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General Relativity and the Tully-Fisher Relation for Rotating Galaxies

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Abstract

The flat limit of rotational velocity (v_φ) approximately equal to the "edge"-velocity of a galaxy is related to the baryonic mass (M_B) via the T-F relationship $M_B \propto v_\varphi^n$ with $n \approx 4$. We explore the connection between mass and the limiting velocity in the framework of general relativity (GR) using the Weyl metric for axially-symmetric galaxies that are supported entirely by their rotational motion. While for small distances from the center, the Newtonian description is accurate as one moves beyond the (baryonic) edge of the galaxy, Lenz's law and non-linearity of the gravitational field inherent in GR not only lead to a flat velocity (obviating its Keplerian fall), but also provide its tight log-log relationship with the enclosed (baryonic) mass.

Keywords

GR, Weyl Metric, Rotating Galaxies, Flat Rotation Curves, TF Relation

1. Introduction

The *rotation curves*, *i.e.*, the rotational velocity as a function of the distance from the center do not appear to follow a Keplerian drop as $1/\sqrt{r}$; instead they tend to reach a constant plateau velocity (v_φ). This experimental fact, discovered by Vera Rubin [1] in the 1980's and confirmed by many later observations, poses one of the most challenging problems in theoretical physics. At present, the most widely accepted explanation is to suppose that the observed (luminous) mass and radius of a galaxy is only a small part of the total; the rest is a vast (spherically symmetric) distribution of hypothetical dark matter (DM), which interacts only through gravitation; this is the basis of the widely accepted ΛCDM model [2] (cosmological constant plus cold dark matter). It is important to note that ΛCDM is itself anchored upon Einstein's general relativity (GR) with a cosmological constant.

The DM model when applied to (rotating) galaxies has its problems. First of all, in spite of extensive searches no trace of this mysterious dark matter has been found. Secondly, there is an empirical but successful relation, the Opik-Tully-Fischer law [3] [4] [5], between the plateau velocity of the gas (v_φ) and the visible, hence baryonic, mass of the galaxy: ($M_{\text{baryonic}} \propto v_\varphi^4$). But if the baryonic mass is supposed to be only a few percent of the total, how does this tiny fraction determine the rotational velocity of the galaxy? (Recall that in DM, an asymptotic v_φ is generated by the *dark mass* not the baryonic mass). Thirdly, there has been no satisfactory explanation offered in DM for the magnitude of the observed (intrinsic) angular momentum (J_z) of a galaxy. By contrast, in general relativity (GR), we can compute J_z in terms of rotation velocity and the baryonic mass-current density that only extends over the visible size of any galaxy [6]. In fact, Salucci's review on DM concludes with a somber note: *It seems impossible to explain the observational evidences gathered so far in a simple dark matter framework* [7].

Building on previous work by other authors [8]-[17], and our own earlier work [18], we propose in the present paper that general relativity (GR), when appropriately applied, is perfectly capable of explaining the observed phenomena above, provided one takes into account the finite size (and a non-spherical mass distribution) of most galaxies and the basic fact that they rotate.

To be concrete, let us consider our own galaxy [19]. The Milky Way has a diameter of 25 Kilo parsec and a thickness of 2 Kilo parsec with a visible baryonic mass of about $(1 \div 2.5) \times 10^{11} M_\odot$. The considerably non-spherical geometry fixes the (stable) axis of rotation and our galaxy acquires a rotational velocity of about 200 km/sec at the edge (of the diameter). As previously noted in Ref. [18], rotations bring about a well-known but often forgotten fundamental difference between the Newtonian theory & GR.

In the Newtonian theory, *there is no dependence of the gravitational field upon the rotation of a body* [20]. In GR, on the other hand, the rotation of a system makes the metric nondiagonal (*i.e.*, the time-space component $g_{oi} \propto A_i$ becomes non-zero and a 3-vector-field A_i is generated). A *preferred* direction (in space) is thus chosen and the *sense* of rotation (clock-wise or anti-clockwise) is established and fixed. This leads to the introduction of parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$)-violating but ($\mathcal{PT}$) conserving terms. Thus, a geo-magnetic field $\boldsymbol{B} = \nabla \wedge \boldsymbol{A}$ emerges (already at the linearized level in GR) that gives rise to the GEM (geo-electromagnetic) theory of Thirring & Lense [21] [22] [23]. (The ensuing Lense-Thirring effect has been beautifully confirmed experimentally in Ref.

[24]). An intrinsic angular momentum J is generated (through the non diagonal term). These issues are discussed in detail in later sections.

Another important fact distinguishing the Newtonian theory from GR is that a non-spherical mass distribution in GR necessarily radiates gravitational waves through its quadrupole moment. In ΛCDM, by contrast, as explicitly noted by Peebles [2], *it assumes... no gravitational waves.* Neglecting gravitational radiation from non-spherical rotating galaxies that have been in existence for several billion years and that have been continuously radiating, would be an extravagant assumption. It is also important to note that (massless) gravitational waves produce no *scalar* curvature [$T(\text{gravitational wave}) = g_{\mu\nu}T^{\mu\nu}(\text{gravitational wave}) \equiv 0$], yet they contribute to the overall energy-momentum balance [$T_{\mu\nu}(\text{gravitational wave}) \neq 0$]. Also, as we shall see later, the length parameter a in the Weyl metric provides a precise relationship between the asymptotic rotation velocity and the radiation field.

The paper is organized as follows. In Section (II), we briefly discuss our previous work [18] that was anchored upon the most general class of stationary, axially-symmetric metrics in GR found by Weyl [25] [26]. In particular, we here reproduce 1) the Einstein equations valid in the vacuum (*i.e.*, outside the galaxy); 2) motion of a test particle outside of the galaxy; 3) Exact Weyl constraints in the vacuum; 4) choice of the matter energy-momentum density appropriate for a galaxy that is supported entirely by rotations with zero pressure; 5) the nature of the solutions of the Einstein equations for the matter within the galaxy and 6) obtain Ludwig's extended GEM theory from the exact Weyl metric upon truncating the scalar potential U to linear order in the constraint equations but keeping the exact non-linearity in the rotation field intact. In Section (3), we reemphasize a key role that Lenz's law plays in always boosting the rotation velocity up. In Section (4), we consider the rotation velocity and the TF law. In Section (5), we show that both the Weyl class of metrics and the Kerr metric possess an intrinsic angular momentum. It is worthy of note that the Schwarzschild metric has zero (intrinsic) angular momentum simply because it is spherical and thus lacks a vector field fixing a direction in space. A simple phenomenological analysis using an analytic, factorized mass density is applied to obtain the rotation velocity and the intrinsic angular momentum for our own galaxy and compared with experimental data in Section (7). The paper concludes in Section (8) with a summary of results obtained, work in progress and future prospects.

2. The Weyl Metric

In various subsections below, we list results relevant for the present paper from our previous work [18]:

1) Einstein equations for the Weyl metric outside the galaxy;

2) Motion of a test particle outside of the galaxy;

3) Exact Weyl constraints in the vacuum;

4) Matter energy-momentum density for galaxies supported entirely by rota-

tions with zero pressure;

5) Einstein equation constraints within the galaxy;

6) Ludwig's extended GEM theory results from the Einstein equations.

2.1. Weyl Metric and Einstein Equations outside the Galaxy

The axially-symmetric Weyl metric for a cylindrically symmetric space-time [27], with coordinates (ct,φ,ρ,z), including explicitly the rotation term (see, for example [20]) may be written as:

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}\left(c\mathrm{d}t - a\mathrm{d}\varphi\right)^2 + \mathrm{e}^{-2U}\rho^2\mathrm{d}\varphi^2 + \mathrm{e}^{2\nu-2U}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right), \tag{2.1}$$

$$g_{\mu\nu} = \begin{pmatrix} -\mathrm{e}^{2U} & \mathrm{e}^{2U}a & 0 & 0 \\ \mathrm{e}^{2U}a & -\mathrm{e}^{2U}a^2 + \mathrm{e}^{-2U}\rho^2 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{2\nu-2U} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2\nu-2U} \end{pmatrix};$$

$$g = \det g_{\mu\nu} = -\mathrm{e}^{4\nu-4U}\rho^2;$$

the inverse metric has the form:

$$g^{\mu\nu} = \begin{pmatrix} \frac{\mathrm{e}^{2U}a^2}{\rho^2} - \mathrm{e}^{-2U} & \frac{\mathrm{e}^{2U}a}{\rho^2} & 0 & 0 \\ \frac{\mathrm{e}^{2U}a}{\rho^2} & \frac{\mathrm{e}^{2U}}{\rho^2} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{2U-2\nu} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2U-2\nu} \end{pmatrix};$$

and the invariant (spatial) volume element reads

$$\begin{aligned} &\mathrm{d}V = \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z\sqrt{-g} = \mathrm{e}^{-2(U-\nu)}\left(\rho\mathrm{d}\rho\mathrm{d}z\mathrm{d}\varphi\right); \\ &\mathrm{d}V \ge \mathrm{d}V_{flat} \end{aligned} \tag{2.2}$$

Below, we list some salient aspects of the above axially-symmetric metric:

- 1: U, a & ν are functions only of $\rho = \sqrt{x^2 + y^2}$ and *z*. independent of φ. Hence, there are two Killing vectors; one time-like and the other space-like (outside of the horizon) of the system.
- 2: The function U is related to the Newtonian potential Φ through $\mathrm{e}^{2U} = 1 + 2\frac{\Phi}{c^2}$.
- 3: The function *a* would be related to the angular momentum of the system.
- 4: The gravito-magnetic potential-field $A_\phi = \frac{ca}{\rho}$, is a vector potential $\boldsymbol{A} = \left(0, \frac{ca}{\rho}, 0\right)$.
- 5: The three potential fields (U, a & ν) characterizing the metric are not all independent. The Einstein equations in the vacuum, that is outside the boundaries of a confined system such as a galaxy, impose the following *exact* non-linear differential constraints on these functions [20]:

$$
\begin{aligned}
&R_{\mu\nu}=0;\ \text{in the vacuum of the system implies:}\\
&\frac{\partial^2 U}{\partial\rho^2}+\frac{\partial U}{\rho\partial\rho}+\frac{\partial^2 U}{\partial z^2}=-\frac{e^{4U}}{2\rho^2}\left[\left(\frac{\partial a}{\partial\rho}\right)^2+\left(\frac{\partial a}{\partial z}\right)^2\right];\text{(i)}\\
&\frac{\partial}{\partial z}\left(\frac{e^{4U}}{\rho}\frac{\partial a}{\partial z}\right)+\frac{\partial}{\partial\rho}\left(\frac{e^{4U}}{\rho}\frac{\partial a}{\partial\rho}\right)=0;\text{(ii)}\\
&\text{and }\frac{\partial\nu}{\rho\partial\rho}=\left[\left(\frac{\partial U}{\partial\rho}\right)^2-\left(\frac{\partial U}{\partial z}\right)^2\right]-\frac{e^{4U}}{4\rho^2}\left[\left(\frac{\partial a}{\partial\rho}\right)^2-\left(\frac{\partial a}{\partial z}\right)^2\right];\text{(iii)}\\
&\frac{\partial\nu}{\rho\partial z}=2\frac{\partial U}{\partial\rho}\frac{\partial U}{\partial z}-\frac{e^{4U}}{2\rho^2}\frac{\partial a}{\partial\rho}\frac{\partial a}{\partial z};\text{(iv)}
\end{aligned}
\tag{2.3}
$$

N.B.: Since U and a begin at order G, ν begins at second order (*i.e.*, is of order G^2). Once U & a satisfy the top two equations relating them Equation (2.3(i), (ii)), a solution for ν exists since the last two equations Equations (2.3(iii), (iv)) become the integrability conditions for it; $\nu\to 0$ as $\rho\to 0$ for any z.

- 6: The inequality in Equation (2.2) that tells us that the invariant spatial volume element is larger than its value in the flat-limit is useful for proving bounds on integrals of (positive definite) integrands, in gravitational asymptotic perturbation theory such as that developed by Landau-Lifshitz [27] & by Weinberg [6].

2.2. Motion of a Test Particle outside the Galaxy

A test particle in this axially symmetric metric would have two constants of motion, that we indicate as $p_0=E/c$ for time translations, $p_\phi=J/c$ for rotational motion in the x-y plane. We shall write $E=\gamma mc^2$, or $E=mc^2+\mathcal{E}_{NR}$ to study the non-relativistic limit.

We now write the geodesic equation for a test particle of mass m for the above metric. The simplest formalism that extends to a Riemannian space blessed with a metric is through the action principle. Calling the action S, m the mass and τ the proper time τ, we have

$$
\begin{aligned}
&\mathrm{d}S=-\left(mc^2\right)\mathrm{d}\tau;\ \left(\mathrm{d}S\right)^2=\left(mc\right)^2\left(c\,\mathrm{d}\tau\right)^2;\\
&\text{Let } p_\mu=\frac{\partial S}{\partial x^\mu};\text{Hamilton-Jacobi Eqn. implies: } g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}=-\left(mc\right)^2;\\
&\text{We have } p_\mu p_\nu g^{\mu\nu}=-\left(mc\right)^2
\end{aligned}
\tag{2.4}
$$

As stated earlier, an axially symmetric system has two conserved quantities: the energy E and the component of angular momentum J_z say, for rotational motion in the xy-plane. Hence, the dependence on time-interval (t) and that on φ can be prescribed as

$$
\begin{aligned}
&S\left(ct;\rho;\varphi;z\right)=-Et+J\varphi+\hat{S}\left(\rho;z\right);\\
&-\frac{\partial S}{\partial ct}=E/c;\ \frac{\partial S}{\partial\varphi}=J;\ \frac{\partial S}{\partial\rho}=p_\rho;\ \frac{\partial S}{\partial z}=p_z
\end{aligned}
\tag{2.5}
$$

Hence, for the Weyl metric, we have

$$\begin{aligned}&(mc)^2=\left(\frac{E}{c}\right)^2\left[\mathrm{e}^{-2U}-\left(\frac{a}{\rho}\right)^2\mathrm{e}^{2U}\right]-2\frac{a}{\rho}\frac{J}{\rho}\frac{E}{c}\mathrm{e}^{2U}-\left(\frac{J}{\rho}\right)^2\mathrm{e}^{2U}-\mathrm{e}^{2(U-\nu)}\left[p_\rho^2+p_z^2\right];\text{(i)}\\&\left(\frac{E}{c}\right)^2\mathrm{e}^{-2U}-\left[\frac{J}{\rho}+\frac{a}{\rho}\frac{E}{c}\right]^2\mathrm{e}^{2U}=(mc)^2+\mathrm{e}^{2(U-\nu)}\left[p_\rho^2+p_z^2\right];\text{(ii)}\\&\text{Or}:\left[\frac{E}{c}\left\{\mathrm{e}^{-U}+\frac{a}{\rho}\mathrm{e}^{U}\right\}+\frac{J}{\rho}\mathrm{e}^{U}\right]\left[\frac{E}{c}\left\{\mathrm{e}^{-U}-\frac{a}{\rho}\mathrm{e}^{U}\right\}-\frac{J}{\rho}\mathrm{e}^{U}\right]=(mc)^2+\mathrm{e}^{2(U-\nu)}\left(p_\rho^2+p_z^2\right);\text{(iii)}\end{aligned}\tag{2.6}$$

Let $E=mc^2\gamma$ and as both E & J are constants of motion, we can define a reduced (a-dimensional) angular momentum, *i.e.*, angular momentum per unit energy per unit ρ (the perpendicular distance or, the impact parameter): $j\equiv Jc/E\rho$; and through it a *rotational* velocity $v_\varphi\equiv jc$. Similarly, the rotational parameter *a* from the metric, can be employed to define a *vector potential*: $A_\varphi\equiv ca/\rho$ that has the dimensions of a velocity. With these definitions, Equation (2.6(ii)) reads:

$$\begin{aligned}&J=\rho\frac{E}{c}j;\ v_\varphi=cj;\ a=\rho\frac{A_\varphi}{c};\ \pi_\varphi\equiv v_\varphi+A_\phi;\\&\gamma^2\left[\mathrm{e}^{-2U}-\left(\frac{\pi_\varphi}{c}\right)^2\mathrm{e}^{2U}\right]=1+\mathrm{e}^{2(U-\nu)}\frac{p_\rho^2+p_z^2}{(mc)^2}\end{aligned}\tag{2.7}$$

For a galaxy supported totally by rotations along φ, that is the focus of this paper, we set $p_z=0$ & $p_\rho=0$. Then the above equation is reduced to

$$\begin{aligned}&\gamma=\frac{1}{\sqrt{\mathrm{e}^{-2U}-\left(\pi_\varphi/c\right)^2\mathrm{e}^{2U}}};\\&\text{Keeping leading terms only}:\gamma\approx\frac{1}{\sqrt{1-2U-\left(\pi_\varphi/c\right)^2}};\\&\text{Test particle energy}:E=\gamma\left(mc^2\right)\approx mc^2+\mathcal{E}_{NR};\\&\mathcal{E}_{NR}=m\Phi+\frac{m}{2}\pi_\varphi^2;\ \pi_\varphi=v_\varphi+A_\varphi;\text{(ii)}\end{aligned}\tag{2.8}$$

Equation (2.8(ii)) shows clearly what the Newtonian theory leaves out that GR supplies: viz., the vector potential A_φ. That in turn generates the GEM magnetic field. The lack of the dynamics generated by mass current density in the Newtonian theory is a serious lacuna that has important consequences. We discuss one such important improvement that GR provides.

As $U<0$, the particle will remain bound so long as $\left|v_\varphi+A_\varphi\right|<\sqrt{-2\Phi}$ and not $v_\varphi<\sqrt{-2\Phi}$ (their values at the coordinates ρ,z in question) as the Newtonian theory asserts.

This leads to the well known quandary when one computes, using Newtonian gravity, the escape velocity of our Sun were it to escape from our Galaxy. The mean rotational velocity of our Sun is about 220 km/sec and it is approximately 8.2 Kilo-parsec away from the center of our Galaxy. There is apparently very little (baryonic) mass beyond this distance. Thus, Newtonian theory for the Sun's escape velocity predicts $\sqrt{2}\times220\approx310$ km/sec [28] in the vicinity of our Sun, experimental astrophysicists estimate the Sun's escape velocity to be between (500 ÷ 550) km/sec.

In GEM, by contrast, the escape velocity reads: $v_{escape} \approx -A_\varphi + \sqrt{-2\Phi}$. As we shall discuss later in more detail, Lenz's law (reminding us that all masses attract so that the GEM magnetic field obeys the *left hand rule*) forces us to have $A_\varphi < 0$, thus boosting the escape velocity up [*vedi* Section (3)]. From the phenomenology of the Milky Way in Section (7), we estimate the magnetic term to add about 200 km/sec, thereby bringing the escape velocity much closer to its estimated experimental value. A quantitative analysis of this matter shall be presented in a later work.

2.3. Exact Weyl Constrains in the Vacuum

Having delineated a few important aspects that distinguish GR from the Newtonian theory regarding the dynamics of a rotation-supported galaxy, let us return to a discussion of the exact Weyl constraints.

At first glance, Equations (2.3(i-iv)) appear quite opaque and daunting, but they acquire a physically more appealing aspect through the following *dictionary* in terms of the GEM electric $\boldsymbol{E}$ & magnetic $\boldsymbol{B}$ fields of order *G*, along with a higher order field $\hat{\boldsymbol{B}}$ that is of order G^2. They are defined as follows:

$$
\begin{aligned}
&\boldsymbol{E} = \left(E_\rho, 0, E_z\right) = \left(-\frac{\partial\Phi}{\partial\rho}, 0, -\frac{\partial\Phi}{\partial z}\right) = -\nabla\Phi; \text{(i)}\\
&\boldsymbol{B} = \left(B_\rho, 0, B_z\right) = \left(-\frac{\partial A_\varphi}{\partial z}, 0, \frac{\partial A_\varphi}{\partial\rho}\right) = \nabla\wedge\boldsymbol{A}; \text{(ii)}\\
&\hat{\boldsymbol{B}} = \left(\hat{B}_\rho, 0, \hat{B}_z\right) = \left(-\frac{1}{\rho}\frac{\partial\nu}{\partial z}, 0, \frac{1}{\rho}\frac{\partial\nu}{\partial\rho}\right); \text{(iii)}\\
&\text{Thus, we have}: \hat{\boldsymbol{B}}^2 = \frac{1}{\rho^2}\left(\nu_\rho^2 + \nu_z^2\right); \text{(iv)}\\
&\& \ -\rho\left(\nabla\wedge\hat{\boldsymbol{B}}\right)_\varphi = \nu_{\rho\rho} + \nu_{zz} - \frac{1}{\rho}\nu_\rho; \text{(v)}
\end{aligned}
\tag{2.9}
$$

Before considering the equations they obey, let us pause to say a few words about the genesis of the nomenclature in Equation (2.9). This EM analogy was first noticed and Equations (2.9(i) and (ii)) were used by Thirring. His initial purpose was to compute the gravitational field inside a hollow rotating sphere (in linearized GR). Later with Lense, he extended the analysis of the effect of proper rotation of a central body on the motion of other celestial bodies, which led to the discovery of the Lense-Thirring effect [24]. In a set of four beautiful papers, Ludwig [13] [14] [15] [16] has extended GEM by including additional field energy (that are second order in *G*) and obtained a closed set of non-linear equations for the rotational velocity (v_φ) in terms of the Newtonian velocity (via its acceleration) and the matter distribution within the galaxy. We shall return to discuss them in a later subsection and show that indeed they are reproduced in the appropriate limit.

In terms of the field variables defined in Equation (2.9), the Weyl equations in the vacuum given in Equation (2.3) read:

$$\begin{aligned}
&\nabla\cdot\boldsymbol{E} = -\frac{2}{c^2}\mathrm{e}^{-2U}\boldsymbol{E}^2 + \frac{c^2}{2}\mathrm{e}^{6U}\boldsymbol{B}^2;\text{(i)}\\
&\nabla\wedge\boldsymbol{B} = -\frac{4}{c^2}\left(\boldsymbol{E}\wedge\boldsymbol{B}\right);\text{(ii)}\\
&\hat{\boldsymbol{B}}_\rho = \frac{\rho}{c^2}\left[E_z^2 - E_\rho^2\right] + \frac{\mathrm{e}^{4U}}{4}\left[B_\rho^2 - B_z^2\right];\text{(iii)}\\
&\hat{\boldsymbol{B}}_z = 2\frac{\rho}{c^2}E_\rho E_z + \frac{e^{4U}}{2}B_\rho B_z;\text{(iv)}
\end{aligned} \tag{2.10}$$

Within the galaxy, the *Gauß law* in Equation (2.10(i)) shall get the mass density term on the right-hand side ($-4\pi G\rho_m$). Similarly the *Ampere law* in Equation (2.10(ii)) shall get the mass current density ($-4\pi G\rho_m v_\varphi$) when we continue the solution within the galaxy. On the other hand, Equations (2.10(iii) and (iv)) remain valid both inside and outside of the galaxy, due to our choice of the matter energy-momentum density as discussed in a later subsection.

The various exponentials in these expressions add on higher order polynomials in the Newtonian potential due to the non-linearity of GR. In all the four equations above, the quadratic terms in $\boldsymbol{E}$ & $\boldsymbol{B}$ appear; these are easily interpretable as different components of the field energy-momentum density.

An attentive reader might wonder how (& why) one can possibly succeed in describing the dynamics of a spin-2 gravitational field in terms of just the GEM-electric and magnetic (spin-1 vector) fields? The answer to this question lies in the non-linearity of GR. Already at the second order (in G), there are constraints between the $\boldsymbol{E}$-field (whose longitudinal part is defined through the gradient of the Newtonian potential Φ and whose transverse part arises through the time derivative of the transverse part of the vector potential, $\partial\boldsymbol{A}_T/\partial t$) and there are constraints between them, *vedi* Equations (2.3(i) and (ii)). Further on, at order G^2, a subsidiary field ν appears in the metric as well as in the equations of motion, that is completely constrained by the behavior of the GEM fields and the boundary condition that $\nu(\rho=0;z)\equiv 0$. Thus, in the far field region, once the origin is appropriately chosen, the gravitational field is limited to its two degrees of freedom and its multipole expansion beginning with the quadrupole. Not so, in the near field within or in the vicinity of the galaxy where both longitudinal and transverse fields are present with constraints between them playing a crucial role in limiting the dynamics, as the following discussion illustrates.

The assumption that there is no motion along the (radial) ρ-direction or along the z-direction, brings in constraints for the dynamical system. Weinberg's Equation (9.12) [6] gives the following expression for a particle's (spatial) acceleration $\mathcal{A}^i$ ($i=2,3,4$ with coordinates labeled as x^μ: ($x^1=ct$, $x^2=\varphi$; $x^3=\rho$; $x^4=z$)

$$\begin{aligned}
\mathcal{A}^i = &-\Gamma^i_{1,1} - 2\Gamma^i_{1,j}\frac{\mathrm{d}x^j}{\mathrm{d}t} - \Gamma^i_{j,k}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}\\
&+\frac{\mathrm{d}x^i}{\mathrm{d}t}\left[\Gamma^1_{1,1} + 2\Gamma^1_{1,j}\frac{\mathrm{d}x^j}{\mathrm{d}t} + \Gamma^1_{j,k}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}\right];
\end{aligned} \tag{2.11}$$

Assuming only circular motion (about the z-axis), we have non-vanishing velocity only along the φ-axis: $\mathrm{d}\varphi/\mathrm{d}t = v/\rho$ and $\mathrm{d}x^i/\mathrm{d}t = 0$ for $i = 3,4$. Under this premise, also the accelerations along the 3- & 4-axes must vanish:

$$\begin{aligned}
&\text{(i) } \mathcal{A}^{\rho} = -c^2 e^{4U-2\nu} U_{,\rho} + c e^{4U-2\nu} \frac{v}{\rho}\left[a_{,\rho} + 2aU_{,\rho}\right] \\
&\qquad - e^{-2\nu}\left(\frac{v}{\rho}\right)^2 \left[-\rho + e^{4U} a a_{,\rho} + \rho^2 U_{,\rho} + e^{4U} a^2 U_{,\rho}\right] = 0; \\
&\text{(ii) } \mathcal{A}^{z} = -c^2 e^{4U-2\nu} U_{,z} + c e^{4U-2\nu} \frac{v}{\rho}\left[a_{,z} + 2aU_{,z}\right] \\
&\qquad - e^{-2\nu}\left(\frac{v}{\rho}\right)^2 \left[\rho^2 U_{,z} + e^{4U} a^2 U_{,z} + e^{4U} a a_{,z}\right] = 0;
\end{aligned} \tag{2.12}$$

Equations (2.12) along with Equations (2.3(i, ii)) allow us to obtain an exact non-linear, first order differential equation for the velocity field $\beta(\rho, z=0) = v(\rho, z=0)/c$ on the equatorial plane in terms of the (normalized dimensionless) Newtonian (velocity squared) defined as usual $g(\rho) = (\rho/c^2)(\partial\Phi(\rho,0)/\partial\rho)$, where $\Phi(\rho,0)$ is the Newtonian potential in the equatorial plane. This rather complicated expression can be found in Appendix A of [18]. Here we shall illustrate the strategy employed to derive the result valid to the lowest non-vanishing order. To the desired order of accuracy, Equations (2.12), yield the following expressions for a,ρ & $a_{,z}$:

$$\frac{a,\rho}{\rho} = -\frac{\beta}{\rho} + \left(\frac{1}{\beta} + \beta\right)\frac{\Phi_{,\rho}}{c^2}; \quad \frac{a,z}{\rho} = +\left(\frac{1}{\beta} + \beta\right)\frac{\Phi_{,z}}{c^2}; \tag{2.13}$$

We can thus eliminate $a_{,\rho}; a_{,z}$ in Equation (2.3(ii)), to obtain an expression for the second derivatives of *U*. To the desired order of accuracy:

$$\left[e^{4U}\left(\frac{1}{\beta} + \beta\right)U_{,z}\right]_{,z} + \left[e^{4U}\left\{-\frac{\beta}{\rho} + \left(\frac{1}{\beta} + \beta\right)U_{,\rho}\right\}\right]_{,\rho} = 0; \tag{2.14}$$

Keeping only terms linear in the *U*-field:

$$\left(\frac{1}{\beta} + \beta\right)\left[U_{,\rho,\rho} + U_{,z,z}\right] = \frac{1-\beta^2}{\beta^2}\beta_{,z}U_{,z} + \frac{1-\beta^2}{\beta^2}\beta_{,\rho}U_{,\rho} - \frac{\beta}{\rho^2} + \frac{\beta_{,\rho}}{\rho}; \tag{2.15}$$

Thus:

$$\begin{aligned}
&U_{,\rho,\rho} + U_{,z,z} + \frac{U_{,\rho}}{\rho} \\
&= \frac{1-\beta^2}{\beta(1+\beta^2)}\beta_{,z}U_{,z} - \frac{\beta^2}{\rho^2(1+\beta^2)} + \frac{\rho\beta_{,\rho}}{\rho^2}\frac{\beta^2 + (1-\beta^2)g(\rho,z)}{\beta(1+\beta^2)} + \frac{g(\rho,z)}{\rho^2};\text{(i)}
\end{aligned} \tag{2.16}$$

According to Equation (2.3(i)), lhs is of order G^2, outside the galaxy. Thus, to linear order in G, we have at $z = 0$ upon using the up-down symmetry, for the rate of increase of $\beta(\rho)$ (outside the galaxy)

$$\rho\frac{\partial\beta}{\partial\rho} = \beta\frac{\beta^2 - g(\rho)(1-\beta^2)}{\beta^2 + g\left(\rho(1+\beta^2)\right)}; \tag{2.17}$$

Equation (2.17) is of course only valid outside the galaxy. It agrees exactly with Ludwig's Equation (4.13) [13] when his solution is continued to outside the galaxy where the matter density term $f = 0$.

It is easy to obtain the rate equation inside the galaxy (to linear order) upon including the matter density term on the rhs of Equation (2.3(i)). To lowest order, the (2-dimensional) Laplacian of U receives the matter field contribution ($4\pi G\rho_m$). Explicitly, inside the galaxy, we have

$$\nabla^2 U(\rho,z) = \frac{4\pi G\rho_m(\rho,z)}{c^2} + \text{terms of order } G^2;$$

$$\text{Define for } z=0;\ f(\rho) = \frac{4\pi G\rho_m(\rho,z=0)\rho^2}{c^2};$$

$$\text{Equation (2.14(i))} \ \rightarrow (f-g) + \frac{\beta^2}{1+\beta^2} = \frac{1}{\beta(1+\beta^2)}\rho\frac{\partial\beta}{\partial\rho}\left[\beta^2 + g(1-\beta^2)\right];$$

$$\rho\frac{\partial\beta}{\partial\rho} = \beta\frac{\beta^2 + (1-\beta^2)(f-g)}{\beta^2 + g(1+\beta)^2} \tag{2.18}$$

This essentially reproduces Ludwig's result inside the galaxy and reduces to Equation (2.17) outside the galaxy for which $f = 0$.

2.4. Matter Energy-Momentum Density

Within the boundaries of the galaxy, the dynamics of course changes:

$$E_{\mu\nu}(\rho,z) = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.19}$$

and thus we need a model for the energy-momentum density of the rotating galaxy and a choice for the metric inside. Hoping that no confusion ensues, we shall continue to use the same form of the metric as given in Equation (2.1). The simplest and most commonly used model for matter is that of *free dust* with in general an equation of state relating the mass density to the pressure. We shall assume further that our galaxy has zero-pressure, which implies that it is *totally* supported by rotations around its stable axis, with no further extraneous motion. Choosing the axis of rotation along the z-axis (with an angular velocity $\dot{\varphi}$), our extreme simplifying assumptions, allow us to restrict the matter energy-momentum density to the following form [with coordinates (o,φ,ρ,z)]:

$$\begin{aligned}
&T^{\mu\nu} = \rho_m u^\mu u^\nu;\\
&u^\mu(\rho,z) = \gamma c\left(1,\frac{\beta}{\rho},0,0\right);\\
&u_o = -\gamma c e^{2U}\left[1-\beta\frac{a}{\rho}\right]; u_\varphi = \gamma c\left[e^{2U}a\left(1-\beta\frac{a}{\rho}\right) + \beta\rho e^{-2U}\right]; u_\rho = 0; u_z = 0;\\
&\text{The trace}: T_\mu^\mu = -\rho_m c^2 \Rightarrow \frac{1}{\gamma^2} = \left(1-\beta\frac{a}{\rho}\right)^2 e^{2U} - \beta^2 e^{-2U}
\end{aligned} \tag{2.20}$$

While lack of motion along the ρ (radial) & z (vertical) directions simplify the structure of the matter energy-momentum density tensor from a (4 × 4) matrix to a (2 × 2) matrix form, this simplification also brings some unexpected peculiarities such as:

- 1: Even though the reduced matrix $T_{\mu\nu}$ is real-Hermitean, it is non-diagonal and because it is factorizable its determinant is zero. We recall that in the general case, this matrix has 4 eigenvalues: a positive definite (time-like) mass density with 3 (space-like) pressures (p_1, p_2, p_3 along its principal axes). By setting all pressures p_i to zero, we have made the matrix *singular* with the lone non-vanishing eigenvalue the scalar (generally invariant) mass density $\rho_m c^2$.
- 2: For any finite β, the Lorentz factor γ in Equation (2.20) does not reduce to its expected value $\left(1-\beta^2\right)^{-1/2}$, unless the rotation parameter $a \to 0$. But, if we let $a = 0$, the metric becomes *diagonal*, since then $g_{o\varphi} = 0$ thereby rendering the (matter + field) angular-momentum zero. Clearly, this is unphysical and thus unacceptable. We must have $a \neq 0$ (it can be positive or negative, of course).
- 3. In the expression for γ, the linear term in β induced by a non-vanishing length parameter $a \neq 0$, would exceed the expected β^2 correction unless *for any value of* $\rho \le \rho_{edge}$ *within the galaxy*, $2\left|a(\rho)/\rho\right| < \beta(\rho)$. In short, β cannot be too small if the rotational velocity alone has to support a galaxy with zero internal pressure.
- 4: The metric and its first derivatives must be matched at the boundary for their inside versus outside values.

Thus, β just outside cannot be too small either. A clear indication from GR that Newtonian values for β that are becoming too small at the edge must get supplemented by (the mass current density) contributions to stabilize the system.

2.5. Einstein Constraints within the Galaxy

To emphasize the affinity and the difference between Einstein gravity and electromagnetism, and partly to follow the works by Ludwig [13] [14] [15] [16], it is convenient to write the Einstein equations for this metric in terms of the three vectors $\boldsymbol{E}, \boldsymbol{B}, \hat{\boldsymbol{B}}$ defined earlier. Overall we have a dictionary with which we can write the Einstein equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.21}$$

We have:

$$\begin{aligned} R = g^{\mu\nu} R_{\mu\nu} &= \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} \\ &= \mathrm{e}^{2U-2\nu}\left(2\nabla^2 U + \frac{\mathrm{e}^{4U}}{\rho^2}\left(a_{,\rho}^2 + a_{,z}^2\right) - 2\left(\nu_{,\rho,\rho} + a_{,zz} + U_{rho}^2 + U_{,z}^2\right)\right) \\ &= \mathrm{e}^{2U-2\nu}\left(-2\frac{\mathrm{e}^{-2U}}{c^2}\nabla\cdot\boldsymbol{E} - 4\frac{\mathrm{e}^{-4U}}{c^4}\boldsymbol{E}^2 + 16\frac{\mathrm{e}^{4U}}{c^2}\boldsymbol{B}^2 + 2\rho\left(\nabla\wedge\hat{\boldsymbol{B}}\right)_{\varphi} + \hat{B}_{\rho}\right) \end{aligned}$$

and therefore a "Gauß law"

$$\nabla\cdot\boldsymbol{E} = -4\pi G\rho_m \mathrm{e}^{2\nu}\left(1+\mathrm{e}^{-2U}\left(\beta\gamma\right)^2\right) - 2\frac{\mathrm{e}^{-2U}}{c^2}\boldsymbol{E}^2 + 8\mathrm{e}^{6U}\boldsymbol{B}^2 + \rho c^2\mathrm{e}^{2U}\left(\nabla\wedge\hat{\boldsymbol{B}}\right)_\varphi - \frac{1}{2}c^2 e^{2U}\hat{B}_z \tag{2.22}$$

To single out the non-diagonal part of $E_{\mu\nu}$ in terms of the matter current density $J_m = \rho_m v_\varphi$, we consider the combination

$$\begin{aligned} aE_{ctct} + E_{ct\varphi} &= \frac{8\pi G}{c^2}\left(aT_{ctct} + T_{ct\varphi}\right) = \frac{8\pi G}{c^2}\left(-\left(J_m\right)_\varphi \gamma^2\rho\left(1-\frac{a}{\rho}\beta\right)\right) \\ &= -\frac{1}{2}\mathrm{e}^{4U-2\nu}\left(a_{,\rho,\rho} + a_{,z,z} - \frac{1}{\rho}a_{,\rho} + 4\left(a_{,\rho}U_{,\rho} + a_{,z}U_{,z}\right)\right) \\ &= \frac{2\rho}{c}\mathrm{e}^{4U-2\nu}\left(\left(\nabla\wedge\boldsymbol{B}\right)\right) - 4\left(E\wedge\boldsymbol{B}\right)_\varphi\right) \end{aligned} \tag{2.23}$$

and therefore an "Ampère law" emerges:

$$\nabla\wedge\boldsymbol{B} = \frac{4\pi G}{c}\mathrm{e}^{-4U+2\nu}\left(-\boldsymbol{J}_m\gamma^2\left(1-\frac{a}{\rho}\beta\right)\right) + \frac{c}{2\rho}\mathrm{e}^{-4U+2\nu}\boldsymbol{E}\wedge\boldsymbol{B} \tag{2.24}$$

In Appendix B of Ref. [18], we have reproduced some details of the traditional iterative scheme in GR (developed over a century ago). Anyone interested can readily compare the higher order contributions as they arise from the perturbative scheme with the exact Einstein-Weyl equations.

2.6. Ludwig's Extended GEM Theory Results from Einstein Equations

Neglecting higher order term in G and (special) relativistic corrections, we can summarize Gauß and Ampère law as:

$$\nabla\cdot\boldsymbol{E} = -4\pi G\rho_m, \quad \nabla\wedge\boldsymbol{B} = -\frac{4\pi G}{c}\boldsymbol{J}_m \tag{2.25}$$

It is important to note (and very useful to remember to implement) the negative sign of the matter fields on the rhs of Equations (2.25), especially in the Ampére law that leads to a *left hand rule* for the GEM magnetic field. Precisely because gravitation has only attraction (unlike E & M that has both), the Lenz's law for gravity implies that there is a net boost to the acceleration due to other masses. We illustrate in Section (3) that the model obeying Lenz's law produces a rotation velocity curve consistent with mass-to-luminosity data whereas another model while successful in producing the rotation curve was inconsistent with the light intensity data.

3. Lenz's Law Always Boosts Rotational Velocities for Stable Galaxies

An attentive reader might rightly wonder why there is always a counter rotating GEM magnetic field produced by the velocity-field of material masses. Such is not always the case in Maxwellian electrodynamics due to the fact that both attractive and repulsive forces are generated as both positive and negative charges exist in the electro-magnetic theory of Maxwell. In GEM however, the force is always attractive [29] [30]. For the problem at hand, it is most easily seen in the equation for the GEM magnetic field

$$\nabla \times \boldsymbol{B} = -\frac{4\pi G}{c^2}\rho \boldsymbol{v} + \frac{\partial \boldsymbol{E}}{c^2 \partial t} \tag{3.26}$$

The minus sign in the first term on the right hand side of Equation (6.6) tells us that the magnetic field induced on the left side (due to the velocity field) follows the *left-hand rule* always. In standard electrodynamics with different signs of charge, Lenz's law implies that a negatively charged electron in a beam of co-moving electrons loses momentum due to other negatively charged electrons in the beam. On the other hand, the same Lenz's law implies that an electron gains momentum if there are say positively charged parallel moving protons. In GEM, there is only attraction between masses and thus the situation is similar to that between an electron and a proton. Ergo, Lenz's law implies that there is always an increase in the rotational velocity of galaxies due to GEM. In the following, we shall confirm these results explicitly that the resultant rotational velocity is indeed boosted through a GEM magnetic term $B_z < 0$. We may consider it as a strict *boundary condition* to be imposed for the stability of a galaxy that is supported entirely by rotations.

We have shown in Ref. [18] that the model obeying Lenz's law produces a rotation velocity curve consistent with mass-to-luminosity data whereas another model while successful in producing the rotation curve was inconsistent with the light intensity data.

The example of galaxy NGC 1560 has been discussed at length by Ludwig in Ref. [13] using two different parametrizations, we shall call them model 1 & model II with two different Newtonian g-functions g_I & g_{II}. These are shown in Figure 1 of Ref. [18]. They both produce roughly the same $\beta(\rho)$. The GEM magnetic field is defined as

$$\frac{B_z}{c} = \frac{g(\rho) - \beta^2}{\beta\rho}. \tag{3.27}$$

For model I, $B_z > 0$ and for model II, $B_z < 0$. In Figure 2 of Ref. [18], are shown the magnetic fields, B_{z_I} for model I and $-B_{z,\text{II}}$ for model II. Lenz's law is not obeyed in model I but it is in model II. In Figure 3 of Ref. [18], are shown the corresponding Newtonian velocities

Ludwig's model II obeys Lenz's law and at the same time is also consistent with the mass-to-luminosity data, whereas model I does not agree with the mass-to-luminosity data. This shows the efficacy of Lenz's law in limiting the class of acceptable solutions.

4. Rotation Velocity and the Tully-Fisher Law

As discussed in Section (3), the induced GEM magnetic field $\boldsymbol{B}$ is always counter-rotating (follows the left hand rule) with respect to velocity-field of material masses that produce it. Also, as shown earlier, the Einstein-Weyl equations acquire the form of Gauß-like and Ampère-like laws, even at the linearized level.

Upon assuming that $\boldsymbol{A}_g = A_\varphi \hat{\varphi}; \boldsymbol{v} = v\hat{\varphi}$ and that we are in stationary conditions, the equations (in cylindrical coordinates) read [13]:

$$\begin{aligned} &\phi_g = \frac{\Phi}{c^2}; \\ &\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(r\frac{\partial\phi_g}{\partial\rho}\right) + \frac{\partial^2\phi_g}{\partial z^2} = \nabla^2\phi_g = 4\pi G\rho_m; \\ &\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial\rho}\right) + \frac{\partial^2 A_\varphi}{\partial z^2} = \frac{4\pi G}{c^2}\rho_m v \end{aligned} \tag{4.1}$$

The assumption is that $v(\rho, z)$ describes continuously the motion of the rotating matter inside the galaxy and the motion of the ionized gas that circles round it. While the geodesic equations for the (spatial) acceleration of a particle $\mathcal{A}^i$ have been shown to be non-linear and complicated, however, we here limit our discussion and consider only equatorial circular motion around the z-axis with $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{v}{\rho}$ and $\mathcal{A}^\rho = \mathcal{A}^z = 0$. Under these provisions, to lowest order we have the Lorentz force equations:

$$\begin{aligned} &\frac{\partial\Phi}{\partial\rho} - \frac{v^2}{\rho} = \frac{v}{\rho}\frac{\partial(ca)}{\partial\rho}, \ \frac{\partial\Phi}{\partial z} = \frac{v}{\rho}\frac{\partial(ca)}{\partial z} \leftrightarrow E_z - vB_\rho = 0; \ E_\rho + vB_z = -\frac{v^2}{\rho}; \\ &\text{Define, a magnetic velocity term}: \beta_{mag} \equiv \frac{\rho(-B_z)}{c} \geq 0; \\ &\text{Thus, with } g \text{ the Newtonian velocity squared}: \beta^2 = g + \beta\beta_{mag} \geq g; \text{(i)} \\ &\beta = \frac{1}{2}\left[\beta_{mag} + \sqrt{4g + \beta_{mag}^2}\right]; \text{(ii)} \end{aligned} \tag{4.2}$$

Thus, as we proposed to show in Section (1), GR with its inherent Lenz's law does indeed produce the remarkable result that the rotational velocity always exceeds its Newtonian value: [$\beta^2 \geq g$ Equation (4.2(i))].

The above inequality is a powerful constraint that has been amply confirmed through 2700 data points from 153 SPARC galaxies. For details, we refer the reader to Ref. [5], especially Figure 3 in it.

We have also shown that up to the order of required accuracy, Ludwig's rate equations for the rotation velocity emerge from the Weyl metric, thereby giving strong support to Ludwig's computational program. We shall return to it in Section (6).

We reproduce here from Ref. [18], a simple qualitative argument for constant asymptotic velocity deduced from these equations, with a Newtonian term augmented by the magnetic term. At small distances from the center, the Newtonian term dominates but as one proceeds further towards the edge of the galaxy, the picture changes dramatically due to the onset of the magnetic term.

If we consider our own galaxy, the Newtonian velocity has roughly speaking two bumps and then it goes down in the Keplerian fashion as $1/\sqrt{\rho}$. If we simply add a magnetic term that begins from zero and grows up near the edge to produce a constant (negative) vector potential A_φ in obedience to the Lenz's law, we have the desired result of a constant rotational velocity. Also, the same asymptotically constant vector potential allows us to obtain a reasonable estimate both for the rotation velocity & the angular momentum of our galaxy.

For our galaxy, the maximum of the Newtonian term coincides approximately with the onset of asymptotic velocity, $\beta^2(\infty)=\frac{R_s}{2R_{edge}}$, where the Schwarzschild radius $R_s=2GM/c^2$ with M denoting the baryonic mass (plus that of the gravitational field). For a pillbox like galaxy, $V=\pi R_{edge}^2 h$, $M=\rho_m V$, so that $\beta^2(\infty)\sim\frac{M}{M^{1/2}}\sim M^{1/2}$, reproducing the Tully-Fisher law: $M\propto\beta^4$. As we change the geometry, we expect that $M\propto\beta^n$ with $n=4\pm0.5$. Thus, while the scatter in the index n is to be expected, the log-log relationship would be maintained.

We note that the mass parameter here refers to the baryonic mass (+that of the GR radiation). A linear light-to mass ratio $\Upsilon^*\approx0.5M_\odot/L_\odot$ is employed for all disk galaxies in Ref. [5] to convert the light data from 3.6 μm band of Spitzer into mass. Thus, to use the terminology in Ref. [7], we are discussing the baryonic TF relation.

5. Weyl Class of Metrics & the Particular Kerr Metric

We briefly note here the similarities and differences between the large distance behaviour of the Weyl class of metrics to the particular one of the Kerr solution of the Einstein equations [20]. This solution apparently describes a rotating black hole in terms of a mass M and a (constant) length parameter a that is known to be linearly related to its angular momentum.

Taking $\hat{z}$ as axis of rotation, $g_{\mu\nu}\equiv\eta_{\mu\nu}+h_{\mu\nu}$, at large distance, the Kerr metric asymptotic behaviour is given by ([6] pg. 240):

$$h_{ij}\to-\frac{R_s}{r^3}x_i x_j,\quad h_{0i}\to\frac{R_s}{r^2}\left(x_i+\frac{1}{r}(\boldsymbol{a}\wedge\boldsymbol{x})_i\right),\quad R_s\equiv\frac{2GM}{c^2},\ \boldsymbol{a}=(0,0,a),\ i,j=1,2,3 \tag{5.3}$$

As amply discussed in an Appendix in Ref. [18], this is quite generally all that one needs to calculate the total mass and angular momentum. For the Kerr metric (5.3), it yields $E_{tot}=Mc^2$, $\boldsymbol{J}=Mc\boldsymbol{a}$, as expected. If $a=0$ the Kerr metric coincides with the Schwarzschild metric and $J=0$. We can see that for the system to have a finite intrinsic angular momentum, it is crucial that the space-time part of $h_{\mu\nu}$ does not vanish asymptotically beyond $1/r^2$.

Let us now consider the general class of Weyl's axially-symmetric metrics as in Section (2) focusing on their space-time part in the equatorial plane (*i.e.*, at $z=0$ so that $\rho=r$) and we have:

$$
\begin{aligned}
&g_{o\varphi}(r) = \frac{a(r)}{c}\mathrm{e}^{2U(r)}, \text{can be written in pseudo-Euclidean} \\
&\text{coordinatesas the special case of } g_{oi} = \varepsilon_{ijk} a_j x_k \frac{\mathrm{e}^{2U}}{r^2}; \\
&\text{with Weyl's being the special case } \boldsymbol{a} = (0,0,a); \\
&\text{Expanding in perturbation theory:} \\
&g_{oi} = g_{oi}^{(1)} + g_{oi}^{(2)} = \varepsilon_{ijk}\frac{a_j x_k}{r^2}\left[1+2U(r)+\cdots\right], \\
&\text{with } g_{oi}^{(1)} = \varepsilon_{ijk}\frac{a_j x_k}{r^2}; \& \ g_{oi}^{(2)} = \varepsilon_{ijk}\frac{a_j x_k}{r^2}2U(r)
\end{aligned}
\tag{5.4}
$$

We are interested in the second part ($g_{oi}^{(2)}$) that relates to the angular momentum ($\boldsymbol{J}$) of the system. Asymptotically, we have (*vedi* Ref. [6]) for the second term,

$$
g_{oi}^{(2)} = \frac{2G}{r^3}(\boldsymbol{r}\times\boldsymbol{J})_i; \ \text{ we find } J_z = Mca \tag{5.5}
$$

exactly the same as that for the Kerr metric provided we associate the (constant) Kerr length parameter *a* with the (asymptotic) Weyl length parameter *a*.

The implication is that a finite value of the total (material + that of the gravitational field) angular momentum of the galaxy requires that the rotational velocity asymptote to a constant value and *vice versa*.

A mental picture of what is happening may be formed through the following rough guide about the Weyl parameter *a*. For small *r*, *a* increases from zero linearly until the edge, beyond which, while continuous at the edge, it eventually becomes a constant. At very large *r*, as expected the GEM magnetic field ($-B_z \to 1/r$), as all radiation fields do.

6. Ludwig's Non-Linear Differential Equation for the Velocity Field

While in Section (4) Equation (4.2) we have tried to keep our equations *linear* by keeping both the Newtonian and the magnetic contributions at the same level, the strategy followed by Ludwig [13] (see also Ref. [16] [17]) has been to eliminate the magnetic term entirely, at the expense of course of ending up with a non-linear equation for the velocity field. Below we follow his formalism to pinpoint a few aspects.

As stated in the last paragraph, we can use Equation (4.1) to eliminate A_φ from the expression of the Ampère law, that becomes

$$
\frac{\partial}{\partial\rho}\left(\frac{1}{v}\frac{\partial\phi}{\partial\rho}-\frac{v}{\rho}\right)+\frac{\partial}{\partial z}\left(\frac{1}{v}\frac{\partial\phi}{\partial z}\right)=\frac{4\pi G}{c^2}\rho_m v. \tag{6.6}
$$

This equation multiplied by v and subtracted from the expression of Gauß' law given earlier, eliminates the double derivatives and yields:

$$
4\pi G\rho_m\left(1-\frac{v^2}{c^2}\right)=\left(\frac{1}{\rho}+\frac{1}{v}\frac{\partial v}{\partial\rho}\right)\frac{\partial\phi_g}{\partial\rho}+\frac{1}{v}\frac{\partial v}{\partial z}\frac{\partial\phi_g}{\partial z}+v\frac{\partial}{\partial\rho}\frac{v}{\rho} \tag{6.7}
$$

a non linear first order differential equation for $v(\rho, z)$ for given $\rho(\rho, z)_m$, $\Phi_g(\rho, z)$. In the equatorial plane $z = 0$ by the up-down symmetry we can drop the $\frac{\partial \phi_g}{\partial z}$; then:

$$\left(\beta^2 + \rho\frac{\partial\varphi}{\partial\rho}\right) r\frac{\partial\beta}{\partial\rho} = \frac{\beta}{\rho}\left(\left(\beta^2 - r\frac{\partial\varphi}{\partial\rho}\right) + \frac{4\pi G\rho_m}{c^2}\rho^2\left(1-\beta^2\right)\right);$$

$$\beta = \frac{v(\rho,0)}{c}, \varphi = \frac{\phi_g}{c^2}$$

Outside the galaxy, where $\rho(\rho,0)_m = 0$, the equation becomes

$$\frac{\rho}{\beta}\frac{\partial\beta}{\partial\rho} = \frac{\beta^2 - \rho\frac{\partial\varphi}{\partial\rho}}{\beta^2 + \rho\frac{\partial\varphi}{\partial\rho}} = \frac{\beta^2 - g(\rho)}{\beta^2 + g(\rho)} \tag{6.8}$$

This equation shows the key role played by the GEM magnetic field, that is now:

$$\frac{B_z}{c} = \frac{\rho\frac{\partial\varphi}{\partial\rho} - \beta^2}{\beta\rho} = \frac{g(\rho) - \beta^2}{\beta\rho}. \tag{6.9}$$

Equation (6.8) is an elegant rate equation for the velocity outside the galaxy. However, in any phenomenology, care must be taken to ensure that the GEM magnetic field employed (*vedi* Equation (6.9)) $B_z < 0$ is indeed negative. A realistic example confirming this fact has already been provided in Section (3).

7. Rotation Velocity and Angular Momentum for the Milky Way

Our own galaxy the Milky Way is presumably the one we ought to know the best and yet it is most arduous to discuss it realistically given its rings and spiral arms that belie our assumption of axial symmetry as its structure in no way can be considered independent of the angle φ[1]. In the Weyl formalism under consideration in this paper, rings and spiral arms can occur due to instabilities generated by the motion of the interstellar medium (ISM). See, for example, Ref. [31]. Following a hollowed theoretical custom, presently we shall ignore these as of no consequence and proceed with confidence that the Einstein theory with an extended Weyl metric and a pressure-less source is applicable to it and we shall be satisfied if our description is even approximately successful for the angular momentum and rotational velocity of this massive bar like object in terms of its known diameter (about 25 kpc); thickness (about 2 kpc) and its baryonic mass; that is, use only the *visible* part of the galaxy in trying to understand it. After all, we do feel less guilty in our maneuvers in that we are not assuming that our galaxy consists of a vast (over an order of magnitude more massive) amount of *unseen* dark matter (of *unknown* origin) spread out (over a radius of 380 kpc)

[1]To say nothing of a massive black hole of mass ($4\times10^6 M_\odot$) and a Schwarzschild radius (1.2×10^7 km) that exists at the center of the galaxy.

rotating with perfect *spherical symmetry* obeying *Newtonian* mechanics [32].

To begin our phenomenology, we need an input mass density $\rho_m(\rho, z)$ that describes the bulge, the disk and a co-rotating gas surrounding it, a Newtonian potential and the corresponding Newtonian (squared, normalized) velocity $g(\rho, z=0)$ generated from it and an estimate of its baryonic mass (M). Unfortunately, there is less than unanimity as to what this mass is: Allen's astronomical data lists $M_{galaxy} = 1.4\times10^{11} M_{\odot}$ [19]; Trimble quotes $M_{galaxy} = 1\times10^{11} M_{\odot}$ [33]; Nagai-Miyamoto estimate it to be about $2.567\times10^{11} M_{\odot}$ [34]; Lipovka estimate is $2.3\times10^{11} M_{\odot}$ [35]; Sofue obtains for the bulge and the disk mass $M_{b+d} = 7.9\times10^{10} M_{\odot}$, however this analysis also has a DM halo mass of $2.23\times10^{11} M_{\odot}$ (within a DM halo radius $h \sim 22$ kpc) [32]. It is important to note that the Sofue estimates include the DM component to the regular baryonic bulge and disk components in fitting the galaxy rotation curve at *small* distances. The total fraction of baryons from WMAP cosmic value is 17% [36], it is estimated to be 12% as the mean for a group of galaxies, whereas for our own galaxy it is only 5.9% of DM considered spread out up to 380 kpc (chosen arbitrarily as the half-distance between our and M31 galaxy nearby [32].

In view of the above uncertainties, we limited our task to the following: Assume a baryonic mass density $\rho_m(\rho, z)$ spread out only over the visible domain of our galaxy (roughly 25 kpc in diameter and 2 kpc in thickness) whose Newtonian potential provides a reasonable description of the rotation velocity including the two visible bumps in the velocity along with the expected Keplerian fall-off at larger distances. We computed using the GR formalism described in the text: the total mass M (baryonic + radiation); the total angular momentum J and the rotation velocity. As we have stressed, the continuity constraints in GR imply that the *magnetic* contribution that keeps the velocity up at larger distances cannot be ignored since it is related to the Newtonian term. Thus followed the simple illustrative example described in detail in Ref. [18]. We here briefly summarize the results obtained therein.

We chose a convenient analytic (& factorizable) mass distribution due to Lipovka [35] so as to facilitate our computations of the total mass, angular momentum and the Newtonian velocity vs. distance. The geo-magnetic velocity has been chosen to asymptote to a constant as discussed in the text. In units of kpc, it reads

$$V_{mag}(\rho) = \frac{160}{3.09\times10^{16}} \frac{\rho}{\rho+70}, \tag{7.10}$$

where ρ is in units of kpc.

The modified velocity is

$$V_{mod}(\rho) = \frac{V_{mag}(\rho) + \sqrt{4V_{\perp}^2(\rho) + V_{mag}^2(\rho)}}{2}. \tag{7.11}$$

Figure 1 shows the modified velocity. For the angular momentum, we obtain

$$J \simeq 1.155\times10^{67}\,(\text{Joules}\cdot\text{sec});\quad J_{mod} \simeq 1.193\times10^{67}\,(\text{Joules}\cdot\text{sec}). \tag{7.12}$$

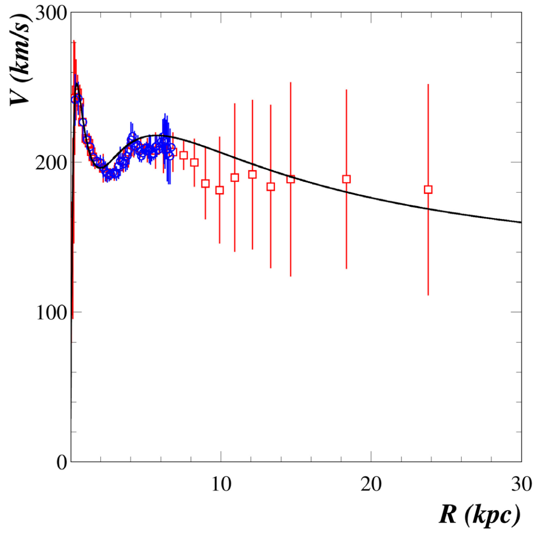

Figure 1. We show rotation velocity for our Milky Way using Equation (7.10).

This estimate can be compared to Trimble's estimate [33] of the angular momentum 6×10^{66} (Joules-sec), obtained using *only the disk* part of the Milky Way.

8. Conclusions & Future Prospects

We first summarize results obtained previously, then describe research in progress and close with prospects for the future.

- Our work began with the Weyl class of axisymmetric metrics in GR for whom solutions to the Einstein-Weyl equations in the vacuum are known in terms of a few differential equations. Even more fortunately, for what we call the extended Weyl class that includes rotations explicitly, exact differential equations are also known.
- Unlike the Kerr metric, Weyl metric can be easily (and has been) continued within the galaxy and physically meaningful results obtained.
- Armed with exact solutions, it became possible to show how Gauß and Ampère laws emerged and under what conditions Ludwig's extended GEM theory and his non-linear rate equations for the rotation velocity field could be deduced.
- Using the century old iterative procedure in GR and further elaborated by Weinberg, we could discuss the value of the mass M (baryonic mass + that of the gravitational field) & that of the intrinsic angular momentum J of a rotationally-supported galaxy. The extended Weyl metric analysis allowed us to conclude rigorously that Weyl's (vectorial) length parameter a must have a finite limit to obtain a finite J. As the same parameter also controls the asymptotic limit of the rotation velocity, we can conclude that GR is indeed capable of obtaining a flat plateau in the rotation velocity.
- We have attempted an alternative strategy to that of Ludwig as far as the phenomenology of the rotation curves is concerned. Ludwig eliminated the

magnetic contribution to obtain his non-linear rate equation for the velocity field in terms of the input from the Newtonian potential and the mass distribution within the galaxy. Instead, we kept the Newtonian input & the magnetic input together; thus our velocity equations remained linear. This allowed us to provide a clearer physical picture: at small distances, the velocity is basically described by the Newtonian term and as it begins to fall off it is supported near the edge by essentially a constant vector potential. It also brought to focus the crucial role of Lenz's law and the left hand rule for the GEM magnetic field.

- As byproducts of our analysis, we were able to deduce a few other practical results: 1) imposition of Lenz's law implies the rigorous inequality: $\beta^2 \geq g$, the Newtonian value. A result supported by 2700 data points from 153 rotating galaxies; 2) a better estimate (≥500 km/sec.) for our Sun's escape velocity from our galaxy; 3) an easy to remember pneumonic for the asymptotic velocity $\beta^2 \approx R_s/(2R_{edge})$; 4) how Tully-Fisher law emerges from a rotating *pill-box* galaxy; 5) simple dimensional analysis implies $J \propto M^{7/4}$ if Tully-Fischer holds.

Our present focus is four-fold: A: A satisfactory GR description of the deflection of light from large galaxies & from galaxy clusters; B: To obtain a better understanding of the TF-law ($M \propto \beta^4$) and the Virginia Trimble law ($J \propto M^{1.9}$), the latter covering data that run over 50 orders of magnitude [33]; C: A comprehensive phenomenology of the rotation curves with realistic densities and more refined Newtonian inputs; D: Testing our conjecture that spiral arms in rotating galaxies such as ours are generated dynamically through non-linear effects inherent in GR.

Further let us hope for yet more brilliant advances in astrophysical observations (for example, via renewed investigations involving Hanbury-Brown-Twiss techniques) to reduce the error bars in rotation curves. Only then, it would be feasible to truly distinguish between different theoretical models.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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