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Emergence of Space-Time and Gravitation

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ABSTRACT

In relativistic quantum mechanics, elementary particles are described by irreducible unitary representations of the Poincaré group. The same applies to the center-of-mass kinematics of a multi-particle system that is not subject to external forces. As shown in a previous article, for spin-1/2 particles, irreducibility leads to a correlation between the particles that has the structure of the electromagnetic interaction, as described by the perturbation algorithm of quantum electrodynamics. The present article examines the consequences of irreducibility for a multi-particle system of spinless particles. In this case, irreducibility causes a gravitational force, which in the classical limit is described by the field equations of conformal gravity. The strength of this force has the same order of magnitude as the strength of the empirical gravitational force.

Keywords: Conformal Gravity; Quantum Gravity; Emergence of Space-Time

1. Introduction

As a general rule of relativistic quantum mechanics, not only elementary particles, but also compound systems of particles are described by irreducible unitary representations of the Poincaré group, as long as no external forces act on the system.

Within a two-particle state, irreducibility of the representation that describes the center-of-mass kinematics, causes a correlation of the individual particle momenta. In a previous article [1], the author has shown that for spin-1/2 particles, the quantum mechanical formulation of this correlation takes on the structure of the electromagnetic interaction, as described by the perturbation algorithm of quantum electrodynamics. The coupling constant, derived from the geometrical properties of this correlation, was found to be in excellent agreement with the experimental value of the electromagnetic fine-structure constant. This agreement emphasizes the crucial role of irreducibility for the kinematics of quantum mechanical multi-particle systems.

Irreducible representations of the Poincaré group are labeled by the values of two Casimir operators P and W (see, e.g., [2])

$$P = p_\mu p^\mu, \tag{1}$$

where p_μ is the total 4-momentum of the system, and

$$W = -w^\mu w_\mu, \text{ with } w_\sigma = \frac{1}{2}\varepsilon_{\sigma\mu\nu\lambda}M^{\mu\nu}p^\lambda, \tag{2}$$

which refers to the angular momentum of the system.

Whereas the previous article was primarily based on the first Casimir operator P, the present article will concentrate on the second Casimir operator W. This operator is related to the intrinsic angular momentum of the two-particle system, generated by the relative motion of the particles.

Let p_1 and p_2 be the 4-momenta of two particles, for simplicity with equal masses m, so that

$$p = p_1 + p_2 \tag{3}$$

denotes the total momentum and

$$q = p_1 - p_2, \tag{4}$$

the relative momentum. Then p and q satisfy

$$pq = 0. \tag{5}$$

Based on Equation (5), a two-particle system can be described by a total momentum p and a spacelike momentum q, perpendicular to the timelike vector p. "Perpendicular to a timelike vector" means that q is allowed to rotate by the action of an SO(3) subgroup of the Lorentz group. So the kinematics of the relative momentum is restricted to a 3-dimensional subspace of space-time.

For an irreducible two-particle representation, we obtain from the constancy of the Casimir operator P

$$p^2 = (p_1 + p_2)^2 = M^2, \tag{6}$$

where M is the "mass" of the two-particle system, and

$$q^2 = 4m^2 - M^2 \le 0, \qquad (7)$$

and further

$$2p_1 p_2 = M^2 - 2m^2 \qquad (8)$$

and

$$2p_1 p = 2p_2 p = M^2. \qquad (9)$$

Equations (8) and (9) correlate the particle momenta by fixing the angle between them and with respect to the total momentum p. Rotations with rotational axis p preserve these angles. Since these rotations leave p invariant, they can be related to an independent, internal degree of freedom, described by an action of SO(2) on the relative momentum q.

In the quantum mechanical description, this SO(2) symmetric degree of freedom corresponds to the internal angular momentum of the two-particle system. In an irreducible representation, the second Casimir operator W has a well defined value, which requires that also the value of this angular momentum is well-defined. From the quantum mechanics of angular momentum, we know that for large quantum numbers the property of spherical symmetry does not fade away, but is preserved in the sense of an SO(2) symmetry. In [3] we find an approximation to the spherical harmonics for large angular momenta l, which for $m = l$ results in the probability distribution

$$\int \left| Y_{ll}(\theta, \phi) \right|^2 \mathrm{d}\phi \cong A\left(\sin(\theta)^{2l} \right), \qquad (10)$$

describing a clearly defined closed circular orbit.

Therefore, within an irreducible two-particle representation of the Poincaré group, the existence of a classical "Newtonian" limit, where the relative motion of the individual particles is straight and uniform, must be put into question.

In the sense of Newton's first law [4], *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*, a circular orbit is to be understood as the result of an attractive force between the particles. Such a force of apparently universal character has not been seen in the experiments of particle physics—or perhaps, for some reason, it has been ignored.

This article is intended to find out more about this force, which obviously is the outcome of a combination of quantum mechanics and relativistic invariance.

2. Parameter Space-Time vs. Physical Space-Time

Our analysis starts with a review of the role of space-time within the formalism of quantum mechanics.

Given an elementary particle, described by an irreducible representation of the Poincaré group in a state space with eigenstates $|p\rangle$ of the 4-momentum p, then states "in space-time" can be defined by superposing these momentum eigenstates:

$$|\boldsymbol{x}, t\rangle = (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}^3 \boldsymbol{p}}{2|p_0|} e^{-\frac{ipx}{\hbar}} |p\rangle, \ k = 1, 2, 3, \qquad (11)$$

with parameters $x = (\boldsymbol{x}, t)$. A detailed discussion of these states can be found in [5]. See also [6].

The parameters x form a parameter space (x space) with the same metric as the energy-momentum space (p space). The states $|\boldsymbol{x}, t\rangle$ are "localized" (within a Compton wave length) at time $t = x_0$ at the point $\boldsymbol{x}$ of three-dimensional space. So we can say that the x space has also a "physical" meaning in the sense that it is a space in which (isolated) particles can be physically placed.

Note that Definition (11) does not require a prior existence of space-time. It rather defines *space-time* on the basis of the momentum eigenstates. We also define a position operator in three-dimensional space by

$$X_k = -i\hbar \frac{\partial}{\partial p^k}. \qquad (12)$$

The definition of a corresponding "time" operator does not make sense, because the states (11) cannot be "localized in time." Therefore, time is not an observable, but merely a parameter. By Definition (11), space-time is derived as a property of matter, just as momentum is considered a property of matter.

The relation between x space and p space contains Planck's constant h. This is the result of having independent scales for x and p. We can avoid this constant by replacing p by the *wave vector* k, defined by $p = \hbar k$, which in this context may be a more natural choice.

Now consider two elementary particles, described by an irreducible two-particle representation of the Poincaré group. Because of the constraints from the two-particle mass shell relation Equation (6), it is not possible to simultaneously construct localized states for each particle. Therefore, when two or more particles are considered, the possibility of individually placing the particles in x space may be lost, but x space still can serve as a useful parameter space, e.g., for *wave functions*. So we have to be careful not to mix up parameter space-time with physical space-time. In the following, physical space-time will be understood in the sense of the *expectation value* of the position operator of Equation (12).

As a pure mathematical construct, (parameter) x space is not limited by any "physical" scale, such as the Planck length. So it does not make sense to try its

"quantization" at Planck scales, in the hope of finding a road to quantum gravity. On the other hand, physical space-time is quantized right from the beginning, as it has been defined by the expectation values of the position operator. This means that the classical concept of space-time may break down at scales where quantum effects become noticeable, and this happens not at Planck scales, but already at atomic scales.

There is a wide-spread opinion that the difficulties of a quantum theory of gravitation result from the fact that quantum mechanics is defined on space-time, while in quantum gravity, this very space-time continuum "must be quantized." This opinion, obviously, does not make the necessary distinction between parameter space-time and physical space-time.

In contrast to parameter space-time, physical space-time has a natural scale. A scale is, e.g., given by the Bohr radius

$$\frac{\hbar}{c m_e \alpha} \tag{13}$$

of the electron in a hydrogen atom. This scale is determined by the electromagnetic interaction, which in [1] was shown to be a property of the irreducible representations of the Poincaré group, and by the electron mass m_e. So this mass takes over the role of the (hypothetical) Planck mass in characterizing a "smallest length."

3. Geometry of Physical Space-Time

Within an irreducible two-particle representation, the motion of the particles relative to each other is determined by a well-defined angular momentum. The associated Casimir operator W is a constant of the motion. Quantum mechanics describes this angular momentum in (parameter) space-time by spherical functions, which in the limit of large quantum numbers describe probability distributions with the shape of circular orbits.

The circular orbits of a quasi-classical two-particle system, resulting from a well-defined angular momentum, can be described by the semi-classical expression

$$x_i p_j - x_j p_i = n\hbar \tag{14}$$

or by

$$p_t = \frac{n\hbar}{r}, \tag{15}$$

where p_t is the momentum in the tangential direction. In words, the tangential momentum is proportional to the curvature of the orbit. Since there are no external forces to keep the particles on these orbits, we are led to the alternative interpretation that physical space-time, in contrast to parameter space-time, has in general a curved metric.

This curvature is not obtained by an active defor-

mation of a predefined space-time continuum, but by the *ab initio* construction of (physical) space-time from (an entangled superposition of) momentum eigenstates within an irreducible two-particle representation. Viewed in this way, it appears more or less trivial that the distribution of energy-momentum in space-time should be reflected in the metric of physical space-time, and it would be surprising if it were not.

The connection between energy-momentum and space-time is given by the factor $e^{-ipx/\hbar}$ in the states (11). This factor is invariant under two simultaneous conformal transformations

$$x \to \lambda^{-1} x \tag{16}$$

and

$$p \to \lambda p. \tag{17}$$

By these transformations, not only parameter space-time, but also physical space-time, are subjected to a scaling that changes any probability distribution in space-time by a scaling factor λ^{-1}, but keeps the form of this distribution invariant. The symmetry defined by these transformations means that the linear size of a structure in space-time is inversely proportional to the magnitude of the energy-momentum that defines this structure. Accordingly, a curvature associated with this structure is directly proportional to the energy-momentum.

This especially applies to the curvature of the quasi-classical orbit of two particles within an irreducible representation of the Poincaré group. Following Newton's first law, we can describe this orbit as the result of a force that acts perpendicularly to the velocity vectors of the particles. This force generates a space-like linear momentum perpendicular to their actual velocities. (Remember that the kinematics of the relative momentum is a matter of a 3-dimensional subspace of p space). Such a momentum is described by the momentum flux $T^{ik}, i \neq k \; (i, k = 1, 2, 3)$ of the energy-momentum tensor $T^{\mu\nu}$. The diagonal elements T^{ii} obviously do not contribute to the centripetal force. Therefore, the deviation of the particles' kinematics from a straight uniform motion can, in principle, be deduced from the traceless part of the energy momentum tensor. (Although Lorentz transformations may transform the components of T^{ik} into T^{00} and T^{ii}, these transformations leave the trace of $T^{\mu\nu}$ invariant). Because the total linear momentum is conserved, the second particle must contribute a flux of linear momentum that is opposed to the flux of the first. Metaphorically speaking, both particles exchange momentum.

With this in mind, we now try to express the centripetal forces by a non-Euclidean metric of space-time. Consequently, we have to look for a relation between the curvature of space-time and the traceless part of the energy-momentum tensor, as the "cause" of the curvature.

(Einsteinian gravity, which was set up with the goal of replacing Newtonian gravity, uses the trace of the energy-momentum tensor instead. Both approaches are in a sense complementary, as far as spherically symmetric solutions are concerned [7]). According to what has been said above about conformal scaling, the curvature must be proportional to the scaling of the momentum. Therefore, the curvature experienced by the second particle must be proportional to the traceless part of the energy-momentum tensor of the first particle, and vice versa.

A curvature tensor that can be set proportional to a traceless energy-momentum tensor, must itself be traceless too. Such a tensor is the Weyl tensor $C^{\mu\nu\sigma\tau}$, which is the traceless part of the Riemann curvature tensor $R^{\mu\nu\sigma\tau}$. From the Weyl tensor, a traceless "gravitation tensor" $W^{\mu\nu}$ can be derived [8]. This tensor can then be put into relation with the traceless part of the energy-momentum tensor $T^{\mu\nu}$.

Examples of traceless energy-momentum tensors, based on different models of matter, can be found in [8]. Here we simply subtract the trace from the energy-momentum tensor to make it traceless. This leads to the field equations of *conformal gravity*

$$W^{\mu\nu} = G_{conf}\left(T^{\mu\nu} - \frac{1}{2}T_\alpha^\alpha g^{\mu\nu}\right) \qquad (18)$$

with a "gravitational constant" G_{conf}.

Conformal gravity has gained interest in recent years because it may solve the problems usually associated with "dark matter" and "dark energy" [7,8] without additional *ad hoc* assumptions. Within the scale of our solar system, conformal gravity is known to deliver the same results as Einstein's theory of general relativity, which is based on the Riemann curvature tensor, rather than on the Weyl tensor [7]. The problem of "ghosts," which has been encountered in "quantized" versions of conformal gravity [9,10], does not exist for the classical version.

4. The Gravitational Constant

In [1], the electromagnetic coupling constant α was calculated from the geometry of the parameter space associated with an irreducible two-particle state space of spin-1/2 particles. The same calculation, done for spinless particles, results in a coupling constant of $\alpha/4$.

There is, however, a crucial difference between quantum electrodynamics and gravitation. Whereas in quantum electrodynamics it makes sense to consider an isolated two-particle system, this is an unrealistic configuration in gravitation. There is no way to set up a "neutral" environment or to "shield" gravitation. Therefore, an experimental setup for a "scattering experiment" in analogy to electron-electron scattering must always take into account the whole environment. This means we have to take into account at least 10^{80} heavy particles, which is the estimated number of protons in the (observable) universe [11].

A gravitational scattering experiment of an (electrically neutral) particle of, say, the mass of the proton, includes at first the *selection* of a second particle from 10^{80} available particles. This is followed by a *transition* from the "incoming" two-particle pure product state to an irreducible (entangled) two-particle state. Finally, we have a *transition* to an "outgoing" two-particle pure product state, which is the quantum mechanical description of measuring the individual momenta of the particles after the scattering has taken place. Note that there are two transitions between pure product states and entangled state, but only one selection.

The following is an attempt to quantum mechanically describe the "selection process." The selection of a partner particle will be considered as a "transition" from an "incoming" one-particle state (of the first particle) to a two-particle state. For the first particle, there are 10^{80} independent ways to form a two-particle state. Let us describe the corresponding quantum mechanical transition amplitude by a state in a 10^{80}-dimensional state space. Then the states of this state space have to be normalized by the factor $1/10^{40}$.

This normalization ensures that the total transition probability from a specific incoming (one-particle) state to an outgoing (one-particle) state, through any intermediate two-particle state, equals unity. On the other hand, the field equations (18) describe the contribution of only a specific second particle, characterized by its energy-momentum tensor at a point x, to the curvature of space-time. Accordingly, the scattering process contains only the transitions up to the outgoing two-particle product state. For this reason, the "selection amplitude" enters only once. The normalization factor in this amplitude leads to an additional factor to the two-particle coupling constant $\alpha/4$ of $1/10^{40}$.

This results in an estimate of the "gravitational coupling constant." It matches the empirical strength of the gravitational interaction, which, between two protons, is 37 orders of magnitude weaker than the electromagnetic interaction (or 43 orders, between two electrons) [12]. This weakness explains why in the experiments of particle physics the gravitational interaction can be ignored.

5. Quantum Gravity

The field equations (18) describe a classical theory of gravitation. What, then, is their quantum mechanical analogue? Since we just have sketched a connection between quantum theory and classical conformal gravity, we are able to give an answer to this question: The quantum mechanical basis of conformal gravity is nothing

other than an irreducible two-particle representation of the Poincaré group. In other words, there is no specific "quantum gravity" apart from the common rules of relativistic quantum mechanics. The situation is similar to quantum electrodynamics, as discussed in [1]: Gravity emerges from the restrictions on the two-particle state space imposed by the condition of irreducibility.

6. Conclusions

Reasons have been given as to why gravitation can be understood as a basic property of relativistic quantum mechanics, more precisely, as a property of the irreducible two-particle representations of the Poincaré group. Gravitation is not provided by "coupling" to an "external field." Rather it is the outcome of correlations within the quantum mechanical state-space of matter resulting from the condition of irreducibility. These correlations lead to the equations of classical conformal gravity. In short, gravitation is a quantum mechanical property of matter.

Physical space-time turns out to be just another quantum mechanical property of matter. Its geometry in the large is determined by the equations of conformal gravity. Its scale in the small is defined by the electromagnetic interaction and by the masses of the particles involved in this interaction. Together, the electromagnetic and gravitational interactions provide the basis for building extended atoms, molecules, and macroscopic bodies, to fill up space-time. The electromagnetic interaction provides photons, which can be used to unveil the geometry of space-time to an observer. Needless to say, the electromagnetic interaction establishes a causal structure in space-time. It is these interactions that make the difference between parameter space-time and physical space-time. Therefore, the emergence of physical space-time goes in parallel with the emergence of interactions.

The validity of classical space-time ends at scales where quantum mechanics becomes effective. These scales are related to the electron mass, rather than to the Planck mass. There is no room for the latter, because it is not possible to construct a mass from $\hbar$, c, and G_{conf}.

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