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From Differential Sequences to Black Holes

Jean-Francois Pommaret

CERMICS, Ecole des Ponts ParisTech, Paris, France
Email: jean-francois.pommaret@wanadoo.fr

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Abstract

When E. Beltrami introduced in 1892 the six stress functions $\Phi_{ij} = \Phi_{ji}$ wearing his name and allowing to parametrize the Cauchy stress equations of elasticity theory in space, he surely did not know he was using the Einstein operator introduced by A. Einstein in 1915 for general relativity in space-time, both ignoring that it was self-adjoint in the framework of differential double duality and confusing therefore stress functions with the variation $\Omega_{ij} = \Omega_{ji}$ of the metric ω. When I proved in 1995 that the Einstein equations in vacuum could not be parametrized like the Maxwell equations, solving thus negatively a 1000 dollars challenge of J. Wheeler in 1970, I did not imagine that such a purely mathematical result could also prove that the equations of the gravitational waves were not coherent with differential homological algebra. The purpose of this paper is to prove that this result is also showing that black holes cannot exist, not for a problem of detection but because their existence should contradict the link existing between the only two canonical differential sequences existing in the literature, namely the Janet sequence and the Spencer sequence, a result showing that the important object is not the metric but its group of invariance. Indeed, the last sequence is isomorphic to the tensor product of the Poincaré sequence by a Lie algebra of extremely small dimension when dealing with the differential resolutions of Killing vector fields while using successively the Minkowski (M), the Schwarzschild (S) and Kerr (K) metrics with respective parameters $(0,0),(m,0)$ and (m,a). The comparison with other explicit motivating examples also provided needs no comment.

Keywords

Killing Operator, Riemann Operator, Ricci Operator, Einstein Operator, Bianchi Operator, Cauchy Operator, Gravitational Waves, Black Holes

1. Introduction

When M. Janet introduced in 1920 the first finite length differential sequence as a footnote of his paper [1], he surely did not know about the possibility to use such a sequence in elasticity theory along the way introduced by the brothers E. and F. Cosserat in 1909 [2]. Taking the risk in 1970 to become a visiting student of D. C. Spencer (1912-2001) at Princeton University, I discovered that he was not even knowing the mathematical foundations of general relativity (GR) studied by his close friend J. A. Wheeler (1911-2008) who was offering 1000 dollars at that time to anybody finding a potential for Einstein equations in vacuum, similar to the well known one existing for Maxwell equations in electromagnetism (EM), usually defined by $dA = F$ while introducing the exterior derivative. When I discovered in 1995 the negative solution of this challenge, contrary to the general belief of the GR community, I never imagined that Wheeler and a few collaborators should refuse to accept this mathematical result and block up for publication in journals any paper in which his name was appearing. As a byproduct, the GR community is still ignoring such a result that can only be found in books of control theory [3].

Let me now tell about a personal experience that has oriented all my recent scientific research. As a former student of A. Lichnerowicz, I attended to the HIGH MASS held in Paris (2015) for the centenary of gravitational waves (GW). It was a VERY unpleasant atmosphere because EVERYBODY knew that sponsors should stop funding. One day, while listening to the invited talk "ARE BLACK HOLES REAL" by S. Klainermann, my neighbour, a young foreign student, turned towards me saying "Such talks should not have been accepted, do you know him?". I just answered I was listening to such a talk for the first time but that he had already delivered it elsewhere [4]-[6]. The idea is that there are three types of reality: Virtual reality, Physical reality and Mathematical reality. As the meaning of the two first definitions is rather clear, he "defined" the third as the possibility to write down a physical paper without any mathematical mistake and that black holes were belonging to such a category! 6 months later, LIGO announced to have detected GW produced by a couple of merging black holes and this event, highly spread in newspapers, has been followed by the diffusion of pictures of black holes [7]. Since that time, I started to have doubts and, being specialist of control theory, I decided to use my knowledge for studying *at least* the origin of GW. In 2017, I discovered why GW cannot exist because Einstein, copying Beltrami, both ignoring that the Einstein operator, linearization of the Einstein tensor over the M metric, was surprisingly self-adjoint [8] [10]. I started to have doubts, not about the proper DETECTION but mainly about the defining EQUATIONS. Then, I started to have more serious doubts when LIGO did stop for 3 years and I don't speak about the lack of results for KAGRA after spending 250 millions of dollars. It is at this moment that I decided to care about black holes while taking into account a few recent papers I wrote about the comparison of the M, S and K metrics [11]-[13] but also as a way to disagree with the approach used by L. Andersson and collaborators met while lecturing at the Albert Einstein Institute (AEI) of Potsdam (October 23-27, 2017) [14]-[16].

In the Special Relativity paper of Einstein (1905), only a footnote provides a reference to the conformal group of space-time for the Minkowski metric ω but there is no proof that the conformal factor should be equal to 1. The Cauchy stress equations (1823), the Cosserat couple-stress equations (1909), the Clausius virial equation (1870), the Maxwell (1873) and Weyl (1918) equations are among the most famous partial differential equations that can be found today in any textbook dealing with elasticity theory, continuum mechanics, thermodynamics or electromagnetism. Over a manifold of dimension n, their respective numbers are $n, n(n-1)/2, 1, n$ with a total of $N=(n+1)(n+2)/2$, that is 15 when $n=4$ for space-time [17]. This is also the number of parameters of the Lie group of conformal transformations with n translations, $n(n-1)/2$ rotations, 1 dilatation and n highly non-linear elations introduced by E. Cartan in 1922. The purpose of this paper is to prove that the form of these equations only depends on the structure of the conformal group for an arbitrary $n \geq 1$ because they are described *as a whole* by the (formal) adjoint of the first Spencer operator existing in the Spencer differential sequence. Such a group theoretical implication is obtained by applying totally new differential geometric methods in field theory. In particular, when $n=4$, the main idea is not to *shrink* the group from 10 down to 4 or 2 parameters by using the Schwarzschild or Kerr metrics instead of the Minkowski metric, but to *enlarge* the group from 10 up to 11 or 15 parameters by using the Weyl or conformal group instead of the Poincaré group of space-time. Contrary to the Einstein equations, these equations can be all parametrized by the adjoint of the second Spencer operator through $Nn(n-1)/2$ potentials. These results bring the need to revisit the mathematical foundations of both General Relativity and Gauge Theory according to a clever but rarely quoted paper of H. Poincaré (1901) [18]. They strengthen the comments we already made about the dual confusions made by Einstein (1915) while following Beltrami (1892), both using the same Einstein operator but ignoring it is self-adjoint in the framework of differential double duality. They also question the origin and existence of black holes.

FIRST MOTIVATING EXAMPE: (*Macaulay*) With $m=1, n=2, q=2$, consider the second order system $R_2 \subset J_2(E)$ written $Q\xi \equiv d_{22}\xi = \eta^2$, $P\xi \equiv d_{12}\xi - a d_{11}\xi = \eta^1$ or $\mathcal{D}\xi = \eta$ with a constant parameter a and ground differential field $K = \mathbb{Q}(a)$. We may differentiate it once and obtain the third order system $R_3 \subset J_3(E)$ with corresponding Janet tabular. We have two cases:

- $a=0$: As $R_2 \subset J_2(E)$ is involutive with 4 parametric jets $(z^1, \cdots, z^4) = (\xi, \xi_1, \xi_2, \xi_{11})$, we may consider the first order involutive system $R_3 \subset J_1(R_2)$ defined by 7 equations:

$$\left\{\begin{array}{lcl|cc|} d_{22}\xi & = & 0 & 1 & 2 \\ d_{12}\xi & = & 0 & 1 & \bullet \end{array}\right. \Leftrightarrow \left\{\begin{array}{lcl|cc|} d_2 z^4 & = & 0 & 1 & 2 \\ d_2 z^3 & = & 0 & 1 & 2 \\ d_2 z^2 & = & 0 & 1 & 2 \\ d_2 z^1 - z^3 & = & 0 & 1 & 2 \\ d_1 z^3 & = & 0 & 1 & \bullet \\ d_1 z^2 - z^4 & = & 0 & 1 & \bullet \\ d_1 z^1 - z^2 & = & 0 & 1 & \bullet \end{array}\right.$$

The Janet approach does not bring *anything* more that the Spencer approach as we have:

$$\xi=\alpha x^2+\beta\left(x^1\right) \Leftrightarrow z^1=\alpha x^2+\beta\left(x^1\right), z^2=\partial_1\beta\left(x^1\right), z^3=\alpha, z^4=\partial_{11}\beta\left(x^1\right)$$

We have the only first order CC $d_1\eta^2-d_2\eta^1=0$ and we have the exact Janet sequence:

$$\boxed{0\rightarrow\Theta\rightarrow 1\underset{2}{\overset{\mathcal{D}}{\rightarrow}}2\underset{1}{\overset{\mathcal{D}_1}{\rightarrow}}1\rightarrow 0}$$

Now, we have the commutative and exact diagram allowing to construct the Spencer operator $D_1: C_0=R_2\rightarrow C_1$ and let the reader construct similarly $D_2: C_1\rightarrow C_2$ [19]-[21]:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_3 & \rightarrow & T^*\otimes R_2 & \rightarrow & C_1 & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & R_3 & \rightarrow & J_1\left(R_2\right) & \rightarrow & C_1 & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_2 & = & R_2 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The *Fundamental Diagram I* links the upper Spencer sequence with the lower Janet sequence:

$$\boxed{\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Theta & \overset{j_2}{\rightarrow} & 4 & \underset{1}{\overset{D_1}{\rightarrow}} & 7 & \underset{1}{\overset{D_2}{\rightarrow}} & 3 & \rightarrow & 0\\
 & & & & & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \rightarrow & 1 & \underset{2}{\overset{j_2}{\rightarrow}} & 6 & \underset{1}{\overset{D_1}{\rightarrow}} & 8 & \underset{1}{\overset{D_2}{\rightarrow}} & 3 & \rightarrow & 0\\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Theta & \rightarrow & 1 & \underset{2}{\overset{\mathcal{D}}{\rightarrow}} & 2 & \underset{1}{\overset{\mathcal{D}_1}{\rightarrow}} & 1 & \rightarrow & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & \\
 & & & & & & 0 & & 0 & & & &
\end{array}}$$

Finally, multiplying the CC by the test function λ, we obtain the adjoint operator $ad\left(\mathcal{D}_1\right):\lambda\rightarrow\left(d_2\lambda=\mu^1,-d_1\lambda=\mu^2\right)$ and similarly the adjoint operator $ad\left(\mathcal{D}\right):\mu\rightarrow d_{22}\mu^2+d_{12}\mu^1=\nu$ while the CC of $ad\left(\mathcal{D}_1\right)$ providing the dual differential sequence:

$$\boxed{0\leftarrow 1\underset{2}{\overset{ad(\mathcal{D})}{\leftarrow}}\boxed{2}\underset{1}{\overset{ad(\mathcal{D}_1)}{\leftarrow}}1}$$

not exact at $\boxed{2}$ because the only generating CC of $ad\left(\mathcal{D}_1\right)$ is $d_1\mu^1+d_2\mu^2=\nu'$ with $d_2\nu'=\nu$.

Introducing the commutative ring of differential operators $D=\mathbb{Q}[d_1,d_2]$ and applying $hom_D(\bullet,D)$, we let the reader prove that the dual differential sequence:

$$\boxed{0 \leftarrow 4 \underset{2}{\overset{ad(D_1)}{\leftarrow}} \boxed{7} \underset{1}{\overset{ad(D_2)}{\leftarrow}} 3}$$

is also not exact at $\boxed{7}$ because $ad(D_1)$ does not generate the CC of $ad(D_2)$ by using the fact that the extension modules do not depend on the resolution (Janet or Spencer) used, a result highly not evident at first sight, even on such an elementary academic example.

- $a\neq 0$: We may set $a=1$ for example because the formal properties of the sequences and diagrams will not depend on the value of a provided that $a\neq 0$.

$$\left\{\begin{array}{lcl} d_{222}\xi & = & d_2\eta^2 \\ d_{122}\xi & = & d_1\eta^2 \\ d_{112}\xi & = & d_1\eta^2-d_2\eta^1 \\ d_{111}\xi & = & d_1\eta^2-d_2\eta^1-d_1\eta^1 \\ d_{22}\xi & = & \eta^2 \\ d_{12}\xi-d_{11}\xi & = & \eta^1 \end{array}\right. \quad \begin{array}{|cc|} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}$$

In this case, using the Janet tabular or the fact that $PQ-QP=0$, we have the only second order CC $d_{22}\eta^1-d_{12}\eta^2+d_{11}\eta^2=0$ or $\mathcal{D}_1\eta=0$ in the following exact differential sequence with Euler-Poincaré characteristic $1-2+1=0$:

$$\boxed{0 \rightarrow \Theta \rightarrow 1 \underset{2}{\overset{\mathcal{D}}{\rightarrow}} 2 \underset{2}{\overset{\mathcal{D}_1}{\rightarrow}} 1 \rightarrow 0}$$

which is nevertheless far from being a Janet sequence because $\mathcal{D}$ is formally integrable (FI) but *not* involutive with symbol $g_3=0$. We notice that $ad(\mathcal{D})$ generates the only CC of $ad(\mathcal{D}_1)$ and we have the dual exact sequence made by the adjoint operators:

$$\boxed{0 \leftarrow 1 \underset{2}{\overset{ad(\mathcal{D})}{\leftarrow}} 2 \underset{2}{\overset{ad(\mathcal{D}_1)}{\leftarrow}} 1}$$

which is *not* a Janet sequence. Hence, the *only* way to have a Janet sequence is to use the *full* Janet tabular of the involutive system $R_3\subset J_3(E)$ already exhibited while setting $a=1$ and counting the number of single dots (7) or the number of couples (2).

For helping the reader, we recall that basic elementary combinatorics arguments are giving $dim(S_qT^*)=q+1$ while $dim(J_q(E))=(q+1)(q+2)/2$ because $n=2$ and $m=dim(E)=1$.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & g_2 & \rightarrow & S_2T^*\otimes E & \rightarrow & F_0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 \rightarrow & & R_2 & \rightarrow & J_2(E) & \rightarrow & F_0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & R_1 & \rightarrow & J_1(E) & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 1 & \rightarrow & 3 & \rightarrow & 2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \parallel & & \\
0 & \rightarrow & 4 & \rightarrow & 6 & \rightarrow & 2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 3 & \rightarrow & 3 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_3 & \rightarrow & S_3T^*\otimes E & \rightarrow & T^*\otimes F_0 & \rightarrow & h_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \rightarrow & R_3 & \rightarrow & J_3(E) & \rightarrow & J_1(F_0) & \rightarrow & Q_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_2 & \rightarrow & J_2(E) & \rightarrow & F_0 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 4 & \rightarrow & 4 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 10 & \rightarrow & 6 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 6 & \rightarrow & 2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & g_4 & \rightarrow & S_4T^*\otimes E & \rightarrow & S_2T^*\otimes F_0 & \rightarrow & h_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \rightarrow & R_4 & \rightarrow & J_4(E) & \rightarrow & J_2(F_0) & \rightarrow & Q_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_3 & \rightarrow & J_3(E) & \rightarrow & J_1(F_0) & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 15 & \rightarrow & 12 & \rightarrow & 1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 4 & \rightarrow & 10 & \rightarrow & 6 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

Using these diagrams, we obtain successively, till we stop, that the number of generating CC of order 1 is zero and the number of generating CC of strict order two is 1 in a coherent way. Also, setting $F_1 = Q_2$ with $dim(Q_2) = 1$, we obtain the commutative and exact diagram:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & g_5 & \rightarrow & S_5T^* \otimes E & \rightarrow & S_3T^* \otimes F_0 & \rightarrow & T^* \otimes F_1 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & R_5 & \rightarrow & J_5(E) & \rightarrow & J_3(F_0) & \rightarrow & J_1(F_1) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & R_4 & \rightarrow & J_4(E) & \rightarrow & J_2(F_0) & \rightarrow & F_1 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

with dimensions:

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \rightarrow & 6 & \rightarrow & 8 & \rightarrow & 2 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & 4 & \rightarrow & 21 & \rightarrow & 20 & \rightarrow & 3 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & 4 & \rightarrow & 15 & \rightarrow & 12 & \rightarrow & 1 & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

As a byproduct we have the exact sequences: $\forall r \geq 0$:

$$0 \rightarrow R_{r+4} \rightarrow J_{r+4}(E) \rightarrow J_{r+2}(F_0) \rightarrow J_r(F_1) \rightarrow 0$$

Such a result can be checked directly through the identity:

$$4-(r+5)(r+6)/2+2(r+3)(r+4)/2-(r+1)(r+2)/2=0$$

We obtain therefore the formally exact sequence we were looking for, namely:

$$\boxed{0 \rightarrow \Theta \rightarrow E \underset{2}{\overset{\mathcal{D}}{\rightarrow}} F_0 \underset{2}{\overset{\mathcal{D}_1}{\rightarrow}} F_1 \rightarrow 0}$$

The *surprising fact* is that, *in this case*, $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$. Indeed, multiplying by the Lagrange multiplier test function λ and integrating by parts, we obtain the second order operator $\lambda \rightarrow (d_{22}\lambda = \mu^1, -d_{12}\lambda + d_{11}\lambda = \mu^2)$ and thus $d_{112}\lambda = d_1\mu^1 + d_2\mu^2$. Substituting, we finally get the only second order CC $d_{22}\mu^2 + d_{12}\mu^1 - d_{11}\mu^1 = 0$.

In the differential module framework over the commutative ring $D = K[d_1, d_2]$ of differential operators with coefficients in the trivially differential field $K = \mathbb{Q}(a)$, we have the free resolution:

$$\boxed{0 \rightarrow D \underset{2}{\overset{\mathcal{D}_1}{\rightarrow}} D^2 \underset{2}{\overset{\mathcal{D}}{\rightarrow}} D \rightarrow M \rightarrow 0}$$

of the differential module M with Euler-Poincaré characteristic $rk_D(M) = 1 - 2 + 1 = 0$. We recall that $R = R_\infty = hom_K(M, K)$ is a differential module for the Spencer operator $d : R \to T^* \otimes R : R_{q+1} \to T^* \otimes R_q$ (See [19]-[21] for more details). *Only "fingers" could have been used!*

Setting $(\theta_\tau(x)) = \left(1, x^1, x^2, \frac{1}{2}(x^1)^2 + x^1 x^2\right)$ with $\tau = 1,2,3,4$ as a basis of 4 solutions, We may introduce the general section of R_3, namely $\xi_\mu = \lambda^\tau(x)\partial_\mu \theta_\tau(x)$ and obtain for the Spencer operator:

$$(d\xi_3)_{\mu,i}(x) = \partial_i \xi_\mu(x) - \xi_{\mu+1_i}(x) = \left(\partial_i \lambda^\tau(x)\right)\partial_\mu \theta_\tau(x)$$

a result showing that the upper Spencer sequence in the following *Fundamental Diagram I* is isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a vector space $\mathcal{V}$ of dimension 4 over C but a similar situation can be found with the infinitesimal generators of any Lie group G acting on X by using the Lie algebra $\mathcal{G} = T_e(G)$ as in the recent [17]. Using now the involutive system R_3 instead of R_2, we get:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_3} & 4 & \underset{1}{\xrightarrow{D_1}} & 8 & \underset{1}{\xrightarrow{D_2}} & 4 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 1 & \underset{3}{\xrightarrow{j_3}} & 10 & \underset{1}{\xrightarrow{D_1}} & 15 & \underset{1}{\xrightarrow{D_2}} & 6 & \to & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta & \to & 1 & \underset{3}{\xrightarrow{\mathcal{D}}} & 6 & \underset{1}{\xrightarrow{\mathcal{D}_1}} & 7 & \underset{1}{\xrightarrow{\mathcal{D}_2}} & 2 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

In each sequence, the Euler-Poncaré alternate sum of dimensions is indeed vanishing. Taking the adjoint of each operator and inverting the arrows, we obtain the commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \leftarrow & 4 & \underset{1}{\xleftarrow{ad(D_1)}} & 8 & \underset{1}{\xleftarrow{ad(D_2)}} & 4 & & \\
 & & \uparrow & & \uparrow & & \uparrow & \nwarrow & \uparrow & & \\
0 & \leftarrow & 1 & \underset{3}{\xleftarrow{ad(j_3)}} & 10 & \underset{1}{\xleftarrow{ad(D_1)}} & 15 & \underset{1}{\xleftarrow{ad(D_2)}} & 6 & \leftarrow & 0 \\
 & & \| & & \uparrow & \nwarrow & \uparrow & & \uparrow & & \\
0 & \leftarrow & 1 & \underset{3}{\xleftarrow{ad(\mathcal{D})}} & 6 & \underset{1}{\xleftarrow{ad(\mathcal{D}_1)}} & 7 & \underset{1}{\xleftarrow{ad(\mathcal{D}_2)}} & 2 & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

which is not formally exact because a delicate chase allows to prove that the cohomology $H = Z/B$ at $\boxed{7}$ is isomorphic to the kernel of $ad(D_2)$ and is thus

$ext^2(M) \neq 0$ because $n=2$ though $ext^1(M)=0$. Cutting the last diagram vertically after $\boxed{7}$, we notice that Z is the kernel of the north west arrow. Indeed, starting with $a \in 6$ killed by the upper north west arrow, we get $b \in 15$ coming from a *unique* $c \in 7$ killed by the lower north west arrow and thus killed by $ad(\mathcal{D}_1)$, that is $c \in Z$. Such a result is allowing to obtain the following commutative and exact diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \uparrow & & & & \\
 & & & & 8 & \xleftarrow{ad(D_2)} & 4 & \leftarrow & ker(ad(D_2)) & \leftarrow & 0 \\
 & & & & \uparrow & \nwarrow & \uparrow & & & & \\
 & & 0 & \leftarrow & 6 & = & 6 & \leftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \\
0 & \leftarrow & H & \leftarrow & Z & \leftarrow & B & \leftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

A snake chase finally provides the desired isomorphism. We also notice that the two central exact sequences of these diagrams both split. Such a situation is one of the rare ones encountered in the study of exact canonical Spencer/Janet sequences. As a direct checking, $ker(ad(\mathcal{D}_1))$ is defined by the second order EDP $d_{22}\lambda = 0$, $d_{12}\lambda - d_{11}\lambda = 0$ which is a 4-dimensional vector space over the constants in a coherent way with $ker(ad(D_2))$. However, $ad(D_1)$ generates the CC of $ad(D_2)$ and a chase is thus proving that $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$ for any resolution of Θ used because $ext^1(M) = ext^1_D(M,D) = 0$. A similar but more delicate study of another example, also provided by Macaulay, can be found for the dimension $n=3$ with the second order system defined by the PD equations $\{d_{33}\xi = 0, d_{23}\xi - d_{11}\xi = 0, d_{22}\xi = 0\}$ which are among the rare examples known with a non-vanishing 2-acyclic symbol like in the conformal situation [9] [17].

SECOND MOTIVATING EXAMPLE: With $m=1, n=2, q=2$, consider the slightly different new second order system $Q\xi \equiv d_{22}\xi = \eta^2$, $P\xi \equiv d_{12}\xi - a d_1\xi = \eta^1$ with a constant parameter a. We may differentiate it once and keep only the first equations, namely the previous ones and $a^2 d_1\xi = 0$. Hence, if $a=0$ we find the same system as before but if $a \neq 0$, we obtain the new system $R_2^{(1)} \subset R_2$ defined by $d_{22}\xi = 0$, $d_{12}\xi = 0$ $, d_1\xi = 0$ *not containing the parameter any longer*. Differentiating once more ore differentiating the initial system twice, we obtain the new involutive system $R_2^{(2)}$ defined by $d_{22}\xi = 0$, $d_{12}\xi = 0$, $d_{11}\xi = 0$, $d_1\xi = 0$ and the strict inclusions $R_2^{(2)} \subset R_2^{(1)} \subset R_2 \subset J_2(E)$ with $2 < 3 < 4 < 6$. Setting $z^1 = \xi, z^2 = \xi_2$, the two equivalent Janet tabulars become:

$$
\left\{
\begin{array}{lcl|cc|}
d_{22}\xi & = & 0 & 1 & 2 \\
d_{12}\xi & = & 0 & 1 & \bullet \\
d_{11}\xi & = & 0 & 1 & \bullet \\
d_1\xi & = & 0 & \bullet & \bullet
\end{array}
\right.
\quad \Leftrightarrow \quad
\left\{
\begin{array}{lcl|cc|}
d_2 z^2 & = & 0 & 1 & 2 \\
d_2 z^1 - z^2 & = & 0 & 1 & 2 \\
d_1 z^2 & = & 0 & 1 & \bullet \\
d_1 z^1 & = & 0 & 1 & \bullet
\end{array}
\right.
$$

leading to the exact Janet sequence:

$$\boxed{0 \to \Theta \to 1 \underset{2}{\overset{\mathcal{D}}{\to}} 4 \underset{1}{\overset{\mathcal{D}_1}{\to}} 4 \underset{1}{\overset{\mathcal{D}_2}{\to}} 1 \to 0}$$

and a basis of solutions $\{\theta_\tau(x) \mid \tau = 1, 2\} = \{1, x^2\}$. We obtain thus the *Fundamental Diagram I* allowing to link the upper exact Spencer sequence with the lower exact Janet sequence [19] [20]:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_2} & 2 & \underset{1}{\overset{D_1}{\to}} & 4 & \underset{1}{\overset{D_2}{\to}} & 2 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 1 & \underset{2}{\overset{j_2}{\to}} & 6 & \underset{1}{\overset{D_1}{\to}} & 8 & \underset{1}{\overset{D_2}{\to}} & 3 & \to & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta & \to & 1 & \underset{2}{\overset{\mathcal{D}}{\to}} & 4 & \underset{1}{\overset{\mathcal{D}_1}{\to}} & 4 & \underset{1}{\overset{\mathcal{D}_2}{\to}} & 1 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

SUCH A DIAGRAM DOES NOT DEPEND ON THE PARAMETER ANY LONGER!

Finally, as $PQ = QP$, we have also $ext^1(M) = 0$ as in the previous example.

THIRD MOTIVATING EXAMPLE: Having in mind the situation existing for the M, S, and K metrics, we shall add one more constant parameter b and consider the new second order system $R_2 \subset J_2(E)$ written $d_{22}\xi - bx^2 d_1\xi = \eta^2$, $d_{12}\xi - d_{11}\xi = \eta^1$ as an operator $\mathcal{D}\xi = \eta$ with coefficients in the ground differenial field $K = \mathbb{Q}(b)(x^2)$. Of course, if $b = 0$, we find back the previous example and will set up $b = 1$ from now on in order to prove that the previous example is similar to the S metric while this is rather similar to the K metric. Differentiating once, we obtain the third order system $R_3 \subset J_3(E)$ with corresponding Janet tabular:

$$\left\{\begin{array}{lcl}
d_{222}\xi - x^2 d_{12}\xi - d_1\xi & = & d_2\eta^2 \\
d_{122}\xi - x^2 d_{11}\xi & = & d_1\eta^2 \\
d_{112}\xi - x^2 d_{11}\xi & = & d_1\eta^2 - d_2\eta^1 \\
d_{111}\xi - x^2 d_{11}\xi & = & d_1\eta^2 - d_2\eta^1 - d_1\eta^1 \\
d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\
d_{12}\xi - d_{11}\xi & = & \eta^1
\end{array}\right. \quad \begin{array}{|cc|}
\hline
1 & 2 \\
1 & \bullet \\
1 & \bullet \\
1 & \bullet \\
\bullet & \bullet \\
\bullet & \bullet \\
\hline
\end{array}$$

Though the symbol $g_3 = 0$ is trivially involutive, like in the preceding example, we shall discover that the study of such a system is *much more* difficult than previously because this system is *not* even formally integrable (FI). Indeed, trying all the dots, we discover that we have the strict inclusions $R_2^{(2)} \subset R_2^{(1)} = R_2 \subset J_2(E)$ with respective dimension $3 < 4 = 4 < 6$. After a few tricky substitutions and eliminations, we obtain the new second order PD equation:

$$\boxed{A \equiv d_{11}\xi = d_{22}\eta^1 - d_{12}\eta^2 + d_{11}\eta^2 - x^2 d_1\eta^1 \in j_2(\eta)}$$

The hard step is to look for generating CC in the form of an operator $\mathcal{D}_1\eta = \zeta$. Using diagram chasing, we obtain the first prolongation commutative and exact diagram: as before

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & g_3 & \to & S_3T^*\otimes E & \to & T^*\otimes F_0 & \to & h_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \to & R_3 & \to & J_3(E) & \to & J_1(F_0) & \to & Q_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R_2 & \to & J_2(E) & \to & F_0 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 4 & \to & 4 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 4 & \to & 10 & \to & 6 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 4 & \to & 6 & \to & 2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Hence, there is no first order CC and we may use the next prolongation to obtain the new diagram:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & g_4 & \to & S_4T^*\otimes E & \to & S_2T^*\otimes F_0 & \to & h_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \to & R_4 & \to & J_4(E) & \to & J_2(F_0) & \to & Q_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R_3 & \to & J_3(E) & \to & J_1(F_0) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \to & 5 & \to & 6 & \to & 1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & 3 & \to & 15 & \to & 12 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & 4 & \to & 10 & \to & 6 & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

providing the long exact connecting sequence:

$$0 \to R_4 \xrightarrow{\pi_3^4} R_3 \to h_2 \to 0 \qquad 0 \to 3 \to 4 \to 1 \to 0$$

It follows that there cannot exist *any* second order CC and we may start afresh with the new system $R_2' = R_2^{(2)} \subset R_2 \subset J_2(E)$ and its prolongation R_3' which are easily seen to be trivially involutive by checking all the dots in the respective Janet tabulars: For example we have

$$\left\{\begin{array}{lcl|cc|}
d_{222}\xi - d_1\xi & = & 0 & 1 & 2 \\
d_{122}\xi & = & 0 & 1 & \bullet \\
d_{112}\xi & = & 0 & 1 & \bullet \\
d_{111}\xi & = & 0 & 1 & \bullet \\
d_{22}\xi - x^2 d_1\xi & = & 0 & \bullet & \bullet \\
d_{12}\xi & = & 0 & \bullet & \bullet \\
d_{11}\xi & = & 0 & \bullet & \bullet
\end{array}\right.$$

We obtain therefore at once the *Fundamental Diagram I* in which the upper sequence is the Spencer sequence for R_3, the central hybrid sequence is the Janet sequence for j_3 and the bottom sequence is the Janet sequence for $\mathcal{D}'$ as follows for R_3':

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_3} & 3 & \underset{1}{\xrightarrow{D_1}} & 6 & \underset{1}{\xrightarrow{D_2}} & 3 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 1 & \underset{3}{\xrightarrow{j_3}} & 10 & \underset{1}{\xrightarrow{D_1}} & 15 & \underset{1}{\xrightarrow{D_2}} & 6 & \to & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta & \to & 1 & \underset{3}{\xrightarrow{\mathcal{D}'}} & 7 & \underset{1}{\xrightarrow{\mathcal{D}'_1}} & 9 & \underset{1}{\xrightarrow{\mathcal{D}'_2}} & 3 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

and for R_2' which is also involutive:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_2} & 3 & \underset{1}{\xrightarrow{D_1}} & 6 & \underset{1}{\xrightarrow{D_2}} & 3 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \to & 1 & \underset{2}{\xrightarrow{j_2}} & 6 & \underset{1}{\xrightarrow{D_1}} & 8 & \underset{1}{\xrightarrow{D_2}} & 3 & \to & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta & \to & 1 & \underset{2}{\xrightarrow{\mathcal{D}'}} & 3 & \underset{1}{\xrightarrow{\mathcal{D}'_1}} & 2 & \to & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & & & \\
 & & & & & & 0 & & 0 & & & &
\end{array}$$

It is important to notice that in both cases we have isomorphic Spencer sequences while the bottom Janet sequences are *completely different*.

It finally remains to find out the generating CC for the initial second order operator $\mathcal{D}\xi=\eta$ which is neither FI nor involutive. With $A\in j_2(\eta)$, we have the trivially involutive system:

$$\left\{\begin{array}{lcl} d_{22}\xi - x^2 d_1\xi & = & \eta^2 \\ d_{12}\xi & = & A+\eta^1 \\ d_{11}\xi & = & A \end{array}\right. \quad \boxed{\begin{array}{cc} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \end{array}}$$

Checking the two dots separately we have the two *third order* (!) CC:

$$\boxed{B_1 \equiv d_1A - x^2A - d_1\eta^2 + d_2\eta^1 + d_1\eta^1 = 0, \quad B_2 \equiv d_2A - x^2A - d_1\eta^2 + d_2\eta^1 = 0}$$

$$d_2A - d_1A - d_1\eta^1 \equiv d_{222}\eta^1 - d_{122}\eta^2 - d_{122}\eta^1 + 2d_{112}\eta^2 - d_{111}\eta^2 - x^2 d_{12}\eta^1 - x^2 d_{11}\eta^1 = 0$$

Exactly like in [13], we now provide the link existing between these third order CC and the Spencer operator. Indeed, using R_3 we obtain at once:

$$\boxed{\begin{array}{lclcl} d_1\xi_{11} - \xi_{111} & = & d_1A - x^2A - d_1\eta^2 + d_2\eta^1 + d_1\eta^2 & = & B_1 \\ d_2\xi_{11} - \xi_{112} & = & d_2A - x^2A - d_1\eta^2 + d_2\eta^1 & = & B_2 \end{array}}$$

a result not evident a priori that we shall obtain now just by diagram chasing:

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & g_5 & \to & S_5T^*\otimes E & \to & S_3T^*\otimes F_0 & \to & h_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & \searrow & \downarrow & & \\ 0 & \to & R_5 & \to & J_5(E) & \to & J_3(F_0) & \to & Q_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R_4 & \to & J_4(E) & \to & J_2(F_0) & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

$$\begin{array}{ccccccccccc} & & & & 0 & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \to & 6 & \to & 8 & \to & 2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & \searrow & \parallel & & \\ 0 & \to & 3 & \to & 21 & \to & 20 & \to & 2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 3 & \to & 15 & \to & 12 & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

We may thus define $F_1=Q_2$ with $dim(F_1)=2$ and proceed similarly in order to define F_2 with $dim(F_2)=1$ by the long exact sequence:

$$0\to R_6\to J_6(E)\to J_4(F_0)\to J_1(F_1)\to F_2\to 0$$
$$0\to 3\to 28\to 30\to 6\to 1\to 0$$

We have indeed $d_1B_2 - d_2B_1 = 0$ and the exact differential sequence which is *not* a Janet sequence:

$$\boxed{0 \to \Theta \to 1 \underset{2}{\overset{\mathcal{D}}{\to}} 2 \underset{3}{\overset{\mathcal{D}_1}{\to}} 2 \underset{1}{\overset{\mathcal{D}_2}{\to}} 1 \to 0}$$

Using finally the basis $\{\theta_\tau(x) \mid 1 \le \tau \le 3\} = \left\{1, x^2, x^1 + \frac{1}{6}(x^2)^3\right\}$ for the vector space $\mathcal{V}$ over the constants, the Spencer sequence of the *Fundamental Diagram I* is the tensor product by $\mathcal{V}$ of the Poincaré sequence $1 \overset{d}{\to} 2 \overset{d}{\to} 1 \to 0$ for the exterior derivative when $n = 2$.

FOURTH MOTIVATING EXAMPLE: (*Janet*) With $m = 1, n = 3, q = 2$ and the ground differential field $K = \mathbb{Q}(x^2)$, we consider the second order system $R_2 \subset J_2(E)$ defined by the two PD equations $d_{33}\xi - x^2 d_{11}\xi = 0$, $d_{22}\xi = 0$ introduced by M. Janet in 1920 [1]. We have already proved in many references that the minimum involutive system with the same solutions is $R'_5 = R_5^{(2)}$ with $r = 3, s = 2 \Rightarrow r + s = 5$ and a linear space $\mathcal{V}$ of solutions generated over the constants by the 12 solutions

$$\{\theta_\tau(x) \mid 1 \le \tau \le 12\} = \left\{1, x^1, x^2, x^3, x^1x^2, x^1x^3, x^2x^3, x^1, x^2x^3, \frac{1}{2}(x^1)^2 x^3 + \frac{1}{6}x^2(x^3)^3, \right.$$
$$\left. \frac{1}{2}(x^1)^2 + x^2(x^3)^2, \frac{1}{2}x^1x^2(x^3)^2 + \frac{1}{6}(x^1)^3, \frac{1}{6}x^1x^2(x^3)^3 + \frac{1}{6}(x^1)^4 x^2 \right\}.$$

Accordingly, the *Fundamental Diagram I* becomes:

$$\boxed{\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \overset{j_5}{\to} & 12 & \underset{1}{\overset{D_1}{\to}} & 36 & \underset{1}{\overset{D_2}{\to}} & 36 & \underset{1}{\overset{D_3}{\to}} & 12 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 1 & \underset{5}{\overset{j_5}{\to}} & 56 & \underset{1}{\overset{D_1}{\to}} & 140 & \underset{1}{\overset{D_2}{\to}} & 120 & \underset{1}{\overset{D_3}{\to}} & 35 & \to & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta & \to & 1 & \underset{5}{\overset{\mathcal{D}'}{\to}} & 44 & \underset{1}{\overset{\mathcal{D}'_1}{\to}} & 104 & \underset{1}{\overset{\mathcal{D}'_2}{\to}} & 84 & \underset{1}{\overset{\mathcal{D}'_3}{\to}} & 23 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}}$$

while the minimum resolution is the exact differential sequence ([20] p 3+4):

$$\boxed{0 \to \Theta \to 1 \underset{2}{\overset{\mathcal{D}}{\to}} 2 \underset{6}{\overset{\mathcal{D}_1}{\to}} 2 \underset{4}{\overset{\mathcal{D}_2}{\to}} 1 \to 0}$$

It remains now to prove that the situation existing in GR is quite similar but even worst from a purely computational point of view.

FIFTH MOTIVATING EXAMPLE: ([22] p. 346) With again $m = 1$, $n = 3$, $q = 3$, $K = \mathbb{Q}(x)$, the system $R_2 \subset J_2(E)$ defined by $d_{222}\xi + x^2\xi = 0$, $d_{11}\xi + d_2\xi = 0$ has the *only* solution $\xi = 0$ because $R_4^{(5)} = 0$ with $r = 1, s = 5$ and only the central Janet sequence for j_4 is left:

$$\boxed{0 \to 1 \xrightarrow{j_4} 35 \xrightarrow{D_1} 84 \xrightarrow{D_2} 70 \xrightarrow{D_3} 20 \to 0}$$

IMPORTANT REMARK: Coming back to the previous motivating examples and mixing them, we may consider anew the second order operator $\mathcal{D}$ defined by $d_{22}\xi - bx^2 d_1\xi = \eta^2$, $d_{12}\xi - ad_{11}\xi = \eta^1$ with two constant parameters (a,b). When $(a,b)=(0,0)$ it is involutive with is a single first order generating CC $\mathcal{D}_1$ while, when $(a,b)=(1,0)$ it is formally integrable (FI) but not involutive with a single second order CC $\mathcal{D}_1$ and when $(a,b)=(1,1)$, it is not even FI with two third order CC $\mathcal{D}_1$ having a single first order CC $\mathcal{D}_2$. Such a situation, *having nothing to do with physics*, is nevertheless quite similar to that of the first order Killing system $R_1 \subset J_1(T)$ allowing to define the first order Killing operator $\mathcal{D}: T \to S_2T^* : \xi \to \mathcal{L}(\xi)\omega = \Omega$ through the Lie derivative of a non-degenerate metric ω. Such an operator is not involutive because the symbol $g_2 = R_2 \cap S_2T^* \otimes T$ is vanishing but it is FI only when the metric has a constant Riemannian curvature [19] [20] [23], for example in the case of the Minkowski metric (M), but is far from being FI in the cases of the Schwarzschild (S) or Kerr metrics (K) [13]. The *Prolongation/Projection* (PP) procedure may provide convenient integers (r,s) leading to a FI system $R_{q+r}^{(s)} = \pi_{q+r}^{q+r+s}(R_{q+r+s}) \subset R_{q+r}$ and to the strict inclusions $R_1^{(3)} \subset R_1^{(2)} \subset R^{(1)} = R_1$ for the Killing system and its various prolongations and projections that *must* be done. Using now the Lie algebra $\mathcal{G}$ with dimension 10 for M, 4 for S and 2 for S instead of $\mathcal{V}$, the previous result is thus also showing that The Spencer sequence is *always* isomorphic to the following differential sequence:

$$\boxed{0 \to \Theta \to \wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^3 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^4 T^* \to 0}$$

which is the tensor product of the Poincaré sequence for the exterior derivative by a Lie algebra of *very small dimension*. It follows that, in the *Fundamental Diagram I*:

$\boxed{\text{THE IMPORTANT OBJECT IS NOT THE METRIC BUT ITS GROUP OF INVARIANCE}}$

In particular, the FACT that third order generating CC for the Killing operator may exist has no physical meaning as nobody is knowing a way to select a best candidate among the possible explicit solutions of Einstein equations in vacuum, a mathematical result questioning the origin and existence of black holes as we shall see! We also notice the fact that the PP procedure is highly depending on the various parameters involved, namely the only parameter m for the S metric which is reduced to the M metric when $m=0$ while the K metric depends on the two parameters (m,a) and is reduced to the S metric when $a=0$. We study now this comment.

2. Differential Tools

2.1. From Lie Groups to Differential Sequences

Let G be a Lie group with coordinates $(a^\rho) = (a^1, \cdots, a^p)$ acting on a manifold

X with a local action map $y = f(x,a)$. According to the *second fundamental theorem* of Lie, if $\theta_1,\cdots,\theta_p$ are the infinitesimal generators of the effective action of a lie group G on X, then $[\theta_\rho,\theta_\sigma] = c^\tau_{\rho\sigma}\theta_\tau$ where the $c = (c^\tau_{\rho\sigma} = -c^\tau_{\sigma\rho})$ are the *structure constants* of a Lie algebra of vector fields which can be identified with $\mathcal{G} = T_e(G)$ the tangent space to $\mathcal{G}$ at the identity $e \in G$ by using the action.

More generally, if X is a manifold and G is a lie group (*not acting necessarily on* X), let us consider *gauging* maps $a : X \to G : (x) \to (a(x))$. If $x + dx$ is a point of X close to x, then $T(a)$ will provide a point $a + da = a + \frac{\partial a}{\partial x}dx$ close to a on G. We may bring a back to e on G by acting on a with a^{-1}, *on the left*, getting therefore a 1-form $a^{-1}da = A \in T^* \otimes \mathcal{G}$ and the *curvature* 2-form $F = (\partial_i A^\tau_j(x) - \partial_j A^\tau_i(x) - c^\tau_{\rho\sigma}A^\rho_i(x)A^\sigma_j(x) = F^\tau_{ij}(x)) \in \wedge^2 T^*$ in the *non-linear gauge sequence*:

$$\boxed{\begin{array}{ccccc} X \times G & \to & T^* \otimes \mathcal{G} & \to & \wedge^2 T^* \otimes \mathcal{G} \\ a & \to & a^{-1}da = A & \to & dA - [A,A] = F \end{array}}$$

In 1956, at the birth of *gauge theory* (GT), the above notations were coming from the EM potential A and EM field $dA = F$ of relativistic Maxwell theory. Accordingly, $G = U(1)$ (unit circle in the complex plane) $\to dim(G) = 1$) was the *only possibility* to get a 1-form A and a 2-form F with vanishing structure constants $c = 0$.

Choosing now a "close" to e, that is $a(x) = e + t\lambda(x) + \cdots$ and linearizing as usual, we obtain the linear operator
$d : \wedge^0 T^* \otimes \mathcal{G} \to \wedge^1 T^* \otimes \mathcal{G} : (\lambda^\tau(x)) \to (\partial_i \lambda^\tau(x))$ and the *linear gauge sequence*:

$$\boxed{\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} \cdots \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \to 0}$$

which is the tensor product by $\mathcal{G}$ of the Poincaré sequence for the exterior derivative.

Considering now a Lagrangian on $T^* \otimes \mathcal{G}$, that is an *action* $W = \int w(A)dx$ where $dx = dx^1 \wedge \cdots \wedge dx^n$, we may vary it. With $A = a^{-1}da$ we may introduce $\lambda = a^{-1}\delta a \in \mathcal{G} = \wedge^0 T^* \otimes \mathcal{G}$ and get $\delta A^\tau_i = \partial_i \lambda^\tau - c^\tau_{\rho\sigma}A^\rho_i \lambda^\sigma$ ([20], pp. 180-185). Setting $\partial w/\partial A = \mathcal{A} = (\mathcal{A}^i_\tau) \in \wedge^{n-1} T^* \otimes \mathcal{G}^*$, we obtain the Poincaré equations
$\partial_i \mathcal{A}^i_\tau + c^\sigma_{\rho\tau} A^\rho_i \mathcal{A}^i_\sigma = 0$ as the adjoint of the previous operator (up to sign) [18]. Setting now $(\delta a)a^{-1} = \mu \in \mathcal{G}$, we get the *adjoint representation*
$\lambda = a^{-1}((\delta a)a^{-1})a = Ad(a)\mu$ while, introducing $\mathcal{B}$ such that $\mathcal{B}\mu = \mathcal{A}\lambda$, we get the divergence-like equations $\partial_i \mathcal{B}^i_\sigma = 0$.

In a different setting, if G acts on X, let $\{\theta_\tau \mid 1 \le \tau \le p = dim(G)\}$ be a basis of infinitesimal generators of the action. If $\mu = (\mu_1,\cdots,\mu_n)$ is a multi-index of length $|\mu| = \mu_1 + \cdots + \mu_n$ and $\mu + 1_i = (\mu_1,\cdots,\mu_{i-1},\mu_i + 1,\mu_{i+1},\cdots,\mu_n)$, we may introduce the Lie algebroid $R_q \subset J_q(T)$ with sections defined by
$\xi^k_\mu(x) = \lambda^\tau(x)\partial_\mu \theta^k_\tau(x)$ for an arbitrary section $\lambda \in \wedge^0 T^* \otimes \mathcal{G}$ and the trivially involutive operator $j_q : T \to J_q(T) : \theta \to (\partial_\mu \theta, 0 \le |\mu| \le q)$ of order q. We fi-

nally obtain the *Spencer operator* $d : R_{q+1} \to T^* \otimes Rq$ through the chain rule for derivatives [17]:

$$\boxed{\left(d\xi_{q+1}\right)^k_{\mu,i}(x) = \partial_i \xi^k_\mu(x) - \xi^k_{\mu+1_i}(x) = \partial_i \lambda^\tau(x)\partial_\mu \theta^k_\tau(x)}$$

When q is large enough to have an isomorphism $R_{q+1} \simeq R_q \simeq \wedge^0 T^* \otimes \mathcal{G}$ and the following *linear Spencer sequence* in which the operators D_r are induced by d as above:

$$\boxed{0 \to \Theta \xrightarrow{j_q} R_q \xrightarrow{D_1} T^* \otimes R_q \xrightarrow{D_2} \wedge^2 T^* \otimes R_q \xrightarrow{D_3} \cdots \xrightarrow{D_n} \wedge^n T^* \otimes R_q \to 0}$$

is isomorphic to the linear gauge sequence but with a completely different meaning because G is now acting on X and $\Theta \subset T$ is such that $[\Theta, \Theta] \subset \Theta$. Surprisingly, these results have NEVER been used in the study of the M, S and K metrics [13].

2.2. Lie Algebroids

If $R_q \subset J_q(E)$ is a system of order q on E, then $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is called the *r-prolongation* of R_q. In actual practice, if the system is defined by PDE $\Phi^\tau \equiv a^{\tau\mu}_k(x)\xi^k_\mu = 0$ the first prolongation is defined by adding the PDE $d_i \Phi^\tau \equiv a^{\tau\mu}_k(x)\xi^k_{\mu+1_i} + \partial_i a^{\tau\mu}_k(x)\xi^k_\mu = 0$. Accordingly, $\xi_q \in R_q \Leftrightarrow a^{\tau\mu}_k(x) f^k_\mu(x) = 0$ and $\xi_{q+1} \in R_{q+1} \Leftrightarrow a^{\tau\mu}_k(x)\xi^k_{\mu+1_i}(x) + \partial_i a^{\tau\mu}_k(x)\xi^k_\mu(x) = 0$ as identities on X or at least over an open subset $U \subset X$. Differentiating the first relation with respect to x^i and subtracting the second, we finally obtain:

$$a^{\tau\mu}_k(x)\left(\partial_i \xi^k_\mu(x) - \xi^k_{\mu+1_i}(x)\right) = 0 \Rightarrow d\xi_{q+1} \in T^* \otimes R_q$$

and the Spencer operator restricts to $d : R_{q+1} \to T^* \otimes R_q$.

DEFINITION 2.B.1 We set $R^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}\left(R_{q+r+s}\right)$.

DEFINITION 2.B.2: The *symbol* of R_q is the family $g_q = R_q \cap S_q T^* \otimes E$ of vector spaces over X. The symbol g_{q+r} of $\mathcal{R}_{q+r}$ only depends on g_q by a direct prolongation procedure. We may define the vector bundle F_0 over $\mathcal{R}_q$ by the short exact sequence $0 \to R_q \to J_q(E) \to F_0 \to 0$ and we have the exact induced sequence $0 \to g_q \to S_q T^* \otimes E \to F_0$.

When $|\mu| = q$, we obtain:

$$g_q = \left\{ v^k_\mu \in S_q T^* \otimes E \mid a^{\tau\mu}_k(x) v^k_\mu = 0 \right\}, |\mu| = q$$

$$\Rightarrow g_{q+r} = \rho_r(g_q) = \left\{ v^k_{\mu+\nu} \in S_{q+r} T^* \otimes E \mid a^{\tau\mu}_k(x) v^k_{\mu+\nu} = 0 \right\},$$

$$|\mu| = q, |\nu| = r$$

In general, neither g_q nor g_{q+r} are vector bundles over X as can be seen in the simple example $xy_x - y = 0 \Rightarrow xy_{xx} = 0$.

On $\wedge^s T^*$ we may introduce the usual bases $\left\{ dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_s} \right\}$ where we have set $I = (i_1 < \cdots < i_s)$. In a purely algebraic setting, one has:

PROPOSITION 2.B.3: There exists a map

$\delta:\wedge^s T^*\otimes S_{q+1}T^*\otimes E\to\wedge^{s+1}T^*\otimes S_qT^*\otimes E$ which restricts to $\delta:\wedge^s T^*\otimes g_{q+1}\to\wedge^{s+1}T^*\otimes g_q$ and $\delta^2=\delta\circ\delta=0$.

Proof: Let us introduce the family of s-forms $\omega=\left\{\omega_\mu^k=v_{\mu,I}^k dx^I\right\}$ and set $\left(\delta\omega\right)_\mu^k=dx^i\wedge\omega_{\mu+1_i}^k$. We obtain at once $\left(\delta^2\omega\right)_\mu^k=dx^i\wedge dx^j\wedge\omega_{\mu+1_i+1_j}^k=0$ and $a_k^{\tau\mu}\left(\delta\omega\right)_\mu^k=dx^i\wedge\left(a_k^{\tau\mu}\omega_{\mu+1_i}^k\right)=0$.

□

The kernel of each δ in the first case is equal to the image of the preceding δ but this may no longer be true in the restricted case and we set:

DEFINITION 2.B.4: Let $B_{q+r}^s\left(g_q\right)\subseteq Z_{q+r}^s\left(g_q\right)$ and $H_{q+r}^s\left(g_q\right)=Z_{q+r}^s\left(g_q\right)/B_{q+r}^s\left(g_q\right)$ with $H^s\left(g_q\right)=H_q^s\left(g_q\right)$ be the coboundary space $im\left(\delta\right)$, cocycle space $ker\left(\delta\right)$ and cohomology space at $\wedge^sT^*\otimes g_{q+r}$ of the restricted δ-sequence which only depend on g_q and may not be vector bundles. The symbol g_q is said to be s-*acyclic* if $H_{q+r}^1=\cdots=H_{q+r}^s=0,\forall r\ge 0$, *involutive* if it is n-acyclic and *finite type* if $g_{q+r}=0$ becomes trivially involutive for r large enough. In particular, if g_q is involutive *and* finite type, then $g_q=0$. Finally, $S_qT^*\otimes E$ is involutive for any $q\ge 0$ if we set $S_0T^*\otimes E=E$.

A first point, *not known by physicists*, is provided by the following useful but technical results. As we do not want to provide details about groupoids, we shall introduce a "copy" Y (*target*) of X (*source*) and define simply a Lie pseudogroup $\Gamma\subseteq aut\left(X\right)$ as a group of transformations solutions of a (in general nonlinear) system $\mathcal{R}_q$, such that, whenever $y=f\left(x\right),z=g\left(y\right)\in\Gamma$ can be composed, then $z=g\circ f\left(x\right)\in\Gamma$, $x=f^{-1}\left(y\right)\in\Gamma$ and $y=id\left(x\right)=x\in\Gamma$. Setting $y=x+t\xi\left(x\right)+\cdots$ and passing to the limit when $t\to 0$, we may linearize the later system and obtain a (linear) system $R_q\subset J_q\left(T\right)$ such that $\left[\Theta,\Theta\right]\subset\Theta$. We may use the Frobenius theorem in order to find a generating fundamental set of *differential invariants* $\left\{\Phi^\tau\left(y_q\right)\right\}$ up to order q which are such that $\Phi^\tau\left(\bar{y}_q\right)=\Phi^\tau\left(y_q\right)$ whenever $\bar{y}=g\left(y\right)\in\Gamma$. We obtain the *Lie form* $\Phi^\tau\left(y_q\right)=\Phi_\tau\left(id_q\left(x\right)\right)=\Phi^\tau\left(j_q\left(id\right)\left(x\right)\right)=\omega^\tau\left(x\right)$ of $\mathcal{R}_q$.

Of course, in actual practice *one must use sections of* R_q *instead of solutions* and we now prove why the use of the Spencer operator becomes crucial for such a purpose. Indeed, we may define:

$$\left\{j_{q+1}\left(\xi\right),j_{q+1}\left(\eta\right)\right\}=j_q\left(\left[\xi,\eta\right]\right),\forall\xi,\eta\in T\quad\left(\text{algebraic bracket}\right)$$

We may obtain by bilinearity a bracket on $J_q\left(T\right)$ extending the bracket on T:

$$\left[\xi_q,\eta_q\right]=\left\{\xi_{q+1},\eta_{q+1}\right\}+i\left(\xi\right)d\eta_{q+1}-i\left(\eta\right)d\xi_{q+1},\forall\xi_q,\eta_q\in J_q\left(T\right)\quad\left(\text{differential bracket}\right)$$

which does not depend on the respective lifts ξ_{q+1} and η_{q+1} of ξ_q and η_q in $J_{q+1}\left(T\right)$. This bracket on sections satisfies the Jacobi identity:

$$\boxed{\left[\left[\xi_q,\eta_q\right],\zeta_q\right]+\left[\left[\eta_q,\zeta_q\right],\xi_q\right]+\left[\left[\zeta_q,\xi_q\right],\eta_q\right]=0,\forall\xi_q,\eta_q,\zeta_q\in J_q\left(T\right)}$$

and we set [19]-[21]:

DEFINITION 2.B.5: We say that a vector subbundle $R_q\subset J_q\left(T\right)$ is a *system of infinitesimal Lie equations* or a *Lie algebroid* if $\left[R_q,R_q\right]\subset R_q$, that is to say

$\left[\xi_q,\eta_q\right]\in R_q,\forall\xi_q,\eta_q\in R_q$. Such a definition can be tested by means of computer algebra. We shall also say that R_q is *transitive* if we have the short exact sequence $0\to R_q^0\to R_q\xrightarrow{\pi_0^q}T\to 0$.

THEOREM 2.B.6: The bracket is compatible with prolongations:

$$\left[R_q,R_q\right]\subset R_q \quad\Rightarrow\quad \left[R_{q+r},R_{q+r}\right]\subset R_{q+r},\forall r\geq 0$$

Proof: When $r=1$, we have $\rho_1\left(R_q\right)=R_{q+1}=\left\{\xi_{q+1}\in J_{q+1}\left(T\right)\mid\xi_q\in R_q,d\xi_{q+1}\in T^*\otimes R_q\right\}$ and we just need to use the following formulas showing how d acts on the various brackets if we set $L\left(\xi_1\right)\zeta=\left[\xi,\zeta\right]+i\left(\zeta\right)d\xi_1$ (See [20] and [23] for more details):

$$i\left(\zeta\right)d\left\{\xi_{q+1},\eta_{q+1}\right\}=\left\{i\left(\zeta\right)d\xi_{q+1},\eta_q\right\}+\left\{\xi_q,i\left(\zeta\right)d\eta_{q+1}\right\},\quad\forall\zeta\in T$$

$$\begin{aligned}i\left(\zeta\right)d\left[\xi_{q+1},\eta_{q+1}\right]=&\left[i\left(\zeta\right)d\xi_{q+1},\eta_q\right]+\left[\xi_q,i\left(\zeta\right)d\eta_{q+1}\right]\\&+i\left(L\left(\eta_1\right)\zeta\right)d\xi_{q+1}-i\left(L\left(\xi_1\right)\zeta\right)d\eta_{q+1}\end{aligned}$$

The right member of the second formula is a section of R_q whenever $\xi_{q+1},\eta_{q+1}\in R_{q+1}$. The first formula may be used when R_q is formally integrable.

$\square$

COROLLARY 2.B.7: The bracket is compatible with the PP procedure:

$$\left[R_q,R_q\right]\subset R_q \quad\Rightarrow\quad \left[R_{q+r}^{(s)},R_{q+r}^{(s)}\right]\subset R_{q+r}^{(s)},\forall r,s\geq 0$$

EXAMPLE 2.B.8: When $n=1$, the components at order zero, one, two and three are defined by the unusual successive formulas:

$$\left[\xi,\eta\right]=\xi\partial_x\eta-\eta\partial_x\xi$$

$$\left(\left[\xi_1,\eta_1\right]\right)_x=\xi\partial_x\eta_x-\eta\partial_x\xi_x$$

$$\left(\left[\xi_2,\eta_2\right]\right)_{xx}=\xi_x\eta_{xx}-\eta_x\xi_{xx}+\xi\partial_x\eta_{xx}-\eta\partial_x\xi_{xx}$$

$$\left(\left[\xi_3,\eta_3\right]\right)_{xxx}=2\xi_x\eta_{xxx}-2\eta_x\xi_{xxx}+\xi\partial_x\eta_{xxx}-\eta\partial_x\xi_{xxx}$$

That can be used for linear ($\xi_x=0$), affine ($\xi_{xx}=0$) or projective ($\xi_{xxx}=0$) transformations.

EXAMPLE 2.B.9: With $n=m=2$ and $q=1$, let us consider the Lie pseudodogroup $\Gamma\subset aut\left(X\right)$ of finite transformations $y=f\left(x\right)$ such that $y^2dy^1=x^2dx^1=\alpha$. Setting $y=x+t\xi\left(x\right)+\cdots$ and linearizing, we get the Lie operator $\mathcal{D}\xi=\mathcal{L}\left(\xi\right)\alpha$ where $\mathcal{L}$ is the Lie derivative and the system $R_1\subset J_1\left(T\right)$ of linear infinitesimal Lie equations:

$$x^2d_1\xi^1+\xi^2=0,\quad d_2\xi^1=0$$

Replacing $j_1\left(\xi\right)$ by a section $\xi_1\in J_1\left(T\right)$, we have:

$$\xi_1^1+\frac{1}{x^2}\xi^2=0,\quad\xi_2^1=0$$

Let us choose the two sections:

$$\xi_1=\left\{\xi^1=0,\xi^2=-x^2,\xi_1^1=1,\xi_2^1=0,\xi_1^2=0,\xi_2^2=0\right\}\in R_1$$

$$\eta_1 = \left\{ \eta^1 = x^2, \eta^2 = 0, \eta_1^1 = 0, \eta_2^1 = -x^2, \eta_1^2 = 0, \eta_2^2 = 1 \right\} \in R_1$$

We let the reader check that $\left[\xi_1, \eta_1\right] \in R_1$. However, we have the strict inclusion $R_1^{(1)} \subset R_1$ defined by the additional equation $\xi_1^1 + \xi_2^2 = 0$ and thus $\xi_1, \eta_1 \notin R_1^{(1)}$ though we have indeed $\left[R_1^{(1)}, R_1^{(1)}\right] \subset R_1^{(1)}$, a result not evident because ξ_1 and η_1 have *nothing to do* with solutions.

2.3. Janet and Spencer Sequences

Let us prove that the interpretation of the Spencer sequence is coherent with mechanics and electromagnetism both with their well known couplings [24] [25]. In a word, the problem we have to solve is to get a 2-form in $\wedge^2 T^*$ from a 1-form in $T^* \otimes R_q$ for a certain $R_q \subset J_q(T)$.

For this purpose, introducing the *Spencer map* $\delta : \wedge^s T^* \otimes S_{q+1} T^* \otimes E \to \wedge^{s+1} T^* \otimes S_q T^* \otimes E$ defined by $(\delta\omega)_\mu^k = dx^i \wedge \omega_{\mu+1_i}^k$, we recall from [19] [20] the definition of the *Janet bundles* $F_r = \wedge^r T^* \otimes J_q(E) \big/ \left(\wedge^r T^* \otimes R_q + \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E\right)\right)$ and the *Spencer bundles* $C_r = \wedge^r T^* \otimes R_q \big/ \delta\left(\wedge^{r-1} T^* \otimes g_{q+1}\right)$ or $C_r(E) = \wedge^r T^* \otimes J_q(E) \big/ \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E\right)$ with $C_r \subset C_r(E)$. When $R_q \subset J_q(E)$ is an involutive system on E, we have the commutative and exact *Fundamental Diagram I* where each operator involved is first order apart from $\mathcal{D} = \Phi \circ j_q$, generates the CC of the preceding one and is induced by the extension $D : \wedge^r T^* \otimes J_{q+1}(E) \to \wedge^{r+1} T^* \otimes J_q(E) : \alpha \otimes \xi_{q+1} \to d\alpha \otimes \xi_q + (-1)^r \alpha \wedge D\xi_{q+1}$ of the Spencer operator $D : J_{q+1}(E) \to T^* \otimes J_q(E) : \xi_{q+1} \to j_1(\xi_q) - \xi_{q+1}$. The upper sequence is the *Spencer sequence* while the lower sequence is the *Janet sequence* [7] [26] and the sum $dim(C_r) + dim(F_r) = dim(C_r(E))$ does not depend on the system while the epimorphisms Φ_r are induced by $\Phi = \Phi_0$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2} \cdots \xrightarrow{D_n} & C_n & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & E & \xrightarrow{j_q} & C_0(E) & \xrightarrow{D_1} & C_1(E) & \xrightarrow{D_2} \cdots \xrightarrow{D_n} & C_n(E) & \to & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_n & & \\
0 & \to & \Theta & \to & E & \xrightarrow[q]{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} & F_n & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}$$

For later computations, the sequence $J_{q+1}(E) \xrightarrow{d} T^* \otimes J_q(E) \xrightarrow{d} \wedge^2 T^* \otimes J_{q-1}(E)$ can be described by the images $\partial_i \xi_\mu^k - \xi_{\mu+1_i}^k = X_{\mu,i}^k$ leading to the *identities*::

$$\boxed{\partial_i X_{\mu,j}^k - \partial_j X_{\mu,i}^k + X_{\mu+1_j,i}^k - X_{\mu+1_i,j}^k = 0}$$

We also recall that the linear Spencer sequence for a Lie group of transformations $G \times X \to X$, which *essentially* depends on the action because infinitesimal generators are needed, is locally isomorphic to the linear gauge sequence which does not depend on the action any longer as it is the tensor product of the Poincaré sequence by the Lie algebra $\mathcal{G}$ of G.

The main idea will be to introduce and compare the three Lie groups of transformations:

- The *Poincare group* of transformations with 10 parameters leading to the *Killing system* R_2:

$$\Omega_{ij} \equiv \left(L(\xi_1)\omega\right)_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r\partial_r\omega_{ij}(x) = 0$$

$$\Gamma_{ij}^k \equiv \left(L(\xi_2)\gamma\right)_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r\partial_r\gamma_{ij}^k(x) = 0$$

- The *Weyl group* of transformations with 11 parameters leading to the *Weyl system* $\tilde{R}_2$:

$$\Omega_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r\partial_r\omega_{ij}(x) = 2A(x)\omega_{ij}$$

$$\Gamma_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r\partial_r\gamma_{ij}^k(x) = 0$$

- The *conformal group* of transformations with 15 parameters (4 translations + 6 rotations + 1 dilatation + 4 elations) leading to the *conformal Killing system* $\hat{R}_2$:

$$\Omega_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r\partial_r\omega_{ij}(x) = 2A(x)\omega_{ij}(x)$$

$$\begin{aligned}\Gamma_{ij}^k &\equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r\partial_r\gamma_{ij}^k(x)\\ &= \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}(x)\omega^{kr}(x)A_r(x)\end{aligned}$$

where one has to eliminate the arbitrary function $A(x)$ and 1-form $A_i(x)dx^i$ for finding sections, replacing the *ordinary Lie derivative* $\mathcal{L}(\xi)$ by the *formal Lie derivative* $L(\xi_q)$, that is replacing $j_q(\xi)$ by ξ_q when needed. When $n=4$, $\hat{R}_2$ is FI but $\hat{g}_2$ is only 2-acyclic while $\hat{g}_3 = 0$ and we have for the involutive $\hat{R}_3 \simeq \hat{R}_2$ (See [27] [28] for details and counterexamples):

$$\begin{array}{ccccccccccccccc}
 & & & & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & \hat{\Theta} & \xrightarrow{j_3} & 15 & \xrightarrow{D_1} & 60 & \xrightarrow{D_2} & 90 & \xrightarrow{D_3} & 60 & \xrightarrow{D_4} & 15 & \to 0\\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & 4 & \underset{3}{\xrightarrow{j_3}} & 140 & \xrightarrow{D_1} & 420 & \xrightarrow{D_2} & 504 & \xrightarrow{D_3} & 280 & \xrightarrow{D_4} & 60 & \to 0\\
 & & & \| & & \downarrow\Phi_0 & & \downarrow\Phi_1 & & \downarrow\Phi_2 & & \downarrow\Phi_3 & & \downarrow\Phi_4 & \\
0\to & \hat{\Theta} & \to & 4 & \underset{3}{\xrightarrow{\hat{\mathcal{D}}}} & 125 & \underset{1}{\xrightarrow{\hat{\mathcal{D}}_1}} & 360 & \underset{1}{\xrightarrow{\hat{\mathcal{D}}_2}} & 414 & \underset{1}{\xrightarrow{\hat{\mathcal{D}}_3}} & 220 & \underset{1}{\xrightarrow{\hat{\mathcal{D}}_4}} & 45 & \to 0\\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

The top Spencer sequence is the tensor product of the Poincaré sequence by the Lie algebra $\hat{\mathcal{G}}$ of dimension 15 and we may use the inclusions $R_2 \subset \tilde{R}_2 \subset \hat{R}_2 \subset J_2(T)$ with $10 < 11 < 15 < 60$. Working by induction, the minimum formally exact resolution on the jet level is:

$$\boxed{0 \rightarrow \hat{\Theta} \rightarrow 4 \underset{1}{\rightarrow} 9 \underset{2}{\rightarrow} 10 \underset{2}{\rightarrow} 9 \underset{1}{\rightarrow} 4 \rightarrow 0}$$

with "*up and down*" orders that must be compared to the above canonical Janet sequence.

Finding such numbers has been done by my former PhD student A. Quadrat (INRIA) by means of computer algebra (arXiv: 1603.05030) but it is not possible to prove that such a sequence is formally exact as it involves *enormous* matrices (up to 840×1134!!!) and can only be achieved using the Spencer δ-cohomology, still *never* introduced in GR or conformal geometry [8] [26].

When ω is the M metric, it follows that $\gamma = 0$ and we obtain therefore:

$$X^r_{ri,j} - X^r_{rj,i} = \partial_i X^r_{r,j} - \partial_j X^r_{r,i} = n\partial_i\left(\partial_j A - A_j\right) - n\partial_j\left(\partial_i A - A_j\right) = n\left(\partial_i A_j - \partial_j A_i\right)$$

Dividing by n, we may thus obtain $\left(F_{ij} = \partial_i A_j - \partial_j A_i\right) \in \wedge^2 T^*$ from $X^r_{rj,i} \in T^* \otimes \hat{g}_2 \subset C_1$ with $dF = 0$ because $\hat{g}_3 = 0$ and thus $\xi^k_{rij} = 0$ in $S_3 T^* \otimes T$.

This result is solving the dream of H. Weyl for exhibiting the conformal origin of electromagnetism in [29]. It is however completely contradicting the standard approach of classical gauge theory based on the group $U(1)$ which is not acting on space-time. In addition, the EM field F is a section of the first Spencer bundle C_1 in the image of D_1 because $\left(A, A_i\right) \in C_0 = \hat{R}_3 \simeq \hat{R}_2$.

For a later use, we provide a few additional results on the linearization procedure which is only a part of the so-called *vertical machinery* of Spencer. First of all, the Riemann tensor is:

$$\rho^k_{l,ij} = \partial_i \gamma^k_{lj} - \partial_j \gamma^k_{li} + \gamma^r_{lj}\gamma^k_{ri} - \gamma^r_{li}\gamma^k_{rj}$$

Now, as the linearization $\Gamma \in S_2 T^* \otimes T$ of γ is a tensor, the linearization R of ρ becomes:

$$\begin{aligned}
R^k_{l,ij} &= d_i\Gamma^k_{lj} - d_j\Gamma^k_{li} + \gamma^r_{lj}\Gamma^k_{ri} - \gamma^r_{li}\Gamma^k_{rj} + \gamma^k_{ri}\Gamma^r_{lj} - \gamma^k_{rj}\Gamma^r_{li} \\
&= \left(d_i\Gamma^k_{lj} - \gamma^r_{li}\Gamma^k_{rj} + \gamma^k_{ri}\Gamma^r_{lj}\right) - \left(d_j\Gamma^k_{li} - \gamma^r_{lj}\Gamma^k_{ri} + \gamma^k_{rj}\Gamma^r_{li}\right) \\
&= \left(d_i\Gamma^k_{lj} - \gamma^r_{li}\Gamma^k_{rj} - \gamma^r_{ji}\Gamma^k_{lr} + \gamma^k_{ri}\Gamma^r_{lj}\right) - \left(d_j\Gamma^k_{li} - \gamma^r_{lj}\Gamma^k_{ri} - \gamma^r_{ij}\Gamma^k_{lr} + \gamma^k_{rj}\Gamma^r_{li}\right) \\
&= \nabla_i\Gamma^k_{lj} - \nabla_j\Gamma^k_{li}
\end{aligned}$$

by introducing the covariant derivative ∇. We recall that $\nabla_r\omega_{ij} = 0, \forall r, i, j$ or, equivalently, that $\omega_{sj}\gamma^s_{ir} + \omega_{is}\gamma^s_{jr} = \partial_r\omega_{ij}$, a result allowing to move down the index k in the previous formulas.

We may thus take into account the Bianchi identities implied by the cyclic sums on (ijr)

$$\beta_{kl,ijr} \equiv \nabla_r\rho_{kl,ij} + \nabla_i\rho_{kl,jr} + \nabla_j\rho_{kl,ri} = 0 \quad \Leftrightarrow \quad \beta \equiv \sum_{cycl}\left(\partial\rho - \gamma\rho\right) = 0$$

and their respective linearizations $B_{kl,ijr} = 0$. In fact, β and B are sections of the vector bundle F_2 defined by the short exact sequence:

$$0 \to F_2 \to \wedge^3 T^* \otimes g_1 \xrightarrow{\delta} \wedge^4 T^* \otimes T \to$$

$$dim(F_2) = (n(n-1)(n-2)/6)(n(n-1)/2) - (n(n-1)(n-2)(n-3)/24)n$$
$$= n^2(n^2-1)(n-2)/24$$

because $dim(g_1) = n(n-1)/2$ for *any* nondegenerate metric, that is $24 - 4 = 20$ when $n = 4$.

Such results cannot be even imagined by somebody not aware of the δ-acyclicity ([10] [11] [18]).

We have the linearized cyclic sums of covariant derivatives both with their respective symbolic descriptions, not to be confused with the non-linear corresponding ones:

$$B_{kl,rij} \equiv \nabla_r R_{kl,ij} + \nabla_i R_{kl,jr} + \nabla_j R_{kl,ri} = 0 \ mod(\Gamma) \quad \Leftrightarrow \quad \sum_{cycl}(dR - \gamma R - \rho\Gamma) = 0$$
$$\Leftrightarrow \quad B \equiv \sum_{cycl}(\nabla R) = \sum_{cycl}(\rho\Gamma)$$

In order to recapitulate these new concepts obtained after one, two or three prolongations, we have successively $\omega \to \gamma \to \rho \to \beta$ and the respective linearizations $\Omega \to \Gamma \to R \to B$.

3. Applications

We shall study together and similarly the Minkowski, the Schwarzschild and the Kerr metrics.

In the Boyer-Lindquist (BL) coordinates $(t,r,\theta,\phi) = (x^0,x^1,x^2,x^3)$, the Schwarzschild metric is $\omega = A(r)dt^2 - (1/A(r))dr^2 - r^2 d\theta^2 - r^2\sin^2(\theta)d\phi^2$ and $\xi = \xi^i d_i \in T$, let us introduce $\xi_i = \omega_{ri}\xi^r$ and the 4 *formal derivatives* $(d_0 = d_t, d_1 = d_r, d_2 = d_\theta, d_3 = d_\phi)$. With speed of light $c = 1$ and $A = 1 - \frac{m}{r}$ where m is a constant, the metric can be written in the diagonal form $(A, -1/A, -r^2, -r^2\sin^2(\theta))$ with a surprisingly simple determinant $det(\omega) = -r^4\sin^2(\theta)$. Using the notations of jet theory, we may consider the infinitesimal Killing equations:

$$\Omega_{ij} \equiv \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 0$$

and the Christoffel symbols γ through the standard Levi-Civita isomorphism $j_1(\omega) \simeq (\omega,\gamma)$ while setting $A' = \partial_r A$ in the differential field K of coefficients. We obtain:

$$\xi_0^0 = -\frac{A'}{2A}\xi^1, \xi_1^1 = +\frac{A'}{2A}\xi^1, \xi_2^2 = -\frac{1}{r}\xi^1, \xi_3^3 = -\frac{1}{r}\xi^1 - \cot(\theta)\xi^2$$
$$\Rightarrow \xi_0^0 + \xi_1^1 = 0, \xi_2^2 + \xi_3^3 = -\cot(\theta)\xi^2$$

Let us now introduce the Riemann tensor $(\rho_{l,ij}^k) \in \wedge^2 T^* \otimes T^* \otimes T$ and use the metric in order to raise or lower the indices in order to obtain the purely covariant tensor $(\rho_{kl,ij}) \in \wedge^2 T^* \otimes T^* \otimes T^*$. Then, using r as an implicit summation index, we may consider the first order equations:

$$R_{kl,ij} \equiv \rho_{rl,ij}\xi_k^r + \rho_{kr,ij}\xi_l^r + \rho_{kl,rj}\xi_i^r + \rho_{kl,ir}\xi_j^r + \xi^r\partial_r\rho_{kl,ij} = 0$$

that can be considered as an infinitesimal variation. As for the Ricci tensor $(\rho_{ij}) \in S_2T^*$, we notice that $\rho_{ij} = \rho_{i,rj}^r = 0 \Rightarrow R_{ij} \equiv \rho_{rj}\xi_i^r + \rho_{ir}\xi_j^r + \xi^r\partial_r\rho_{ij} = 0$.

The 6 non-zero components of the Riemann tensor are known to be:

$$\boxed{\begin{array}{lll} \rho_{01,01} = +\dfrac{m}{r^3}, & \rho_{02,02} = -\dfrac{mA}{2r}, & \rho_{03,03} = -\dfrac{mA\sin^2(\theta)}{2r} \\ \rho_{12,12} = +\dfrac{m}{2rA}, & \rho_{13,13} = +\dfrac{m\sin^2(\theta)}{2rA}, & \rho_{23,23} = -mr\sin^2(\theta) \end{array}}$$

However, *as we are dealing with sections*, $\xi^1 = 0$ implies $\xi_0^0 = 0$, $\xi_1^1 = 0$, $\xi_2^2 = 0$ ... but *NOT* (*care*) $\xi_0^1 = 0$, these later condition being only brought by one additional prolongation and we have the strict inclusions $R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)} = R_1 \subset J_1(T)$ with dimensions $4 < 5 < 10 = 10 < 20$, determined exactly like we did in the Introduction. Indeed, we have already proved in [13] that two prolongations bring the five new equations:

$$\boxed{\xi^1 = 0,\ \xi_2^1 = 0,\ \xi_3^1 = 0,\ \xi_2^0 = 0,\ \xi_3^0 = 0}$$

and a new prolongation *only* brings the single equation $\boxed{\xi_0^1 = 0}$, leading to $dim\left(R_1^{(3)}\right) = 4$.

The group of invariance is thus made by the time translation and the three space rotations.

As $R_2^{(3)} \subset J_2(T)$ is involutive and does not depend any longer on m, the Spencer sequence is:

$$\boxed{0 \rightarrow \Theta \xrightarrow{j_2} 4 \xrightarrow{D_1} 16 \xrightarrow{D_2} 24 \xrightarrow{D_3} 16 \xrightarrow{D_4} 4 \rightarrow 0}$$

Using the Spencer operator and the fact that $\xi^1 \in j_2(\Omega)$, we obtain the 3 third order CC:

$$\boxed{d_1\xi^1 - \xi_1^1 = 0,\ d_2\xi^1 - \xi_2^1 = 0,\ d_3\xi^1 - \xi_3^1 = 0}$$

in which we have to use $\xi_1^1 = \dfrac{A'}{2A}\xi^1 \in j_2(\Omega), \xi_2^1 \in j_2(\Omega), \xi_3^1 \in j_2(\Omega)$.

We now write the Kerr metric in Boyer-Lindquist coordinates:

$$ds^2 = \frac{\rho^2 - mr}{\rho^2}dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{2amr\sin^2(\theta)}{\rho^2}dtd\phi$$
$$-\left(r^2 + a^2 + \frac{mra^2\sin^2(\theta)}{\rho^2}\right)\sin^2(\theta)d\phi^2$$

where we have set $\Delta = r^2 - mr + a^2, \rho^2 = r^2 + a^2\cos^2(\theta)$ as usual and we check that we recover the Schwarschild metric when $a = 0$. We notice that t or ϕ do not appear in the coefficients of the metric. We shall change the coordinate system in order to confirm theses results by using computer algebra and the idea is to use the so-called *"rational polynomial"* coefficients as follows:

$$\left(x^0=t, x^1=r, x^2=c=\cos(\theta), x^3=\phi\right)$$
$$\Rightarrow dx^2=-\sin(\theta)d\theta \Rightarrow \left(dx^2\right)^2=\left(1-c^2\right)d\theta^2$$

We obtain over the differential field $K=\mathbb{Q}(a,m)(t,r,c,\phi)=\mathbb{Q}(a,m)(x)$:

$$ds^2=\frac{\rho^2-mx^1}{\rho^2}\left(dx^0\right)^2-\frac{\rho^2}{\Delta}\left(dx^1\right)^2-\frac{\rho^2}{1-\left(x^2\right)^2}\left(dx^2\right)^2-\frac{2amx^1\left(1-\left(x^2\right)^2\right)}{\rho^2}dx^0dx^3$$
$$-\left(1-\left(x^2\right)^2\right)\left(\left(x^1\right)^2+a^2+\frac{ma^2x^1\left(1-\left(x^2\right)^2\right)}{\rho^2}\right)\left(dx^3\right)^2$$

with now $\Delta=\left(x^1\right)^2-mx^1+a^2=r^2-mr+a^2$ and $\rho^2=\left(x^1\right)^2+a^2\left(x^2\right)^2=r^2+a^2c^2$. For a later use, it is also possible to set $\omega_{33}=-\left(1-c^2\right)\left(\left(r^2+a^2\right)^2-a^2\left(1-c^2\right)\left(a^2-mr+r^2\right)\right)\Big/\left(r^2+a^2c^2\right)$ and we have $det(\omega)=-\left(r^2+a^2c^2\right)^2$ in a coherent way with the fact that the S metric that can be written $\left(A,-\frac{1}{A},-\frac{r^2}{\sin^2(\theta)},-r^2\sin^2(\theta)\right)$ in the new system of coordinates.

We obtain the Lie algebroid $R_1\subset J_1(T)$:

$$\left[R_q,R_q\right]\subset R_q \Rightarrow \left[R_{q+r}^{(s)},R_{q+r}^{(s)}\right]\subset R_{q+r}^{(s)}, \forall q,r,s\geq 0$$

As $R_1^{(1)}=\pi_1^2\left(R_2\right)=R_1$, it follows that $R_1^{(2)}=\pi_1^3\left(R_3\right)$ is such that $\left[R_1^{(2)},R_1^{(2)}\right]\subset R_1^{(2)}$ with $dim\left(R_1^{(2)}\right)=20-16=4$ because we have obtained a total of 6 *new different* first order equations.

$$\boxed{\xi^1=0,\xi^2=0\Rightarrow \xi_1^1=0,\xi_2^2=0}, \boxed{\xi_3^0=0,\xi_2^1=0\Leftrightarrow \xi_0^3=0,\xi_1^2=0,\xi_0^0=0,\xi_3^3=0}$$

Now, the system of 4 linear equations $R_{01,03}=0, R_{03,23}=0, R_{03,13}=0, R_{0203}=0$ for the 4 jets $\left(\xi_0^1,\xi_0^2,\xi_3^1,\xi_3^2\right)$ has rank 2 for both the S and K metrics thanks to the 2 striking identities:

$$\boxed{R_{03,13}+a\left(1-c^2\right)R_{01,03}=0,\ R_{02,03}+\frac{a}{r^2+a^2}R_{03,23}=0}$$

Similarly to the S metric, two prolongations provide 6 additional equations (instead of 5) that we set on the left side in the following list obtained $mod\left(j_2(\Omega)\right)$:

We have *on sections* (*care*) the 16 (linear) equations $mod\left(j_2(\Omega)\right)$ of $R_1^{(2)}$ as follows ([13]):

$$R_1^{(2)}\subset R_1\subset J_1(T)\begin{cases}\xi^1=0,\xi^2=0 & \Rightarrow \boxed{\omega_{00}\xi_1^0+\omega_{03}\xi_1^3}+\omega_{11}\xi_0^1=0,\ \xi_1^1=0,\ \xi_2^2=0\\ \xi_2^1=0 & \Rightarrow \xi_1^2=0\\ \xi_3^1+lin\left(\xi_0^1,\xi_0^2\right)=0 & \Rightarrow \boxed{\omega_{03}\xi_1^0+\omega_{33}\xi_1^3}+\omega_{11}\xi_3^1=0\\ \xi_3^2+lin\left(\xi_0^1,\xi_0^2\right)=0 & \Rightarrow \boxed{\omega_{00}\xi_2^0+\omega_{03}\xi_2^3}+\omega_{22}\xi_0^2=0,\\ & \quad\ \boxed{\omega_{03}\xi_2^0+\omega_{33}\xi_2^3}+\omega_{22}\xi_3^2=0\\ \xi_3^0=0 & \Rightarrow \xi_0^3=0,\ \xi_0^0=0,\ \xi_3^3=0\end{cases}$$

and the coefficients in the linear equations lin involved depend on the Riemann tensor. Accordingly, we may choose only the 2 parametric jets $\left(\xi_0^1, \xi_0^2\right)$ among $\left(\xi_0^1, \xi_3^1, \xi_0^2, \xi_3^2\right)$ to which we must add $\left(\xi^0, \xi^3\right)$ *in any case* as they are not appearing in the Killing equations.

The system is *not* involutive because its symbol is finite type but non-zero.

Using diagrams like in the motivating examples, we discover that the operator defining R_1 has $10+4=14$ CC of order 2, a result obtained *totally independently of any specific GR technical object* like the *Teukolski scalars* or the *Killing-Yano tensors* introduced in [14]-[16].

Using one more prolongation, all the *sections* (*care again*) vanish but ξ^0 and ξ^3, a result leading to $dim\left(R_1^{(3)}\right)=2$ in a coherent way with the only nonzero Killing vectors $\left\{\partial_t, \partial_\phi\right\}$. We have indeed:

$$\boxed{\xi_0^1=0}, \boxed{\xi_0^2=0} \Leftrightarrow \xi_3^1=0, \xi_3^2=0 \Rightarrow \xi_1^0=0, \xi_1^0=0, \xi_2^0=0, \xi_2^3=0$$

Taking therefore into account that the metric only depends on $\left(x^1=r, x^2=\cos(\theta)\right)$ we obtain *after three prolongations* the first order system:

$$R_1^{(3)} \subset R_1^{(2)} \subset R_1^{(1)}=R_1 \subset J_1(T) \Leftrightarrow 2<4<10=10<20$$

Surprisingly and contrary to the situation found for the S metric, we have now an involutive first order system with only solutions $\left(\xi^0=cst, \xi^1=0, \xi^2=0, \xi^3=cst\right)$ and notice that $R_1^{(3)}$ does not depend any longer on the parameters $(m,a)\in K$. The difficulty is to know what second members must be used along the procedure met for all the motivating examples. In particular, we have again identities to zero like $d_0\xi^1-\xi_0^1=0, d_0\xi^2-\xi_0^2=0$ and thus 6 third order CC coming from the 6 following components of the Spencer operator, namely:

$$\boxed{d_1\xi^1-\xi_1^1=0, d_2\xi^1-\xi_2^1=0, d_3\xi^1-\xi_3^1=0, d_1\xi^2-\xi_1^2=0, d_2\xi^2-\xi_2^2=0, d_3\xi^2-\xi_3^2=0}$$

a result that cannot be even imagined from [14]. Of course, proceeding like in the motivating examples, we must substitute in the right members the values obtained from $j_2(\Omega)$ and set for example $\xi_1^1=-\frac{1}{2\omega_{11}}\xi\partial\omega_{11}$ while replacing ξ^1 and ξ^2 by the corresponding linear combinations of the Riemann tensor already obtained for the right members of the two zero order equations.

The corresponding *Fundamental Diagram I* is no longer depending on (m,a) as follows:

$$\boxed{\begin{array}{ccccccccccccccc}
 & & & & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \rightarrow & \Theta & \xrightarrow{j_1} & 2 & \xrightarrow{D_1} & 8 & \xrightarrow{D_2} & 12 & \xrightarrow{D_3} & 8 & \xrightarrow{D_4} & 2 & \rightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \rightarrow & 4 & \xrightarrow{j_1} & 20 & \xrightarrow{D_1} & 40 & \xrightarrow{D_2} & 40 & \xrightarrow{D_3} & 20 & \xrightarrow{D_4} & 4 & \rightarrow 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0\rightarrow & \Theta & \rightarrow & 4 & \xrightarrow{\mathcal{D}} & 18 & \xrightarrow{\mathcal{D}_1} & 32 & \xrightarrow{\mathcal{D}_2} & 28 & \xrightarrow{\mathcal{D}_3} & 12 & \xrightarrow{\mathcal{D}_4} & 2 & \rightarrow 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}}$$

with the Euler-Poincaré characteristic $4-18+32-28+12-2=0$. However, the only intrinsic concepts associated with a differential sequence are the *"extension modules"* that only depend on the Kerr differential module but *not* on the differential sequence and we repeat once more that:

THE ONLY IMPORTANT CONCEPT IS THE GROUP INVOLVED, NOT THE METRIC.

Needless to say that the group involved in this case has no physical usefulness.

4. Conclusions

When a linear partial differential operator $\mathcal{D}\xi=\eta$ is given, a *direct* problem is to look for the generating compatibility conditions $\mathcal{D}_1\eta=0$ that must be satisfied by η. Similarly, if $\mathcal{D}_1\eta=\zeta$ is given, one may look for CC of the form $\mathcal{D}_2\zeta=0$ and so on. The mathematical community (and we do not speak about the physical community!) is of course aware of such a "*step by step*" way but is *not at all* aware of the existence of another "*as a whole*" procedure allowing to define the various differential operators of the differential sequence thus obtained apart from the very specific situation of the Poincaré (in France!) sequence for the exterior derivative that admits a unique defining formula for each operator *separately*. The best known case is that of Riemannian geometry and its application to general relativity with the successive Killing, Riemann and Bianchi operators of first, second and third order respectively. In particular, we may ask "*Who knows about the Spencer operator and the corresponding Spencer sequence?"* at the heart of this paper.

In the Introduction, we have explained and illustrated through five motivating examples that, when a second order differential operator $\mathcal{D}$ is depending on constant or variable coefficients, its generating compatibility conditions (CC) may be of first, second, third and even sixth or higher order, a result largely depending on the parameters. In the meantime, we have shown that the solution of this problem for a system of order q *cannot* be obtained without bringing such a system to an involutive form or *at least* to a formally integrable form of order $q+r$ after differentiating $r+s$ times the equations while keeping only the equations left up to order $q+r$ in such a way that the order of the CC is *at most* $r+s+1$.

From a completely different point of view, the Spencer differential sequence is obtained by bringing any involutive system $R_q\subset J_q(E)$ to a *first order* involutive system $R_{q+1}\subset J_1(R_q)$ having an isomorphic space of solutions or, with a more precise language, allowing to define a differential module isomorphic to the differential module M defined by the initial system. The quotient of the Spencer sequence for the first order trivially involutive first order system

$J_{q+1}(E)\subset J_1(J_q(E)$ by the previous Spence sequence which is induced by the inclusion $R_q\subset J_q(E)$ is the well defined finite length differential Janet sequence introduced by M. Janet as a footnote in 1920 which is thus providing another resolution of the same space of solutions or of the differential module M already defined. According to a very difficult theorem of (differential) homological algebra, the *only* objects that do not depend of the resolution used are the (differential) *extension modules* that are measuring the fact that the corresponding dual sequence made by the respective formal adjoint of the operators involved and going thus " *backwards*" (that is from right to left) may not be exact, that is *each operator may not generate the CC of the preceding one*. It thus follows that the Spencer and Janet sequences will bring the same formal information *as a whole*, even though, *in actual practice*, we proved that they can be completely different.

It may happen, for example with the Schwarzschild and Kerr metrics, with $q=1, r=0, s=3$, that the final corresponding FI systems will not depend any longer on the parameters involved initially. Accordingly, the only important object to consider is *not* the metric but its group G of invariance which is used through the fact that the Spencer sequence is the tensor product of the Poincaré sequence by its Lie algebra $\mathcal{G}$, the main formal reason for which black holes cannot exist.

We have thus finally proved that the main idea, along the dream of H. Weyl in 1918, is *not* to shrink the dimension of this group from 10 down to to 4 or 2 parameters by using the S or K metrics instead of the M metric but, *on the contrary*, to enlarge the group from 10 up to 11 or 15 parameters by using the Weyl or conformal group instead of the Poincar' group of space-time while using the adjoint of the respective Spencer sequences [17]. It will follow that the first set of Maxwell equations is obtained by a projection of the second Spencer operator D_2 that can be parametrized by a projection of the first Spencer operator D_1, a result contradicting the basic assumption of classical gauge theory in which they are induced by D_3. The main problem today is that, in the minimum resolution of the conformal Killing operator, the generating CC of the second order Weyl operator are also made by a second order operator, a result confirmed by my PhD student A. Quadrat (INRIA) in 2016 [arXiv: 1603.05030, 26] but still not acknowledged and showing that conformal geometry but me revisited by using the Spencer δ-cohomology.

With more details, if $\mathcal{D}\xi=\eta$ has generating CC $\mathcal{D}_1\eta=0$, then $\mathcal{D}_1\circ\mathcal{D}=0 \Rightarrow ad(\mathcal{D})\circ ad(\mathcal{D}_1)=0$ but $ad(\mathcal{D})$ may not generate the CC of $ad(\mathcal{D}_1)$ while $ad(D_1)$ may not generate all the CC of $ad(D_2)$ as can be seen in the motivating examples (See [30]-[32] for other examples and homological tools).

Finally, in a more general framework, when a Lie group G is acting on a manifold X of dimension n, the Spencer sequence is always locally and formally exact, being isomorphic to the tensor product of the Poincaré sequence by the Lie algebra of G. On the contrary, the corresponding formally exact Janet sequence may have Janet bundles of high dimensions. The last operator $\mathcal{D}_n$ is always surjective while its adjoint is always injective. However, $ad(\mathcal{D}_{n-1})$ may not define all the CC of $ad(\mathcal{D}_n)$ because $ad(D_n)$ may fail to be injective. Applying these methods to the conformal group of transformations when $n=4$, we discovered in the very recent [17] the common conformal origin of the Cauchy, Cosserat, Clausius and Maxwell equations. For example, the EM field F comes from the composition of epimorphisms:

$$\boxed{\hat{C}_1 \to \hat{C}_1/\tilde{C}_1 = \left(T^* \otimes \hat{R}_2\right)/\left(T^* \otimes \tilde{R}_2\right) \simeq T^* \otimes \left(\hat{R}_2/\tilde{R}_2\right) \simeq T^* \otimes \hat{g}_2 \simeq T^* \otimes T^* \xrightarrow{\delta} \wedge^2 T^*}$$

while the EM potential comes from the composition of epimorphisms:

$$\hat{C}_0 \to \hat{C}_0/\tilde{C}_0 \simeq \hat{R}_2/\tilde{R}_2 \simeq \hat{g}_2 \simeq T^*$$

and the parametrization $dA = F$ is induced by D_1 while the Maxwell equation $dF = 0$ is induced by D_2. Using the short exact splitting sequence $0 \to S_2T^* \xrightarrow{\delta} T^* \otimes T^* \xrightarrow{\delta} \wedge^2 T^* \to 0$ leading to the isomorphism $T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^* \simeq \left(R_{ij}\right) \oplus \left(F_{ij}\right)$ that only depends on the elations of the conformal group, one obtains the *Fundamental Diagram II* showing the common conformal origin of electromagnetism and gravitation, already published as early as in … 1983 [33] and referring the reader to [34] for more details or applications and to [35] for a summary. However, no one of these results could have been even imagined by Weyl because the Riemann operator must be replaced by the Weyl operator while the Bianchi operator must be replaced by a second order CC operator when $n = 4$, a result still not known today in the conformal framework [8].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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