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# On Electron Clouds and Light

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## Abstract

The wave equation of the electron, recently improved, allows physics to obtain all the quantum numbers and other results explaining the hydrogen spectrum. The Pauli exclusion principle then gives the description of electron clouds used in chemistry. The relativistic wave equation is associated with a Lagrangian density, thus also with an energy-momentum tensorial density. The wave of an electron cloud adds these energy-momentum densities, while photons in light are precisely those differences between such energy-momentum densities.

## Keywords

Quantum Mechanics, Nonlinear Wave Equation, Relativity, Electron Clouds, Photon, Light

## 1. Introduction

An improved Dirac equation with two mass terms is presented, yielding the electron clouds through addition and light through subtraction.

The relativistic wave of the electron was conceived by de Broglie a century ago [1]. Schrödinger gave a wave equation as early as 1926, and Dirac obtained a relativistic equation in 1928 [2]. This wave equation was solved by C.G. Darwin in the case of the hydrogen atom and for other atoms with only one electron [3]. This solution gave all the quantum numbers needed by Bohr's model of atomic spectra, with precise Sommerfeld energy levels. De Broglie explained why the Dirac equation was better than the previous wave equations [4], by yielding such results as the spin 1/2, selection rules, Zeeman effect and Landé's numbers. He also described the various tensorial densities, the probability current, the gauge invariance and the relativistic invariance of the wave equation.

Despite these brilliant results, the Dirac wave equation was nearly abandoned by quantum mechanics. The main reason was the finding that nonrelativistic wave equations, plus the exclusion principle, were enough to study the waves of the electron clouds that are necessary for chemistry.

The first important results of our improved wave equation were the resolution of the negative energy problem, and a better comprehension of charge conjugation. The improved wave equation yield also exactly the Lorentz force. It also may be extended to describe all particles of one generation. All these results were previously obtained for a single fermion [5], [6]. The wave equation is here extended to describe electron clouds in atoms and emitted and absorbed light.

## 2. The Relativistic Wave Equation of the Electron

The Dirac wave equation, for high velocity, electromagnetic and weak interactions, reads:

$$0=\left[\gamma^{\mu}\left(\partial_{\mu}+iqA_{\mu}\right)+im\right]\psi;\ q:=\frac{e}{\hbar c};\ \hbar:=\frac{h}{2\pi};\ m:=\frac{m_0 c}{\hbar}.$$

$$\psi:=\begin{pmatrix}\xi\\ \eta\end{pmatrix};\ \xi:=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix};\ \eta:=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \tag{1}$$

where we use the usual summing on up and down indices, $\mu=0,1,2,3$, and where $e$ is the charge, $m_0$ is the electron proper mass, $A$ is the space-time vector called the exterior electromagnetic potential, $\xi$ is the right part of the wave and $\eta$ is the left part. The following matrices are used throughout this text:

$$\gamma_j:=\begin{pmatrix}0 & \sigma_j\\ -\sigma_j & 0\end{pmatrix};\ \sigma^j=-\hat{\sigma}^j=\hat{\sigma}_j:=-\sigma_j,\ \ j=1,2,3, \tag{2}$$

$$\gamma_0=\gamma^0:=\begin{pmatrix}0 & I_2\\ I_2 & 0\end{pmatrix};\ I_2=\sigma_0=\sigma^0=\hat{\sigma}^0=\hat{\sigma}_0:=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{3}$$

$$\gamma_5:=i\gamma_1\gamma_2\gamma_3\gamma_0=\begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix};\ \frac{1+\gamma_5}{2}\psi=\begin{pmatrix}\xi\\ 0\end{pmatrix};\ \frac{1-\gamma_5}{2}\psi=\begin{pmatrix}0\\ \eta\end{pmatrix},$$

where $\sigma_j$ are Pauli matrices. The Dirac equation (1) is equivalent to the following system:

$$0=\sigma^{\mu}\left(\partial_{\mu}+iqA_{\mu}\right)\eta+im\xi, \tag{4}$$

$$0=\hat{\sigma}^{\mu}\left(\partial_{\mu}+iqA_{\mu}\right)\xi+im\eta. \tag{5}$$

The algebra generated by Dirac matrices, used in 1928, was replaced by Hestenes much later (1967) with the Clifford algebra $Cl_{1,3}$ [7], and still later with the smaller Pauli algebra $Cl_3$ [8] [9]. This means that we let (see also [5] [6]):

$$\nabla\hat{\phi}i\sigma_3=qA\hat{\phi}+m\phi, \tag{6}$$

$$\phi:=\sqrt{2}\left(\xi\ \ \hat{\eta}\right)=\sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^*\\ \xi_2 & \eta_1^*\end{pmatrix};\ \hat{\phi}=\sqrt{2}\left(\eta\ \ \hat{\xi}\right)=\sqrt{2}\begin{pmatrix}\eta_1 & -\xi_2^*\\ \eta_2 & \xi_1^*\end{pmatrix}, \tag{7}$$

$$\nabla:=\sigma^{\mu}\partial_{\mu};\ \hat{\nabla}:=\hat{\sigma}^{\mu}\partial_{\mu};\ A:=\sigma^{\mu}A_{\mu};\ \hat{A}:=\hat{\sigma}^{\mu}A_{\mu}. \tag{8}$$

With the four complex numbers of the $\psi$ wave is hence built the $\phi$ wave function with value in $Cl_3$, which is also the Pauli algebra. To obtain the relativistic invariance of the Dirac equation, it is enough to consider the **dilator** $M$, a general fixed element in $Cl_3$, which defines the transformation $R$ as:

$$R:\mathrm{x}\mapsto \mathrm{x}'=R(\mathrm{x}):=M\mathrm{x}M^{\dagger};\ M\in Cl_3;\ \xi'=M\xi;\ \eta'=\hat{M}\eta, \tag{9}$$

$$\mathrm{x}=\mathrm{x}^{\mu}\sigma_{\mu}=\mathrm{x}^0+\vec{\mathrm{x}}=\begin{pmatrix}\mathrm{x}^0+\mathrm{x}^3 & \mathrm{x}^1-i\mathrm{x}^2\\ \mathrm{x}^1+i\mathrm{x}^2 & \mathrm{x}^0-\mathrm{x}^3\end{pmatrix};\ \mathrm{x}'=\mathrm{x}'^{\mu}\sigma_{\mu}, \tag{10}$$

$$\hat{\mathrm{x}}=\overline{\mathrm{x}}=\mathrm{x}^0-\vec{\mathrm{x}},\ \det(M)=re^{i\theta},\ r:=\left|\det(M)\right| \tag{11}$$

$$\|\mathrm{x}\|^2=\det(\mathrm{x})=\mathrm{x}\hat{\mathrm{x}}=\mathrm{x}\cdot\mathrm{x}=\left(\mathrm{x}^0\right)^2-\left(\vec{\mathrm{x}}\right)^2. \tag{12}$$

$$\det(\mathrm{x}')=r^2\det(\mathrm{x});\ R_0^0>0;\ \det\left(R_\nu^\mu\right)=r^4;\ \mathrm{x}'^{\mu}=R_\nu^\mu\mathrm{x}^\nu. \tag{13}$$

Relation (9) means that left and right waves are distinguished by the relativistic transformation itself. Relation (10) includes space-time into the $Cl_3$ set, while (12) implies the **parity** transformation ($M\mapsto\hat{M}$) as a part of the space-time pseudo-norm. The transformation $R$ is a **similitude**, the product of a proper Lorentz transformation and a homothety with ratio $r$. Moreover $f:M\mapsto R$ is a group homomorphism from the Lie group $Cl_3^*$, the set of the invertible elements in $Cl_3$, into the similitude group $\mathcal{S}$. The kernel of this homomorphism is the chiral group $U(1)$ of weak interactions. It is also the gauge group of Lochak's theory of magnetic monopoles [10].

## 3. Densities, Identities

The electron wave is not only the wave function; it also allows us the definition of **tensorial densities**:

$$\det(\phi)=S_0:=\eta^{\dagger}\xi=\phi\overline{\phi}=\overline{\phi}\phi=\Omega_1+i\Omega_2=\rho e^{i\beta},\ \overline{\phi}:=\hat{\phi}^{\dagger}, \tag{14}$$

where $\Omega_1$, $\Omega_2$ and $\rho$ are relativistic invariant, $\beta$ is the Yvon-Takabayasi angle, and we have also:

$$\mathrm{D}_\mu:=\phi\sigma_\mu\phi^{\dagger};\ 2\mathrm{D}_\mu\cdot\mathrm{D}_\nu=\delta_{\mu\nu}\rho^2;\ \mathrm{D}_0=\mathrm{D}_R+\mathrm{D}_L, \tag{15}$$

$$\mathrm{D}_R:=\phi\frac{1+\sigma_3}{2}\phi^{\dagger};\ \mathrm{D}_L:=\phi\frac{1-\sigma_3}{2}\phi^{\dagger},\ \mathrm{D}_3=\mathrm{D}_R-\mathrm{D}_L.$$

$\left(\mathrm{D}_0,\mathrm{D}_1,\mathrm{D}_2,\mathrm{D}_3\right)$ is a mobile orthogonal basis of space-time. $\mathrm{D}_0$ and $\mathrm{D}_3$ are the sum and difference of the chiral currents $\mathrm{D}_R$ and $\mathrm{D}_L$. $\mathrm{D}_1$ and $\mathrm{D}_2$ are not gauge invariant. Next we have:

$$S_\mu:=\phi\sigma_\mu\overline{\phi},\ \mu=0,1,2,3;\ S_1-iS_2=2S_L; \tag{16}$$

$$S_1+iS_2=2S_R;\ S_R=\phi\left(\sigma_1+i\sigma_2\right)\overline{\phi};\ S_L=\phi\left(\sigma_1-i\sigma_2\right)\overline{\phi},$$

$$\mathrm{D}_0\hat{\mathrm{D}}_0=\hat{\mathrm{D}}_0\mathrm{D}_0=\rho^2;\ \mathrm{v}:=\mathrm{D}_0/\rho;\ 1=\mathrm{v}\hat{\mathrm{v}}=\hat{\mathrm{v}}\mathrm{v};\ \mathrm{v}^{-1}=\hat{\mathrm{v}},$$

$$\mathrm{v}\eta=e^{-i\beta}\xi;\ \hat{\mathrm{v}}\xi=e^{i\beta}\eta;\ \mathrm{v}\hat{\phi}=e^{-i\beta}\phi;\ \hat{\mathrm{v}}\phi=e^{i\beta}\hat{\phi}. \tag{17}$$

These two last identities are new and we use them to **uncross** the system of equations. The (5) system becomes:

$$
\begin{aligned}
&0=\left(-i\nabla+qA+me^{i\beta}\mathrm{v}\right)\eta;\ 0=\left(-i\nabla+qA\right)\eta+m\xi,\\
&0=\left(-i\hat{\nabla}+q\hat{A}+me^{-i\beta}\hat{\mathrm{v}}\right)\xi;\ 0=\left(-i\hat{\nabla}+q\hat{A}\right)\xi+m\eta.
\end{aligned}
\tag{18}
$$

The Dirac theory is actually a nonlinear theory, since $\beta$ and $\mathrm{v}$ depend on $\phi$. The Dirac theory seems a linear theory if only the $\psi$ wave function is considered, without consideration of the densities, which are quadratic on the wave. Another important density is the Lagrangian density:

$$
\begin{aligned}
0=\mathcal{L}=\frac{1}{2}\Big[&\bar{\psi}\gamma^{\mu}\left(-i\partial_{\mu}+qA_{\mu}\right)\psi+\left(\bar{\psi}\gamma^{\mu}\left(-i\partial_{\mu}+qA_{\mu}\right)\psi\right)^{\dagger}\Big]\\
&+m\bar{\psi}\psi;\ \bar{\psi}:=\psi^{\dagger}\gamma_0,
\end{aligned}
\tag{19}
$$

$$
0=\mathcal{L}=\Re\left[\bar{\phi}\left(\nabla\hat{\phi}\right)\sigma_{21}+\bar{\phi}qA\hat{\phi}+m\bar{\phi}\phi\right].
\tag{20}
$$

It is possible to simplify the Lagrangian density and the uncrossed system by suppressing the nonlinear term containing the Yvon-Takabayasi angle $\beta$ [11]:

$$
\begin{aligned}
&0=\left(-i\nabla+qA+m\mathrm{v}\right)\eta;\ 0=\left(-i\nabla+qA\right)\eta+me^{-i\beta}\xi,\\
&0=\left(-i\hat{\nabla}+q\hat{A}+m\hat{\mathrm{v}}\right)\xi;\ 0=\left(-i\hat{\nabla}+q\hat{A}\right)\xi+me^{i\beta}\eta.
\end{aligned}
\tag{21}
$$

$$
0=\mathcal{L}=\Re\left[\bar{\phi}\left(\nabla\hat{\phi}\right)\sigma_{21}+\bar{\phi}qA\hat{\phi}+m\rho\right].
\tag{22}
$$

This wave equation is similar to the wave equation of the magnetic monopole [10]. The study of their tracks [12] [13] brings the idea of a possible double mass: a proper mass for the right wave, and another for the left. This equation has three equivalent forms, the completely invariant form:

$$
0=\bar{\phi}\left(\nabla\hat{\phi}\right)\sigma_{21}+\bar{\phi}qA\hat{\phi}+\rho\mathbf{m};\ \mathbf{m}=\begin{pmatrix}\mathbf{l}&0\\0&\mathbf{r}\end{pmatrix},
\tag{23}
$$

The usual form reads:

$$
0=\nabla\hat{\phi}\sigma_{21}+qA\hat{\phi}+e^{-i\beta}\phi\mathbf{m}.
\tag{24}
$$

And we obtain the wave equation as a system by replacing the unique mass $m$ of the previous system (18) with the mass $\mathbf{l}$ for the left wave and the mass $\mathbf{r}$ for the right:

$$
\begin{aligned}
&0=\left(-i\nabla+qA+\mathbf{l}\mathrm{v}\right)\eta,\\
&0=\left(-i\hat{\nabla}+q\hat{A}+\mathbf{r}\hat{\mathrm{v}}\right)\xi.
\end{aligned}
\tag{25}
$$

Using the Planck length $l_P$ the Lagrangian density reads:

$$
\begin{aligned}
&0=\mathcal{L}=\frac{m}{k\mathbf{l}}\mathcal{L}_L+\frac{m}{k\mathbf{r}}\mathcal{L}_R;\ \mathcal{L}_L=\Re\left[\eta^{\dagger}\left(-i\nabla\eta+qA\eta+\mathbf{l}\mathrm{v}\eta\right)\right],\\
&l_P^3=k\hbar c;\ \mathcal{L}_R=\Re\left[\xi^{\dagger}\left(-i\hat{\nabla}\xi+q\hat{A}\xi+\mathbf{r}\hat{\mathrm{v}}\xi\right)\right],
\end{aligned}
\tag{26}
$$

This means that the Lagrangian density is the harmonic mean of the left and the right densities. Thus the electron has not a single but actually **two** energy-momentum tensorial densities: Tetrode's $T$ and Costa de Beauregard's $V$ [14]:

$$T_\nu^\mu := \frac{m}{k\mathbf{r}} T_{R\nu}^\mu + \frac{m}{k\mathbf{l}} T_{L\nu}^\mu ;\ T_{L\nu}^\mu := \Re\left[-i\left(\eta^\dagger \sigma^\mu d_\nu^L \eta\right)\right], \tag{27}$$

$$V_\nu^\mu := \frac{m}{k\mathbf{r}} T_{R\nu}^\mu - \frac{m}{k\mathbf{l}} T_{L\nu}^\mu ;\ T_{R\nu}^\mu := \Re\left[-i\left(\xi^\dagger \hat{\sigma}^\mu d_\nu^R \xi\right)\right], \tag{28}$$

$$d_\mu^L := -i\partial_\mu + qA_\mu + \mathbf{l}\mathrm{v}_\mu ;\ d_\mu^R := -i\partial_\mu + qA_\mu + \mathbf{r}\mathrm{v}_\mu .$$

When the Yvon-Takabayasi $\beta$ angle is zero or negligible, the wave equation is reduced to:

$$0 = \overline{\phi}\left(\nabla\hat{\phi}\right)\sigma_{21} + \overline{\phi} qA\hat{\phi} + \overline{\phi}\phi\mathbf{m}. \tag{29}$$

Its usual form reads:

$$0 = \nabla\hat{\phi}\sigma_{21} + qA\hat{\phi} + \phi\mathbf{m}. \tag{30}$$

We obtain this wave equation as a system simply by replacing the unique mass $m$ of the previous system (18) with the mass $\mathbf{l}$ for the left wave and the mass $\mathbf{r}$ for the right:

$$\begin{aligned} 0 &= \left(-i\nabla + qA\right)\eta + \mathbf{l}\xi, \\ 0 &= \left(-i\hat{\nabla} + q\hat{A}\right)\xi + \mathbf{r}\eta. \end{aligned} \tag{31}$$

This linear system is a linear approximation of (25) and it is thus different from our improved equation. But it must also be thought of as a non-linear system:

$$\begin{aligned} 0 &= \left(-i\nabla + qA + \mathbf{l}e^{i\beta}\mathrm{v}\right)\eta, \\ 0 &= \left(-i\hat{\nabla} + q\hat{A} + \mathbf{r}e^{-i\beta}\hat{\mathrm{v}}\right)\xi. \end{aligned} \tag{32}$$

The Lagrangian density thus reads:

$$0 = \mathcal{L} = \frac{m}{k\mathbf{l}}\mathcal{L}_L + \frac{m}{k\mathbf{r}}\mathcal{L}_R ;\ \mathcal{L}_L = \Re\left[\eta^\dagger\left(-i\nabla\eta + qA\eta + \mathbf{l}e^{i\beta}\mathrm{v}\eta\right)\right], \tag{33}$$

$$\mathcal{L}_R = \Re\left[\xi^\dagger\left(-i\hat{\nabla}\xi + q\hat{A}\xi + \mathbf{r}e^{-i\beta}\hat{\mathrm{v}}\xi\right)\right],$$

Tetrode's $T$ and Costa de Beauregard's $V$ [14] are now expressed as:

$$T_\nu^\mu := \frac{m}{k\mathbf{r}} T_{R\nu}^\mu + \frac{m}{k\mathbf{l}} T_{L\nu}^\mu ;\ T_{L\nu}^\mu := \Re\left[-i\left(\eta^\dagger \sigma^\mu d_\nu^L \eta\right)\right], \tag{34}$$

$$V_\nu^\mu := \frac{m}{k\mathbf{r}} T_{R\nu}^\mu - \frac{m}{k\mathbf{l}} T_{L\nu}^\mu ;\ T_{R\nu}^\mu := \Re\left[-i\left(\xi^\dagger \hat{\sigma}^\mu d_\nu^R \xi\right)\right], \tag{35}$$

$$d_\mu^L := \partial_\mu + iqA_\mu + ie^{i\beta}\mathrm{v}_\mu ;\ d_\mu^R := \partial_\mu + iqA_\mu + ie^{-i\beta}\mathrm{v}_\mu .$$

The electron wave is governed by its wave equation, and also by another rule, a nonlocal one: the global energy of the electron, which is the mass-energy $m_0c^2 = mc\hbar$, must be equal to the sum over all space of the local energy of its wave (the equivalence principle):

$$\mathbf{J} := \frac{m}{k\mathbf{l}}\mathrm{D}_L + \frac{m}{k\mathbf{r}}\mathrm{D}_R ; \tag{36}$$

$$\iiint \mathrm{d}v \frac{\mathbf{J}^0}{\hbar c} = 1 \Leftrightarrow E = \iiint \mathrm{d}v T_0^0 . \tag{37}$$

## 4. The Electron in a Hydrogen Atom

While C.G. Darwin could use only the algebra of the Dirac matrices, a method of solution by separation of variables was obtained by H. Krüger [11] [15] (see also [5] [6] Chapter C). With:

$$x^1 := r\sin\theta\cos\varphi;\ x^2 := r\sin\theta\sin\varphi;\ x^3 := r\cos\theta, \tag{38}$$

the following notations are used:

$$i_1 := \sigma_{23} = i\sigma_1;\ i_2 := \sigma_{31} = i\sigma_2;\ i_3 := \sigma_{12} = i\sigma_3, \tag{39}$$

$$S := \exp\left(-\frac{\varphi}{2}i_3\right)\exp\left(-\frac{\theta}{2}i_2\right);\ \Omega = \hat{\Omega} := r^{-1}(\sin\theta)^{-\frac{1}{2}}S, \tag{40}$$

$$\vec{\partial}' := \sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi;\ \nabla' := \partial_0 - \vec{\partial}'. \tag{41}$$

H. Krüger obtained the identity [15]:

$$\nabla = \Omega\nabla'\Omega^{-1};\ \Omega^{-1}\nabla\hat{\phi} = \nabla'\left(\Omega^{-1}\hat{\phi}\right). \tag{42}$$

Three similar terms are used in $\vec{\partial}$ and $\vec{\partial}'$. But it is possible to use only two terms, because $\sigma_3\sigma_1 = i_2$. We have:

$$i\vec{\partial}' = i_3\left[\partial_r + \frac{i_2}{r}\left(\partial_\theta + \frac{i_3}{\sin\theta}\partial_\phi\right)\right]. \tag{43}$$

Moreover the wave equation reads:

$$\nabla\hat{\phi}i_3 = qA\hat{\phi} + \phi\mathbf{m}. \tag{44}$$

Here also, only the $i_3$ term is used. And this term is both a factor of the time derivative and of the derivative along the $\varphi$ angle. It is hence possible to separate together the $t$ and $\varphi$ variables from $r$ and $\theta$ variables by letting:

$$\phi =: \Omega X e^{\left(\lambda\varphi - E\mathrm{x}^0\right)i_3};\ \hat{\phi} = \Omega\hat{X}e^{\left(\lambda\varphi - E\mathrm{x}^0\right)i_3},$$

$$X := Up + i_2Vq;\ \hat{X} = U\hat{p} + i_2V\hat{q}, \tag{45}$$

$$p := p_r\frac{1+\sigma_3}{2} + p_l\frac{1-\sigma_3}{2};\ q := q_r\frac{1+\sigma_3}{2} + q_l\frac{1-\sigma_3}{2}, \tag{46}$$

where $p_r$, $p_l$, $q_r$ and $q_l$ are functions with value in $\mathbb{C}$ of the radial variable $r$, with $U$ and $V$ being real functions of the angular variable $\theta$. $\hbar cE$ is the energy of the electron and $\lambda$ is the magnetic quantum number. Terms $p$ and $q$ are the general terms commuting with $i_3$. They do not commute with $i_2$, since:

$$pi_2 = \left(p_r\frac{1+\sigma_3}{2} + p_l\frac{1-\sigma_3}{2}\right)i_2 = i_2\left(p_r\frac{1-\sigma_3}{2} + p_l\frac{1+\sigma_3}{2}\right) = i_2\overline{p},$$

$$\overline{p} = \hat{p}^\dagger;\ \hat{p} = p_r^*\frac{1-\sigma_3}{2} + p_l^*\frac{1+\sigma_3}{2}. \tag{47}$$

The wave equation (44) uses:

$$\partial_\varphi\left(\hat{X}e^{\left(\lambda\varphi - E\mathrm{x}^0\right)i_3}\right) = \hat{X}\lambda i_3 e^{\left(\lambda\varphi - E\mathrm{x}^0\right)i_3}, \tag{48}$$

$$\begin{aligned}\frac{i_3}{\sin\theta}\partial_\varphi\left(\hat{X}e^{\left(\lambda\varphi-Ex^0\right)i_3}\right)&=\frac{\lambda}{\sin\theta}i_3\left(U\hat{p}+i_2V\hat{q}\right)i_3e^{\left(\lambda\varphi-Ex^0\right)i_3}\\&=\frac{\lambda}{\sin\theta}\left(-U\hat{p}+i_2V\hat{q}\right)e^{\left(\lambda\varphi-Ex^0\right)i_3},\end{aligned}\tag{49}$$

$$\begin{aligned}&\left(\partial_\theta+\frac{i_3}{\sin\theta}\partial_\varphi\right)\left(\hat{X}e^{\left(\lambda\varphi-Ex^0\right)i_3}\right)\\&=\left[\left(U'-\frac{\lambda}{\sin\theta}U\right)\hat{p}+i_2\left(V'+\frac{\lambda}{\sin\theta}V\right)\hat{q}\right]\mathrm{e}^{\left(\lambda\varphi-Ex^0\right)i_3}.\end{aligned}\tag{50}$$

If $U$ and $V$ are solutions of the following system:

$$U'-\frac{\lambda U}{\sin\theta}=-\kappa V;\ V'+\frac{\lambda V}{\sin\theta}=\kappa U,\tag{51}$$

where $\kappa$ is a real number, we have:

$$\begin{aligned}\frac{i_2}{r}\left(\partial_\theta+\frac{i_3}{\sin\theta}\partial_\varphi\right)\left(\hat{X}e^{\left(\lambda\varphi-Ex^0\right)i_3}\right)&=\left(-\frac{\kappa}{r}U\hat{q}-\frac{\kappa}{r}i_2V\hat{p}\right)e^{\left(\lambda\varphi-Ex^0\right)i_3},\\-\vec{\partial}'\left[\hat{X}e^{\left(\lambda\varphi-Ex^0\right)i_3}\right]i_3&=\sigma_3\left[U\left(\hat{p}'-\frac{\kappa}{r}\hat{q}\right)+i_2V\left(-\hat{q}'-\frac{\kappa}{r}\hat{p}\right)\right]i_3e^{\left(\lambda\varphi-Ex^0\right)i_3}\\&=\left[Ui\left(\hat{p}'-\frac{\kappa}{r}\hat{q}\right)+i_2Vi\left(\hat{q}'+\frac{\kappa}{r}\hat{p}\right)\right]e^{\left(\lambda\varphi-Ex^0\right)i_3},\end{aligned}\tag{52}$$

$$\partial_0\left[\Omega^{-1}\hat{\phi}i_3\right]=E\hat{X}e^{\left(\lambda\varphi-Ex^0\right)i_3}.\tag{53}$$

The equation (44) gives, with a $qA=-\alpha/r$ potential:

$$\Omega^{-1}\nabla\hat{\phi}i_3=-\frac{\alpha}{r}\Omega^{-1}\hat{\phi}+\Omega^{-1}\phi\mathbf{m},\tag{54}$$

$$\left(\partial_0-\vec{\partial}'\right)\left(\hat{X}e^{\left(\lambda\varphi-Ex^0\right)i_3}\right)i_3=\left[-\frac{\alpha}{r}\hat{X}+X\mathbf{m}\right]e^{\left(\lambda\varphi-Ex^0\right)i_3}.\tag{55}$$

Hence multiplying on the left side by $\Omega$ and on the right side by $e^{-\left(\lambda\varphi-Ex^0\right)i_3}$, we obtain:

$$0=\left(E+\frac{\alpha}{r}\right)\left(U\hat{p}+i_2V\hat{q}\right)+i\left[U\left(\hat{p}'-\frac{\kappa}{r}\hat{q}\right)+i_2V\left(-\hat{q}'+\frac{\kappa}{r}\hat{p}\right)\right]+\left(\hat{U}p+i_2\hat{V}q\right)\mathbf{m}.\tag{56}$$

And since $U=\hat{U}$ and $V=\hat{V}$ the wave equation splits into:

$$\begin{aligned}0&=-\left(E+\frac{\alpha}{r}\right)\hat{p}+i\left(\hat{p}'-\frac{\kappa}{r}\hat{q}\right)+p\mathbf{m},\\0&=-\left(E+\frac{\alpha}{r}\right)\hat{q}+i\left(-\hat{q}'+\frac{\kappa}{r}\hat{p}\right)+q\mathbf{m}.\end{aligned}\tag{57}$$

And we obtain:

$$\phi=\sqrt{2}\begin{pmatrix}\xi_1&-\eta_2^*\\\xi_2&\eta_1^*\end{pmatrix}=\frac{1}{r\sqrt{\sin\theta}}e^{-\frac{\varphi}{2}i_3}e^{-\frac{\theta}{2}i_2}\left(Up+i_2Vq\right)e^{\left(\lambda\varphi-Ex^0\right)i_3},\tag{58}$$

which gives:

$$\xi_1=\frac{e^{i\left[\left(\lambda-\frac{1}{2}\right)\varphi-Ex^0\right]}}{r\sqrt{2\sin\theta}}\left[\left(\cos\frac{\theta}{2}\right)Up_r+\left(\sin\frac{\theta}{2}\right)Vq_r\right],\tag{59}$$

$$\xi_2=\frac{e^{i\left[\left(\lambda+\frac{1}{2}\right)\varphi-Ex^0\right]}}{r\sqrt{2\sin\theta}}\left[\left(\sin\frac{\theta}{2}\right)Up_r-\left(\cos\frac{\theta}{2}\right)Vq_r\right],\tag{60}$$

$$-\eta_2^*=\frac{e^{-i\left[\left(\lambda+\frac{1}{2}\right)\varphi-Ex^0\right]}}{r\sqrt{2\sin\theta}}\left[\left(\cos\frac{\theta}{2}\right)Vq_l-\left(\sin\frac{\theta}{2}\right)Up_l\right],\tag{61}$$

$$\eta_1^*=\frac{e^{-i\left[\left(\lambda-\frac{1}{2}\right)\varphi-Ex^0\right]}}{r\sqrt{2\sin\theta}}\left[\left(\sin\frac{\theta}{2}\right)Vq_l+\left(\cos\frac{\theta}{2}\right)Up_l\right].\tag{62}$$

The wave function is well-defined only if the functions of $\varphi$ have simply one value: $\lambda\pm\frac{1}{2}$ must be an integer number, thus $\lambda$ must be a half-odd integer number. At the second order we get:

$$0=U''+\left(\kappa^2-\frac{\lambda^2}{\sin^2\theta}\right)U+\lambda\frac{\cos\theta}{\sin^2\theta}U,\tag{63}$$

$$0=V''+\left(\kappa^2-\frac{\lambda^2}{\sin^2\theta}\right)V-\lambda\frac{\cos\theta}{\sin^2\theta}V,\tag{64}$$

$$0=\partial^2_{\theta\theta}X+\left(\kappa^2-\frac{\lambda^2}{\sin^2\theta}\right)X-\lambda\frac{\cos\theta}{\sin^2\theta}\sigma_{12}X\sigma_{12}.\tag{65}$$

The wave may be normalized only if the infinite series of the angular (51) and radial (57) systems are actually finite. This gives polynomial functions whose degrees are the integer numbers awaited by spectroscopy. Thus the $\phi$ wave is normalized only if $\kappa$ is a nonzero integer and then the $j=|\kappa|-\frac{1}{2}$ number satisfies (65), and thus $j(j+1)$ is a proper value of the total angular momentum $J^2$. $\lambda$ is a proper value of the $J_3$ operator. Hence the only possible values of $j$ are 1/2, 3/2, 5/2, ... while the only possible values of $\lambda$ are $-j$, $-j+1,\ \cdots,\ j-1,\ j$. With the Gegenbauer polynomial with degree $n$ satisfying:

$$|\lambda|+n=\left|\kappa+\frac{1}{2}\right|,\tag{66}$$

the $C$ function such that:

$$C(\theta)=\sum_{n=0}^{\infty}\frac{\left(|\lambda|-\kappa-\frac{1}{2}\right)_n\left(|\lambda|+\kappa+\frac{1}{2}\right)_n}{\left(\frac{1}{2}+|\lambda|\right)_n n!}\sin^{2n}\left(\frac{\theta}{2}\right),\tag{67}$$

is a finite sum with $n+1$ terms. Then, if $\lambda>0$ we obtain:

$$\begin{aligned}U&:=\sin^\lambda\theta\left[\sin\left(\frac{\theta}{2}\right)C'-\left(\kappa+\frac{1}{2}-\lambda\right)\cos\left(\frac{\theta}{2}\right)C\right],\\V&:=\sin^\lambda\theta\left[\cos\left(\frac{\theta}{2}\right)C'+\left(\kappa+\frac{1}{2}-\lambda\right)\sin\left(\frac{\theta}{2}\right)C\right].\end{aligned}\tag{68}$$

while if $\lambda < 0$ we obtain:

$$U := \sin^{-\lambda}\theta\left[\cos\left(\frac{\theta}{2}\right)C' + \left(\kappa + \frac{1}{2} + \lambda\right)\sin\left(\frac{\theta}{2}\right)C\right],$$
$$V := \sin^{-\lambda}\theta\left[-\sin\left(\frac{\theta}{2}\right)C' + \left(\kappa + \frac{1}{2} + \lambda\right)\cos\left(\frac{\theta}{2}\right)C\right]. \tag{69}$$

We proved in [16] that for opposite values of $\kappa$, the functions $U$ and $V$ are unchanged. Using now the following notations with the radial functions:

$$A := p_r;\ B := -q_l^*;\ C := -q_r;\ D := p_l^*, \tag{70}$$

the radial system (57) is equivalent to:

$$\begin{aligned}
0 &= i\left(E + \frac{\alpha}{r}\right)D + D' + \frac{\kappa}{r}B - i\mathbf{l}A,\\
0 &= -i\left(E + \frac{\alpha}{r}\right)C - C' - \frac{\kappa}{r}A + i\mathbf{r}B,\\
0 &= i\left(E + \frac{\alpha}{r}\right)B - B' - \frac{\kappa}{r}D - i\mathbf{l}C,\\
0 &= -i\left(E + \frac{\alpha}{r}\right)A + A' + \frac{\kappa}{r}C + i\mathbf{r}D.
\end{aligned} \tag{71}$$

We now let:

$$x := mr;\ \varepsilon := \frac{E}{m};\ a(x) := A(r) = A\left(\frac{x}{m}\right), \tag{72}$$
$$b(x) := B(r);\ c(x) := C(r);\ d(x) := D(r).$$

The system (71) becomes:

$$\begin{aligned}
-\left(\varepsilon + \frac{\alpha}{x}\right)d + id' + i\frac{\kappa}{x}b &= -\frac{\mathbf{l}}{m}a,\\
-\left(\varepsilon + \frac{\alpha}{x}\right)c + ic' + i\frac{\kappa}{x}a &= -\frac{\mathbf{r}}{m}b,\\
-\left(\varepsilon + \frac{\alpha}{x}\right)b - ib' - i\frac{\kappa}{x}d &= -\frac{\mathbf{l}}{m}c,\\
-\left(\varepsilon + \frac{\alpha}{x}\right)a - ia' - i\frac{\kappa}{x}c &= -\frac{\mathbf{r}}{m}d.
\end{aligned} \tag{73}$$

The necessity of obtaining a probability current implies that radial functions must be, as in the angular system, polynomials, instead of infinite series. We thus let:

$$\begin{aligned}
a &:= e^{-\Lambda x}x^s\left(a_0 + a_1x + \cdots + a_nx^n\right),\\
b &:= e^{-\Lambda x}x^s\left(b_0 + b_1x + \cdots + b_nx^n\right),\\
c &:= e^{-\Lambda x}x^s\left(c_0 + c_1x + \cdots + c_nx^n\right),\\
d &:= e^{-\Lambda x}x^s\left(d_0 + d_1x + \cdots + d_nx^n\right),
\end{aligned} \tag{74}$$

where $\Lambda$ and $s$ are two a real constants. With:

$$\nu := \frac{\sqrt{\mathbf{lr}}}{m} = \frac{m_g}{m} = \frac{m_a}{m_g}, \tag{75}$$

where $m_g$ is the geometric mean of the masses and $m_a$ their arithmetic mean. The system (73) is equivalent to:

$$\begin{aligned} 0 = &-\varepsilon\left(\ +a_0x+\cdots+a_{n-1}x^n+a_nx^{n+1}\right)-\alpha\left(a_0+a_1x+\cdots+a_nx^n\right)\\ &+i\Lambda\left(\ +a_0x+\cdots+a_{n-1}x^n+a_nx^{n+1}\right)-is\left(a_0+a_1x+\cdots+a_nx^n\right)\\ &-i\left(\ +a_1x+\cdots+na_nx^n\right)-i\kappa\left(c_0+c_1x+\cdots+c_nx^n\right)\\ &+\frac{\mathbf{r}}{m}\left(\ +d_0x+\cdots+d_{n-1}x^n+d_nx^{n+1}\right). \end{aligned} \tag{76}$$

$$\begin{aligned} 0 = &-\varepsilon\left(\ +b_0x+\cdots+b_{n-1}x^n+b_nx^{n+1}\right)-\alpha\left(b_0+b_1x+\cdots+b_nx^n\right)\\ &+i\Lambda\left(\ +b_0x+\cdots+b_{n-1}x^n+b_nx^{n+1}\right)-is\left(b_0+b_1x+\cdots+b_nx^n\right)\\ &-i\left(\ +b_1x+\cdots+nb_nx^n\right)-i\kappa\left(d_0+d_1x+\cdots+d_nx^n\right)\\ &+\frac{\mathbf{l}}{m}\left(\ +c_0x+\cdots+c_{n-1}x^n+c_nx^{n+1}\right). \end{aligned} \tag{77}$$

$$\begin{aligned} 0 = &-\varepsilon\left(\ +c_0x+\cdots+c_{n-1}x^n+c_nx^{n+1}\right)-\alpha\left(c_0+c_1x+\cdots+c_nx^n\right)\\ &-i\Lambda\left(\ +c_0x+\cdots+c_{n-1}x^n+c_nx^{n+1}\right)+is\left(c_0+c_1x+\cdots+c_nx^n\right)\\ &+i\left(\ +c_1x+\cdots+nc_nx^n\right)+i\kappa\left(a_0+a_1x+\cdots+a_nx^n\right)\\ &+\frac{\mathbf{r}}{m}\left(\ +b_0x+\cdots+b_{n-1}x^n+b_nx^{n+1}\right). \end{aligned} \tag{78}$$

$$\begin{aligned} 0 = &-\varepsilon\left(\ +d_0x+\cdots+d_{n-1}x^n+d_nx^{n+1}\right)-\alpha\left(d_0+d_1x+\cdots+d_nx^n\right)\\ &-i\Lambda\left(\ +d_0x+\cdots+d_{n-1}x^n+d_nx^{n+1}\right)+is\left(d_0+d_1x+\cdots+d_nx^n\right)\\ &+i\left(\ +d_1x+\cdots+nd_nx^n\right)+i\kappa\left(b_0+b_1x+\cdots+b_nx^n\right)\\ &+\frac{\mathbf{l}}{m}\left(\ +a_0x+\cdots+a_{n-1}x^n+a_nx^{n+1}\right). \end{aligned} \tag{79}$$

We thus obtain three kinds of systems: with index 0, with index between 0 and $n$, and with index $n$. With the zero index, the system depends only on $\alpha$, $\kappa$ and $s$:

$$\begin{gathered} 0=\left(-\alpha-is\right)a_0-i\kappa c_0;\ 0=\left(-\alpha-is\right)b_0-i\kappa d_0,\\ 0=i\kappa a_0+\left(-\alpha+is\right)c_0;\ 0=i\kappa b_0+\left(-\alpha+is\right)d_0. \end{gathered} \tag{80}$$

This system is identical to the Dirac equation case. It is made of two sub-systems. A nonzero solution is obtained only if the determinant of each sub-system is null, hence if $s$ is such that:

$$0=\left(-\alpha-is\right)\left(-\alpha+is\right)-\kappa^2;\ \kappa^2=s^2+\alpha^2;\ s=\sqrt{\kappa^2-\alpha^2}. \tag{81}$$

Letting:

$$e^{i\gamma} := \frac{s+i\alpha}{|\kappa|}, \tag{82}$$

The systems (80) give:

$$c_0 = \frac{i\alpha - s}{\kappa} a_0 = -\frac{\kappa}{|\kappa|} e^{-i\gamma} a_0;\; d_0 = \frac{i\alpha - s}{\kappa} b_0 = -\frac{\kappa}{|\kappa|} e^{-i\gamma} b_0. \tag{83}$$

With the $n$ index we obtain the following systems:

$$0 = (\varepsilon - i\Lambda) a_n - \frac{\mathbf{r}}{m} d_n;\; 0 = (\varepsilon - i\Lambda) b_n - \frac{\mathbf{l}}{m} c_n,$$

$$0 = (\varepsilon + i\Lambda) d_n - \frac{\mathbf{l}}{m} a_n;\; 0 = (\varepsilon + i\Lambda) c_n - \frac{\mathbf{r}}{m} b_n. \tag{84}$$

We thus obtain two similar sub-systems, whose determinants are transposed and have same value. A nonzero solution exists only if this value is zero:

$$0 = \begin{vmatrix} \varepsilon - i\Lambda & -\dfrac{\mathbf{r}}{m} \\ -\dfrac{\mathbf{l}}{m} & \varepsilon + i\Lambda \end{vmatrix} = \varepsilon^2 + \Lambda^2 - \nu^2,$$

$$\nu^2 = \varepsilon^2 + \Lambda^2, \tag{85}$$

Letting:

$$e^{i\delta} := \frac{\varepsilon + i\Lambda}{\nu}, \tag{86}$$

The systems in (84) are reduced to:

$$d_n = \sqrt{\frac{\mathbf{l}}{\mathbf{r}}} e^{-i\delta} a_n, \tag{87}$$

$$c_n = \sqrt{\frac{\mathbf{r}}{\mathbf{l}}} e^{-i\delta} b_n. \tag{88}$$

When $n$ is nonzero, the system with index $n-1$ reads:

$$0 = -(\varepsilon + i\Lambda) d_{n-1} + \frac{\mathbf{l}}{m} a_{n-1} + i(n + s + i\alpha) d_n + i\kappa b_n, \tag{89}$$

$$0 = -(\varepsilon + i\Lambda) c_{n-1} + \frac{\mathbf{r}}{m} b_{n-1} + i(n + s + i\alpha) c_n + i\kappa a_n, \tag{90}$$

$$0 = -(\varepsilon - i\Lambda) b_{n-1} + \frac{\mathbf{l}}{m} c_{n-1} - i(n + s - i\alpha) b_n - i\kappa d_n, \tag{91}$$

$$0 = -(\varepsilon - i\Lambda) a_{n-1} + \frac{\mathbf{r}}{m} d_{n-1} - i(n + s - i\alpha) a_n - i\kappa c_n. \tag{92}$$

Multiplying (89) by $(\varepsilon - i\Lambda)$, (92) by $\dfrac{\mathbf{l}}{m}$ and subsequently adding them, or multiplying (90) by $(\varepsilon - i\Lambda)$ and (91) by $\dfrac{\mathbf{r}}{m}$ and subsequently adding them, we obtain after simplification:

$$\left(n+s+i\alpha\right)e^{-i\delta}=\left(n+s-i\alpha\right)e^{i\delta}. \tag{93}$$

We thus obtain:

$$e^{2i\delta}=\frac{\left(n+s+i\alpha\right)^2}{\left(n+s\right)^2+\alpha^2}, \tag{94}$$

$$e^{i\delta}=\frac{\varepsilon+i\Lambda}{\nu}=\frac{n+s+i\alpha}{\sqrt{\left(n+s\right)^2+\alpha^2}}, \tag{95}$$

$$\varepsilon=\frac{\nu}{\sqrt{1+\dfrac{\alpha^2}{\left(n+s\right)^2}}}, \tag{96}$$

which is Sommerfeld's formula for the energy levels.

If $n=0$, that means when the polynomials are constant functions, the radial system is reduced to (83), (87) and (88); we thus obtain:

$$d_0=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{-i\delta}a_0=-\frac{\kappa}{|\kappa|}e^{-i\gamma}b_0, \tag{97}$$

$$c_0=\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}e^{-i\delta}b_0=-\frac{\kappa}{|\kappa|}e^{-i\gamma}a_0. \tag{98}$$

This gives the following system:

$$\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{-i\delta}a_0+\frac{\kappa}{|\kappa|}e^{-i\gamma}b_0=0, \tag{99}$$

$$\frac{\kappa}{|\kappa|}e^{-i\gamma}a_0+\sqrt{\frac{\mathbf{r}}{\mathbf{l}}}e^{-i\delta}b_0=0.$$

This system has a nonzero solution if and only if its determinant is null, and this gives:

$$0=e^{-2i\delta}-e^{-2i\gamma};\ \delta=\gamma \bmod \pi;\ \frac{\varepsilon+i\Lambda}{\nu}=\pm\frac{s+i\alpha}{|\kappa|}. \tag{100}$$

The positive sign of $\Lambda$, $\nu$, $\varepsilon$, $s$ and $\alpha$ implies that:

$$\frac{\varepsilon+i\Lambda}{\nu}=\frac{s+i\alpha}{|\kappa|}. \tag{101}$$

And we have:

$$D=\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}e^{-i\gamma}A;\ C=-e^{-i\gamma}\frac{\kappa}{|\kappa|}A;\ B=-\sqrt{\frac{\mathbf{l}}{\mathbf{r}}}\frac{\kappa}{|\kappa|}A. \tag{102}$$

If $\kappa=1$ and $\lambda=1/2$, we have:

$$U=-\sqrt{\sin\theta}\cos\frac{\theta}{2};\ V=-\sqrt{\sin\theta}\sin\frac{\theta}{2}. \tag{103}$$

With:

$$a_1 := |a_0|;\ a_1 e^{ia} := a_0, \tag{104}$$

$$\vec{\mathrm{x}} := \mathrm{x}^1\sigma_1 + \mathrm{x}^2\sigma_2 + \mathrm{x}^3\sigma_3;\ u := \frac{\vec{\mathrm{x}}}{r};\ N := \begin{pmatrix} \sqrt{\dfrac{m}{\mathbf{l}}} & 0 \\ 0 & \sqrt{\dfrac{m}{\mathbf{r}}} \end{pmatrix}, \tag{105}$$

we have:

$$X = \begin{pmatrix} UA & -VB^* \\ VC & UD^* \end{pmatrix} = -|A|\sqrt{\sin\theta}\begin{pmatrix} \cos\dfrac{\theta}{2} & \sin\dfrac{\theta}{2} \\ -e^{-i\gamma}\sin\dfrac{\theta}{2}\sqrt{\dfrac{\mathbf{l}}{\mathbf{r}}} & e^{i\gamma}\cos\dfrac{\theta}{2}\sqrt{\dfrac{\mathbf{l}}{\mathbf{r}}} \end{pmatrix} e^{ai_3}, \tag{106}$$

$$|A| = a_1(mr)^s e^{-\Lambda mr}\sqrt{\frac{m}{\mathbf{l}}}, \tag{107}$$

$$\phi = -\frac{|A|}{r}\left(\frac{1+u}{2} + \frac{1-u}{2}e^{-\gamma i_3}\right)e^{\left(a-E\mathrm{x}^0\right)i_3}N. \tag{108}$$

If $\kappa = 1$ and $\lambda = -1/2$, the radial system does not change and we obtain:

$$U = -\sqrt{\sin\theta}\sin\frac{\theta}{2};\ V = \sqrt{\sin\theta}\cos\frac{\theta}{2}. \tag{109}$$

That gives:

$$X = \begin{pmatrix} UA & -VB^* \\ VC & UD^* \end{pmatrix} = -|A|\sqrt{\sin\theta}\begin{pmatrix} \sin\dfrac{\theta}{2} & -\cos\dfrac{\theta}{2} \\ e^{-i\gamma}\sqrt{\dfrac{\mathbf{l}}{\mathbf{r}}}\cos\dfrac{\theta}{2} & e^{i\gamma}\sqrt{\dfrac{\mathbf{l}}{\mathbf{r}}}\sin\dfrac{\theta}{2} \end{pmatrix} e^{ai_3}, \tag{110}$$

$$\phi = \frac{|A|}{r}\left(\frac{1+u}{2} + \frac{1-u}{2}e^{\gamma i_3}\right)i_2 e^{\left(a-E\mathrm{x}^0\right)i_3}N. \tag{111}$$

Following the equivalence principle, which is the basis of general relativity, the gravitational mass-energy—linked to the frequency of the wave—must be equal to the inertial mass-energy, and thus to that of the wave extended to the whole space, which is the sum over all space of the energy density of the wave, because we have:

$$T_0^0 = E\frac{\mathbf{J}^0}{\hbar c};\ \mathbf{J} = \frac{m}{k\mathbf{l}}\mathrm{D}_L + \frac{m}{k\mathbf{r}}\mathrm{D}_R = \mathbf{J}^\mu\sigma_\mu, \tag{112}$$

$$k\hbar c = l_P^3;\ \mathrm{D}_R = \phi\frac{1+\sigma_3}{2}\phi^\dagger;\ \mathrm{D}_L = \phi\frac{1-\sigma_3}{2}\phi^\dagger, \tag{113}$$

where $l_P$ is the Planck length. The normalization condition with its usual form thus reads:

$$\iiint \mathrm{dv}\frac{\mathbf{J}^0}{\hbar c} = 1. \tag{114}$$

Then, this condition is used to complete the solution of the wave equation: the last constant to be calculated, $a_1$, satisfies:

$$1=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}r\, r^2\sin\theta\frac{\mathbf{J}^0}{\hbar c},$$

$$\frac{\mathbf{J}}{\hbar c}=\frac{1}{k\hbar c}\left(\frac{m}{\mathbf{l}}\mathrm{D}_L+\frac{m}{\mathbf{r}}\mathrm{D}_R\right) \tag{115}$$

We have:

$$\frac{m}{\mathbf{l}}\mathrm{D}_L+\frac{m}{\mathbf{r}}\mathrm{D}_R=\frac{1}{\nu^2}\underline{\phi}\underline{\phi}^\dagger;\underline{\phi}:=\phi N^{-1}, \tag{116}$$

$$\begin{aligned}\underline{\phi}\underline{\phi}^\dagger&=\frac{AA^*}{r^2}\left(\frac{1+u}{2}+\frac{1-u}{2}e^{-\gamma i_3}\right)\left(\frac{1+u}{2}+e^{\gamma i_3}\frac{1-u}{2}\right)\\&=\frac{a_1^2}{r^2}(mr)^{2s}e^{-2\Lambda mr}\left[1+\alpha(\sigma_3\times u)\right].\end{aligned} \tag{117}$$

We then have:

$$\frac{\mathbf{J}}{\hbar c}=\frac{a_1^2}{l_P^3\nu^2r^2}(mr)^{2s}e^{-2\Lambda mr}\left[1+\alpha(\sigma_3\times u)\right], \tag{118}$$

$$\frac{\mathbf{J}^0}{\hbar c}=\frac{a_1^2}{l_P^3\nu^2r^2}(mr)^{2s}e^{-2\Lambda mr}. \tag{119}$$

So besides the probability density, a probability current exists around the third axis, with maximal intensity in the equatorial plane ( $x^3=0$ ). This current is responsible for the magnetism of the electron. We then obtain:

$$1=\frac{a_1^2}{l_p^3\nu^2}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\sin\theta\int_0^{\infty}\mathrm{d}r(mr)^{2s}e^{-2\Lambda mr}, \tag{120}$$

$$a_1=\sqrt{\frac{\nu^2l_P^3(2\Lambda m)^{2s+1}}{4\pi\Gamma(2s+1)}};\ \frac{\mathbf{J}^0}{\hbar c}=\frac{(2\Lambda m)^{2s+1}}{4\pi\Gamma(2s+1)}r^{2s-2}e^{-2\Lambda mr}, \tag{121}$$

$$\frac{\mathbf{J}}{\hbar c}=\frac{(2\Lambda m)^{2s+1}}{4\pi\Gamma(2s+1)}r^{2s-2}e^{-2\Lambda mr}\left[1+\alpha(\sigma_3\times u)\right]. \tag{122}$$

For the other state, with $\kappa=1$, $\lambda=-1/2$ and $n=0$, we obtain similar results:

$$\frac{\mathbf{J}^0}{\hbar c}=\frac{a_1^2}{l_p^3\nu^2}m^{2s}r^{2s-2}e^{-2\Lambda mr};\ a_1=\sqrt{\frac{l_P^3\nu^2(2\Lambda m)^{2s+1}}{4\pi\Gamma(2s+1)}}, \tag{123}$$

$$\frac{\mathbf{J}}{\hbar c}=\frac{(2\Lambda m)^{2s+1}}{4\pi\Gamma(2s+1)}r^{2s-2}e^{-2\Lambda mr}\left[1-\alpha(\sigma_3\times u)\right],$$

$$\frac{\mathbf{J}^0}{\hbar c}=\frac{(2\Lambda m)^{2s+1}}{4\pi\Gamma(2s+1)}r^{2s-2}e^{-2\Lambda mr}. \tag{124}$$

## 5. Electron Clouds

The calculation of the energy levels in the case of the $\mathrm{He}^+$ ion was considered as a major success for the Bohr model, where an electron is modeled as a point particle moving in the electric field created by the double charge of the nucleus.

When the quantum wave was considered, the calculation became much more difficult, because it was also required the explanation of why two electrons in a helium atom are unable to occupy the same quantum state. This exclusion principle was moreover generalized by Pauli: the list of quantum numbers characterizing each state was proper, different for each electron of an atomic cloud, in any atom. This was accounted for by saying that the wave of a system of two electrons was the anti-symmetrical product $\psi = \psi_1\psi_2 - \psi_2\psi_1$ of the wave of each particle [17]. Even in the case of a wave function with value in $\mathbb{C}$, for nonrelativistic electrons, this induces difficulties, because the $\psi$ function becomes a function of seven variables, the three spatial coordinates of each particle plus time. And when the spin 1/2 is considered the wave function has value in $\mathbb{C}^2$ or in $\mathbb{C}^4$: it is much more complicated, because it is first necessary to define an adequate product.

We must first recall that quantum numbers were correctly obtained only from the Dirac equation, with a wave which is a function of space-time into a $\mathbb{C}^4$, or $\mathbb{R}^8$ or $Cl_3$ set, as seen in the previous sections. These states never are functions of several points in space. It is illogical to first have recourse to the Dirac wave in obtaining the true set of quantum numbers and the true energy levels, only to abruptly abandon this wave and eventually resort to another wave equation in calculating the waves of electron systems! In any case, such a calculation is never completely achieved with this "tensor product", not being truly defined.

The solution giving the quantization itself does not use only the $\phi$ wave, but considers also $\mathbf{J}^0$, the probability density, and it is the need for normalization which induces quantization. This normalization comes from a physical necessity: the equivalence principle. The physical meaning of normalization is the unity of mass-energy. When two electrons form the cloud around a helium nucleus, they also **add** their energies, at least as first approximation. The physical object which is inside an atom and participates in an electron cloud, must also allow us to understand what happens when an electron is ejected out of this cloud. Even though they are indistinguishable from one another when they are in a cloud, they are perfectly distinct when exiting or entering such a cloud. De Broglie explained (in [17]: p. 181) that the rays of a helium atom, with usual conditions, correspond with a simple excitation of one of the two electrons going out the K level. It is thus more sensible to think that something of each electron remains in an electron cloud. So we will suppose that the wave of such a cloud is simply the **sum** of the waves of each electron; calling the waves $\phi_1$ and $\phi_2$ we suppose that the wave equations of the system including these two waves are:

$$\begin{gathered} 0 = \nabla\hat{\phi}_1\sigma_{21} + qA\hat{\phi}_1 + \phi_1\mathbf{m}_1, \\ 0 = \nabla\hat{\phi}_2\sigma_{21} + qA\hat{\phi}_2 + \phi_2\mathbf{m}_2, \\ 0 = \nabla\hat{\phi}_{12}\sigma_{21} + qA\hat{\phi}_{12} + \phi_{12}\mathbf{m}_{12};\ \phi_{12} = \phi_1 + \phi_2. \end{gathered} \tag{125}$$

And we will not forget the normalization:

$$\iiint \mathrm{d}v \frac{\mathbf{J}_1^0}{\hbar c} = \iiint \mathrm{d}v \frac{\mathbf{J}_2^0}{\hbar c} = 1; \ \iiint \mathrm{d}v \frac{\mathbf{J}_{12}^0}{\hbar c} = 2. \tag{126}$$

This result of 2 means that the cloud contains two electrons. These conditions may trivially be generalized to any number of electrons in the case of atoms with more protons. The sum is commutative, $\phi_1$ and $\phi_2$ waves are indistinguishable. And since the normalization is nonlinear on the wave, we cannot obtain a wave $\phi_{12}$ in any case, except for the rare case where (125) and (126) are both satisfied: this is the Pauli principle, that we now examine. We have:

$$\frac{m}{\mathbf{l}}\mathrm{D}_L + \frac{m}{\mathbf{r}}\mathrm{D}_R = \frac{m}{\mathbf{l}}\phi\frac{1-\sigma_3}{2}\phi^\dagger + \frac{m}{\mathbf{r}}\phi\frac{1+\sigma_3}{2}\phi^\dagger,$$

$$\mathbf{J}^0 = \frac{m}{\mathbf{l}}\mathrm{D}_L^0 + \frac{m}{\mathbf{r}}\mathrm{D}_R^0 = \frac{m}{\mathbf{l}}\eta^\dagger\eta + \frac{m}{\mathbf{r}}\xi^\dagger\xi, \tag{127}$$

$$\eta_{12} = \eta_1 + \eta_2; \ \xi_{12} = \xi_1 + \xi_2, \tag{128}$$

$$\eta_{12}^\dagger\eta_{12} = \eta_1^\dagger\eta_1 + \eta_1^\dagger\eta_2 + \eta_2^\dagger\eta_1 + \eta_2^\dagger\eta_2. \tag{129}$$

Normalization leads to the definition of a product:

$$\langle \phi_1 | \phi_2 \rangle := \iiint \mathrm{d}v \left( \frac{m}{\mathbf{l}}\eta_1^\dagger\eta_2 + \frac{m}{\mathbf{r}}\xi_1^\dagger\xi_2 \right), \tag{130}$$

With this definition, (126) reads:

$$\langle \phi_1 | \phi_1 \rangle = 1; \ \langle \phi_2 | \phi_2 \rangle = 1; \ \langle \phi_{12} | \phi_{12} \rangle = 2. \tag{131}$$

This means:

$$\begin{aligned} 2 &= \langle (\phi_1 + \phi_2) | (\phi_1 + \phi_2) \rangle \\ &= \langle \phi_1 | \phi_1 \rangle + \langle \phi_1 | \phi_2 \rangle + \langle \phi_2 | \phi_1 \rangle + \langle \phi_2 | \phi_2 \rangle \\ &= 1 + \langle \phi_1 | \phi_2 \rangle + \langle \phi_2 | \phi_1 \rangle + 1, \end{aligned} \tag{132}$$

$$\langle \phi_1 | \phi_2 \rangle = -\langle \phi_2 | \phi_1 \rangle. \tag{133}$$

Thus the anti-symmetrical product required by the Pauli principle is now correctly defined as this simple product (130). Since each state of an atom with only one electron satisfies the normalization condition, these states may be added and give the electron states of each cloud.

## 6. The Photon as Subtraction

Since in the previous section we obtained the electron cloud by adding quantum waves, what happens by subtraction?

The real part of the wave equation, with its completely invariant form, is the Lagrangian density (33) (see also [5] [6] 1.9). To this Lagrangian density is associated with the energy-momentum density (35). The electric current $j = q\mathbf{J}$ is linked to the chiral currents $\mathrm{D}_L = LL^\dagger$, $L = \phi\frac{1-\sigma_3}{2}$, $\mathrm{D}_R = RR^\dagger$, $R = \phi\frac{1+\sigma_3}{2}$. In $Cl_3$, all of Maxwell's laws are reduced to (see [5] [6] A.3.6):

$$F = \nabla\hat{A}; \ \hat{F} = \hat{\nabla}A; \ \nabla\hat{F} = j, \tag{134}$$

and thus at the second order:

$$\nabla\left(\hat{\nabla}A\right)=\left(\nabla\hat{\nabla}\right)A=\Box A=\mathrm{j}=q\mathbf{J}=\frac{e}{\hbar c}\left(\frac{m}{k\mathbf{l}}\mathrm{D}_L+\frac{m}{k\mathbf{r}}\mathrm{D}_R\right). \tag{135}$$

Since $\mathbf{J}$ is a linear combination of chiral currents, we study two fields:

$$F_L:=\nabla\hat{\mathrm{D}}_L;\ \hat{F}_R:=\hat{\nabla}\mathrm{D}_R. \tag{136}$$

The left field $F_L$ satisfies

$$\begin{aligned}F_L=\vec{E}_L+i\vec{H}_L&=\left(\partial_0-\vec{\partial}\right)\left(\mathrm{D}_L^0-\vec{\mathrm{D}}_L\right)\\&=\partial_\mu\mathrm{D}_L^\mu-\partial_0\vec{\mathrm{D}}_L-\vec{\partial}\mathrm{D}_L^0+i\vec{\partial}\times\vec{\mathrm{D}}_L,\end{aligned} \tag{137}$$

and we thus obtain:

$$0=\partial_\mu\mathrm{D}_L^\mu \tag{138}$$

$$\vec{E}_L=-\partial_0\vec{\mathrm{D}}_L-\vec{\partial}\mathrm{D}_L^0;\ \vec{H}_L=\vec{\partial}\times\vec{\mathrm{D}}_L. \tag{139}$$

The right field $F_R$ satisfies

$$\begin{aligned}\hat{F}_R=-\vec{E}_R+i\vec{H}_R&=\left(\partial_0+\vec{\partial}\right)\left(\mathrm{D}_R^0+\vec{\mathrm{D}}_R\right)\\&=\partial_\mu\mathrm{D}_R^\mu+\partial_0\vec{\mathrm{D}}_R+\vec{\partial}\mathrm{D}_R^0+i\vec{\partial}\times\vec{\mathrm{D}}_R,\end{aligned} \tag{140}$$

and we thus obtain:

$$0=\partial_\mu\mathrm{D}_R^\mu \tag{141}$$

$$\vec{E}_R=-\partial_0\vec{\mathrm{D}}_R-\vec{\partial}\mathrm{D}_R^0;\ \vec{H}_R=\vec{\partial}\times\vec{\mathrm{D}}_R. \tag{142}$$

We now start from the wave equation, which gives, with $e^{i\beta}=\underline{c}+i\underline{s}$ :

$$\begin{aligned}\partial_0\eta+\sigma^1\partial_1\eta&=\sigma_2\partial_2\eta+\sigma_3\partial_3\eta-iq\left(A_0+A_1\sigma^1+A_2\sigma^2+A_3\sigma^3\right)\eta\\&\quad-i\mathbf{l}\left(\underline{c}+i\underline{s}\right)\left(\mathrm{v}_0+\mathrm{v}_1\sigma^1+\mathrm{v}_2\sigma^2+\mathrm{v}_3\sigma^3\right)\eta,\end{aligned} \tag{143}$$

$$\begin{aligned}\partial_0\xi+\hat{\sigma}^1\partial_1\xi&=\hat{\sigma}_2\partial_2\xi+\hat{\sigma}_3\partial_3\xi-iq\left(A_0+A_1\hat{\sigma}^1+A_2\hat{\sigma}^2+A_3\hat{\sigma}^3\right)\xi\\&\quad-i\mathbf{r}\left(\underline{c}-i\underline{s}\right)\left(\mathrm{v}_0+\mathrm{v}_1\hat{\sigma}^1+\mathrm{v}_2\hat{\sigma}^2+\mathrm{v}_3\hat{\sigma}^3\right)\xi.\end{aligned} \tag{144}$$

Multiplying (143) on the left side by $\eta^\dagger\sigma^1$ and (144) by $\xi^\dagger\hat{\sigma}^1$ we obtain:

$$\begin{aligned}&\eta^\dagger\sigma^1\partial_0\eta+\eta^\dagger\partial_1\eta\\&=-i\eta^\dagger\sigma^3\partial_2\eta+i\eta^\dagger\sigma^2\partial_3\eta-iq\left(A_0\mathrm{D}_L^1+A_1\mathrm{D}_L^0-iA_2\mathrm{D}_L^3+iA_3\mathrm{D}_L^2\right)\\&\quad+\left(-i\mathbf{l}\underline{c}+\mathbf{l}\underline{s}\right)\left(\mathrm{v}_0\mathrm{D}_L^1+\mathrm{v}_1\mathrm{D}_L^0-i\mathrm{v}_2\mathrm{D}_L^3+i\mathrm{v}_3\mathrm{D}_L^2\right),\end{aligned} \tag{145}$$

$$\begin{aligned}&\xi^\dagger\hat{\sigma}^1\partial_0\xi+\xi^\dagger\partial_1\xi\\&=-i\xi^\dagger\hat{\sigma}^3\partial_2\xi+i\xi^\dagger\hat{\sigma}^2\partial_3\xi-iq\left(A_0\mathrm{D}_R^1+A_1\mathrm{D}_R^0-iA_2\mathrm{D}_R^3+iA_3\mathrm{D}_R^2\right)\\&\quad+\left(-i\mathbf{r}\underline{c}-\mathbf{r}\underline{s}\right)\left(\mathrm{v}_0\mathrm{D}_R^1+\mathrm{v}_1\mathrm{D}_R^0-i\mathrm{v}_2\mathrm{D}_R^3+i\mathrm{v}_3\mathrm{D}_R^2\right).\end{aligned} \tag{146}$$

Using the adjoint, followed by addition, we obtain:

$$\begin{aligned}\partial_0\mathrm{D}_L^1+\partial_1\mathrm{D}_L^0&=i\left(\partial_2\eta^\dagger\right)\sigma^3\eta-i\eta^\dagger\sigma^3\partial_2\eta-2qA_2\mathrm{D}_L^3-2\mathbf{l}\underline{c}\mathrm{v}_2\mathrm{D}_L^3\\&\quad-i\left(\partial_3\eta^\dagger\right)\sigma^2\eta+i\eta^\dagger\sigma^2\partial_3\eta+2qA_3\mathrm{D}_L^2+2\mathbf{l}\underline{c}\mathrm{v}_3\mathrm{D}_L^2\\&\quad+2\mathbf{l}\underline{s}\left(\mathrm{v}_0\mathrm{D}_L^1+\mathrm{v}_1\mathrm{D}_L^0\right),\end{aligned} \tag{147}$$

$$\begin{aligned}\partial_0 \mathrm{D}_R^1 + \partial_1 \mathrm{D}_R^0 = {} & i\left(\partial_2 \xi^\dagger\right)\hat{\sigma}^3 \xi - i\xi^\dagger \hat{\sigma}^3 \partial_2 \xi - 2qA_2 \mathrm{D}_R^3 - 2\mathbf{r}\underline{c}\mathrm{v}_2 \mathrm{D}_R^3 \\ & - i\left(\partial_3 \xi^\dagger\right)\hat{\sigma}^2 \xi + i\xi^\dagger \hat{\sigma}^2 \partial_3 \xi + 2qA_3 \mathrm{D}_R^2 + 2\mathbf{r}\underline{c}\mathrm{v}_3 \mathrm{D}_R^2 \\ & - 2\mathbf{r}\underline{s}\left(\mathrm{v}_0 \mathrm{D}_R^1 + \mathrm{v}_1 \mathrm{D}_R^0\right).\end{aligned} \tag{148}$$

And since we have:

$$2T_{L\nu}^\mu = -i\eta^\dagger \sigma^\mu \partial_\nu \eta + i\left(\partial_\nu \eta^\dagger\right)\sigma^\mu \eta + 2\left(qA_\nu + \mathbf{l}\underline{c}\mathrm{v}_\nu\right)\mathrm{D}_L^\mu, \tag{149}$$

$$2T_{R\nu}^\mu = -i\xi^\dagger \hat{\sigma}^\mu \partial_\nu \xi + i\left(\partial_\nu \xi^\dagger\right)\hat{\sigma}^\mu \xi + 2\left(qA_\nu + \mathbf{r}\underline{c}\mathrm{v}_\nu\right)\mathrm{D}_R^\mu, \tag{150}$$

we obtain:

$$\begin{aligned}\partial_0 \mathrm{D}_L^1 + \partial_1 \mathrm{D}_L^0 &= 2T_{L2}^3 - 2T_{L3}^2 + 2\mathbf{l}\underline{s}\left(\mathrm{v}_0 \mathrm{D}_L^1 + \mathrm{v}_1 \mathrm{D}_L^0\right) \\ \partial_0 \mathrm{D}_R^1 + \partial_1 \mathrm{D}_R^0 &= 2T_{R2}^3 - 2T_{R3}^2 - 2\mathbf{r}\underline{s}\left(\mathrm{v}_0 \mathrm{D}_R^1 + \mathrm{v}_1 \mathrm{D}_R^0\right).\end{aligned} \tag{151}$$

Thus letting:

$$A := \frac{m}{k\mathbf{l}}\mathrm{D}_L - \frac{m}{k\mathbf{r}}\mathrm{D}_R, \tag{152}$$

the electric field $\vec{E}$ satisfies:

$$E^1 = -\partial_0 A^1 - \partial_1 A^0 = \frac{m}{k\mathbf{l}}\left(-\partial_0 \mathrm{D}_L^1 - \partial_1 \mathrm{D}_L^0\right) - \frac{m}{k\mathbf{r}}\left(-\partial_0 \mathrm{D}_R^1 - \partial_1 \mathrm{D}_R^0\right)$$

$$= -\frac{m}{k\mathbf{l}}\left[2T_{L2}^3 - 2T_{L3}^2 + 2\mathbf{l}\underline{s}\left(\mathrm{v}_0 \mathrm{D}_L^1 + \mathrm{v}_1 \mathrm{D}_L^0\right)\right] \tag{153}$$

$$+\frac{m}{k\mathbf{r}}\left[2T_{R2}^3 - 2T_{R3}^2 + 2\mathbf{r}\underline{s}\left(\mathrm{v}_0 \mathrm{D}_R^1 + \mathrm{v}_1 \mathrm{D}_R^0\right)\right] \tag{154}$$

$$= 2\left(V_3^2 - V_2^3\right) - \frac{2m\underline{s}}{k}\left[\mathrm{v}_0\left(\mathrm{D}_L^1 + \mathrm{D}_R^1\right) + \mathrm{v}_1\left(\mathrm{D}_L^0 + \mathrm{D}_R^0\right)\right]. \tag{155}$$

And we have:

$$\begin{aligned}&\mathrm{v}_0\left(\mathrm{D}_L^1 + \mathrm{D}_R^1\right) + \mathrm{v}_1\left(\mathrm{D}_L^0 + \mathrm{D}_R^0\right) \\ &= \mathrm{v}_0\left(\rho \mathrm{v}^1\right) + \mathrm{v}_1\left(\rho \mathrm{v}^0\right) = \rho\left(\mathrm{v}^0 \mathrm{v}^1 - \mathrm{v}^1 \mathrm{v}^0\right) = 0.\end{aligned} \tag{156}$$

Thus the electric field satisfies (with a circular permutation of indices):

$$\begin{aligned}E^1 &= 2\left(V_3^2 - V_2^3\right), \\ E^2 &= 2\left(V_1^3 - V_3^1\right), \\ E^3 &= 2\left(V_2^1 - V_1^2\right).\end{aligned} \tag{157}$$

Starting again from (145) and (146), taking adjoints and now subtracting, we obtain:

$$2T_{L0}^1 + 2T_{L1}^0 = -\partial_2 \mathrm{D}_L^3 + \partial_3 \mathrm{D}_L^2 + 2\mathbf{l}\underline{s}\left(-\mathrm{v}_2 \mathrm{D}_L^3 + \mathrm{v}_3 \mathrm{D}_L^2\right), \tag{158}$$

$$2T_{R0}^1 + 2T_{R1}^0 = -\partial_2 \mathrm{D}_R^3 + \partial_3 \mathrm{D}_R^2 + 2\mathbf{r}\underline{s}\left(\mathrm{v}_2 \mathrm{D}_L^3 - \mathrm{v}_3 \mathrm{D}_L^2\right), \tag{159}$$

$$\begin{aligned}2\left(V_0^1 + V_1^0\right) &= -\partial_2 A^3 + \partial_3 A^2 + 2m\underline{s}\left(-\mathrm{v}_2 \rho \mathrm{v}^3 + \mathrm{v}_3 \rho \mathrm{v}^2\right) \\ &= -\partial_2 A^3 + \partial_3 A^2.\end{aligned} \tag{160}$$

Next dividing by $i$ and permuting indices, we obtain:

$$H^1 = -2V_0^1 - 2V_1^0,$$
$$H^2 = -2V_0^2 - 2V_2^0, \qquad (161)$$
$$H^3 = -2V_0^3 - 2V_3^0.$$

Thus at the second order we have:

$$\Box \mathrm{D}_L = q\mathrm{D}_L;\ \Box \mathrm{D}_R = -q\mathrm{D}_R. \qquad (162)$$

Letting:

$$\mathbf{A} := \frac{m}{k\mathbf{l}}\left(\mathrm{D}_L - q\Box^{-1}\mathrm{D}_L\right) - \frac{m}{k\mathbf{r}}\left(\mathrm{D}_R + q\Box^{-1}\mathrm{D}_R\right), \qquad (163)$$

$$\mathbf{F} := \vec{\mathbf{E}} + i\vec{\mathbf{H}} = \nabla\hat{\mathbf{A}}, \qquad (164)$$

Since $q = e/\hbar c$, the field $\mathbf{F}$ is the electromagnetic field, satisfying:

$$\begin{aligned}\Box\mathbf{A} &= \hat{\nabla}\left(\nabla\mathbf{A}\right)\\ &= \Box\left[\frac{m}{k\mathbf{l}}\left(\mathrm{D}_L - q\Box^{-1}\mathrm{D}_L\right) - \frac{m}{k\mathbf{r}}\left(\mathrm{D}_R + q\Box^{-1}\mathrm{D}_R\right)\right] \qquad (165)\\ &= \frac{m}{k\mathbf{l}}\left(\Box\mathrm{D}_L - q\mathrm{D}_L\right) - \frac{m}{k\mathbf{r}}\left(\Box\mathrm{D}_R + q\mathrm{D}_R\right) = 0,\end{aligned}$$

$$\Box\mathbf{A} = 0;\ \vec{\mathbf{E}} + i\vec{\mathbf{H}} = \nabla\hat{\mathbf{A}};\ \hat{\nabla}\left(\vec{\mathbf{E}} + i\vec{\mathbf{H}}\right) = 0, \qquad (166)$$

which are the Maxwell laws.

## 7. Conclusions

The Dirac equation is not only the relativistic wave equation of a single electron; it is also the wave equation of an electron cloud, in any atom.

The photon emitted by an electron which was occupying the state $\phi_1$ and which shall occupy the lower state $\phi_2$ is exactly and simply the difference between the energy-momentum densities $V_1$ and $V_2$ of these states:

$$\mathbf{A} = \mathbf{A}_2 - \mathbf{A}_1; \vec{\mathbf{E}} + i\vec{\mathbf{H}} = \nabla\hat{\mathbf{A}};\ \hat{\nabla}\left(\vec{\mathbf{E}} + i\vec{\mathbf{H}}\right) = 0, \qquad (167)$$

which are Maxwell laws in the void. The same photon, absorbed somewhere by an electron in the state $\phi_2$, is exactly the energy-impulse necessary to allow an electron to become the state $\phi_1$.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

## References

[1] de Broglie, L. (1924) Recherches sur la Théoriedes Ouanta. 1-104.

https://fondationlouisdebroglie.org/AFLB-171/these.pdf

[2] Dirac, P.A.M. (1928) *Proceedings of the Royal Society of London*, **117**, 610-624. https://doi.org/10.1098/rspa.1928.0023

[3] Darwin, C.G. (1928) *Proceedings of the Royal Society of London*, **118**, 654-680. https://doi.org/10.1098/rspa.1928.0076

[4] de Broglie, L. (1934) L'électron magnétique. Hermann, Paris.

[5] Daviau, C., Bertrand, J., Socroun, T. and Girardot, D. (2022) Developing the Theory of Everything. 1-313. https://fondationlouisdebroglie.org/MEMOS/ToE_Eng.pdf

[6] Daviau, C., Bertrand, J., Socroun, T. and Girardot, D. (2023) Vers une unification de toutes les interactions. 1-320. https://fondationlouisdebroglie.org/MEMOS/ToE_fran.pdf

[7] Hestenes, D. (1967) *Journal of Mathematical Physics*, **8**, 798-808. https://doi.org/10.1063/1.1705279

[8] Daviau, C. (1997) *Annales de la Fondation Louis de Broglie*, **22**, 87-103.

[9] Daviau, C. (2001) *Annales de la Fondation Louis de Broglie*, **26**, 149-171.

[10] Lochak, G. (1983) *Annales de la Fondation Louis de Broglie*, **8**, 345-370.

[11] Daviau, C. (1993) Equation de Dirac non linéaire. PhD thesis, Université de Nantes, Nantes.

[12] Priem, D., Daviau, C. and Racineux, G. (2009) *Annales de la Fondation Louis de Broglie*, **34**, 103.

[13] Daviau, C., Fargue, D., Priem, D. and Racineux, G. (2013) *Annales de la Fondation Louis de Broglie*, **38**, 139-153.

[14] Costa de Beauregard, O. (1989) *Annales de la Fondation Louis de Broglie*, **14-3**, 335-342.

[15] Krüger, H. (1991) New Solutions of the Dirac Equation for Central Fields. In: Hestenes, D. and Weingartshofer, A., Ed., *The Electron*, Kluwer, Dordrecht, 49-81. https://doi.org/10.1007/978-94-011-3570-2_4

[16] Daviau, C. and Bertrand, J. (2021) *International Journal of Modern Physics*, **12**, 483-512. https://doi.org/10.4236/jmp.2021.124033

[17] de Broglie, L. (1950) La mécanique ondulatoire des systèmes de corpuscules. Gauthier-Villars, Paris.