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Quantization of Action for Elementary Particles and the Principle of Least Action

Shuming Wen

Faculty of Science, Kunming University of Science and Technology, Kunming, China
Email: shmwen@126.com

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Abstract

The uncertainty principle is a fundamental principle of quantum mechanics, but its exact mathematical expression cannot obtain correct results when used to solve theoretical problems such as the energy levels of hydrogen atoms, one-dimensional deep potential wells, one-dimensional harmonic oscillators, and double-slit experiments. Even after approximate treatment, the results obtained are not completely consistent with those obtained by solving Schrödinger's equation. This indicates that further research on the uncertainty principle is necessary. Therefore, using the de Broglie matter wave hypothesis, we quantize the action of an elementary particle in natural coordinates and obtain the quantization condition and a new deterministic relation. Using this quantization condition, we obtain the energy level formulas of an elementary particle in different conditions in a classical way that is completely consistent with the results obtained by solving Schrödinger's equation. A new physical interpretation is given for the particle eigenfunction independence of probability for an elementary particle: an elementary particle is in a particle state at the space-time point where the action is quantized, and in a wave state in the rest of the space-time region. The space-time points of particle nature and the wave regions of particle motion constitute the continuous trajectory of particle motion. When an elementary particle is in a particle state, it is localized, whereas in the wave state region, it is nonlocalized.

Keywords

Elementary Particle, Quantization of Action, Deterministic Relation, Inherent State Nonprobabilistic Interpretation, Localization Region, Nonlocalization Region

1. Introduction

Quantum mechanics has achieved great success as one of the pillars of modern physics. Its theoretical calculations are highly consistent with experiments, and it is considered to be the most accurate theory for describing microscopic particles. However, some fundamental issues regarding the physical essence of quantum mechanics have long puzzled people, being topics of debate for decades and still controversial today [1] [2]. No consensus has been reached on counterintuitive quantum mechanical phenomena such as quantum superposition, wave function collapse, the uncertainty principle, and the double-slit experiment. From a practical perspective, as long as the calculation results are correct, there is no need to argue about the physical mechanism. However, from the perspective of understanding nature, revealing the essence of the material world is also the fundamental task of physics. Therefore, further research on the physical essence of quantum mechanics and in-depth exploration of the nature of the material world are still necessary.

The precise expression of the uncertainty principle [1] is $\Delta p \Delta x \geq \hbar/2$, where Δp and Δx are the uncertainties in momentum and position, respectively. The physical meaning is that it is impossible to determine the momentum and position of a particle with precision at the same time. Similarly, it is also impossible to determine the energy and time of a particle with precision at the same time. Heisenberg's uncertainty principle became one of the fundamental principles of quantum mechanics. However, physicists represented by Einstein, Schrödinger, and de Broglie had objections to the uncertainty principle and did not believe that the material world itself is uncertain, they considered the uncertainty principle to be incomplete. I agree with this viewpoint held by Einstein and other physicists. For example, when using this particular mathematical expression of the uncertainty principle to solve the energy levels of hydrogen atoms, only when approximating it as $pr = \hbar$ can we obtain the formula for the ground state energy of hydrogen atoms, but this does not give the formulas for other energy levels. This approximation has no mathematical basis and insufficient physical basis. When dealing with the double-slit experiment using the uncertainty relationship, only when approximating it as $\Delta p_y \Delta y = h$ can we obtain the formula for the first bright fringe of the double-slit experiment, but this does not give the formulas for other bright fringes. Similarly, when dealing with one-dimensional harmonic oscillators, one-dimensional deep potential wells, and other problems, similar approximate treatments are also needed to obtain the energy formulas for zero-point energy and ground state energy, but they still cannot obtain the expressions for other energy levels. This fully demonstrates that the uncertainty principle is not complete. In response to the imperfections of the uncertainty principle, the physicists such as Einstein and Schrödinger engaged in extensive debates and research. However, their efforts merely highlighted these imperfections; the proposed hidden variable theory failed to resolve the issues with the uncertainty principle.

In 2013, Hossenfelder and others proposed the entropic uncertainty relation, which primarily discusses the relationship between the amount of information obtainable through quantum measurement and the perturbation of the system's state. This research focuses on the lower bound of information obtained from measurements and does not involve the physical essence of the motion of elementary particles, marking a fundamental difference from Heisenberg's uncertainty principle.

The state of a fundamental particle is described by a wave function. Before measurement, the wave function is in a superposition state. When an operator acts on this superposition state, the wave function collapses to an eigenstate and simultaneously obtains the eigenvalue and eigenfunction of the corresponding mechanical quantity. The square modulus of the wave function represents the probability of finding a particle at a certain space-time position, which is the probabilistic interpretation of the wave function [3]. However, when this probabilistic interpretation is applied to the eigenfunction of a fundamental particle, there are sometimes counterintuitive phenomena that are difficult to understand and violate physical logic.

For example, the double-slit experiment based on the principle of state superposition suggests that when a photon or particle arrives at slits A and B, it has two possible states, ψ_A and ψ_B, respectively. As the photon or particle passes through slits A and B, it is in a superposition state $\psi = \psi_A + \psi_B$. When the photon or particle reaches the receiving screen, the wave function collapses and its landing position is determined by the square modulus of the wave function, *i.e.* probability. Because $|\psi|^2 = |\psi_A + \psi_B|^2 = |\psi_A|^2 + |\psi_B|^2 + \psi_A\psi_B^* + \psi_B\psi_A^*$, the cross term $\psi_B\psi_A^* + \psi_A\psi_B^*$ is caused by the interference of probability waves. Therefore, the cross term is also the reason for the formation of multiple bright fringes. However, quantum mechanics requires that wave functions ψ_A and ψ_B be independent of each other, *i.e.* orthogonal. Because of the properties of orthogonal wave functions, $\psi_B\psi_A^* = 0$ and $\psi_A\psi_B^* = 0$, *i.e.* the cross term does not actually exist. Even if ψ_A and ψ_B are not orthogonal, Young's interference fringe formula cannot be obtained through the interference term. Therefore, the probability interpretation has not been able to accurately explain the results of the double-slit experiment for photons or particles so far.

In another example, the wave function of a hydrogen atom is obtained by solving the Schrödinger equation. When the atom is in its ground state, the square modulus of its wave function is not equal to 1 at the ground state radius r_1, and it is not equal to 0 in the regions $0 < r < r_1$ and $r_1 < r < r_2$. This indicates that an electron can be found anywhere in the region $0 < r < r_2$. If this is true, the hydrogen atom spectrum should be continuous. This is obviously inconsistent with the conclusions of Bohr, Heisenberg, de Broglie, and Schrödinger that the ground state electron can only be at $r = r_1$, and also does not conform to the fact that the atomic hydrogen spectrum is discontinuous. Therefore, the probabilistic interpretation of wave functions is imperfect.

Another example is the wave function of a one-dimensional harmonic oscillator, which is also obtained by solving the Schrödinger equation [4]. When the oscillator is in its ground state, quantum mechanics defines it as vacuum. The square modulus of the ground state wave function of the harmonic oscillator is not zero in vacuum. According to the probabilistic interpretation, this means the particle can be found in vacuum, which contradicts the definition of vacuum.

Feynman introduced the action principle into quantum mechanics and established the path integral form of quantum mechanics, which was proved to be equivalent to Schrödinger's wave mechanics, and successfully applied in fields such as quantum field theory and statistical physics. This method also used the probabilistic interpretation but did not solve the above problems. Bohr solved the atomic hydrogen energy levels using various assumptions and arrived at the angular momentum quantization hypothesis. Sommerfeld extended Bohr's quantization of angular momentum to elliptic orbits and introduced the angular quantum number. This quantization of angular momentum has the dimensions of action, but angular momentum and action are two completely different physical quantities, so they cannot solve the problems above.

Physicists [1] have conducted extensive research to resolve a series of issues violating physical logic brought about by the probabilistic interpretation. They have proposed multiple interpretations such as the Many-Worlds Interpretation, Nonlocal Hidden Variables Interpretation, Quantum Bayesianism, Quantum Darwinism, Transactional Interpretation, and Relational Interpretation, among others. While some of these studies have provided mathematical formulations, not all interpretations are falsifiable, and some fail to explain any experimental phenomena, leaving the problems with the probabilistic interpretation unresolved.

Since 2016, Steven Weinberg [2] has repeatedly echoed the concerns of physicists like Einstein by questioning the integration of probability into physics, which troubles physicists. However, he argues that the real difficulty with quantum mechanics is not the probability itself, but rather the origin of this probability. The widespread application of probability brings many counterintuitive issues.

Using the de Broglie material wave hypothesis and classical action principle, we can quantize the action of a fundamental particle in a natural coordinate system. If this can solve some problems in quantum mechanics, it will be important for completing quantum mechanics.

2. Quantization of Action for an Elementary Particle and the Principle of Least Action

For an elementary particle like the photon or electron (hereafter referred to as a particle), the Copenhagen School of Quantum Mechanics believes that there is no definite position before measurement, and the position obtained through measurement is imprecise. Therefore, the particle does not have a definite trajectory. However, for convenience, we still assume that the particle moves along path x, which can be a straight line, curve, or other shape. We establish a path coordinate system (natural coordinate system) on the particle's path, with the origin of the coordinate being the position where we start studying the particle, and the direction of the particle's motion along path x being the positive x direction. At any point on the coordinate axis x, the velocity of the particle is v_x, its acceleration is a_x, its momentum is p_x, and its energy is E.

The action of a physical system is the integral of the Lagrangian with respect to time, and its mathematical expression $A=\int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t=\int_{t_1}^{t_2}(T-U)\mathrm{d}t$ indicates that when the Lagrangian does not change over time or changes only slightly, the action has an approximate proportional relationship between the product of energy and time and the product of momentum and displacement. In 1744, Pierre Louis Moreau [5] [6] published an article titled "The agreement between the different laws of Nature that had, until now, seemed incompatible," in which he defined the action as the product of mass, velocity, and displacement.

For fundamental particles, to facilitate our study, we define the action as the product of momentum and path, which is also equal to the product of energy and time, just like Pierre Louis Moreau. The principle of action in the macroscopic world is a universal fundamental principle that we believe is also applicable to microscopic particles. The action of a particle is denoted by A and defined in natural coordinates as

$$A=p_x x=Et \tag{1}$$

According to Einstein's quantum hypothesis of light [7] [8], de Broglie proposed that material particles also have wave-like properties [9], meaning that a free particle with energy E and momentum p_x has a frequency ν and wavelength λ according to

$$\nu=E/h \tag{2}$$

and

$$\lambda=h/p_x\,. \tag{3}$$

The state of the particle can be described by a plane wave:

$$\psi=\mathrm{e}^{i(p_x x-Et)/\hbar}\,. \tag{4}$$

In these equations, h and $\hbar$ are the Planck constant and reduced Planck constant, respectively; x and t represent position and time; ψ represents the amplitude of the particle at the space-time location (x, t), and is known as the state function or wave function of the particle.

According to Equations (1), (2) and (3), we have $ET=h$ and $p_x\lambda=h$, where T is the period. Setting $x=n\lambda$ and $t=nT$ gives

$$A=Et=EnT=nET=nh\,, \tag{5}$$

$$A=p_x x=p_x n\lambda=np_x\lambda=nh\,. \tag{6}$$

If n in Equations (5) and (6) could take any value, then the result would be $\psi\equiv1$ in Equation (4), which suggests that the particle does not exhibit wave-like behaviour, contradicting de Broglie's assumption of wave-like particles. Therefore, n cannot take arbitrary values. We will now discuss the range of possible values for n.

For a particle emitted from a particle source with momentum p_x and energy E, we detect the particle at a distance x away from the source. To ensure the possibility of detecting the particle, the momentum operator $-i\hbar\frac{\partial}{\partial x}$ must be Hermitian according to quantum mechanics, which requires the state function of the particle to satisfy periodic boundary conditions, *i.e.* $\psi_p\left(-\frac{x}{2}\right)=\psi_p\left(\frac{x}{2}\right)$. Assuming the momentum eigenfunction is $\psi_p(x)\sim e^{\frac{i}{\hbar}p_x x}$ and performing periodic extension, we have $e^{-\frac{i}{\hbar}p_x x/2}=e^{\frac{i}{\hbar}p_x x/2}$. This simplifies to $e^{\frac{i}{\hbar}p_x x}=1$, thus establishing the equation

$$p_x x = nh,\quad n=1,2,3,\cdots \tag{7}$$

Because $A=p_x x=Et$, the following equation is also satisfied:

$$Et = nh,\quad n=1,2,3,\cdots \tag{8}$$

Therefore, to satisfy the Hermitian property of the operator, n in the expressions for the action of the particle's motion along path *x* in Equations (5) and (6) can only take positive integer values. This means that the action of the particle's motion is not continuous but quantized. The quantization condition is given by

$$A = p_x x = Et = nh,\quad n=1,2,3,\cdots \tag{9}$$

When *n* takes the minimum value of 1 in Equation (9), the minimum action of the particle's motion is given by

$$A_{\min} = h. \tag{10}$$

Based on the path coordinate system, Pierre Louis Moreau's definition of action, Einstein's light quantum hypothesis, de Broglie's matter wave conjecture, and the hermiticity of the momentum operator, rigorous mathematical deductions have been made to obtain the quantization conditions for the action of fundamental particle motion and the minimum action.

In other words, the minimum action for the fundamental particle's motion cannot be zero but rather is the Planck constant *h*. Therefore, I believe that the Planck constant *h* is not just a coefficient in the energy quantization expression of quantum mechanics, but rather an exact and independent physical quantity. Its precise physical meaning is the minimum action of a fundamental particle. This is the principle of minimum action for the fundamental particle's motion.

3. Experimental and Theoretical Verification of the Quantization Condition for Particle Action

Although the action quantization condition for particle motion is obtained through mathematical derivation and has a certain physical basis, it needs to be further verified through experiments or by explaining existing experimental phenomena. Applying it to solving quantum mechanics problems such as atomic hydrogen energy levels and obtaining correct results would also verify this theory well.

3.1. Derivation of Double-Slit Formulas

In the early 19th century, Thomas Young [10] first introduced the double-slit interference experiment on light. For more than a hundred years, countless double-slit interference experiments have obtained the same rules. The experimental principle is shown in Figure 1(a), where a beam of white or red light is shot at the slits A and B over a distance d, and the photons continue to move towards the receiving screen OP at a distance L from the double slit. The interference patterns on the receiving screen for light of different wavelengths are shown in Figure 1(b) and Figure 1(c). Similarly, a group of electrons prepared under certain conditions are shot at slits A and B, and the electrons continue to move towards the receiving screen OP after passing through the double slit. The interference pattern on the receiving screen is shown in Figure 1(d). When the electrons arrive at the receiving screen one by one, there are few electrons in the early stage, and the receiving screen displays several irregular points. As the number of arriving electrons increases, the pattern that gradually appears is almost the same as that when all the electrons arrive at the receiving screen together, as shown in Figure 1(e).

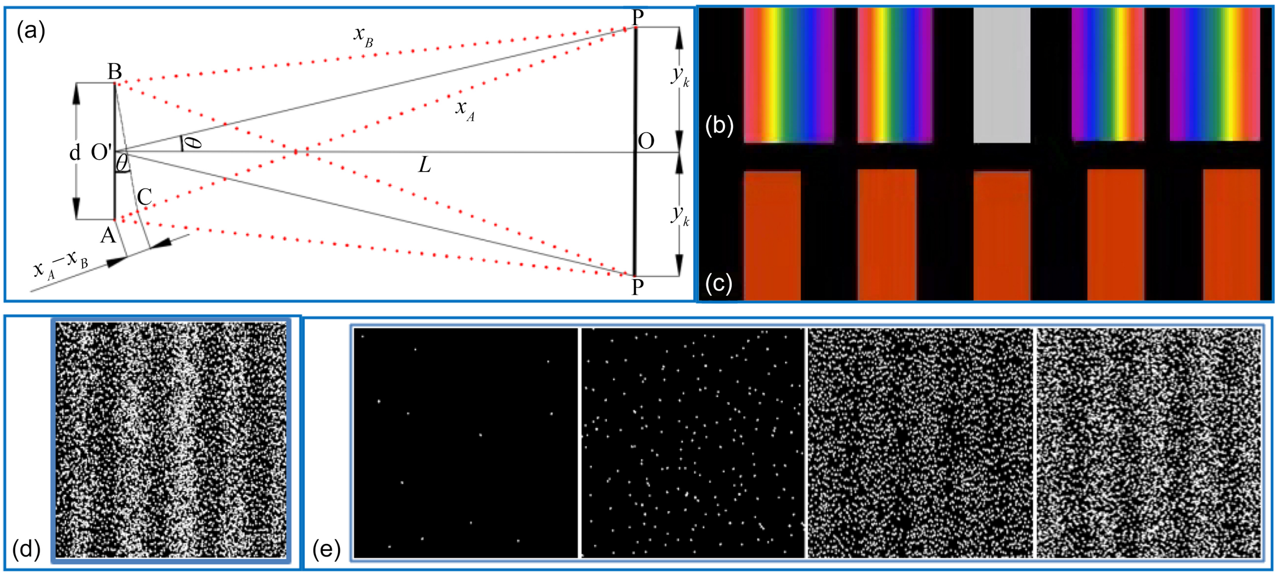

Figure 1. Schematic of the double-slit experiment with light or particles. (a) Geometric relationships of the Young double-slit experiment; (b) Simulation result for white light passing through the double slit; (c) Simulation result for red light passing through the double slit; (d) Simulation result for electrons arriving at the receiving screen together; (e) Simulation result for electrons arriving at the receiving screen one by one.

It is believed that Young's double-slit interference experiment on light has been satisfactorily explained through the wave nature of light, specifically, that the positions where bright fringes appear are derived from the superposition and interference of waves according to the equation [11]

$$d\sin\theta_k = \pm k\lambda, \quad k = 0,1,2,3,\cdots \tag{11}$$

In this equation, d is the distance between the two slits, θ_k is the angular deviation of the position of a bright fringe on the receiving screen from the centre of the receiving screen, and λ is the wavelength of the light.

However, no theoretical formula consistent with Equation (11) has yet been obtained on the basis of the probabilistic interpretation of quantum mechanics. Now, without considering the interference of waves, we will derive a new expression for Equation (11) based on the quantum condition of particle action (9). As shown for the double-slit experiment in **Figure 1(a)**, a photon or particle passes through one of the slits, A, and then moves along path *x*. On its path, there is a receiving screen that receives it. The following quantum condition for the particle's action must hold for the receiving screen to be able to receive the particle:

$$p_A x_A = n_A h, \quad E_A t_A = n_A h, \quad n_A = 1, 2, 3, \cdots \tag{12}$$

We can see that only when the space-time conditions satisfy equation (12) can the receiving screen detect photons or particles. If two photons or particles with the same numerical momentum and energy ($p_A = p_B = p$, $E_A = E_B = E$)are coming from slits A and B, respectively, and they are shot towards the receiving screen as measuring probes, these photons or particles can be detected on the receiving screen when their paths upon reaching the screen are given by

$$x_A = n_A h/p, \tag{13}$$

$$x_B = n_B h/p. \tag{14}$$

If both photons or particles reach the same point P on the receiving screen, the path difference is given by

$$x_A - x_B = (n_A - n_B) h/p. \tag{15}$$

According to the approximate geometric relationship of the photon or particle path, the following two equations hold:

$$x_A - x_B = d \sin\theta_k, \tag{16}$$

$$y_k = L \sin\theta_k. \tag{17}$$

Considering $n_A - n_B = \pm k$, $h/p = \lambda$, we can deduce that

$$d \sin\theta_k = \pm k\lambda, \quad k = 0, 1, 2, 3, \cdots \tag{18}$$

and

$$y_k = \frac{L}{d} \cdot (\pm k) \lambda, \quad k = 0, 1, 2, 3, \cdots \tag{19}$$

must hold.

Equations (18) and (19) are the new quantum mechanical formulas for the double-slit experiment on photons or particles, and are exactly the same as the formula (11) for the double-slit interference experiment on light. At a space position that satisfies Equations (13), (14), (18), and (19) simultaneously, two photons or particles can fall on the same position, forming a bright spot. If both slits are standard one-dimensional lines, the bright fringes will also be one-dimensional lines without width. However, actual slits always have a certain width, so the bright fringes also have a certain width. Photons or particles in different states (λ) will be detected in different regions of the bright fringes. Particles can be detected in the region that only satisfies Equations (13) and (14), and this region forms a faintly lit background. The dark fringe region is composed of the background region and the region where particles cannot be detected.

From the process of the double-slit experiment, we can see that whether photons or particles arrive at the receiving screen one by one or together, or whether they come from the double slits, a single slit, or any two different points, a photon or particle does not need to pass through both slits at the same time or interfere with itself as long as Equations (13), (14), (18), and (19) are satisfied simultaneously. The alternating bright and dark fringes will appear.

3.2. Deterministic and Indeterministic Particle Relations

The Heisenberg uncertainty principle is a well-known basic principle of quantum mechanics, and its strict mathematical expression [12] is $\Delta p_x \Delta x \geq \hbar/2$. The physical meaning of this expression remains controversial. Although the expression has been approximated and rewritten in many ways to obtain the formulas of the first-order fringe of double-slit interference, the ground state energy formula of the hydrogen atom, the ground state or zero-point energy of the one-dimensional harmonic oscillator, and the ground state energy of the one-dimensional square potential well, these various approximations do not seem to have strict physical and mathematical bases, and they are no longer the original uncertainty relationship.

We can see from the quantization of particle action (Equation (9)) that when a particle is emitted from a source and its momentum p_x is known, it appears in a particle state at $x_n = nh/p_x$ along the path *x*, and we do not need to measure it to know its definite position is x_n. In other words, a particle has a definite momentum and position at the space-time position where it appears in a particle state. Similarly, at the space-time position where it appears in a particle state, a particle has a definite energy and time. Therefore, we also call the quantization condition of particle action represented by Equation (9) the deterministic relation for particles.

When the momentum and energy of particle motion change, the action of the particle motion changes, which means that the particle jumps from one particle state to another. According to the principles of quantum mechanics, we cannot know the specific process by which a particle jumps from one particle state to another, but the changes in energy ΔE, momentum Δp_x, path Δx, and time Δt experienced during this process can be obtained by analysing observables such as spectra. Therefore, the energy level transition of a particle must follow the principle of minimum action in the equation $\Delta p_x \Delta x = \Delta E \Delta t = A_{\min} = h$, which we call the deterministic relation when the particle's energy and momentum change. In this way, the deterministic relations can be summarized as

$$\begin{cases} p_x x = Et = nh, n = 1, 2, 3, \cdots \\ \Delta p_x \Delta x = \Delta E \Delta t = A_{\min} = h \end{cases}. \tag{20}$$

In other space-time regions where a particle is not in a particle state, these are called uncertainty regions because there are no physical quantities corresponding to the particle and no results will be obtained from measuring its particle state quantities. These regions are represented by the equation

$$\begin{cases} (n-1)h < p_x x < nh \\ (n-1)h < Et < nh \end{cases}, \quad n = 1, 2, 3, \cdots \tag{21}$$

Equation (21) is the new uncertainty relation for particles.

From Equations (20) and (21), we can see that the deterministic relation and the new uncertainty relation are completely different from Heisenberg's uncertainty relation. The deterministic relation, as a condition for action quantization, has been used to derive a bright fringe formula for the double-slit experiment that is completely consistent with the formula proposed by Young.

3.3. Derivation of Energy Levels for Hydrogen Atoms and Quantization of Angular Momentum

Combining the new deterministic relation (20) in which $p_x x = nh(n = 1, 2, 3, \cdots)$ (here x represents the circular path) with the classical balance relationship between energy and force, we obtain the relations

$$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}, \tag{22}$$

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}, \tag{23}$$

$$p_x x = p_x \cdot 2\pi r = mv \cdot 2\pi r = nh. \tag{24}$$

Using Equations (22)-(24), we can directly solve the formulas of the hydrogen atom's energy levels and the position radius of the electron outside the nucleus of the hydrogen atom, and obtain the quantization condition of angular momentum.

Simplifying Equation (23) yields the kinetic energy formula for extranuclear electrons, Equation (23a).

$$\frac{1}{2}mv^2 = \frac{1}{8\pi\varepsilon_0}\frac{e^2}{r}, \tag{23a}$$

From Equation (24), we can derive the velocity formula for extranuclear electrons, Equation (24a).

$$v = \frac{nh}{2\pi mr}, \tag{24a}$$

Substituting Equation (24a) into Equation (23) yields the positional radius of extranuclear electrons, Equation (25). By substituting Equations (23a) and (25) into Equation (22), we can derive the energy level formula for hydrogen atoms, Equation (26).

According to the definition of angular momentum, when an electron moves in a circular orbit outside the nucleus, its angular momentum $L = r \times p_x = \frac{x}{2\pi}p_x = \frac{1}{2\pi}p_x x = \frac{nh}{2\pi} = n\hbar$, this leads to the condition for the quantization of angular momentum, Equation (27).

$$r = \frac{\varepsilon_0 h^2}{\pi m e^2} n^2, \quad (25)$$

$$E = -\frac{1}{n^2}\left(\frac{me^4}{8\varepsilon_0^2 h^2}\right), \quad (26)$$

$$L = r \times p_x = n\hbar. \quad (27)$$

The resulting equations are completely consistent with those obtained by solving Schrödinger's equation.

These results are obtained directly from the action quantization condition or the new deterministic relation without relying on the Bohr correspondence principle, electron transition, Planck–Einstein relation, or assumption of quantized angular momentum (L).

3.4. Derivation of Energy Levels for One-Dimensional Harmonic Oscillators and Vacuum Zero-Point Energy

For a one-dimensional harmonic oscillator, we assume its equilibrium position $x = 0$, the particle's momentum is p_x, and its displacement is $\pm x$. According to the de Broglie matter wave hypothesis, its wavelength $\lambda = h/p_x$. Furthermore, considering that a one-dimensional harmonic oscillator, when at the extreme positions on either side of the coordinate origin, should maintain its particle nature, meaning the momentum operator must satisfy Hermiticity. Through periodic extension, we have $A = p_x 2x = (2n+1)h$ based on the action quantization condition or the new deterministic relation, that is, the displacement of the particle from the starting point to the end point should be equal to the base number of wavelengths. Then the particle's momentum $p_x = (2n+1)h/2x$. We know that the total energy of a one-dimensional harmonic oscillator is $E = \frac{p_x^2}{2m} + \frac{1}{2}m\nu^2 x^2$. By substituting $p_x = (2n+1)h/2x$ into this, we obtain $E = \frac{(2n+1)^2 h^2}{8mx^2} + \frac{1}{2}m\nu^2 x^2$. When $\frac{\mathrm{d}E}{\mathrm{d}(x^2)} = 0$, $x^2 = \left(n+\frac{1}{2}\right)\frac{h}{m\nu}$ and $x = \pm\sqrt{\left(n+\frac{1}{2}\right)\frac{h}{m\nu}}$. At this x position, the harmonic oscillator has a stable energy $E_n = \left(n+\frac{1}{2}\right)h\nu = \left(n+\frac{1}{2}\right)\hbar\omega_0$, which is the energy level of a one-dimensional harmonic oscillator. When n equals zero, $E_0 = \frac{1}{2}\hbar\omega_0$, which is the zero-point vacuum energy. These results are completely consistent with those obtained by solving Schrödinger's equation.

3.5. Derivation of Energy Levels for One-Dimensional Deep Potential Wells

For a particle moving in a potential well, we set the origin of coordinates at the wall of the potential well, where $x = 0$ and the width of the potential well is a. The particle's momentum is p_x, and its displacement is x. When $x = a$, the particle is at the other side wall of the potential well, at which point it must be in a particle state. Therefore, based on the boundary condition that the momentum operator must satisfy Hermiticity, when the coordinate origin is determined to be at a boundary, through periodic extension, we have $A = p_a 2a = nh$ according to the action quantization condition or the new deterministic relation, and then the particle's momentum $p_a = nh/2a$. The total energy of a particle in a potential well is its kinetic energy, that is,

$$E = \frac{p_x^2}{2m}. \tag{28}$$

Substituting $p_a = nh/2a$ into Equation (28) gives us

$$E_n = n^2 \frac{\hbar^2 \pi^2}{2ma^2}. \tag{29}$$

Equation (29) is the formula for the energy levels of a one-dimensional deep potential well, which is completely consistent with that obtained by solving Schrödinger's equation.

The correctness of these results shows that the action quantization of particle motion and the principle of least action are correct, and they are the most basic quantum properties of particles.

4. Application of Particle Action Quantization and the Principle of Least Action

4.1. New Properties of Particle Motion

Action quantization and the principle of least action reveal new properties of particle motion along a path x:

1) When the momentum and kinetic energy of a particle are constant, the space-time position of the particle is given by ($x_n = nh/p_x$, $t_n = nh/E$) ($n = 1, 2, 3, \cdots$), indicating that the particle motion in space-time is discontinuous and jump-like.

2) At any determined space-time position (x, t), the particle's momentum and energy are respectively given by $p_n = nh/x$ ($n = 1, 2, 3, \cdots$) and $E_n = nh/t$ ($n = 1, 2, 3, \cdots$), indicating that at a determined space-time position, the allowed momenta and energies of the particle are also discontinuous and quantized.

3) When the momentum and kinetic energy of a particle change, the action of the particle's motion changes, which means that the particle jumps from one particle state to another. The energy level transition of a particle must follow the principle of minimum action in the equation $\Delta p_x \Delta x = \Delta E \Delta t = A_{\min} = h$.

4.2. New Physical Interpretation of Particle Eigenfunctions

On the basis of particle action quantization and the principle of least action combined with the definition of de Broglie matter waves, it is easy to conclude that the eigenfunction ψ_j of a particle j has a new physical meaning that does not depend on probability. That is, for a particle with a definite momentum and energy, its eigenfunction ψ_j is the mathematical expression of its wave–particle duality, indicating that the particle is in a particle state at the space-time point ($x = nh/p_x$, $t = nh/E$) ($n = 1,2,3,\cdots$) and in a wave state in the space-time region [$(n-1)h/p_x \le x \le nh/p_x$, $(n-1)h/E \le t \le nh/E$] ($n = 1,2,3,\cdots$). The particle nature is discontinuous, whereas the wave nature is continuous.

When the momentum and energy of a particle change between (p_j , E_j) and (p_k , E_k), the particle is in a superposition state between the two states ψ_j and ψ_k . According to the principle of wave superposition, the state function of the particle at this time is $\psi_{jk} = C_{p_j}\psi_j + C_{p_k}\psi_k$. We also call the state corresponding to ψ_{jk} the true superposition state or transition state (coherent state), where $k \ne j$; $j = 1,2,3,\cdots$; $k = 1,2,3,\cdots$; and $\left|C_{p_j}\right|^2 + \left|C_{p_k}\right|^2 = 1$. The time required for energy level transitions follows the deterministic relation given by Equation (20).

4.3. Experimental Geometric Image of Wave-Particle Duality

According to the new physical interpretation of the particle eigenfunction, at the particle-like space-time points in the motion path, the particle exhibits particle-like behaviour, and throughout the entire space-time region, the particle also exhibits wave-like behaviour. Thus, the particle has wave-particle duality.

If the particle is at a particle-like space-time point, it exhibits particle-like behaviour and its particle characteristics can be observed; if the particle is in a wave-like space-time region, it exhibits wave-like behaviour and its particle characteristics cannot be observed.

In the double-slit experiment, photons or particles periodically exhibit particle-like behaviour. When they are in a particle state at the receiving screen, the receiving screen can detect the photons or particles, and the particle-like behaviour is observed as bright fringes on the receiving screen and bright regions in the background (**Figure 1**). When they are not in a particle state at the receiving screen, the screen cannot detect the presence of photons or particles, and wave-like behaviour is observed as dark fringes on the screen (**Figure 1**). In the eigenfunction of photons or particles, the size of the spatial region where they exhibit wave-like behaviour is the wavelength λ. This wavelength is very small and generally cannot be observed under macroscopic conditions. However, through the double-slit experiment, the wavelength region λ where photons or particles exhibit wave-like behaviour on the receiving screen is magnified by $k \cdot L/d$, which can be clearly observed under macroscopic conditions. Therefore, the results of double-slit, single-slit, and crystal diffraction experiments are all geometric images of the new wave-particle duality of photons or particles magnified by $k \cdot L/d$.

4.4. Time of a Particle Energy Transition

Since quantum mechanics was developed, it has been known that the transition of a particle from one state to another does not occur instantaneously and requires time, but it is not yet clear what formula the time required follows. Using the action quantization condition or deterministic relation for particle motion, we can predict the time for this transition.

According to the deterministic relation Equation (20), when the energy of a particle changes by ΔE, the time required to cause this change is Δt, that is,

$$\Delta t = h/\Delta E \tag{30}$$

Since the energy level transition of a particle is accompanied by the release and absorption of energy, which can be measured, we can calculate the time required for the energy level transition using Equation (30) once we have obtained the change in energy ΔE.

For the electron in a hydrogen atom, based on the formula for the energy levels of a hydrogen atom, we can predict the time required for an electron to transition from level j to level k as given in Equation (31).

$$\Delta t_{jk} = \frac{k^2 j^2}{j^2 - k^2}\left(\frac{8\varepsilon_0^2 h^3}{me^4}\right). \tag{31}$$

From Equation (31) and the Balmer formula, we can see that Δt_{jk} is exactly the period of the electromagnetic wave corresponding to a photon released when an electron transitions from energy level j to energy level k. Therefore, the time required for an electron energy transition can be obtained by measuring the frequency of light emitted by a hydrogen atom.

During the transition of an electron from energy level j to energy level k, according to our new physical interpretation of the intrinsic state of an elementary particle, its state is in a real superposition represented by $\psi_{jk} = C_{p_j}\psi_j + C_{p_k}\psi_k$. This means that the transition process is unobservable. Equation (31) provides the duration of this unobservable transition process.

4.5. Apparent Particle Trajectory

To date, fundamental particles can only be represented by a single point. If the position determined by the deterministic relation (20) is represented by a point, the dotted line formed by these points becomes the apparent trajectory in **Figure 2** of particle motion. This means the particle-like space-time points and wave-like space-time regions of a particle form its continuous trajectory of motion. The points on the dotted line, where the particle exhibits particle-like behaviour, can be observed. The region between two points has a length of wavelength λ, where the particle exhibits wave-like behaviour and no mechanical quantities belonging to the particle can be observed, so the trajectory cannot be observed. Mathematically, we treat fundamental particles as points, but in actual measurements, instruments obtain information about mechanical quantities through interactions with particles. This interaction does not entirely depend on direct contact. Therefore, depending on different measurement methods and accuracies, particles will always exhibit certain geometric scales. When the measurement scale of a particle is greater than or equal to the wave-like spatial range λ in Equation (21), we can always "see" the particle macroscopically. At this time, the trajectory of the particle appears as a solid line.

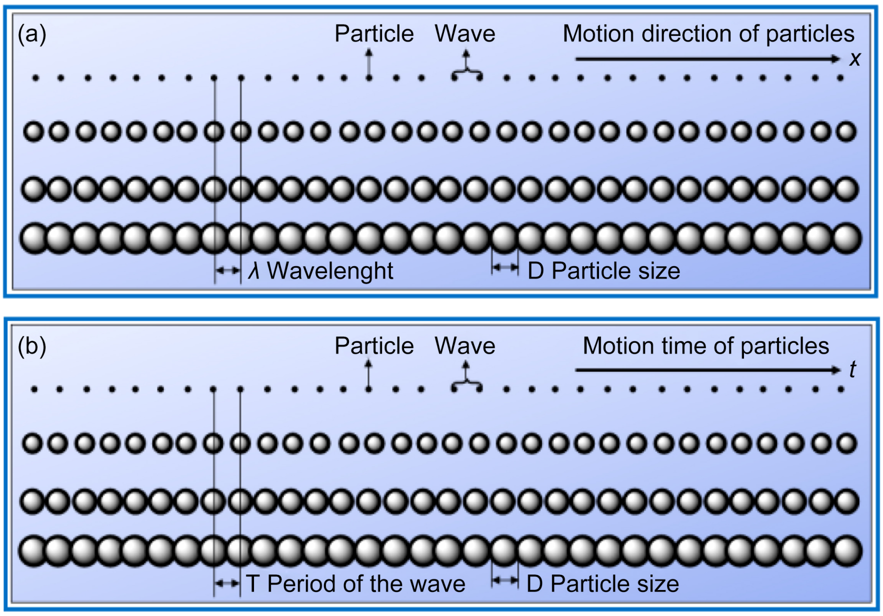

Figure 2. Apparent particle trajectories. (a) Apparent trajectories in space; (b) Apparent trajectories in time.

Based on the trajectory of particle motion, we can understand the movement of a particle as the fundamental particle orderly creation and annihilation on discrete spacetime points.

4.6. Space-Time Range Where Bell's Inequality Holds

Einstein was confident that particles are always localized realities, possessing definite observable physical quantities at any spacetime position. In contrast, the Copenhagen school, led by Bohr, believed that particles exist probabilistically and non-locally when not measured, and only exhibit observable particle properties after the wave function collapses due to measurement. Bell agreed with Einstein's localized realism and proposed the Bell inequality as a criterion for the validity of localized realism. However, numerous subsequent experiments have shown that the Bell inequality does not hold, indicating that Einstein's localized realism is incorrect.

On the basis of the new deterministic and indeterministic relations, particles are localized at space-time points of Equation (20) in a particle state and nonlocalized in space-time regions of Equation (21) in a wave state. It is clear that the nonlocalized space-time region where a particle exists in a wave state is much larger than the localized space-time point where it exists in a particle state. Perhaps this is why the Bell inequality [13] [14] has not been established experimentally so far.

Under what conditions does the Bell inequality hold? If we consider two particles A and B, their state functions are represented by ψ_A and ψ_B, respectively. When the two particles interact and then separate, and they are still mutually correlated, the state function of the two particles can be expressed as

$\psi_{AB} = C_A\psi_A + C_B\psi_B$, where $|C_A|^2 + |C_B|^2 = 1$. At this time, both particles are in a true superposition state (coherent state) represented by the state ψ_{AB}. When $C_A = 1$, particle A is in the state ψ_A, at which time particle B must be in the state ψ_B. When $C_A = 0$, particle A is in the state ψ_B, at which time particle B must be in the state ψ_A.

When $C_A = 1$ or $C_A = 0$, particles A and B both have definite eigenfunctions ψ_A and ψ_B, and when the two particles are in a particle state at the same time, all their mechanical quantities are determined. For this situation, we consider a certain mechanical quantity F and use a Cartesian coordinate system. The components of this mechanical quantity in the X, Y, and Z directions for particles A and B are F_{AX}, F_{AY}, F_{AZ}, F_{BX}, F_{BY}, and F_{BZ}. Because the mechanical quantity F is determined, these components must also have definite values. At this time, we can mathematically derive the Bell inequality $|P_{XZ} - P_{ZY}| \le 1 + P_{XY}$.

When $C_A \ne 1$ or $C_A \ne 0$, particles A and B are both in the true superposition state (coherent state) ψ_{AB}, and the condition of quantization of particle action is not satisfied. At this time, the particle does not have a definite mechanical quantity F, and we cannot determine the values of all components of this mechanical quantity. Therefore, the Bell inequality does not exist and of course does not hold.

5. Conclusions

1) The action principle is a universal principle of the material world and can be used to describe the motion of elementary particles. the action of elementary particle motion is quantized according to the condition $A = p_x x = Et = nh$ with $n = 1, 2, 3, \cdots$. The minimum action of elementary particles is the Planck constant h, and their motion follows the principle of minimum action. The exact physical significance of the Planck constant his the minimum action of a fundamental particle.

2) The quantization of the action of particle motion and the principle of minimum action yield new deterministic and indeterministic relations. The new deterministic relation (the quantization of action and the principle of minimum action) can be directly used to solve the exact formulas for the positions of bright fringes in the double-slit experiment, the energy levels of hydrogen atoms, the quantization of angular momentum, the energy levels of one-dimensional harmonic oscillators, vacuum zero-point energy, the energy levels of one-dimensional deep potential wells, and the transition time between energy levels .

3) Particle action quantization and the principle of minimum action yield a new physical interpretation of the particle eigenfunction: the eigenfunction ψ_j of particle j is the mathematical expression of its wave-particle duality, indicating that the particle is in a particle state at the space-time point ($x = nh/p_x$, $t = nh/E$) ($n = 1, 2, 3, \cdots$), and in a wave state in the space-time region [$(n-1)h/p_x \le x \le nh/p_x$, $(n-1)h/E \le t \le nh/E$] ($n = 1, 2, 3, \cdots$). The particle nature is discontinuous, whereas the wave nature is continuous.

4) The bright and dark fringes in the double-slit experiment are a geometric

image of the new wave-particle duality of particles, magnified by $k \cdot L/d$. The space-time points of particle nature and the space-time regions of wave nature constitute the continuous trajectory of particle motion. When two particles are correlated, the Bell inequality holds at the space-time point of a particle state, but does not hold in the space-time region of the wave state outside the particle state.

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Data Availability

The content of this manuscript falls within the realm of theoretical physics. Given the constraints on manuscript length, some simple mathematical derivations that general scientific personnel could complete based on their own mathematical knowledge have been omitted in the manuscript.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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