to J_{ab} of $SO(1,3)$

$$g_s^{-1}T_s g_s = J_{ab} \tag{133}$$

- The role of the $SU(3)$ part:

$$g_s^{-1}\left(A_\mu^s T_s\right)g_s = A_\mu^s \cdot Ad_{g_s}\left(T_s\right) = A_\mu^s J_{jk} \tag{134}$$

- Lorentz transformation of $SO(1,3)$: The Lorentz matrix Λ of the gravitational group element also acts on the strong force generator

$$g_{grav}^{-1}\left(A_\mu^s J_{jk}\right)g_{grav} = A_\mu^s \cdot \Lambda_j^{j'}\Lambda_k^{k'}J_{j'k'} \tag{135}$$

- Contribution of the Maurer-Cartan term: Strong force transformation parameter $\theta^s(x)$ modifies the gravitational term

$$g_s^{-1}dg_s \supset \theta_\mu^s J_{ab}dx^\mu \tag{136}$$

- Strong force part after transformation: The strong force gauge potential is absorbed as the gravitational term

$$\omega_V^{strong} = \left(A_\mu^s\Lambda_j^{j'}\Lambda_k^{k'} + \theta_\mu^s\right)J_{ab}dx^\mu \tag{137}$$

6) Unified gauge potential after comprehensive transformation

Combining all terms, the transformed connection is:

$$\omega_V = \underbrace{\left(A_\mu^{ab} + A_\mu^W\Lambda_j^{j'}\Lambda_k^{k'} + A_\mu^s\Lambda_j^{j'}\Lambda_k^{k'} + \theta_\mu^W + \theta_\mu^s\right)J_{ab}dx^\mu}_{\text{Unified gravitational term}} + \underbrace{A_\mu^{EM}T_{EM}dx^\mu}_{\text{Residual electromagnetic term}} \tag{138}$$

7) Verification of the covariance of the field intensity tensor

Calculate field strength $F_V = d\omega_V + \omega_V \wedge \omega_V$:

- Derivative term:

$$d\omega_V = \partial_\nu\left(A_\mu^{ab} + A_\mu^W\Lambda_j^{j'}\Lambda_k^{k'} + A_\mu^s\Lambda_j^{j'}\Lambda_k^{k'} + \cdots\right)J_{ab}dx^\mu \wedge dx^\nu \tag{139}$$

- Connection product terms:

$$\omega_V \wedge \omega_V = \left(A_\mu^{ab} J_{ab}\right) \wedge \left(A_\nu^{cd} J_{cd}\right) dx^\mu \wedge dx^\nu \tag{140}$$

where we can use the commutation relation $\left[J_{ab}, J_{cd}\right] = \eta_{ad} J_{bc} - \eta_c J_{bd} + \cdots$ of so(1,3), and the product term generates the standard gravitational curvature term.

- Gauge covariance: The field strength satisfies

$$F_V = g_{UV}^{-1} F_U g_{UV} \tag{141}$$

where F_U is the original field strength, which proves the covariance of the transformation, see Appendix B for details.

8) Physical meaning and analysis of extreme conditions

- Rigorous treatment of non-Abelian gauge potentials: The conversion between weak and strong gauge potentials is achieved through adjoint action, and the components A_μ^i as scalar fields maintain geometric consistency. The Lorentz transformation matrix Λ explicitly acts on the generator index to ensure spacetime covariance.
- Physical manifestations under extreme conditions: Conventional weak field: each interaction is independent, and the gauge potential components remain separated. Strong gravitational field: weak and strong gauge potentials are absorbed into equivalent gravitational potentials, and the cross-effects are absorbed by the geometric structure.
- Mathematical basis of unified theory: Through independent transformations and generator mappings of the direct product group, a unified description of gravity and quantum gauge potential is constructed. The non-Abelian corrections to the Maurer-Cartan term reflect the locality of gauge symmetry.

So the conclusion we finally came to is that through the direct product structure of the generalized gauge transformation group and the embedding mapping of the generator:

- It is mathematically self-consistent: the weak and strong gauge potentials are naturally transformed into the gravitational gauge potential through group action, satisfying gauge covariance.
- Physical unity: Under extreme gravitational conditions, non-gravitational interactions manifest as geometric properties of spacetime, providing a gauge framework for unified theory.
- Extended significance: This model lays the foundation for exploring the unified behavior of interactions in high-energy scenarios such as black hole physics and the early universe.

Finally, the expression of the unified gauge potential is: $\omega_V =$ (gravitational term + weak force conversion term + strong force conversion term) + electromagnetic term; the conversion terms are naturally merged through the generalized gauge transformation, without the need for explicit cross terms,

$$F_V = d\omega_V + \omega_V \wedge \omega_V = g^{-1}\left(d\omega_U + \omega_U \wedge \omega_U\right) g = g^{-1} F_U g \tag{142}$$

Therefore, by strictly treating the signs and associative order of the cross terms and using the antisymmetry of Lie algebraic values and the compatibility of gauge transformations, we verify that the field strength holds.

9. Conclusions and Perspectives

This study systematically investigates the mathematical and physical foundations of the gravitational-electromagnetic gauge field theory, with a rigorous analysis of the definitions of Generalized Gauge Equation (GGE) for connections and curvatures, and their relationship to the structural group $G = SO(1,3) \times U(1) \times SU(2) \times SU(3)$. By extracting the gauge potentials A_{μ}^{ab} and gauge tensors $F_{\mu\nu}^{ab}$ of the gravitational gauge field, and contrasting them with the connection (ω) and curvature (Ω) in GGE formulations, we validate the compatibility of gravitational gauge fields with quantum gauge theories under GGE constraints. Furthermore, we extend the framework to gravitational-electromagnetic fields, demonstrating their universal adherence to GGEs for both connections and curvatures. In Appendix B, we propose a unified transformation framework for gravitational, weak, and strong interactions, laying the groundwork for integrating the four fundamental forces.

Through concrete applications, we illustrate the conversion of optical solitons into gravitational solitons, confirming the efficacy of GGE, and derive a profound relationship between the electromagnetic tensor and the Weyl tensor, revealing its intrinsic link to solitonic transformations. Crucially, we rigorously prove the gauge invariance of principal bundle connections and curvatures in the gravitational-electromagnetic field—a property that unifies gravity, electromagnetism, weak, and strong forces while providing theoretical foundations for constructing a unified cosmic gauge field.

The results are compelling: the gravitational-electromagnetic gauge theory elucidates how the structural group G governs gravity, electromagnetism, weak, and strong forces via $SO(1,3)$, $U(1)$, $SU(2)$, and $SU(3)$, respectively. The central contribution lies in constructing gauge potentials and tensors that satisfy GGE, enabling cross-interaction conversions between electromagnetic and gravitational fields. The "generalized" nature of GGE manifests in their capacity to transcend distinct fundamental interactions, facilitating mutual transformations in overlapping regions through principal bundle theory while preserving gauge invariance. We propose that the four fundamental interactions are merely projected components of a unified cosmic gauge field across different cosmic domains, with their quantum or classical nature determined by the choice of gauge representations (*i.e.*, structural group selections). This perspective offers a novel philosophical and physical lens to re-examine the quantization of gravity.

Looking ahead, while this work establishes robust mathematical and physical foundations for the gravitational-electromagnetic gauge theory, several avenues warrant further exploration. First, the unified transformation framework for gravitational, weak, and strong interactions could be refined to identify observable effects in high-energy experiments. Second, the soliton conversion model may be tested against astronomical observations (e.g., gravitational wave data) to validate

GGE predictions. Additionally, extending the gauge-invariant cosmic unification framework to cosmology could unveil geometric origins of dark matter and dark energy. We anticipate that these endeavors will yield new insights into the ultimate unified theory of fundamental interactions, advancing our understanding of the universe's deepest structures.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. GGE and $F^{cd}_{\mu\nu}\left(R^{-1}\right)^a_c\left(R^{-1}\right)^b_d$

This appendix focuses on the core derivation of the curvature (or gauge tensor) transformation in the generalized gauge theory (GGE), specifically how to derive the transformation formula of the gauge tensor components $F'^{ab}_{\mu\nu}=\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b F^{cd}_{\mu\nu}$ from $\Omega_V=g^{-1}_{UV}\Omega_U g_{UV}$. This is a key step that verifies the consistency of the gauge tensor $F^{ab}_{\mu\nu}$ in our extended gravitational gauge field framework with the GGE curvature Ω_U in terms of form and transformation law. The following is a detailed calculation process to make each step of the derivation from $\Omega_V=g^{-1}_{UV}\Omega_U g_{UV}$ to $F'^{ab}_{\mu\nu}=\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b F^{cd}_{\mu\nu}$ clear.

1) Curvature GGE:

$$\Omega_V=g^{-1}_{UV}\Omega_U g_{UV} \tag{A1}$$

Here the curvature is defined as:

$$\Omega_U=\frac{1}{2}F^r_{\mu\nu}e_r dx^\mu\wedge dx^\nu \tag{A2}$$

where $F^r_{\mu\nu}$ is the curvature component; e_r is the Lie algebra basis (corresponding to J_{ab} of $so(1,3)$). Then the gauge tensor of the extended gravitational gauge field framework is

$$F_{\mu\nu}=F^{ab}_{\mu\nu}J_{ab} \tag{A3}$$

$$F^{ab}_{\mu\nu}=\partial_\mu A^{ab}_\nu-\partial_\nu A^{ab}_\mu+\left[A_\mu,A_\nu\right]^{ab} \tag{A4}$$

$$\left[A_\mu,A_\nu\right]^{ab}=A^{ac}_\mu A_{\nu c}{}^b-A^{ac}_\nu A_{\mu c}{}^b \tag{A5}$$

Its Lie algebraic basis is transformed as:

$$g^{-1}_{UV}J_{ab}g_{UV}=\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b J_{cd} \tag{A6}$$

where $g_{UV}=g\in SO(1,3)$ is the c-th row and a-th column element of the matrix g^{-1}.

The target formula we want to prove is:

$$F'^{ab}_{\mu\nu}=\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b F^{cd}_{\mu\nu} \tag{A7}$$

The task is:

- Detailed derivation from $\Omega_V=g^{-1}_{UV}\Omega_U g_{UV}$ to $F'^{ab}_{\mu\nu}=\left(g^{-1}\right)^c_a\left(g^{-1}\right)^d_b F^{cd}_{\mu\nu}=\left(g^{-1}\right)^a_c\left(g^{-1}\right)^b_d F^{cd}_{\mu\nu}$;
- Explain how the derivation reflects the formal consistency of $F^{ab}_{\mu\nu}$ and Ω_U.

2) Detailed derivation process

We define the gravitational curvature of region U as

$$\Omega_U=\frac{1}{2}F^r_{\mu\nu}e_r dx^\mu\wedge dx^\nu \tag{A8}$$

Because in the $SO(1,3)$ gauge theory, the Lie algebra base e_r corresponds to J_{ab}, where r is the composite index, representing (ab), so we have:

$$\Omega_U=\frac{1}{2}F^{ab}_{\mu\nu}J_{ab}dx^\mu\wedge dx^\nu \tag{A9}$$

where $F^{ab}_{\mu\nu}$ are the components of the gauge tensor:

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A_{\nu c}{}^{b} - A^{ac}_\nu A_{\mu c}{}^{b} \tag{A10}$$

The curvature in region V is:

$$\Omega_V = \frac{1}{2} F'^{ab}_{\mu\nu} J_{ab} dx^\mu \wedge dx^\nu \tag{A11}$$

where $F'^{ab}_{\mu\nu}$ is the transformed gauge tensor component. Next, from the curvature GGE transformation (A1), we need to calculate $g^{-1}_{UV}\Omega_U g_{UV}$ and extract $F'^{ab}_{\mu\nu}$.

3) Calculation $g^{-1}_{UV}\Omega_U g_{UV}$

Substituting into Ω_U (A8) we have

$$g^{-1}_{UV}\Omega_U g_{UV} = g^{-1}_{UV}\left(\frac{1}{2} F^{ab}_{\mu\nu} J_{ab} dx^\mu \wedge dx^\nu\right) g_{UV} \tag{A12}$$

Because $dx^\mu \wedge dx^\nu$ is the 2-form of the bottom manifold, a scalar property, and is not affected by g_{UV}, we have

$$g^{-1}_{UV}\left(dx^\mu \wedge dx^\nu\right) g_{UV} = dx^\mu \wedge dx^\nu \tag{A13}$$

where $F^{ab}_{\mu\nu}$ is a tensor component (scalar coefficient) and is not directly affected by the matrix of g_{UV}, so the key is the transformation of the Lie algebra basis J_{ab}:

$$g^{-1}_{UV} J_{ab} g_{UV} = \left(R^{-1}\right)^c_a \left(R^{-1}\right)^d_b J_{cd} \tag{A14}$$

hence:

$$g^{-1}_{UV}\Omega_U g_{UV} = \frac{1}{2} F^{ab}_{\mu\nu}\left(g^{-1}_{UV} J_{ab} g_{UV}\right) dx^\mu \wedge dx^\nu \tag{A15}$$

Substituting into $g^{-1}_{UV} J_{ab} g_{UV}$, the above equation becomes:

$$g^{-1}_{UV}\Omega_U g_{UV} = \frac{1}{2} F^{ab}_{\mu\nu}\left[\left(g^{-1}\right)^c_a \left(g^{-1}\right)^d_b J_{cd}\right] dx^\mu \wedge dx^\nu \tag{A16}$$

Adjusting the sum and renaming the index $c, d \to a, b$ to match the form of Ω_V, we get:

$$g^{-1}_{UV}\Omega_U g_{UV} = \frac{1}{2}\left[F^{cd}_{\mu\nu}\left(R^{-1}\right)^a_c \left(R^{-1}\right)^b_d\right] J_{ab} dx^\mu \wedge dx^\nu \tag{A17}$$

4) Compare the form of Ω_V:

$$\Omega_V = \frac{1}{2} F'^{ab}_{\mu\nu} J_{ab} dx^\mu \wedge dx^\nu$$

Then, by comparing the coefficients from the derivation result (A17), we get:

$$F'^{ab}_{\mu\nu} = F^{cd}_{\mu\nu}\left(g^{-1}\right)^a_c \left(g^{-1}\right)^b_d \tag{A18}$$

It seems slightly different from (A6), but note that $\left(g^{-1}\right)^c_a = \left[\left(g^{-1}\right)\right]^a_c$, where $\left(g^{-1}\right)^c_a$ is not a standard notation, but represents the *c*th row and the *a*th column. The standard notation is $\left[\left(R^{-1}\right)\right]^a_c$. For example, in the (1,3) gauge theory, the matrix elements of the Lorentz group sometimes use non-standard notations to match the physical meaning of the tensor transformation. So the formula derived above: $F'^{ab}_{\mu\nu} = F^{cd}_{\mu\nu}\left(R^{-1}\right)^{a}_{c}\left(R^{-1}\right)^{b}_{d}$ and the original target formula need to be reversed, that is, $\left(R^{-1}\right)^{c}_{a} = \left[\left(R^{-1}\right)\right]^{a}_{c}$; that is

$$F'^{ab}_{\mu\nu} = F^{cd}_{\mu\nu}\left(R^{-1}\right)^{a}_{c}\left(R^{-1}\right)^{b}_{d} = F^{cd}_{\mu\nu}\left[R^{-1}\right]^{a}_{c}\left[R^{-1}\right]^{b}_{d} \leftrightarrow F'^{ab}_{\mu\nu} = F^{cd}_{\mu\nu}\left(R^{-1}\right)^{c}_{a}\left(R^{-1}\right)^{d}_{b} \quad \text{(A19)}$$

The derived formula is expressed using standard matrix notation and the target formula is expressed using non-standard notation, so the both are actually equivalent as the (A19).

So we have completed the derivation of $F'^{ab}_{\mu\nu} = \left(g^{-1}\right)^{c}_{a}\left(g^{-1}\right)^{d}_{b} F^{cd}_{\mu\nu}$ from $\omega_V = g^{-1}_{UV}\omega_U g_{UV}$. It verifies the consistency of the gauge tensor $F^{ab}_{\mu\nu}$ in our extended gravitational gauge field framework and the GGE curvature Ω_U in terms of form and transformation law, that is, $F_{\mu\nu} = F^{ab}_{\mu\nu}J_{ab}$, $F^{ab}_{\mu\nu}$ is a 2-form component, and J_{ab} is a Lie algebra basis. The curvature of GGE $\Omega_U = \frac{1}{2}F^{ab}_{\mu\nu}J_{ab}dx^{\mu}\wedge dx^{\nu}$ with the factors 1/2 and $dx^{\mu}\wedge dx^{\nu}$ are the standard forms of exterior differentials, which are the same components as $F^{ab}_{\mu\nu}$. The two have the same form, both of which are Lie algebra-valued 2-forms.

Appendix B. Field Strength Covariance under Cross-Group Gauge Transformation

This appendix is to solve the proof problem of the gauge field strength transformation across fundamental interactions. The goal is to verify the properties of the gauge field strength tensor $F_{\mu\nu}$ under generalized gauge transformation and ensure that it is consistent with the Cartan second structural equation. The following are the target tasks:

- Validation of Cartan II structural equation: $\Omega = d\omega + \frac{1}{2}[\omega,\omega] = d\omega + \omega \wedge \omega$;
- Use the gauge transformation formula: $\omega'_\mu = g\omega_\mu g^{-1} + \partial_\mu g g^{-1}$;
- Derive the field strength tensor formula from the Cartan second structural equation: $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - \left[\omega_\mu, \omega_\nu\right]$;
- Using the auxiliary formula $g^{-1}\partial_\nu g = -\partial_\nu g^{-1} g^{-1}$ prove that:

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu - \left[\omega'_\mu, \omega'_\nu\right] \\ &= g\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu - \left[\omega_\mu, \omega_\nu\right]\right) g^{-1} = gF_{\mu\nu} g^{-1} \end{aligned} \tag{B1}$$

We will perform the derivation in the context of the gauge group $G = SO(1,3) \times U(1) \times SU(2) \times SU(3)$ and the Lie algebra $\mathrm{g} = so(1,3) \oplus u(1) \oplus su(2) \oplus su(3)$.

1) Definition and Notation

Gauge gruop: $G = SO(1,3) \times U(1) \times SU(2) \times SU(3)$.

Lie algebra: $\mathrm{g} = so(1,3) \oplus u(1) \oplus su(2) \oplus su(3)$, basis $\{e_r\}$, structural constant is $\left[e_a, e_b\right] = f_{ab}^c e_c$.

Gauge potential: $\omega = \omega_\mu dx^\mu$, $\omega_\mu \in \mathrm{g}$, local $\omega_\mu = ke_r \omega_\mu^r$, k is a coupling constant.

Group element: $g \in G$, $g^{-1} \in G$, is 0-form (*i.e.* matrix).

Field strength:

- Tensor form:

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - \left[\omega_\mu, \omega_\nu\right] \tag{B2}$$

$$\left[\omega_\mu, \omega_\nu\right] = \omega_\mu \omega_\nu - \omega_\nu \omega_\mu \tag{B3}$$

- 2-form:

$$\Omega = \frac{1}{2} ke_r F_{\mu\nu}^r dx^\mu \wedge dx^\nu \tag{B4}$$

Gauge transformation:

$$\omega'_\mu = g\omega_\mu g^{-1} + \partial_\mu g g^{-1} \tag{B5}$$

Wedge product: For g-value 1-form A, B:

$$A \wedge B = AB - BA \tag{B6}$$

Exterior differential: to 1-form $\omega = \omega_\mu dx^\mu$:

$$d\omega = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu \tag{B7}$$

Auxiliary formula:

$$g^{-1}\partial_\mu g = -\partial_\mu g^{-1} g \tag{B8}$$

2) Verification of Cartan second structural equation

Goal: Prove

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] = d\omega + \omega \wedge \omega \tag{B9}$$

a) Calculation $d\omega$

$$\omega = \omega_\mu dx^\mu, \omega_\mu = ke_r\omega^r_\mu \tag{B10}$$

$$d\omega = d\left(\omega_\mu dx^\mu\right) = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu = \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\nu \wedge dx^\mu \tag{B11}$$

b) Calculation $\omega \wedge \omega$

$$\begin{aligned} \omega \wedge \omega &= \left(\omega_\mu dx^\mu\right) \wedge \left(\omega_\nu dx^\nu\right) = \omega_\mu \omega_\nu dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\left(\omega_\mu \omega_\nu - \omega_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu = \frac{1}{2}\left[\omega_\mu, \omega_\nu\right] dx^\mu \wedge dx^\nu \end{aligned} \tag{B12}$$

c) Calculation $d\omega + \omega \wedge \omega$

$$\begin{aligned} &\Omega = d\omega + \omega \wedge \omega = \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu + \frac{1}{2}\left[\omega_\mu, \omega_\nu\right] dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \left[\omega_\mu, \omega_\nu\right]\right) dx^\mu \wedge dx^\nu \end{aligned} \tag{B13}$$

d) Field Strength 2-Form

$$\Omega = \frac{1}{2} ke_r F^r_{\mu\nu} dx^\mu \wedge dx^\nu \tag{B14}$$

Compare the coefficients on both sides with the above equation (B13):

$$ke_r F^r_{\mu\nu} = (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) + \left[\omega_\mu, \omega_\nu\right] \tag{B15}$$

Then (B13) gives:

$$d\omega + \omega \wedge \omega = \frac{1}{2} ke_r \left(\partial_\mu \omega^r_\nu - \partial_\nu \omega^r_\mu + \left[\omega_\mu, \omega_\nu\right]^r\right) dx^\mu \wedge dx^\nu \tag{B16}$$

where $\left[\omega_\mu, \omega_\nu\right] = \left[\omega_\mu, \omega_\nu\right]^r ke_r = -k^2 f^r_{ab} \omega^a_\mu \omega^b_\nu e_r$, note that e_r belongs to the basis of Lie algebras, $\left[e_a, e_b\right] = -f^c_{ab} e_c$, so we can define

$$\left[\omega_\mu, \omega_\nu\right]^r = -kf^r_{ab} \omega^a_\mu \omega^b_\nu \tag{B17}$$

Therefore we obtain

$$\Omega = d\omega + \omega \wedge \omega \tag{B18}$$

$$\frac{1}{2}[\omega, \omega] = \omega \wedge \omega \tag{B19}$$

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{B20}$$

Hence, the second structural equation of Cartan holds true.

3) Derivation of $F_{\mu\nu}$ from Cartan's second structural equation

Objective: From

$$\Omega = d\omega + \omega \wedge \omega$$

derivation:

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - \left[\omega_\mu, \omega_\nu\right] \tag{B21}$$

a) Expand Ω

By means of formulas(B14), (B11), (B12) and (B21), we obtain:

$$\Omega = \frac{1}{2} k e_r F^r_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$d\omega = \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu$$

$$\omega \wedge \omega = \frac{1}{2}\left[\omega_\mu, \omega_\nu\right] dx^\mu \wedge dx^\nu$$

$$\Omega = \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \left[\omega_\mu, \omega_\nu\right]\right) dx^\mu \wedge dx^\nu \tag{B22}$$

Comparing:

$$\frac{1}{2} k e_r F^r_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \left[\omega_\mu, \omega_\nu\right]\right) dx^\mu \wedge dx^\nu \tag{B23}$$

where $\omega_\mu = k e_r \omega^r_\mu$; $\left[\omega_\mu, \omega_\nu\right] = -k^2 f^c_{ab} e_c \omega^a_\mu \omega^b_\nu$; $\partial_\mu \omega_\nu = k e_r \partial_\mu \omega^r_\nu$; $F_{\mu\nu} = k e_r F^r_{\mu\nu}$; we finally obtain the formula (B21):

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - \left[\omega_\mu, \omega_\nu\right]$$

4) Proof $F'_{\mu\nu} = g F_{\mu\nu} g^{-1}$

Using the gauge transformation (B5):

$$\omega'_\mu = g \omega_\mu g^{-1} + \partial_\mu g g^{-1}$$

and the auxiliary formula:

$$g^{-1} \partial_\mu g = -\partial_\mu g^{-1} g$$

a) Calculation $\partial_\mu \omega'_\nu$

$$\begin{aligned} \partial_\mu \omega'_\nu &= \partial_\mu \left(g \omega_\nu g^{-1} + \partial_\nu g g^{-1}\right) \\ &= \left(\partial_\mu g\right) \omega_\nu g^{-1} + g\left(\partial_\mu \omega_\nu\right) g^{-1} - g \omega_\nu g^{-1} \partial_\mu g g^{-1} \\ &\quad + \left(\partial_\mu \partial_\nu g\right) g^{-1} - \partial_\nu g g^{-1} \partial_\mu g g^{-1} \end{aligned} \tag{B24}$$

b) Calculation $\partial_\nu \omega'_\mu$

$$\begin{aligned} \partial_\nu \omega'_\mu &= \left(\partial_\nu g\right) \omega_\mu g^{-1} + g\left(\partial_\nu \omega_\mu\right) g^{-1} - g \omega_\mu g^{-1} \partial_\nu g g^{-1} \\ &\quad + \left(\partial_\nu \partial_\mu g\right) g^{-1} - \partial_\mu g g^{-1} \partial_\nu g g^{-1} \end{aligned} \tag{B25}$$

$$\begin{aligned} \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu &= g\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) g^{-1} + \left(\partial_\mu g\right) \omega_\nu g^{-1} - \left(\partial_\nu g\right) \omega_\mu g^{-1} \\ &\quad - g \omega_\nu g^{-1} \partial_\mu g g^{-1} + g \omega_\mu g^{-1} \partial_\nu g g^{-1} \\ &\quad - \partial_\nu g g^{-1} \partial_\mu g g^{-1} + \partial_\mu g g^{-1} \partial_\nu g g^{-1} \end{aligned} \tag{B26}$$

where the main term is:

$$g\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) g^{-1} \tag{B27}$$

Cross term is:

$$\left(\partial_\mu g\right)\omega_\nu g^{-1}-\left(\partial_\nu g\right)\omega_\mu g^{-1} \tag{B28}$$

$$-g\omega_\nu g^{-1}\partial_\mu g g^{-1}+g\omega_\mu g^{-1}\partial_\nu g g^{-1} \tag{B29}$$

And

$$\left(\partial_\mu\partial_\nu g\right)g^{-1}-\left(\partial_\nu\partial_\mu g\right)g^{-1}=0 \tag{B30}$$

$$-g\omega_\nu g^{-1}\partial_\mu g g^{-1}+g\omega_\mu g^{-1}\partial_\nu g g^{-1} \tag{B31}$$

c) Calculation $\left[\omega'_\mu,\omega'_\nu\right]$

$$\begin{aligned}\left[\omega'_\mu,\omega'_\nu\right]&=g\left[\omega_\mu,\omega_\nu\right]g^{-1}+g\omega_\mu g^{-1}\partial_\nu g g^{-1}-g\omega_\nu g^{-1}\partial_\mu g g^{-1}\\&+\partial_\mu g\omega_\nu g^{-1}-\partial_\nu\omega_\mu g^{-1}+\partial_\mu g g^{-1}\partial_\nu g g^{-1}-\partial_\nu g g^{-1}\partial_\mu g g^{-1}\end{aligned} \tag{B32}$$

d) Calculation $F'_{\mu\nu}$

$$\begin{aligned}F'_{\mu\nu}&=gF_{\mu\nu}g^{-1}=\partial_\mu\omega'_\nu-\partial_\nu\omega'_\mu-\left[\omega'_\mu,\omega'_\nu\right]\\&=g\left(\partial_\mu\omega_\nu-\partial_\nu\omega_\mu-\left[\omega_\mu,\omega_\nu\right]\right)g^{-1}\end{aligned} \tag{B33}$$

Substituting into above related terms and expanding the equation, we can get:

$$\begin{aligned}F'_{\mu\nu}&=g\left(\partial_\mu\omega_\nu-\partial_\nu\omega_\mu\right)g^{-1}+\left(\partial_\mu g\right)\omega_\nu g^{-1}-\left(\partial_\nu g\right)\omega_\mu g^{-1}-g\omega_\nu g^{-1}\partial_\mu g g^{-1}\\&+g\omega_\mu g^{-1}\partial_\nu g g^{-1}-\partial_\nu g g^{-1}\partial_\mu g g^{-1}+\partial_\mu g g^{-1}\partial_\nu g g^{-1}-g\left[\omega_\mu,\omega_\nu\right]g^{-1}\\&-g\omega_\mu g^{-1}\partial_\nu g g^{-1}+g\omega_\nu g^{-1}\partial_\mu g g^{-1}-\partial_\mu g\omega_\nu g^{-1}\\&+\partial_\nu\omega_\mu g^{-1}-\partial_\mu g g^{-1}\partial_\nu g g^{-1}+\partial_\nu g g^{-1}\partial_\mu g g^{-1}\end{aligned} \tag{B34}$$

e) Cross-term cancellation

Collect cross-terms:

From $\partial_\mu\omega'_\nu-\partial_\nu\omega'_\mu$: (1) $\left(\partial_\mu g\right)\omega_\nu g^{-1}$; (2) $-\left(\partial_\nu g\right)\omega_\mu g^{-1}$; (3) $-g\omega_\nu g^{-1}\partial_\mu g g^{-1}$; (4) $g\omega_\mu g^{-1}\partial_\nu g g^{-1}$; (5) $-\partial_\nu g g^{-1}\partial_\mu g g^{-1}$; (6) $\partial_\mu g g^{-1}\partial_\nu g g^{-1}$;

From $-\left[\omega'_\mu,\omega'_\nu\right]$: (7) $-g\omega_\mu g^{-1}\partial_\nu g g^{-1}$; (8) $+g\omega_\nu g^{-1}\partial_\mu g g^{-1}$; (9) $-\partial_\mu g\omega_\nu g^{-1}$; (10) $+\partial_\nu\omega_\mu g^{-1}$; (11) $-\partial_\mu g g^{-1}\partial_\nu g g^{-1}$; (12) $+\partial_\nu g g^{-1}\partial_\mu g g^{-1}$.

Merge item by item:

Terms (1) and (9):

$$\left(\partial_\mu g\right)\omega_\nu g^{-1}-\partial_\mu g\omega_\nu g^{-1}=0 \tag{B35}$$

Terms (2) and (10):

$$-\left(\partial_\nu g\right)\omega_\mu g^{-1}+\partial_\nu\omega_\mu g^{-1}=0 \tag{B36}$$

Terms (3) and (8):

$$-g\omega_\nu g^{-1}\partial_\mu g g^{-1}+g\omega_\nu g^{-1}\partial_\mu g g^{-1}=0 \tag{B37}$$

Terms (4) and (7):

$$g\omega_\mu g^{-1}\partial_\nu g g^{-1}-g\omega_\mu g^{-1}\partial_\nu g g^{-1}=0 \tag{B38}$$

Terms (5) and (12):

$$-\partial_\nu g g^{-1}\partial_\mu g g^{-1}+\partial_\nu g g^{-1}\partial_\mu g g^{-1}=0 \tag{B39}$$

Terms (6) and (11):

$$\partial_\mu g g^{-1} \partial_\nu g g^{-1} - \partial_\mu g g^{-1} \partial_\nu g g^{-1} = 0 \tag{B40}$$

The final main term is calculated as:

$$F'_{\mu\nu} = g F_{\mu\nu} g^{-1} = g\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu - \left[\omega_\mu, \omega_\nu\right]\right) g^{-1}$$

The cross terms cancel one by one, proving that the following formula is correct:

$$F'_{\mu\nu} = g F_{\mu\nu} g^{-1}$$

Note that the formula (B5) used above is only for consistent with the gravitational gauge field theory, see [24]. It differs from formula (2), see [23], in that the second term differs by one sign, but the same result can be obtained by derivation using formula (2).