Journal of Modern Physics, 2024, 15, 2446-2488
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Reflection and Refraction at Isotropic and Anisotropic Media in Coordinate-Invariant Treatment

Alfred Wünsche

Institut für Physik, Humboldt-Universität, Berlin, Germany
Email: alfred.wuensche@physik.hu-berlin.de

How to cite this paper: Wünsche, A. (2024) Reflection and Refraction at Isotropic and Anisotropic Media in Coordinate-Invariant Treatment. *Journal of Modern Physics*, **15**, 2446-2488.
https://doi.org/10.4236/jmp.2024.1513101

Received: November 1, 2024
Accepted: December 27, 2024
Published: December 30, 2024

Abstract

The article develops coordinate-invariant methods to calculate reflection and refraction of plane monochromatic waves at the plane boundary between two isotropic and an isotropic and an anisotropic medium. The vectorial wave equation for the electric field is used to determine polarization vectors to known refraction vectors and this is applied to uniaxial media. Then it is shortly shown how the boundary conditions can be derived using the Heaviside step function and its derivatives which are the delta function and its derivatives. As preparation to the anisotropic case, there are calculated in coordinate-invariant way the amplitude relations for the reflection and refraction between two isotropic media and then in analogous way, the case of reflection and refraction between an isotropic and an anisotropic medium. This is then specialized for perpendicular incidence. It is shown that negative refraction such as discussed in last twenty-five years is impossible.

Keywords

Refraction and Polarization Vector, Projection Operators, Operator of Vectorial Wave Equation, Hamilton-Cayley Identity, Uniaxial Media, Negative Refraction

Notations

Three-dimensional vectors: $\boldsymbol{x}, \boldsymbol{y}, \cdots$ (bold letters).

Three-dimensional linear operators (equivalent to matrices): $\mathsf{A}, \mathsf{B}, \cdots$ (serif-less letters).

Invariants of three-dimensional linear operators: $\langle \mathsf{A} \rangle, [\mathsf{A}], |\mathsf{A}|$ (see Appendix A).

Scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$: $\mathbf{xy}$.

Dyadic product of two vectors $\mathbf{x}$ and $\mathbf{y}$: $\mathbf{x}\cdot\mathbf{y}$.

Three-dimensional anti-symmetric operator dual to vector $\mathbf{x}$: $[\mathbf{x}]$; or $[\mathbf{x}]_{ik}=\epsilon_{ijk}x_j$, $x_j=\frac{1}{2}\epsilon_{ijk}[\mathbf{x}]_{ik}$,

Vector product of two vectors $\mathbf{x}$ and $\mathbf{y}$: $[\mathbf{x},\mathbf{y}]$, ($\mathbf{x}[\mathbf{y}]=[\mathbf{x},\mathbf{y}]=[\mathbf{x}]\mathbf{y}$).

Volume product of three vectors $\mathbf{x},\mathbf{y},\mathbf{z}$: $[\mathbf{x},\mathbf{y},\mathbf{z}]$, ($\mathbf{x}[\mathbf{y},\mathbf{z}]=\mathbf{x}[\mathbf{y}]\mathbf{z}=[\mathbf{x},\mathbf{y},\mathbf{z}]=[\mathbf{x},\mathbf{y}]\mathbf{z}$).

Doubled vector products: $[\mathbf{x},[\mathbf{y},\mathbf{z}]]=\mathbf{x}(\mathbf{z}\cdot\mathbf{y}-\mathbf{y}\cdot\mathbf{z})$, $[[\mathbf{x},\mathbf{y}],\mathbf{z}]=(\mathbf{y}\cdot\mathbf{x}-\mathbf{x}\cdot\mathbf{y})\mathbf{z}$.

Since "Latex" does not provide serif-less letters for Greek characters we denote Greek characters for operators such as for vectors. This concerns, in particular, the letters $\boldsymbol{\varepsilon},\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ contrary to Greek scalars ε,μ and σ.

1. Introduction

Most problems of the optics of anisotropic media (or crystal optics) require a coordinate-invariant treatment. In many problems of this topic, two or more coordinate systems offer their services which, however, are not compatible. In reflection and refraction problems at a plane boundary, these are the system of two coordinates within the plane boundary and one coordinate in direction of the normal to the boundary. This coordinate system is, in general, incompatible with the coordinate system of the principal axes of the permeability or permittivity tensors in general position to the first mentioned system. Some problems such as the reflection and refraction of plane monochromatic waves at the plane boundary from an isotropic to an anisotropic medium with general position of the axes of the anisotropic medium are so difficult that a lengthy solution is almost impossible to write down in coordinates and if nevertheless it would be successful then the solution in coordinate form does not possess some use. A way out of this dilemma is in certain sense the application of coordinate-invariant methods from the statement of the problem up to its solution and the final coordinate-invariant representation of the results. The initiator and pioneer for coordinate-invariant treatment of many problems is F.I. Fyodorov [1]-[3].

Coordinate-invariant methods calculate directly with the involved vectors and operators and calculate in an abstract form with them in such way as one usually calculates with numbers and without resolving them to a coordinate form. This requires the knowledge of modern linear algebra which provides the mostly basic theorems but some methods, for example, the coordinate-invariant calculation of inverse operators to a given linear operator in this way are little known up to now. Let us illustrate this at the example of reflection and refraction of light waves at a plane boundary. First one has to derive from the basic Maxwell equations of macroscopic optics a vectorial wave equation for the electric or magnetic field from which in a first step a dispersion equation is derived from the linear operator of the wave equation. In a second step, one has to determine polarization vectors of the electric and (or) the magnetic field for which one needs the inverse operator to this equation to form projection operators. Then in reflection and refraction problems, one has to determine all involved refraction vectors in dependence on their common tangential components with respect to the boundary plane that in general case of an anisotropic medium leads already to a fourth-degree equation for their normal components that is hardly to do without a computer. In a last step one has to write down the boundary conditions in coordinate-invariant form and has to determine from them by elimination processes the amplitudes of the reflected and refracted waves in dependence on the vectorial amplitudes of the incident waves. This is usually the most difficult problem which is to manage (having in mind general cases).

The last very detailed representation up to now of problems of refraction and reflection at anisotropic media mainly in coordinate representation is the encyclopedia article of Szivessy [4] [5]. Other well written representations of the problems one may find, e.g., in Landau and Lifshits [6], Born and Wolf [7] and Lekner [8] and we also published long ago some articles on these problems [9] [10]. In present article after remarks to the derivation of vectorial wave equations (Section 2), we make preferences for the determination of the polarization vectors of the electric field to known refraction vectors (Sections 3 and 4), to a coordinate-invariant derivation of boundary conditions (Section 5) and discuss also the coordinate-invariant derivation of the amplitude relations for reflection and refraction at isotropic media (Sections 6-8) to use it for analogies to reflection and refraction at anisotropic media (Sections 9-11). The generalization from the isotropic to the anisotropic case makes the formulae very difficult and lengthy and one has to decompose this in well-defined partial steps which can be implemented by computer. In Section 12, we make a remark to the so-called "negative refraction" which was invented by Pendry [11]. From an article of Alú [12], one may conclude that the intensive discussion of this effect is still not finished and that some authors believe to its realization. However, it cannot be realized in the published form because this contradicts to basic laws of electrodynamics and optics.

In the Appendices A and B, we shortly deal with some general topics of the coordinate-invariant calculation with operators and their invariants. In Appendices C and D, this is applied to the wave-equation operators for the magnetic fields and the electric induction. Finally, in Appendix E we give a short table of correspondences of notations of Fyodorov to ours.

2. Maxwell Equations of Macroscopic Optics

We start from the Maxwell equations of macroscopic optics in the form

$$\left[\nabla,\boldsymbol{E}(\boldsymbol{r},t)\right]+\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{0},\qquad \nabla\boldsymbol{B}(\boldsymbol{r},t)=0,$$

$$\left[\nabla,\boldsymbol{B}(\boldsymbol{r},t)\right]-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r},t)=\boldsymbol{0},\qquad \nabla\boldsymbol{D}(\boldsymbol{r},t)=0, \tag{2.1}$$

where $\boldsymbol{E}(\boldsymbol{r},t)$ is the averaged microscopic electric field and $\boldsymbol{B}(\boldsymbol{r},t)$ the averaged microscopic magnetic field [6]. With $\boldsymbol{D}(\boldsymbol{r},t)$ we denote the electric induction where we widely use the notions of Landau and Lifshits [6][1]. The electric induction $\boldsymbol{D}$ is defined by

$$\boldsymbol{D}(\boldsymbol{r},t) \equiv \boldsymbol{E}(\boldsymbol{r},t) + 4\pi \boldsymbol{P}(\boldsymbol{r},t), \tag{2.2}$$

which is connected with the averaged microscopic current density $\boldsymbol{j}(\boldsymbol{r},t)$ and charge density $\varrho(\boldsymbol{r},t)$ by

$$\frac{\partial}{\partial t}\boldsymbol{P}(\boldsymbol{r},t) \equiv \boldsymbol{j}(\boldsymbol{r},t), \qquad \nabla \boldsymbol{P}(\boldsymbol{r},t) = -\varrho, \tag{2.3}$$

where $\boldsymbol{P}(\boldsymbol{r},t)$ is called the polarization. Thus, it is not taken into account a macroscopic current density and a macroscopic charge density. The continuity equation for microscopic current and charge density which are involved in the polarization $\boldsymbol{P}(\boldsymbol{r},t)$

$$\nabla \boldsymbol{j}(\boldsymbol{r},t) + \frac{\partial}{\partial t}\varrho(\boldsymbol{r},t) = 0, \tag{2.4}$$

is satisfied. The scalar Maxwell Equations in (2.1) are widely redundant. For example, from the first vectorial Maxwell equation by applying the operator ∇ as scalar product onto the equation follows $\frac{\partial}{\partial t}\nabla \boldsymbol{B}(\boldsymbol{r},t) = 0$ whereas the scalar Maxwell equation requires the more strong relation $\nabla \boldsymbol{B}(\boldsymbol{r},t) = 0$.

The most general linear constitution equation of the electric induction $\boldsymbol{D}(\boldsymbol{r},t)$ in dependence on the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ for a homogeneous medium with spatial dispersion can be represented in the form [6] [13] [14]

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{r},t) &= \int \mathrm{d}^3 r \mathrm{d}t\, \hat{\boldsymbol{\varepsilon}}(\boldsymbol{r}',t') \boldsymbol{E}(\boldsymbol{r}-\boldsymbol{r}',t-t') \equiv \boldsymbol{\varepsilon}\left(-\mathrm{i}\nabla, \mathrm{i}\frac{\partial}{\partial t}\right)\boldsymbol{E}(\boldsymbol{r},t), \\ \boldsymbol{\varepsilon}(\boldsymbol{k},\omega) &= \int \mathrm{d}^3 r \mathrm{d}t\, \hat{\boldsymbol{\varepsilon}}(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}(\boldsymbol{kr}-\omega t)}, \end{aligned} \tag{2.5}$$

where $\hat{\boldsymbol{\varepsilon}}(\boldsymbol{r}',t')$ and $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$ are second-rank, in general nonsymmetric, tensor functions connected by a complex Fourier transformation.

The complex Fourier transformation of all fields we make according to the following scheme written for the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ [2]

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{(2\pi)^4}\int \mathrm{d}^3 k \mathrm{d}\omega\, \boldsymbol{E}(\boldsymbol{k},\omega)\mathrm{e}^{\mathrm{i}(\boldsymbol{kr}-\omega t)},$$

$$\boldsymbol{E}(\boldsymbol{k},\omega) = \int \mathrm{d}^3 r \mathrm{d}t\, \boldsymbol{E}(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}(\boldsymbol{kr}-\omega t)}. \tag{2.6}$$

Then the Maxwell Equation (2.1) take on the form

$$\left[\boldsymbol{k}, \boldsymbol{E}(\boldsymbol{k},\omega)\right] - \frac{\omega}{c}\boldsymbol{B}(\boldsymbol{k},\omega) = \boldsymbol{0}, \qquad \boldsymbol{k}\boldsymbol{B}(\boldsymbol{k},\omega) = 0,$$

[1]Since we use here only the concept of spatial dispersion for the constitution equations (see [6] [13] [14] and below) we do not make a difference between $\boldsymbol{B}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$ and call $\boldsymbol{B}(\boldsymbol{r},t)$ the magnetic field which in case of distinction of $\boldsymbol{H}$ and $\boldsymbol{B}$ is usually called magnetic induction [6] [7] [15]. Since $\boldsymbol{E}$ and $\boldsymbol{B}$ are involved in the Lorentz force and not $\boldsymbol{H}$ the name "magnetic induction" instead of "magnetic field" is perhaps only to understand historically [15]. Maxwell calls $\boldsymbol{B}$ "magnetic induction" and $\boldsymbol{H}$ "magnetic force" [16] p. 257.

[2]We do not distinguish the function symbols for the field and for its Fourier transform since it should be clear from the content of formulae in connection with the context.

$$\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right]+\frac{\omega}{c}\boldsymbol{D}(\boldsymbol{k},\omega)=\boldsymbol{0},\qquad \boldsymbol{k}\boldsymbol{D}(\boldsymbol{k},\omega)=0. \tag{2.7}$$

where $\boldsymbol{k}$ is the wave vector and ω the frequency. The constitution Equation (2.5) becomes

$$\boldsymbol{D}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega),\qquad \boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\equiv\int \mathrm{d}^3r'\mathrm{d}t'\hat{\boldsymbol{\varepsilon}}(\boldsymbol{r}',t')\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\boldsymbol{r}'-\omega t')}, \tag{2.8}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$ is the permittivity tensor which is, analogously to $\hat{\boldsymbol{\varepsilon}}(\boldsymbol{r}',t')$, a second-rank, in general, nonsymmetric tensor. Its dependence on ω is called frequency dispersion and its additional dependence on the wave vector $\boldsymbol{k}$ is called spatial dispersion [6] [13] [14]. One may but must not include also linear magnetic properties of the medium in form of a relation $\boldsymbol{B}(\boldsymbol{k},\omega)=\boldsymbol{\mu}(\omega)\boldsymbol{H}(\boldsymbol{k},\omega)$ as effect of spatial dispersion of second order in $\boldsymbol{k}$ and, for example, natural optical activity as effect of first order in $\boldsymbol{k}$ into spatial dispersion [17].

After elimination of the magnetic field $\boldsymbol{B}(\boldsymbol{k},\omega)$ by means of the first vectorial Equation (2.1) and using the constitutive Equation (2.5) the following vectorial wave equation for the electric field ($\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$) is obtained

$$\boldsymbol{0}=\mathsf{L}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega), \tag{2.9}$$

with a linear operator $\mathsf{L}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega)$, of this equation defined by

$$\mathsf{L}(\boldsymbol{k},\omega)\equiv\frac{c^2}{\omega^2}\left(\boldsymbol{k}\cdot\boldsymbol{k}-\boldsymbol{k}^2\mathsf{I}\right)+\boldsymbol{\varepsilon}(\boldsymbol{k},\omega). \tag{2.10}$$

By scalar multiplication of the vectorial wave Equation (2.9) with the vector $\boldsymbol{k}$ follows

$$0=\boldsymbol{k}\mathsf{L}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega)=\boldsymbol{k}\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega)\propto\boldsymbol{k}\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\boldsymbol{e}. \tag{2.11}$$

This equation tells us that the vectorial amplitude of the electric field $\boldsymbol{E}(\boldsymbol{k},\omega)$ (and thus normalized polarization vectors $\boldsymbol{e}$) is perpendicular to the vector $\boldsymbol{k}\boldsymbol{\varepsilon}$ or $\boldsymbol{k}$ perpendicular to $\boldsymbol{D}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega)$. However, in this way, the direction of the electric field is not uniquely determined since this determines only the plane in which the electric field is located.

If one neglects the spatial dispersion and takes into account only the frequency dispersion of the permittivity tensor[3]

$$\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}(\omega), \tag{2.12}$$

it is favorable to introduce the abbreviation

$$\boldsymbol{n}\equiv\frac{c}{\omega}\boldsymbol{k}, \tag{2.13}$$

which is called the refraction vector. The Maxwell equations can be written then

$$\left[\boldsymbol{n},\boldsymbol{E}\right]-\boldsymbol{B}=\boldsymbol{0},\qquad \boldsymbol{n}\boldsymbol{B}=0,$$

$$\left[\boldsymbol{n},\boldsymbol{B}\right]+\boldsymbol{D}=\boldsymbol{0},\qquad \boldsymbol{n}\boldsymbol{D}=0, \tag{2.14}$$

[3]Into the permittivity tensor $\boldsymbol{\varepsilon}(\omega)$ one may include for $\omega\neq 0$ also an electric conductivity tensor $\boldsymbol{\sigma}(\omega)$ of the medium by the formal substitution $\boldsymbol{\varepsilon}(\omega)\to\boldsymbol{\varepsilon}'(\omega)=\boldsymbol{\varepsilon}(\omega)+\mathrm{i}\frac{4\pi}{\omega}\boldsymbol{\sigma}(\omega)$ which causes absorption and is formally equivalent to a generally complex permittivity tensor.

where the field variables which are $(\boldsymbol{n},\omega)$ in this case are not appended explicitly. The vectorial wave equation for the electric field can be written then

$$\mathbf{0} = \mathsf{L}(\boldsymbol{n})\boldsymbol{E}, \tag{2.15}$$

with the vectorial operator of this equation ($\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\omega)$, I three-dimensional identity operator)

$$\mathsf{L}(\boldsymbol{n}) \equiv \boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{n}^2\mathsf{I} + \boldsymbol{\varepsilon} = \mathsf{L}(-\boldsymbol{n}), \tag{2.16}$$

where the dependence of the permittivity tensor $\boldsymbol{\varepsilon}$ on the frequency is not written explicitly.

In the following we widely use coordinate-invariant methods who's initiator was F.I. Fyodorov [1]-[3] as said in the Introduction. Coordinate-invariant methods work only with the involved vectors as a whole characterizing the problem and linear operators together with their invariants in calculating the solution, in present problem, with the refraction vectors $\boldsymbol{n}$ and the permittivity tensor $\boldsymbol{\varepsilon}$ considered as a linear operator and its invariants. Important for these methods are mainly the Hamilton-Cayley identity and the invariants of the operator $\mathsf{L}(\boldsymbol{n})$ of the vectorial wave equation and, furthermore, the complementary operator $\overline{\mathsf{L}}(\boldsymbol{n})$ (see Appendix A and Appendix B). The three independent invariants of $\mathsf{L}(\boldsymbol{n})$ are (the dependence of $\boldsymbol{\varepsilon}$ on the frequency ω is again omitted)

$$\begin{aligned}\langle \mathsf{L}(\boldsymbol{n})\rangle &= -2\boldsymbol{n}^2 + \langle\boldsymbol{\varepsilon}\rangle,\\ [\mathsf{L}(\boldsymbol{n})] &= (\boldsymbol{n}^2)^2 - (\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}^2 + \boldsymbol{n\varepsilon n}) + [\boldsymbol{\varepsilon}],\\ |\mathsf{L}(\boldsymbol{n})| &= \boldsymbol{n}^2(\boldsymbol{n\varepsilon n}) - (\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n} - \boldsymbol{n\varepsilon}^2\boldsymbol{n}) + |\boldsymbol{\varepsilon}|.\end{aligned} \tag{2.17}$$

The complementary operator to $\mathsf{L}(\boldsymbol{n})$

$$\overline{\mathsf{L}}(\boldsymbol{n}) \equiv [\mathsf{L}(\boldsymbol{n})]\mathsf{I} - \langle\mathsf{L}(\boldsymbol{n})\rangle\mathsf{L}(\boldsymbol{n}) + \mathsf{L}^2(\boldsymbol{n}), \tag{2.18}$$

with the property

$$\mathsf{L}(\boldsymbol{n})\overline{\mathsf{L}}(\boldsymbol{n}) = \overline{\mathsf{L}}(\boldsymbol{n})\mathsf{L}(\boldsymbol{n}) = |\mathsf{L}(\boldsymbol{n})|\mathsf{I}, \tag{2.19}$$

is calculated to be

$$\begin{aligned}\overline{\mathsf{L}}(\boldsymbol{n}) &= (\boldsymbol{n}^2)\boldsymbol{n}\cdot\boldsymbol{n} - (\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{\varepsilon n}\cdot\boldsymbol{n} - \boldsymbol{n}\cdot\boldsymbol{n\varepsilon} + (\boldsymbol{n\varepsilon n})\mathsf{I}) + \underbrace{[\boldsymbol{\varepsilon}]\mathsf{I} - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^2}_{\equiv\overline{\boldsymbol{\varepsilon}}}\\ &= \overline{\mathsf{L}}(-\boldsymbol{n}).\end{aligned} \tag{2.20}$$

The general identity (see Appendix A)

$$\langle\overline{\mathsf{L}}(\boldsymbol{n})\rangle = [\mathsf{L}(\boldsymbol{n})], \tag{2.21}$$

is easily checked. Furthermore, one easily affirms from (2.20) the relations

$$\boldsymbol{n\varepsilon}\overline{\mathsf{L}}(\boldsymbol{n}) = \overline{\mathsf{L}}(\boldsymbol{n})\boldsymbol{\varepsilon n} = |\mathsf{L}(\boldsymbol{n})|\boldsymbol{n}, \tag{2.22}$$

with determinant $|\mathsf{L}(\boldsymbol{n})|$ explicitly given in (2.17). In connection with the wave-equation operator $\mathsf{L}(\boldsymbol{n})$ this is the identity

$$\boldsymbol{n\varepsilon}\overline{\mathsf{L}}(\boldsymbol{n})\mathsf{L}(\boldsymbol{n}) = |\mathsf{L}(\boldsymbol{n})|(\boldsymbol{n}\mathsf{L}(\boldsymbol{n})) = |\mathsf{L}(\boldsymbol{n})|\boldsymbol{n\varepsilon}, \tag{2.23}$$

or by division with $\left|\mathsf{L}(\boldsymbol{n})\right|$

$$n\mathsf{L}(\boldsymbol{n}) = \boldsymbol{n\varepsilon}, \tag{2.24}$$

which is obvious from (2.16).

With the complementary operator to $\mathsf{L}(\boldsymbol{n})$ in (2.20) it can be determined the inverse operator to $\mathsf{L}(\boldsymbol{n})$ according to (A.11) but also polarization vectors of the electric field that we discuss in detail in next Section. One may also calculate with the wave-equation operator for the magnetic field $\boldsymbol{B}$ or for the electric induction $\boldsymbol{D}$. These operators with their invariants and the complementary operators are shortly considered in Appendix C and Appendix D.

3. Polarization Vectors of the Electric Field to Plane Monochromatic Waves

Necessary condition for a solution of the electric field $\boldsymbol{E}$ of the wave Equation (2.15) is the vanishing of the determinant of the operator (2.16)[4]

$$\left|\mathsf{L}(\boldsymbol{n})\right| = 0. \tag{3.1}$$

If this condition is satisfied one may determine polarization vectors of the electric field using the complementary operator $\overline{\mathsf{L}}(\boldsymbol{n})$ to $\mathsf{L}(\boldsymbol{n})$ given in (2.20). For an arbitrary vector $\boldsymbol{x}$ for which $\boldsymbol{a} \equiv \overline{\mathsf{L}}(\boldsymbol{n})\boldsymbol{x} \neq \boldsymbol{0}$ the vector $\boldsymbol{a}$ is a right-hand eigenvector of $\mathsf{L}(\boldsymbol{n})$ to eigenvalue 0 and thus a solution for the electric field. In the same way, for an arbitrary vector $\tilde{\boldsymbol{x}}$ the vector $\tilde{\boldsymbol{a}} \equiv \tilde{\boldsymbol{x}}\overline{\mathsf{L}}(\boldsymbol{n}) \neq \tilde{\boldsymbol{0}}$ is left-hand eigenvector of $\mathsf{L}(\boldsymbol{n})$ to eigenvalue 0

$$\begin{gathered} \boldsymbol{a} \equiv \overline{\mathsf{L}}(\boldsymbol{n})\boldsymbol{x} \neq \boldsymbol{0}, \quad \tilde{\boldsymbol{a}} \equiv \tilde{\boldsymbol{x}}\overline{\mathsf{L}}(\boldsymbol{n}) \neq \tilde{\boldsymbol{0}}: \ \Rightarrow \\ \mathsf{L}(\boldsymbol{n})\boldsymbol{a} = \mathsf{L}(\boldsymbol{n})\overline{\mathsf{L}}(\boldsymbol{n})\boldsymbol{x} = \left|\mathsf{L}(\boldsymbol{n})\right|\boldsymbol{x} = \boldsymbol{0}, \quad \tilde{\boldsymbol{a}}\mathsf{L}(\boldsymbol{n}) = \tilde{\boldsymbol{x}}\overline{\mathsf{L}}(\boldsymbol{n})\mathsf{L}(\boldsymbol{n}) = \tilde{\boldsymbol{x}}\left|\mathsf{L}(\boldsymbol{n})\right| = \boldsymbol{0}. \end{gathered} \tag{3.2}$$

Moreover, under condition (3.1) and $\left\langle\overline{\mathsf{L}}(\boldsymbol{n})\right\rangle \equiv \left[\mathsf{L}(\boldsymbol{n})\right] \neq 0$ the operator $\frac{\overline{\mathsf{L}}(\boldsymbol{n})}{\left\langle\overline{\mathsf{L}}(\boldsymbol{n})\right\rangle}$ is a projection operator for the determination of right-hand and left-hand eigenvectors (solutions) of the wave-equation operator $\mathsf{L}(\boldsymbol{n})$ to eigenvalue 0 (see also general identity (A.12) in Appendix A)

$$\Pi(\boldsymbol{n}) \equiv \frac{\overline{\mathsf{L}}(\boldsymbol{n})}{\left\langle\overline{\mathsf{L}}(\boldsymbol{n})\right\rangle}, \quad \Pi^2(\boldsymbol{n}) = \Pi(\boldsymbol{n}), \quad \left\langle\Pi(\boldsymbol{n})\right\rangle = 1. \tag{3.3}$$

Then one may set

$$\Pi(\boldsymbol{n}) = \frac{\boldsymbol{a}\cdot\tilde{\boldsymbol{a}}}{\tilde{\boldsymbol{a}}\boldsymbol{a}} \equiv \boldsymbol{e}\cdot\tilde{\boldsymbol{e}}, \quad \Pi^2(\boldsymbol{n}) = \frac{\boldsymbol{a}\cdot\tilde{\boldsymbol{a}}}{\tilde{\boldsymbol{a}}\boldsymbol{a}} = \Pi(\boldsymbol{n}), \quad \left\langle\Pi(\boldsymbol{n})\right\rangle = \tilde{\boldsymbol{e}}\boldsymbol{e} = 1. \tag{3.4}$$

This means that $\boldsymbol{e}$ is a polarization vector of the electric field which is mutually normalized together with an auxiliary vector $\tilde{\boldsymbol{e}}$ by the condition $\tilde{\boldsymbol{e}}\boldsymbol{e} = 1$ and one may check the equivalences between (3.3) and (3.4).

[4]It is called Fresnel equation where the operator $\mathsf{L}(\boldsymbol{n})$ is mostly written in coordinates of the principal axes of the tensor $\boldsymbol{\varepsilon}$ which, in general, are not compatible with natural coordinates in reflection and refraction problems.

If $\langle \bar{\mathsf{L}}(\boldsymbol{n})\rangle \equiv [\mathsf{L}(\boldsymbol{n})]$ in addition to $|\mathsf{L}(\boldsymbol{n})|$ is vanishing then the projection operator (3.3) becomes singular or undetermined and it follows necessarily (see (A.12))

$$\langle \bar{\mathsf{L}}(\boldsymbol{n})\rangle \equiv [\mathsf{L}(\boldsymbol{n})] = 0, \quad \Rightarrow \quad \bar{\mathsf{L}}^2(\boldsymbol{n}) = 0, \tag{3.5}$$

but not automatically $\bar{\mathsf{L}}(\boldsymbol{n}) = 0$. One has to distinguish then two cases $\bar{\mathsf{L}}(\boldsymbol{n}) \neq 0$ and $\bar{\mathsf{L}}(\boldsymbol{n}) = 0$.

1) In case of $\bar{\mathsf{L}}(\boldsymbol{n}) \neq 0, \langle \bar{\mathsf{L}}(\boldsymbol{n})\rangle = 0$ follows

$$0 \neq \bar{\mathsf{L}}(\boldsymbol{n}) \propto \boldsymbol{a}\cdot\tilde{\boldsymbol{a}}, \quad \Rightarrow \quad 0 = \bar{\mathsf{L}}^2(\boldsymbol{n}) \propto (\tilde{\boldsymbol{a}}\boldsymbol{a})\boldsymbol{a}\cdot\tilde{\boldsymbol{a}}, \quad 0 = \langle \bar{\mathsf{L}}(\boldsymbol{n})\rangle = [\mathsf{L}(\boldsymbol{n})] \propto \tilde{\boldsymbol{a}}\boldsymbol{a}. \tag{3.6}$$

The polarization vectors are not normalizable according to (3.4) but $\tilde{\boldsymbol{a}}\boldsymbol{a} = 0$. This case cannot happen for symmetrical and Hermitean permittivity tensors.

2) In case of $\bar{\mathsf{L}}(\boldsymbol{n}) = 0, \langle \bar{\mathsf{L}}(\boldsymbol{n})\rangle = 0$ follows

$$\begin{aligned} &0 = \bar{\mathsf{L}}(\boldsymbol{n}) \equiv [\mathsf{L}(\boldsymbol{n})]\mathsf{I} - \langle \mathsf{L}(\boldsymbol{n})\rangle \mathsf{L}(\boldsymbol{n}) + \mathsf{L}^2(\boldsymbol{n}), \quad \langle \bar{\mathsf{L}}(\boldsymbol{n})\rangle = [\mathsf{L}(\boldsymbol{n})] = 0, \quad \Rightarrow \\ &\bar{\mathsf{L}}(\boldsymbol{n}) = -\mathsf{L}(\boldsymbol{n})\big(\langle \mathsf{L}(\boldsymbol{n})\rangle \mathsf{I} - \mathsf{L}(\boldsymbol{n})\big) = -\big(\langle \mathsf{L}(\boldsymbol{n})\rangle \mathsf{I} - \mathsf{L}(\boldsymbol{n})\big)\mathsf{L}(\boldsymbol{n}) = 0, \end{aligned} \tag{3.7}$$

and one may define a new, in this case two-dimensional, projection operator $\Pi'(\boldsymbol{n})$ by

$$\Pi'(\boldsymbol{n}) \equiv \frac{\langle \mathsf{L}(\boldsymbol{n})\rangle \mathsf{I} - \mathsf{L}(\boldsymbol{n})}{\langle \mathsf{L}(\boldsymbol{n})\rangle}, \quad \Pi'^2(\boldsymbol{n}) = \Pi'(\boldsymbol{n}), \quad \langle \Pi'(\boldsymbol{n})\rangle = 2. \tag{3.8}$$

Explicitly one finds using (2.16) and (2.17)

$$\Pi'(\boldsymbol{n}) = \frac{\big(\boldsymbol{n}^2 - \langle \boldsymbol{\varepsilon}\rangle\big)\mathsf{I} + \boldsymbol{n}\cdot\boldsymbol{n} + \boldsymbol{\varepsilon}}{2\boldsymbol{n}^2 - \langle \boldsymbol{\varepsilon}\rangle}, \quad \Pi'^2(\boldsymbol{n}) = \Pi'(\boldsymbol{n}), \quad \langle \Pi'(\boldsymbol{n})\rangle = 2. \tag{3.9}$$

This is the case of twofold degeneration of the polarization vectors for directions $\boldsymbol{n}$ of optic axes and of the case of isotropic media where all directions of $\boldsymbol{n}$ are optic axes.

The following calculations illustrate the application of the complementary operator (2.20). The choice of the vector $\boldsymbol{x} = [\boldsymbol{n}, \boldsymbol{n\varepsilon}]$ according to (3.2) with the complementary operator (2.20) leads to a non-normalized polarization vector (see (B.3) in Appendix B)

$$\begin{aligned} \boldsymbol{a} &\equiv \bar{\mathsf{L}}(\boldsymbol{n})[\boldsymbol{n}, \boldsymbol{n\varepsilon}] = -(\boldsymbol{n\varepsilon n})[\boldsymbol{n}, \boldsymbol{n\varepsilon}] + \bar{\boldsymbol{\varepsilon}}[\boldsymbol{n}, \boldsymbol{n\varepsilon}] = \big(\bar{\boldsymbol{\varepsilon}} - (\boldsymbol{n\varepsilon n})\mathsf{I}\big)[\boldsymbol{n}, \boldsymbol{n\varepsilon}] \\ &= [\boldsymbol{n\varepsilon}, \boldsymbol{n\varepsilon}^2] + (\boldsymbol{n\varepsilon n})[\boldsymbol{n\varepsilon}, \boldsymbol{n}] = [\boldsymbol{n\varepsilon}, \boldsymbol{n\varepsilon}^2 + \boldsymbol{n\varepsilon n}\cdot\boldsymbol{n}], \quad \Rightarrow \quad \boldsymbol{n\varepsilon a} = 0, \end{aligned}$$

$$\tilde{\boldsymbol{a}} \equiv [\boldsymbol{n}, \boldsymbol{\varepsilon n}]\bar{\mathsf{L}}(\boldsymbol{n}) = [\boldsymbol{n}, \boldsymbol{\varepsilon n}]\big(\bar{\boldsymbol{\varepsilon}} - (\boldsymbol{n\varepsilon n})\mathsf{I}\big) = [\boldsymbol{\varepsilon n}, \boldsymbol{\varepsilon}^2\boldsymbol{n} + \boldsymbol{n\varepsilon n}\cdot\boldsymbol{n}], \quad \Rightarrow \quad \tilde{\boldsymbol{a}}\boldsymbol{\varepsilon n} = 0. \tag{3.10}$$

Thus if the refraction vector $\boldsymbol{n}$ is a solution of the dispersion equation $|\mathsf{L}(\boldsymbol{n})| = 0$ then a polarization vector $\boldsymbol{e}$ of the electric field is proportional to the vector

$$\boldsymbol{e} \propto [\boldsymbol{n\varepsilon}, \boldsymbol{n\varepsilon}^2 + \boldsymbol{n\varepsilon n}\cdot\boldsymbol{n}], \tag{3.11}$$

if the vector on the right-hand side is non-vanishing. The opposite case of vanishing of this vector happens, for example, for extraordinary waves in uniaxial media (see Section 4).

If the vector $\boldsymbol{x}=[\boldsymbol{n},\boldsymbol{n\varepsilon}]$ does not lead to a vector $\boldsymbol{a}\neq\boldsymbol{0}$ then it is to expect that the vector $\boldsymbol{x}=[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{\varepsilon n}]]$ leads to a non-vanishing vector $\boldsymbol{a}$ proportional to a polarization vector of the electric field according to the relations

$$\begin{aligned}\boldsymbol{a}&=\overline{\mathsf{L}}(\boldsymbol{n})[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{\varepsilon n}]]=(\boldsymbol{n\varepsilon}[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{\varepsilon n}]])\boldsymbol{n}-((\boldsymbol{n\varepsilon n})\mathsf{I}-\overline{\boldsymbol{\varepsilon}})[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{\varepsilon n}]]\\&=[\boldsymbol{n\varepsilon},\boldsymbol{n}][\boldsymbol{n},\boldsymbol{\varepsilon n}]\cdot\boldsymbol{n}-\boldsymbol{n\varepsilon n}(\boldsymbol{n\varepsilon n}\cdot\boldsymbol{n}-\boldsymbol{n}^2\cdot\boldsymbol{\varepsilon n})+\overline{\boldsymbol{\varepsilon}}[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{\varepsilon n}]]\\&=[\boldsymbol{n\varepsilon},[\boldsymbol{n}^2\cdot\boldsymbol{\varepsilon n}+\overline{\boldsymbol{\varepsilon}}\boldsymbol{n},\boldsymbol{n}]],\quad\Rightarrow\quad\boldsymbol{n\varepsilon a}=0,\\\tilde{\boldsymbol{a}}&=[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{n\varepsilon}]]\overline{\mathsf{L}}(\boldsymbol{n})=[\boldsymbol{\varepsilon n},[\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon}+\boldsymbol{n}\overline{\boldsymbol{\varepsilon}},\boldsymbol{n}]],\quad\Rightarrow\quad\tilde{\boldsymbol{a}}\boldsymbol{\varepsilon n}=0.\end{aligned}\tag{3.12}$$

In the transformations, we applied a known relation for double vectorial products and identities together with operators of Appendix B (in particular, $\overline{\boldsymbol{\varepsilon}}[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{\varepsilon n}]]=[\boldsymbol{n\varepsilon},[\overline{\boldsymbol{\varepsilon}}\boldsymbol{n},\boldsymbol{n}]]$). Thus, if the refraction vector $\boldsymbol{n}$ is a solution of the dispersion equation $|\mathsf{L}(\boldsymbol{n})|=0$ then a polarization vector $\boldsymbol{e}$ of the electric field may be proportional to the vector

$$\boldsymbol{e}\propto[\boldsymbol{n\varepsilon},[\boldsymbol{n}^2\cdot\boldsymbol{\varepsilon n}+\overline{\boldsymbol{\varepsilon}}\boldsymbol{n},\boldsymbol{n}]],\tag{3.13}$$

if the vector on the right-hand side is non-vanishing. The opposite case of vanishing of this vector happens, for example, for ordinary waves in uniaxial media (see Section 4). Due to symmetry of the wave-equation operator (2.16) and of its complementary operator $\overline{\mathsf{L}}(\boldsymbol{n})$ the polarization vectors to refraction vectors $\boldsymbol{n}$ and $-\boldsymbol{n}$ can be chosen as the same vectors.

As a simple illustration we consider an isotropic medium without spatial dispersion but with frequency dispersion for which the permittivity tensor and the wave-equation operator possess the form

$$\boldsymbol{\varepsilon}=\varepsilon\,\mathsf{I},\quad\mathsf{L}(\boldsymbol{n})=\boldsymbol{n}\cdot\boldsymbol{n}-(\boldsymbol{n}^2-\varepsilon)\mathsf{I},\qquad(\varepsilon\equiv\varepsilon(\omega)),\tag{3.14}$$

from which follows

$$\begin{aligned}&\langle\mathsf{L}(\boldsymbol{n})\rangle=-2\boldsymbol{n}^2+3\varepsilon,\quad[\mathsf{L}(\boldsymbol{n})]=(\boldsymbol{n}^2-\varepsilon)(\boldsymbol{n}^2-3\varepsilon),\quad|\mathsf{L}(\boldsymbol{n})|=\varepsilon(\boldsymbol{n}^2-\varepsilon)^2,\\&\overline{\mathsf{L}}(\boldsymbol{n})=(\boldsymbol{n}^2-\varepsilon)(\boldsymbol{n}\cdot\boldsymbol{n}-\varepsilon\,\mathsf{I}),\qquad\langle\mathsf{L}(\boldsymbol{n})\rangle\mathsf{I}-\mathsf{L}(\boldsymbol{n})=-(\boldsymbol{n}\cdot\boldsymbol{n}+(\boldsymbol{n}^2-2\varepsilon)\mathsf{I}).\end{aligned}\tag{3.15}$$

1) For waves with the dispersion equation

$$\boldsymbol{n}^2-\varepsilon=0,\tag{3.16}$$

one finds

$$\begin{aligned}&\mathsf{L}(\boldsymbol{n})=\boldsymbol{n}\cdot\boldsymbol{n},\quad\langle\mathsf{L}(\boldsymbol{n})\rangle=\boldsymbol{n}^2=\varepsilon,\quad[\mathsf{L}(\boldsymbol{n})]=\langle\overline{\mathsf{L}}(\boldsymbol{n})\rangle=0,\quad\overline{\mathsf{L}}(\boldsymbol{n})=0,\\&\langle\mathsf{L}(\boldsymbol{n})\rangle\mathsf{I}-\mathsf{L}(\boldsymbol{n})=\boldsymbol{n}^2\,\mathsf{I}-\boldsymbol{n}\cdot\boldsymbol{n},\end{aligned}\tag{3.17}$$

and the two-dimensional projection operator for the determination of polarization vectors of the electric field is

$$\Pi'(\boldsymbol{n})=\mathsf{I}-\frac{\boldsymbol{n}\cdot\boldsymbol{n}}{\boldsymbol{n}^2},\quad\Pi'^2(\boldsymbol{n})=\Pi'(\boldsymbol{n}),\quad\langle\Pi'(\boldsymbol{n})\rangle=2.\tag{3.18}$$

Due to this projection operator $\Pi'(\boldsymbol{n})$ these are transversal waves with two independent polarization vectors perpendicular to refraction vector $\boldsymbol{n}$

2) For waves with the dispersion equation

$$\varepsilon = 0, \tag{3.19}$$

this means

$$\mathsf{L}(\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{n} - (\boldsymbol{n}^2)\mathsf{I}, \quad \langle \mathsf{L}(\boldsymbol{n})\rangle = -2\boldsymbol{n}^2, \quad [\mathsf{L}(\boldsymbol{n})] = \langle \overline{\mathsf{L}}(\boldsymbol{n})\rangle = (\boldsymbol{n}^2)^2,$$
$$\overline{\mathsf{L}}(\boldsymbol{n}) = (\boldsymbol{n}^2)\boldsymbol{n}\cdot\boldsymbol{n}, \tag{3.20}$$

and the one-dimensional projection operator for the determination of polarization vectors of the electric field becomes (for $\boldsymbol{n}^2 \neq 0$)

$$\Pi(\boldsymbol{n}) = \frac{\boldsymbol{n}\cdot\boldsymbol{n}}{\boldsymbol{n}^2}, \quad \Pi^2(\boldsymbol{n}) = \Pi(\boldsymbol{n}), \quad \langle \Pi(\boldsymbol{n})\rangle = 1. \tag{3.21}$$

This belongs to longitudinal waves with polarization in direction of the vector $\boldsymbol{n}$.

All these are well-known properties which can be derived without the here developed general formalism but it demonstrates how this formalism acts.

4. Dispersion Equation and Polarization Vectors of Electric Field in Uniaxial Media

In this section, we demonstrate the application of the projection operators for the determination of polarization vectors of the electric field in case of uniaxial media.

The permittivity tensor $\boldsymbol{\varepsilon}$ of uniaxially media is

$$\boldsymbol{\varepsilon} = \varepsilon_e \boldsymbol{c}\cdot\boldsymbol{c} + \varepsilon_o(\mathsf{I} - \boldsymbol{c}\cdot\boldsymbol{c}), \quad \boldsymbol{c}^2 = 1, \tag{4.1}$$

with two (frequency-dependent) parameters ε_e and ε_o and where $\boldsymbol{c}$ is a unit vector in direction of the optic axis. From this follows for the invariants of the permittivity tensor

$$\langle \boldsymbol{\varepsilon}\rangle = \varepsilon_e + 2\varepsilon_o,$$
$$[\boldsymbol{\varepsilon}] = \varepsilon_o(2\varepsilon_e + \varepsilon_o),$$
$$|\boldsymbol{\varepsilon}| = \varepsilon_e\varepsilon_o^2. \tag{4.2}$$

For the complementary operator $\overline{\boldsymbol{\varepsilon}}$ to $\boldsymbol{\varepsilon}$ one finds (see (A.8))

$$\overline{\boldsymbol{\varepsilon}} = \varepsilon_o\left(\varepsilon_o \boldsymbol{c}\cdot\boldsymbol{c} + \varepsilon_e(\mathsf{I} - \boldsymbol{c}\cdot\boldsymbol{c})\right), \quad \boldsymbol{\varepsilon}^{-1} = \frac{\overline{\boldsymbol{\varepsilon}}}{|\boldsymbol{\varepsilon}|}. \tag{4.3}$$

In connection with the refraction vectors $\boldsymbol{n}$ we have (μ is power of $\boldsymbol{\varepsilon}$ and not an index)

$$\begin{aligned} \boldsymbol{n}\boldsymbol{\varepsilon}^\mu &= \boldsymbol{\varepsilon}^\mu \boldsymbol{n} = \varepsilon_e^\mu \boldsymbol{n}\boldsymbol{c}\cdot\boldsymbol{c} + \varepsilon_o^\mu\left[\boldsymbol{c},[\boldsymbol{n},\boldsymbol{c}]\right] \\ &= \varepsilon_o^\mu \boldsymbol{n} + \left(\varepsilon_e^\mu - \varepsilon_o^\mu\right)\boldsymbol{n}\boldsymbol{c}\cdot\boldsymbol{c}, \qquad (\mu = 0,1,2,\cdots). \end{aligned} \tag{4.4}$$

and therefore

$$\boldsymbol{n}^2 = (\boldsymbol{n}\boldsymbol{c})^2 + [\boldsymbol{n},\boldsymbol{c}]^2,$$
$$\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} = \varepsilon_e(\boldsymbol{n}\boldsymbol{c})^2 + \varepsilon_o[\boldsymbol{n},\boldsymbol{c}]^2,$$

$$\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n} = \varepsilon_e^2 (\boldsymbol{nc})^2 + \varepsilon_o^2 [\boldsymbol{n},\boldsymbol{c}]^2. \tag{4.5}$$

Furthermore

$$\boldsymbol{n}^2 \cdot \boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n} - (\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})^2 = (\varepsilon_e - \varepsilon_o)^2 (\boldsymbol{nc})^2 [\boldsymbol{n},\boldsymbol{c}]^2. \tag{4.6}$$

The invariants of the wave equation operator $\mathsf{L}(\boldsymbol{n})$ are

$$\langle \mathsf{L}(\boldsymbol{n}) \rangle = -2\boldsymbol{n}^2 + \varepsilon_e + 2\varepsilon_o,$$

$$\begin{aligned} [\mathsf{L}(\boldsymbol{n})] &= (\boldsymbol{n}^2)^2 - \left(2(\varepsilon_o + \varepsilon_e)(\boldsymbol{nc})^2 + (3\varepsilon_o + \varepsilon_e)[\boldsymbol{n},\boldsymbol{c}]^2\right) + \varepsilon_o (2\varepsilon_e + \varepsilon_o) \\ &= (\boldsymbol{n}^2 - \varepsilon_o)^2 - 2(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \varepsilon_e\varepsilon_o) - (\varepsilon_e - \varepsilon_o)[\boldsymbol{n},\boldsymbol{c}]^2, \end{aligned}$$

$$|\mathsf{L}(\boldsymbol{n})| = (\boldsymbol{n}^2 - \varepsilon_o)(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \varepsilon_e\varepsilon_o). \tag{4.7}$$

The specialization of the complementary operator (2.20) with $\boldsymbol{\varepsilon}$ according to (4.1) leads to the following representation

$$\overline{\mathsf{L}}(\boldsymbol{n}) = (\boldsymbol{n}^2 - \varepsilon_e)\boldsymbol{n}\cdot\boldsymbol{n} + (\varepsilon_e - \varepsilon_o)\left(\boldsymbol{nc}(\boldsymbol{n}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{n}) - \varepsilon_o \boldsymbol{c}\cdot\boldsymbol{c}\right) - (\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \varepsilon_e\varepsilon_o)\mathsf{I}. \tag{4.8}$$

The dispersion equation decomposes in rational way into a product as follows

$$0 = |\mathsf{L}(\boldsymbol{n})| = (\boldsymbol{n}^2 - \varepsilon_o)(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \varepsilon_e\varepsilon_o), \tag{4.9}$$

and one has to consider two special kinds of waves. The dispersion equation for ordinary waves is a sphere

$$\boldsymbol{n}^2 \equiv (\boldsymbol{nc})^2 + [\boldsymbol{n},\boldsymbol{c}]^2 = \varepsilon_o, \quad \Rightarrow \quad \frac{(\boldsymbol{nc})^2 + [\boldsymbol{n},\boldsymbol{c}]^2}{\varepsilon_o} = 1, \tag{4.10}$$

and the dispersion equation for extraordinary waves is a rotational ellipsoid with axis $\boldsymbol{c}$

$$\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} \equiv \varepsilon_e (\boldsymbol{nc})^2 + \varepsilon_o [\boldsymbol{n},\boldsymbol{c}]^2 = \varepsilon_e\varepsilon_o, \quad \Rightarrow \quad \frac{(\boldsymbol{nc})^2}{\varepsilon_o} + \frac{[\boldsymbol{n},\boldsymbol{c}]^2}{\varepsilon_e} = 1, \tag{4.11}$$

with axis lengths $\sqrt{\varepsilon_o}$ in direction of the optic axis and $\sqrt{\varepsilon_e}$ perpendicular to it. For real values of ε_e and ε_o in case of $\varepsilon_o < \varepsilon_e$ the medium is called uniaxial positive and in case of $\varepsilon_o > \varepsilon_e$ uniaxial negative [6] [7]. However, since $\varepsilon_e \equiv \varepsilon_e(\omega)$ and $\varepsilon_o \equiv \varepsilon_o(\omega)$ are frequency-dependent and different from the axes vector $\boldsymbol{c}$ are not fixed by the symmetry of the medium it may happen that the medium changes with frequency from being positive to negative uniaxial or vice versa.

We now determine polarization vectors for the electric field of these two kinds of waves. According to (3.10) polarization vectors of the electric field $\boldsymbol{a}$ are perpendicular to the vector $\boldsymbol{n}\boldsymbol{\varepsilon}$ that means for uniaxial media

$$0 = \boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{a} = \left(\varepsilon_e \boldsymbol{nc}\cdot\boldsymbol{c} + \varepsilon_o \left[\boldsymbol{c},[\boldsymbol{n},\boldsymbol{c}]\right]\right)\boldsymbol{a} = \left((\varepsilon_e - \varepsilon_o)\boldsymbol{nc}\cdot\boldsymbol{c} + \varepsilon_o\boldsymbol{n}\right)\boldsymbol{a}. \tag{4.12}$$

The vector $\boldsymbol{n}\boldsymbol{\varepsilon}$ lies in the plane spanned by the vectors $\boldsymbol{c}$ and $\boldsymbol{n}$ (see (4.4)). Therefore the vector $\boldsymbol{x} = [\boldsymbol{n},\boldsymbol{c}]$ is perpendicular to the vector $\boldsymbol{n}\boldsymbol{\varepsilon}$ and if we insert it into $\boldsymbol{a} \equiv \mathsf{L}(\boldsymbol{n})\boldsymbol{x}$ with $\mathsf{L}(\boldsymbol{n})$ given in (4.8) we find

$$a = \overline{\mathsf{L}}(\boldsymbol{n})[\boldsymbol{n},\boldsymbol{c}] = -(\boldsymbol{n\varepsilon n} - \varepsilon_e\varepsilon_o)[\boldsymbol{n},\boldsymbol{c}] \begin{cases} \neq 0, & \text{if } \boldsymbol{n\varepsilon n} - \varepsilon_e\varepsilon_o \neq 0 \text{ and } [\boldsymbol{n},\boldsymbol{c}] \neq 0, \\ = 0, & \text{if } \boldsymbol{n\varepsilon n} - \varepsilon_e\varepsilon_o = 0 \text{ or } [\boldsymbol{n},\boldsymbol{c}] = 0. \end{cases} \quad (4.13)$$

It is unequal to zero for ordinary waves that means if dispersion Equation (4.10) is satisfied and in addition $[\boldsymbol{n},\boldsymbol{c}] \neq \boldsymbol{0}$ but it is zero if the dispersion Equation (4.11) for extraordinary waves is satisfied and in this case, it is not a polarization vector of the electric field. The case $[\boldsymbol{n},\boldsymbol{c}] = \boldsymbol{0}$ is the degenerated case where the surfaces described by the dispersion equations for ordinary and extraordinary waves touch together and all vectors in the plane perpendicular to $\boldsymbol{n} \propto \boldsymbol{c}$ are possible polarization vectors of the electric field.

From the form (4.8) using the dispersion equation for extraordinary waves (4.11) it is seen that $\mathsf{L}(\boldsymbol{n})\boldsymbol{x}$ for arbitrary vector $\boldsymbol{x}$ is then a superposition of the vectors $\boldsymbol{n}$ and $\boldsymbol{c}$ and since in addition it must be perpendicular to the vector $\boldsymbol{n\varepsilon}$ it should be

$$\boldsymbol{a} \propto [\boldsymbol{n\varepsilon},[\boldsymbol{n},\boldsymbol{c}]] = \boldsymbol{n\varepsilon c}\cdot\boldsymbol{n} - \boldsymbol{n\varepsilon n}\cdot\boldsymbol{c} = \varepsilon_e \boldsymbol{nc}\cdot\boldsymbol{n} - \boldsymbol{n\varepsilon n}\cdot\boldsymbol{c} = \varepsilon_e(\boldsymbol{nc}\cdot\boldsymbol{n} - \varepsilon_o\boldsymbol{c}). \quad (4.14)$$

Thus polarization vectors of the electric field for ordinary waves are (non-normalized and normalized):

$$\boldsymbol{a} \propto [\boldsymbol{n},\boldsymbol{c}], \qquad \boldsymbol{e}_o = \frac{[\boldsymbol{n},\boldsymbol{c}]}{\sqrt{\boldsymbol{n}^2 - (\boldsymbol{nc})^2}} = \frac{[\boldsymbol{n},\boldsymbol{c}]}{\sqrt{\varepsilon_o - (\boldsymbol{nc})^2}}, \quad (4.15)$$

and for extraordinary waves (non-normalized and normalized):

$$\boldsymbol{a} \propto \varepsilon_o\boldsymbol{c} - \boldsymbol{nc}\cdot\boldsymbol{n}, \qquad \boldsymbol{e}_e = \frac{\varepsilon_o\boldsymbol{c} - \boldsymbol{nc}\cdot\boldsymbol{n}}{\sqrt{\left(\varepsilon_o - (\boldsymbol{nc})^2\right)^2 + (\boldsymbol{nc})^2[\boldsymbol{n},\boldsymbol{c}]^2}}. \quad (4.16)$$

The vectors $\boldsymbol{n}$ in (4.15) are refraction vectors for ordinary waves and that in (4.16) for extraordinary waves. One may check (4.15) and (4.16) by inserting them into the wave Equation (2.15)

$$\begin{gathered} \{\boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{n}^2\,\mathsf{I} + \varepsilon_e\boldsymbol{c}\cdot\boldsymbol{c} + \varepsilon_o(\mathsf{I} - \boldsymbol{c}\cdot\boldsymbol{c})\}[\boldsymbol{n},\boldsymbol{c}] = -(\boldsymbol{n}^2 - \varepsilon_o)[\boldsymbol{n},\boldsymbol{c}] = \boldsymbol{0}, \\ \{\boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{n}^2\,\mathsf{I} + \varepsilon_e\boldsymbol{c}\cdot\boldsymbol{c} + \varepsilon_o(\mathsf{I} - \boldsymbol{c}\cdot\boldsymbol{c})\}(\varepsilon_o\boldsymbol{c} - \boldsymbol{nc}\cdot\boldsymbol{n}) \\ = \left(\varepsilon_e\varepsilon_o - \varepsilon_e(\boldsymbol{n},\boldsymbol{c})^2 - \varepsilon_o[\boldsymbol{n},\boldsymbol{c}]^2\right)\boldsymbol{c} = (\varepsilon_e\varepsilon_o - \boldsymbol{n\varepsilon n})\boldsymbol{c} = \boldsymbol{0}. \end{gathered} \quad (4.17)$$

It is also not very difficult to check that the polarization vector (3.11) vanishes for $\boldsymbol{n\varepsilon n} = \varepsilon_e\varepsilon_o$ that means for extraordinary waves and thus can only be a polarization vector for ordinary waves and that the polarization vector (3.13) vanishes for $\boldsymbol{n}^2 = \varepsilon_o$ and thus can only be a polarization vector for extraordinary waves such as directly affirmed by (4.17). In the special case if the refraction vectors of the ordinary and the extraordinary wave are parallel to each other $[\boldsymbol{n}_o,\boldsymbol{n}_e] = \boldsymbol{0}$ the corresponding polarization vectors of the electric field are perpendicular to each other $\boldsymbol{e}_o\boldsymbol{e}_e = 0$. In reflection and refraction problems, however, this is usually not the case since $[\boldsymbol{n}_o,\boldsymbol{n}_e] \neq \boldsymbol{0}$ in general.

Our coordinate-invariant method of derivations of polarization vectors distinguishes from that of Fyodorov [1] by using the complementary operator $\overline{\mathsf{L}}(\boldsymbol{n})$

to the operator of the wave equation $\mathsf{L}(\boldsymbol{n})$ and thus it is nearer to the more generally possible method.

5. Plane Monochromatic Waves at the Plane Boundary Between Two Media

We now consider electromagnetic waves at the plane boundary between two homogeneous media without spatial dispersion but with frequency dispersion. The plane boundary is characterized by a unit vector $\boldsymbol{N}$ perpendicular to it and for simplicity we let it go through the coordinate origin $\boldsymbol{r}=\boldsymbol{0}$ and thus it is determined by (**Figure 1**)

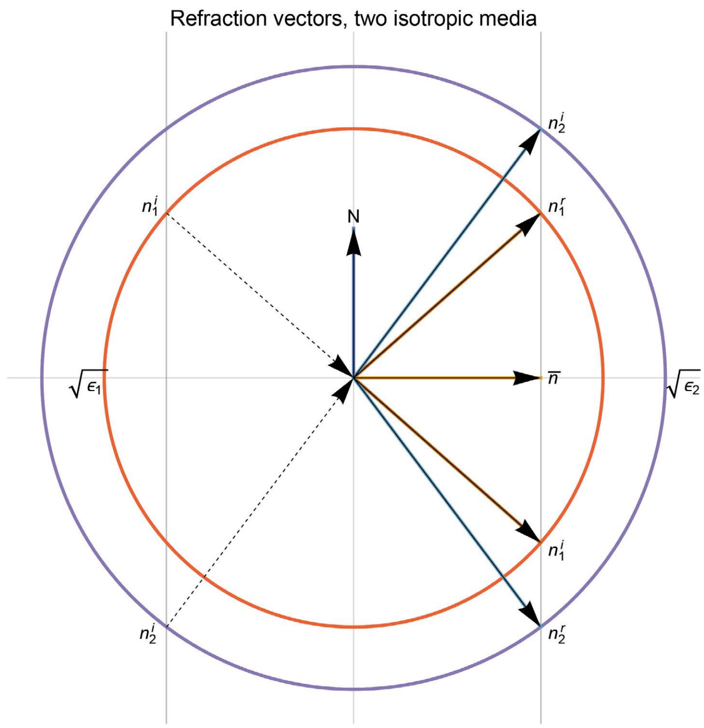

Figure 1. Refraction vectors of incident and reflected and refracted waves at the boundary between two isotropic media. The paper plane is the incidence plane. In general case, we may have two incident waves which contribute to coupled reflected or refracted waves, one from medium 1 and the other from medium 2. $\boldsymbol{N}$ is the normal unit vector to boundary plane and $\bar{\boldsymbol{n}}$ the tangential part common to all refraction vectors of the coupled waves.

$$\boldsymbol{Nr}=0,\quad \left(\boldsymbol{N}^2=1\right). \tag{5.1}$$

We work in the space-frequency picture of variables $(\boldsymbol{r},\omega)$. The two media are called medium 1 and medium 2 with lower indices 1 and 2 at the corresponding quantities. The permittivity tensors of the two media are $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ and the permittivity tensor $\boldsymbol{\varepsilon}$ of the whole (inhomogeneous) medium is

$$\boldsymbol{\varepsilon}(\boldsymbol{r};\omega)=\boldsymbol{\varepsilon}_1(\omega)\theta(\boldsymbol{Nr})+\boldsymbol{\varepsilon}_2(\omega)\theta(-\boldsymbol{Nr}), \tag{5.2}$$

The function $\theta(z)$ is the Heaviside step function ($\theta(z)=0$, $z<0$, $\theta(z)=1$,

$z>0$). For the electric field of the whole medium, we make the supposition

$$\boldsymbol{E}(\boldsymbol{r},\omega)=\boldsymbol{E}_1(\boldsymbol{r},\omega)\theta(\boldsymbol{Nr})+\boldsymbol{E}_2(\boldsymbol{r},\omega)\theta(-\boldsymbol{Nr}), \tag{5.3}$$

It has to satisfy the equation

$$0=\mathsf{L}(\boldsymbol{r};-\mathrm{i}\nabla,\omega)\boldsymbol{E}(\boldsymbol{r},\omega), \tag{5.4}$$

with the operator

$$\mathsf{L}(\boldsymbol{r};\boldsymbol{k},\omega)\equiv\frac{c^2}{\omega^2}\left(\boldsymbol{k}\cdot\boldsymbol{k}-\boldsymbol{k}^2\mathsf{I}\right)+\boldsymbol{\varepsilon}(\boldsymbol{r};\omega). \tag{5.5}$$

The solution of (5.4) with the operator (5.5) consists of the separate solution of the equations for the two (homogeneous) partial media 1 and 2 plus boundary conditions at the plane boundary between the two media. The boundary conditions come from the differentiation of the Heaviside step functions $\theta(\pm\boldsymbol{Nr})$ according to $\nabla\theta(\pm\boldsymbol{Nr})=\pm\boldsymbol{N}\delta(\boldsymbol{Nr})$ which bring into play the delta function $\delta(\boldsymbol{Nr})$ and by two-fold differentiation the derivative of the delta function $\delta^{(1)}(\boldsymbol{Nr})$ and are the vanishing of the vectorial coefficients in front of these functions [18] [19][5].

The boundary conditions for the electric field at the plane boundary $\boldsymbol{Nr}=0$ (vanishing of coefficients in front of $\delta^{(1)}(\boldsymbol{Nr})$) retranslated into the space-time picture are

$$\boldsymbol{0}=\left\{(\mathsf{I}-\boldsymbol{N}\cdot\boldsymbol{N})\left(\boldsymbol{E}_1(\boldsymbol{r},t)-\boldsymbol{E}_2(\boldsymbol{r},t)\right)\right\}_{\boldsymbol{Nr}=0}, \tag{5.6}$$

and for the magnetic field (vanishing of coefficients in front of $\delta(\boldsymbol{Nr})$) retranslated into the space-time picture

$$\boldsymbol{0}=\left\{\left[\boldsymbol{N},\boldsymbol{B}_1(\boldsymbol{r},t)-\boldsymbol{B}_2(\boldsymbol{r},t)\right]\right\}_{\boldsymbol{Nr}=0}, \tag{5.7}$$

The first of these conditions is the continuity of the tangential components of the electric fields at the plane boundary and the second after vectorial multiplication with vector $\boldsymbol{N}$ the continuity of the tangential components of the magnetic fields at the plane boundary. From the continuity of the tangential components of the electric field at the plane boundary follows generally via the first vectorial of the Maxwell Equation (2.1) that also the normal component of the magnetic field with respect to the boundary plane has to be continuous at this plane

$$0=\left\{\boldsymbol{N}\left(\boldsymbol{B}_1(\boldsymbol{r},t)-\boldsymbol{B}_2(\boldsymbol{r},t)\right)\right\}_{\boldsymbol{Nr}=0}. \tag{5.8}$$

This means that the whole magnetic field is continuous in the approximation of neglected spatial dispersion at the boundary

$$0=\left\{\boldsymbol{B}_1(\boldsymbol{r},t)-\boldsymbol{B}_2(\boldsymbol{r},t)\right\}_{\boldsymbol{Nr}=0}. \tag{5.9}$$

Often it is advantageous to use the condition of continuity of the magnetic field

[5]With the publication of the first cited paper, I have had great trouble with Prof. Gustav Richter from Annalen der Physik (Leipzig) and only by objection of Prof. Günter Vojta the second in the order of Members from GDR in the Board of the journal whom I asked for this objection and about him I knew that he was a very able physicist the paper could be published in this journal (see also a footnote in [19]). After this G. Richter rejected two further papers of mine and I understood that I cannot furthermore publish under his leadership of the journal.

in this form with redundancy of one component.

We consider now a plane monochromatic wave with wave vector $\boldsymbol{k}=\frac{\omega}{c}\boldsymbol{n}$ and frequency ω with the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and field amplitude $\boldsymbol{E}$ ("c.c". is "complex conjugated")

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}\exp\left(\mathrm{i}(\boldsymbol{kr}-\omega t)\right)+c.c.=\boldsymbol{E}\exp\left(\mathrm{i}\frac{\omega}{c}(\boldsymbol{nr}-ct)\right)+c.c., \qquad (5.10)$$

and analogously for the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ and the electric induction $\boldsymbol{D}(\boldsymbol{r},t)$. The refraction vectors $\boldsymbol{n}$ and the field amplitudes $\boldsymbol{E}$ are later specialized by indices for the characters of the wave (number of medium, incident or reflected or refracted waves). We separate now any of the involved refraction vectors $\boldsymbol{n}$ and the position vector $\boldsymbol{r}$ into a part parallel to the boundary plane and a part perpendicular to this plane

$$\boldsymbol{n}=\left[[\boldsymbol{N},\boldsymbol{n}],\boldsymbol{N}\right]+\boldsymbol{nN}\cdot\boldsymbol{N}\equiv\overline{\boldsymbol{n}}+\boldsymbol{nN}\cdot\boldsymbol{N},$$

$$\boldsymbol{r}=\left[[\boldsymbol{N},\boldsymbol{r}],\boldsymbol{N}\right]+\boldsymbol{N}\cdot\boldsymbol{Nr}\equiv\overline{\boldsymbol{r}}+\boldsymbol{N}\cdot\boldsymbol{Nr}. \qquad (5.11)$$

Furthermore, we have introduced a notation for the common tangential part of the refraction vectors $\boldsymbol{n}$ with respect to the boundary plane

$$\overline{\boldsymbol{n}}\equiv\left[[\boldsymbol{N},\boldsymbol{n}],\boldsymbol{N}\right],\quad \overline{\boldsymbol{n}}\boldsymbol{N}=0,\quad [\boldsymbol{N},\boldsymbol{n}]=[\boldsymbol{N},\overline{\boldsymbol{n}}],\quad [\boldsymbol{N},\boldsymbol{n}]^2=\boldsymbol{n}^2-(\boldsymbol{nN})^2=\overline{\boldsymbol{n}}^2. \qquad (5.12)$$

Due to the boundary conditions, the vector $\overline{\boldsymbol{n}}$ is the same for all waves coupled in a reflection and refraction problem at the boundary. The phase $\Phi(\boldsymbol{r},t)$ of the wave (5.10) becomes then

$$\Phi(\boldsymbol{r},t)\equiv\frac{\omega}{c}(\boldsymbol{nr}-ct)=\frac{\omega}{c}(\overline{\boldsymbol{n}}\overline{\boldsymbol{r}}-ct)+\frac{\omega}{c}(\boldsymbol{nN}\cdot\boldsymbol{Nr}). \qquad (5.13)$$

The second sum term of the phase in (5.13) vanishes at the boundary plane $\boldsymbol{Nr}=0$ and is only present in the bulk of the medium. Only the first sum term of the phase is non-vanishing at the boundary plane and has to be the same for all coupled waves and therefore usually does not appear in the results for reflection and refraction problems. However, via the dispersion equation, this tangential part of the refraction vectors determines their normal components $\boldsymbol{nN}$ in direction of the normal vector $\boldsymbol{N}$ to the plane (Section 10).

The field amplitudes of the electric fields $\boldsymbol{E}_\nu^s,(\nu=(1,2),\,s=(i,r))$ for isotropic media taking into account $\boldsymbol{n}_\nu^s\boldsymbol{E}_\nu^s=0$ can be decomposed into a part parallel to the incidence plane proportional to $\left[\boldsymbol{n}_\nu^s,[\boldsymbol{N},\boldsymbol{n}]\right]$ but perpendicular to $\boldsymbol{n}_\nu^s$ and a part perpendicular to the incidence plane proportional to $[\boldsymbol{N},\boldsymbol{n}]$ according to

$$\boldsymbol{E}_\nu^s=\underbrace{\frac{\boldsymbol{n}_\nu^s\cdot\boldsymbol{n}_\nu^s}{\boldsymbol{n}_\nu^s\boldsymbol{n}_\nu^s}\boldsymbol{E}_\nu^s}_{=0}+\frac{\boldsymbol{NE}_\nu^s\cdot\left[\boldsymbol{n}_\nu^s,[\boldsymbol{N},\boldsymbol{n}]\right]+\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_\nu^s\right]\cdot[\boldsymbol{N},\boldsymbol{n}]}{[\boldsymbol{N},\boldsymbol{n}]^2},\quad \boldsymbol{n}_\nu^s\boldsymbol{N}\cdot\boldsymbol{NE}_\nu^s=-\overline{\boldsymbol{n}}\boldsymbol{E}_\nu^s, \qquad (5.14)$$

and correspondingly for the other field amplitudes $\boldsymbol{B}$ and $\boldsymbol{D}$. Again, for undetermined $\boldsymbol{n}$ can be inserted an arbitrary of the coupled refraction vectors in a reflection and refraction problem. The considered polarization of each wave can be signified by two free scalar parameters $\boldsymbol{NE}_\nu^s$ or equivalently $\overline{\boldsymbol{n}}\boldsymbol{E}_\nu^s$ and $\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_\nu^s\right]$.

6. Coordinate-Invariant Treatment of Reflection and Refraction of Plane Monochromatic Waves at the Boundary of Two Isotropic Media

In this section, we consider the reflection and refraction of plane monochromatic waves at a plane boundary between two isotropic media in coordinate-invariant treatment. This problem is well known with solution mainly in coordinate treatment but we hope that we may show some advantages of the coordinate-invariant treatment. The method is analogous to a method which we use in Section 9 for the case that the second medium is anisotropic.

With lower indices 1 and 2, we denote the characteristics of the waves in medium 1 and medium 2. The permittivity tensors for the two isotropic media are

$$\boldsymbol{\varepsilon}_1(\omega)=\varepsilon_1(\omega)\mathsf{I},\quad \boldsymbol{\varepsilon}_2(\omega)=\varepsilon_2(\omega)\mathsf{I}, \tag{6.1}$$

The whole electric field in the bulk of both media of considered problem is

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t)&=\mathrm{e}^{\mathrm{i}\frac{\omega}{c}(\overline{\boldsymbol{n}}\boldsymbol{r}-ct)}\left\{\left(\boldsymbol{E}_1^i\mathrm{e}^{\mathrm{i}\frac{\omega}{c}\left(\boldsymbol{n}_1^i\boldsymbol{N}\cdot\boldsymbol{N}\boldsymbol{r}\right)}+\boldsymbol{E}_1^r\mathrm{e}^{\mathrm{i}\frac{\omega}{c}\left(\boldsymbol{n}_1^i\boldsymbol{N}\cdot\boldsymbol{N}\boldsymbol{r}\right)}\right)\theta(\boldsymbol{N}\boldsymbol{r})\right.\\&\quad\left.+\left(\boldsymbol{E}_2^i\mathrm{e}^{\mathrm{i}\frac{\omega}{c}\left(\boldsymbol{n}_2^i\boldsymbol{N}\cdot\boldsymbol{N}\boldsymbol{r}\right)}+\boldsymbol{E}_2^r\mathrm{e}^{\mathrm{i}\frac{\omega}{c}\left(\boldsymbol{n}_2^i\boldsymbol{N}\cdot\boldsymbol{N}\boldsymbol{r}\right)}\right)\theta(-\boldsymbol{N}\boldsymbol{r})\right\}+c.c.\\&\equiv\boldsymbol{E}_1(\boldsymbol{r},t)\theta(\boldsymbol{N}\boldsymbol{r})+c.c.+\boldsymbol{E}_2(\boldsymbol{r},t)\theta(-\boldsymbol{N}\boldsymbol{r})+c.c.\end{aligned} \tag{6.2}$$

The upper indices “ i ” and “ r ” denote “incident” and “reflected” or “refracted” waves, correspondingly, the last with reflected or refracted waves with respect from which side the incident waves are seen. At the boundary $\boldsymbol{N}\boldsymbol{r}=0$ remains from phase only the phase factor $\mathrm{e}^{\mathrm{i}\frac{\omega}{c}(\overline{\boldsymbol{n}}\boldsymbol{r}-ct)}$ which is common to all partial waves due to continuity relations and the vectorial amplitudes. Due to isotropy of the media 1 and 2 one has for the refraction vectors the relations

$$\begin{aligned}&\boldsymbol{n}_\nu^i\boldsymbol{n}_\nu^i=\boldsymbol{n}_\nu^r\boldsymbol{n}_\nu^r=\varepsilon_\nu,\quad \boldsymbol{n}_\nu^s=\overline{\boldsymbol{n}}+\boldsymbol{n}_\nu^s\boldsymbol{N}\cdot\boldsymbol{N},\quad \boldsymbol{n}_\nu^s\boldsymbol{N}=\pm\sqrt{\varepsilon_\nu-\overline{\boldsymbol{n}}^2},\\&\boldsymbol{n}_\nu^r\boldsymbol{N}=-\boldsymbol{n}_\nu^i\boldsymbol{N},\quad \boldsymbol{n}_\nu^r=-\boldsymbol{n}_\nu^i+2\overline{\boldsymbol{n}},\qquad (\nu=(1,2);\,s=(i,r)),\end{aligned} \tag{6.3}$$

where $\overline{\boldsymbol{n}}$ is the tangential part common to all refraction vectors coupled in a reflection and refraction problem. The geometry of the refraction vectors is shown in **Figure 1**.

The relations between the vectorial amplitudes of the electric and magnetic fields are

$$\boldsymbol{B}_\nu^s=\left[\boldsymbol{n}_\nu^s,\boldsymbol{E}_\nu^s\right],\quad \boldsymbol{E}_\nu^s=-\frac{1}{\varepsilon_\nu}\boldsymbol{D}_\nu^s=-\frac{1}{\varepsilon_\nu}\left[\boldsymbol{n}_\nu^s,\boldsymbol{B}_\nu^s\right],\quad (\nu=(1,2);\,s=(i,r)). \tag{6.4}$$

The plane spanned by the vectors $\boldsymbol{N}$ and $\overline{\boldsymbol{n}}$ or, equivalently, $\boldsymbol{N}$ and each of the refraction vectors $\boldsymbol{n}_\nu^s$ is called the incidence plane. The vector $[\boldsymbol{N},\overline{\boldsymbol{n}}]$ is perpendicular to the incident plane and thus parallel to the boundary plane. It holds the identity

$$[\boldsymbol{N},\overline{\boldsymbol{n}}]=[\boldsymbol{N},\boldsymbol{n}],\quad [\boldsymbol{N},\boldsymbol{n}]^2=\overline{\boldsymbol{n}}^2,\qquad (\boldsymbol{n}\equiv\boldsymbol{n}_\nu^s), \tag{6.5}$$

where for $\boldsymbol{n}$ can be inserted an arbitrary of the coupled refraction vectors in a reflection and refraction problem since the normal components $\boldsymbol{n}_\nu^s \boldsymbol{N}\cdot\boldsymbol{N}$ of all refraction vectors are annihilated in this vector product. We have to distinguish two partial cases of polarizations of the involved waves.

6.1. Electric field Polarized Perpendicular to Incidence Plane

The boundary conditions for tangential components of the electric field amplitudes are

$$0=\left[\boldsymbol{N},\boldsymbol{E}_1^i+\boldsymbol{E}_1^r-\boldsymbol{E}_2^i-\boldsymbol{E}_2^r\right], \tag{6.6}$$

and for the vectorial amplitudes of the magnetic field expressed by the vectorial amplitudes of the electric field

$$0=\left[\boldsymbol{n}_1^i,\boldsymbol{E}_1^i\right]+\left[\boldsymbol{n}_1^r,\boldsymbol{E}_1^r\right]-\left[\boldsymbol{n}_2^i,\boldsymbol{E}_2^i\right]-\left[\boldsymbol{n}_2^r,\boldsymbol{E}_2^r\right]. \tag{6.7}$$

Using $\boldsymbol{n}_2^r\boldsymbol{E}_2^r=0$ by vectorial multiplication of (6.7) with $\boldsymbol{n}_2^r$ from (6.7) follows

$$\boldsymbol{E}_2^r=-\frac{1}{\boldsymbol{n}_2^r\boldsymbol{n}_2^r}\left[\boldsymbol{n}_2^r,\left[\boldsymbol{n}_1^i,\boldsymbol{E}_1^i\right]+\left[\boldsymbol{n}_1^r,\boldsymbol{E}_1^r\right]-\left[\boldsymbol{n}_2^i,\boldsymbol{E}_2^i\right]\right]. \tag{6.8}$$

Inserting this into (6.6) one obtains

$$\begin{aligned}&\left[\boldsymbol{N},\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_1^r\right)\mathrm{I}+\boldsymbol{n}_1^r\cdot\boldsymbol{n}_2^r\right)\boldsymbol{E}_1^r\right]\\&=-\left[\boldsymbol{N},\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_1^i\right)\mathrm{I}+\boldsymbol{n}_1^i\cdot\boldsymbol{n}_2^r\right)\boldsymbol{E}_1^i\right]+\left[\boldsymbol{N},\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_2^i\right)\mathrm{I}+\boldsymbol{n}_2^i\cdot\boldsymbol{n}_2^r\right)\boldsymbol{E}_2^i\right].\end{aligned} \tag{6.9}$$

Up to this point last Equation (6.9) still possesses the general form independent of the polarization of the incident waves.

We consider now the case that the electric field amplitudes are perpendicular to the incidence plane that means that they are proportional to the vector $\left[\boldsymbol{N},\overline{\boldsymbol{n}}\right]$ with proportionality factors E_ν^s

$$\boldsymbol{E}_\nu^s=E_\nu^s\left[\boldsymbol{N},\overline{\boldsymbol{n}}\right],\quad\left(\nu=(1,2);\,s=(i,r)\right). \tag{6.10}$$

Then we find from (6.9)

$$\begin{aligned}&\left[\boldsymbol{N},\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_1^r\right)\mathrm{I}+\boldsymbol{n}_1^r\cdot\boldsymbol{n}_2^r\right)\left[\boldsymbol{N},\overline{\boldsymbol{n}}\right]\right]E_1^r\\&=-\left[\boldsymbol{N},\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_1^i\right)\mathrm{I}+\boldsymbol{n}_1^i\cdot\boldsymbol{n}_2^r\right)\left[\boldsymbol{N},\overline{\boldsymbol{n}}\right]\right]E_1^i+\left[\boldsymbol{N},\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_2^i\right)\mathrm{I}+\boldsymbol{n}_2^i\cdot\boldsymbol{n}_2^r\right)\left[\boldsymbol{N},\overline{\boldsymbol{n}}\right]\right]E_2^i,\end{aligned} \tag{6.11}$$

that with

$$\boldsymbol{n}_\nu^s\left[\boldsymbol{N},\overline{\boldsymbol{n}}\right]=\left[\boldsymbol{N},\overline{\boldsymbol{n}},\boldsymbol{n}_\nu^s\right]=\left[\boldsymbol{N},\overline{\boldsymbol{n}},\boldsymbol{n}_\nu^s\boldsymbol{N}\cdot\boldsymbol{N}+\overline{\boldsymbol{n}}\right]=0, \tag{6.12}$$

leads to the relation

$$\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_1^r\right)\right)\overline{\boldsymbol{n}}E_1^r=-\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_1^i\right)\right)\overline{\boldsymbol{n}}E_1^i+\left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r-\boldsymbol{n}_2^i\right)\right)\overline{\boldsymbol{n}}E_2^i. \tag{6.13}$$

By vectorial multiplication of this equation with the vector $\boldsymbol{N}$ and division by the common factor $\boldsymbol{n}_2^r\boldsymbol{N}$

$$\left(\boldsymbol{n}_2^r\boldsymbol{N}-\boldsymbol{n}_1^r\boldsymbol{N}\right)\boldsymbol{E}_1^r=-\left(\boldsymbol{n}_2^r\boldsymbol{N}-\boldsymbol{n}_1^i\boldsymbol{N}\right)\boldsymbol{E}_1^i+\left(\boldsymbol{n}_2^r\boldsymbol{N}-\boldsymbol{n}_2^i\boldsymbol{N}\right)\boldsymbol{E}_2^i. \tag{6.14}$$

This equation can be resolved with respect to $\boldsymbol{E}_1^r$ and by symmetry of the kind of the two media it can be obtained $\boldsymbol{E}_2^r$ from it by interchanging the lower indices "1" and "2". Using (6.3) among some other possible slightly different representations it provides

$$\boldsymbol{E}_1^r = \frac{\boldsymbol{n}_1^i\boldsymbol{N} - \boldsymbol{n}_2^r\boldsymbol{N}}{\boldsymbol{n}_1^i\boldsymbol{N} + \boldsymbol{n}_2^r\boldsymbol{N}}\boldsymbol{E}_1^i + \frac{2\boldsymbol{n}_2^r\boldsymbol{N}}{\boldsymbol{n}_1^i\boldsymbol{N} + \boldsymbol{n}_2^r\boldsymbol{N}}\boldsymbol{E}_2^i,$$

$$\boldsymbol{E}_2^r = \frac{2\boldsymbol{n}_1^r\boldsymbol{N}}{\boldsymbol{n}_2^i\boldsymbol{N} + \boldsymbol{n}_1^r\boldsymbol{N}}\boldsymbol{E}_1^i + \frac{\boldsymbol{n}_2^i\boldsymbol{N} - \boldsymbol{n}_1^r\boldsymbol{N}}{\boldsymbol{n}_2^i\boldsymbol{N} + \boldsymbol{n}_1^r\boldsymbol{N}}\boldsymbol{E}_2^i. \tag{6.15}$$

Using (6.4) one finds from (6.15) relations for the vector amplitudes of the reflected or refracted magnetic waves in medium 1 and 2

$$\boldsymbol{B}_1^r = -\frac{\boldsymbol{n}_1^i\boldsymbol{N} - \boldsymbol{n}_2^r\boldsymbol{N}}{\boldsymbol{n}_1^i\boldsymbol{N} + \boldsymbol{n}_2^r\boldsymbol{N}}\frac{1}{\varepsilon_1}\left[\boldsymbol{n}_1^r,\left[\boldsymbol{n}_1^i,\boldsymbol{B}_1^i\right]\right] - \frac{2\boldsymbol{n}_2^r\boldsymbol{N}}{\boldsymbol{n}_1^i\boldsymbol{N} + \boldsymbol{n}_2^r\boldsymbol{N}}\frac{1}{\varepsilon_2}\left[\boldsymbol{n}_1^r,\left[\boldsymbol{n}_2^i,\boldsymbol{B}_2^i\right]\right],$$

$$\boldsymbol{B}_2^r = -\frac{2\boldsymbol{n}_1^r\boldsymbol{N}}{\boldsymbol{n}_2^i\boldsymbol{N} + \boldsymbol{n}_1^r\boldsymbol{N}}\frac{1}{\varepsilon_1}\left[\boldsymbol{n}_2^r,\left[\boldsymbol{n}_1^i,\boldsymbol{B}_1^i\right]\right] - \frac{\boldsymbol{n}_2^i\boldsymbol{N} - \boldsymbol{n}_1^r\boldsymbol{N}}{\boldsymbol{n}_2^i\boldsymbol{N} + \boldsymbol{n}_1^r\boldsymbol{N}}\frac{1}{\varepsilon_2}\left[\boldsymbol{n}_2^r,\left[\boldsymbol{n}_2^i,\boldsymbol{B}_2^i\right]\right], \tag{6.16}$$

from which follows as it is necessary

$$\boldsymbol{n}_1^r\boldsymbol{B}_1^r = 0, \qquad \boldsymbol{n}_2^r\boldsymbol{B}_2^r = 0. \tag{6.17}$$

In formulae (6.15) all involved field amplitudes of the electric field by proposition possess the same direction perpendicular to the plane of incidence with scalar proportionality factors of reflection and transmission. In (6.16) the corresponding relations for the vectorial amplitudes of the magnetic field which are parallel to the plane of incidence but no more parallel to each other and the relations become vectorial ones. The vectorial amplitudes of the incident waves $\boldsymbol{E}_1^i$ from medium 1 and $\boldsymbol{E}_2^i$ from medium 2 are independent and can be chosen arbitrarily and analogously in (6.16) the vectorial amplitudes $\boldsymbol{B}_1^i$ and $\boldsymbol{B}_2^i$. In particular, one may choose one or both of these vector amplitudes of the incident waves as vanishing and obtain correct results for the reflected or refracted waves. We discuss this more in detail in Section 7.

6.2. Magnetic Field Polarized Perpendicular to Incidence Plane

If the vectorial amplitudes of the electric field are polarized parallel to the incident plane then all vectorial amplitudes of the magnetic field are polarized perpendicular to the incident plane that is in direction of the vector $\left[\boldsymbol{N},\bar{\boldsymbol{n}}\right]$ and it is simpler to begin the calculations with the last.

The boundary conditions for magnetic field amplitudes are

$$\boldsymbol{0} = \boldsymbol{B}_1^i + \boldsymbol{B}_1^r - \boldsymbol{B}_2^i - \boldsymbol{B}_2^r, \tag{6.18}$$

and for the tangential components of the electric field expressed by the magnetic field

$$\boldsymbol{0} = \left[\boldsymbol{N},\frac{1}{\varepsilon_1}\left(\left[\boldsymbol{n}_1^i,\boldsymbol{B}_1^i\right] + \left[\boldsymbol{n}_1^r,\boldsymbol{B}_1^r\right]\right) - \frac{1}{\varepsilon_2}\left(\left[\boldsymbol{n}_2^i,\boldsymbol{B}_2^i\right] + \left[\boldsymbol{n}_2^r,\boldsymbol{B}_2^r\right]\right)\right]. \tag{6.19}$$

Eliminating $\boldsymbol{B}_2^r$ from this condition using (6.18) one arrives at a relation which can be written

$$\left[\boldsymbol{N},\left[\frac{\boldsymbol{n}_1^r}{\varepsilon_1} - \frac{\boldsymbol{n}_2^r}{\varepsilon_2},\boldsymbol{B}_1^r\right]\right] = -\left[\boldsymbol{N},\left[\frac{\boldsymbol{n}_1^i}{\varepsilon_1} - \frac{\boldsymbol{n}_2^r}{\varepsilon_2},\boldsymbol{B}_1^i\right]\right] + \left[\boldsymbol{N},\left[\frac{\boldsymbol{n}_2^i - \boldsymbol{n}_2^r}{\varepsilon_2},\boldsymbol{B}_2^i\right]\right]. \tag{6.20}$$

It is a general relation which is not restricted by the kind of polarization of the magnetic field amplitudes.

We consider now the case that the magnetic field amplitudes are perpendicular to the incidence plane that means that they are proportional to the vector $\left[\boldsymbol{N},\bar{\boldsymbol{n}}\right]\equiv\left[\boldsymbol{N},\boldsymbol{n}\right]$ (see also (6.5))

$$\boldsymbol{B}_s^\nu = B_s^\nu\left[\boldsymbol{N},\boldsymbol{n}\right],\quad \left(\nu=\left(i,r\right);\, s=\left(1,2\right)\right). \tag{6.21}$$

Then one finds from (6.20)

$$\left[\boldsymbol{N},\left[\frac{\boldsymbol{n}_1^r}{\varepsilon_1}-\frac{\boldsymbol{n}_2^r}{\varepsilon_2},\left[\boldsymbol{N},\boldsymbol{n}\right]\right]\right]B_1^r = -\left[\boldsymbol{N},\left[\frac{\boldsymbol{n}_1^i}{\varepsilon_1}-\frac{\boldsymbol{n}_2^r}{\varepsilon_2},\left[\boldsymbol{N},\boldsymbol{n}\right]\right]\right]B_1^i + \left[\boldsymbol{N},\left[\frac{\boldsymbol{n}_2^i-\boldsymbol{n}_2^r}{\varepsilon_2},\left[\boldsymbol{N},\boldsymbol{n}\right]\right]\right]B_2^r, \tag{6.22}$$

which with

$$\left[\boldsymbol{N},\left[\boldsymbol{n}_s^\nu,\left[\boldsymbol{N},\boldsymbol{n}\right]\right]\right] = -\boldsymbol{n}_s^\nu\boldsymbol{N}\cdot\left[\boldsymbol{N},\boldsymbol{n}\right],\quad \left(\nu=\left(i,r\right);\, s=\left(1,2\right)\right), \tag{6.23}$$

is equivalent to

$$\left(\frac{\boldsymbol{n}_1^r\boldsymbol{N}}{\varepsilon_1}-\frac{\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2}\right)\boldsymbol{B}_1^r = -\left(\frac{\boldsymbol{n}_1^i\boldsymbol{N}}{\varepsilon_1}-\frac{\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2}\right)\boldsymbol{B}_1^i + \frac{\boldsymbol{n}_2^i\boldsymbol{N}-\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2}\boldsymbol{B}_2^i. \tag{6.24}$$

The resolution of this equation with respect to $\boldsymbol{B}_1^r$ using (4) and adding by symmetry $\boldsymbol{B}_2^r$ can be written

$$\boldsymbol{B}_1^r = \frac{\varepsilon_2\boldsymbol{n}_1^i\boldsymbol{N}-\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2\boldsymbol{n}_1^i\boldsymbol{N}+\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}\boldsymbol{B}_1^i + \frac{2\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2\boldsymbol{n}_1^i\boldsymbol{N}+\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}\boldsymbol{B}_2^i,$$

$$\boldsymbol{B}_2^r = \frac{2\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}{\varepsilon_1\boldsymbol{n}_2^i\boldsymbol{N}+\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}\boldsymbol{B}_1^i + \frac{\varepsilon_1\boldsymbol{n}_2^i\boldsymbol{N}-\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}{\varepsilon_1\boldsymbol{n}_2^i\boldsymbol{N}+\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}\boldsymbol{B}_2^i. \tag{6.25}$$

It is not very difficult to affirm with the solutions (6.25) the boundary condition (6.18) in the form $\boldsymbol{B}_1^r-\boldsymbol{B}_2^r=-\boldsymbol{B}_1^i+\boldsymbol{B}_2^i$.

For the vectorial amplitude of the electric field of the reflected or refracted waves in medium 1 one finds from this using (6.4)

$$\boldsymbol{E}_1^r = -\frac{\varepsilon_2\boldsymbol{n}_1^i\boldsymbol{N}-\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2\boldsymbol{n}_1^i\boldsymbol{N}+\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}\frac{1}{\varepsilon_1}\left[\boldsymbol{n}_1^r\left[\boldsymbol{n}_1^i,\boldsymbol{E}_1^i\right]\right] - \frac{2\boldsymbol{n}_2^r\boldsymbol{N}}{\varepsilon_2\boldsymbol{n}_1^i\boldsymbol{N}+\varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}}\left[\boldsymbol{n}_1^r,\left[\boldsymbol{n}_2^i,\boldsymbol{E}_2^i\right]\right],$$

$$\boldsymbol{E}_2^r = -\frac{2\boldsymbol{n}_1^r\boldsymbol{N}}{\varepsilon_1\boldsymbol{n}_2^i\boldsymbol{N}+\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}\left[\boldsymbol{n}_2^r,\left[\boldsymbol{n}_1^i,\boldsymbol{E}_1^i\right]\right] - \frac{\varepsilon_1\boldsymbol{n}_2^i\boldsymbol{N}-\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}{\varepsilon_1\boldsymbol{n}_2^i\boldsymbol{N}+\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N}}\frac{1}{\varepsilon_2}\left[\boldsymbol{n}_2^r\left[\boldsymbol{n}_2^i,\boldsymbol{E}_2^i\right]\right], \tag{6.26}$$

from which follows as it is necessary

$$\boldsymbol{n}_1^r\boldsymbol{E}_1^r = 0,\qquad \boldsymbol{n}_2^r\boldsymbol{E}_2^r = 0. \tag{6.27}$$

The amplitude relations for the reflection and refraction of plane monochromatic wave at the boundary plane between two isotropic media in scalar form are called the Fresnel equations.

7. Further Discussion of the Coordinate-Invariant Amplitude Relation

In the derived formulae for the amplitude relations all involved refraction vectors $\boldsymbol{n}_\nu^s$ and vector amplitudes $\boldsymbol{E}_\nu^s$ and $\boldsymbol{B}_\nu^s$, $\left(\nu=\left(1,2\right),\, s=\left(i,r\right)\right)$ are in general complex vectors with real and imaginary part (e.g., absorption cases, total reflection) and this makes it difficult to introduce amplitudes (moduli) and phases or angles between the vectors. Our intention is to write all connected formulae in a way where this general case of complex vectors remains preserved. The derived formulae for the amplitude relations provide correct results if one inserts the vectorial amplitudes with the restrictions which they have to satisfy generally and by proposition.

All vectorial amplitudes of the electric and magnetic field $\boldsymbol{E}_\nu^s$ and $\boldsymbol{B}_\nu^s$ are perpendicular to the corresponding refraction vectors $\boldsymbol{n}_\nu^s$, $(\nu=(1,2), s=(i,r))$

$$0=\boldsymbol{n}_\nu^s\boldsymbol{E}_\nu^s=\left(\boldsymbol{n}_\nu^s\boldsymbol{N}\cdot\boldsymbol{N}+\bar{\boldsymbol{n}}\right)\boldsymbol{E}_\nu^s, \quad \Rightarrow \quad \boldsymbol{N}\boldsymbol{E}_\nu^s=-\frac{\bar{\boldsymbol{n}}\boldsymbol{E}_\nu^s}{\boldsymbol{n}_\nu^s\boldsymbol{N}}, \quad \boldsymbol{E}\rightleftharpoons\boldsymbol{B}. \tag{7.1}$$

Their scalar components $\boldsymbol{N}\boldsymbol{E}_\nu^s$ and $\bar{\boldsymbol{n}}\boldsymbol{E}_\nu^s$ are not independent from each other. With these restrictions, the vectorial amplitudes possess the general form (see also (5.14))

$$\begin{aligned}\boldsymbol{E}_\nu^s &\to \left(\mathrm{I}-\frac{\boldsymbol{n}_\nu^s\cdot\boldsymbol{n}_\nu^s}{\boldsymbol{n}_\nu^s\boldsymbol{n}_\nu^s}\right)\boldsymbol{E}_\nu^s=\frac{1}{\varepsilon_\nu}\left[\boldsymbol{n}_\nu^s,\left[\boldsymbol{E}_\nu^s,\boldsymbol{n}_\nu^s\right]\right]\\ &=\frac{\boldsymbol{N}\boldsymbol{E}_\nu^s\cdot\left[\boldsymbol{n}_\nu^s,[\boldsymbol{N},\boldsymbol{n}]\right]+\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_\nu^s\right]\cdot[\boldsymbol{N},\boldsymbol{n}]}{[\boldsymbol{N},\boldsymbol{n}]^2}\equiv\boldsymbol{E}_{\nu\parallel}^s+\boldsymbol{E}_{\nu\perp}^s, \quad \boldsymbol{E}\rightleftharpoons\boldsymbol{B}.\end{aligned} \tag{7.2}$$

The coefficients of the incident waves $\boldsymbol{N}\boldsymbol{E}_\nu^i$ and $\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_\nu^i\right]$ are freely choosable which then determine automatically $\boldsymbol{N}\boldsymbol{E}_\mu^r$ and $\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_\mu^r\right]$ with the correct polarization. We consider now in this regard the derived amplitude relations. Before this let us write down an auxiliary formula which we need for the case of polarizations parallel to the incidence plane

$$\begin{aligned}-\frac{1}{\varepsilon_\nu}\left[\left[\boldsymbol{n}_\mu^r,\boldsymbol{n}_\nu^i\right]\right]\left[\boldsymbol{n}_\nu^i,[\boldsymbol{N},\boldsymbol{n}]\right] &\equiv -\frac{1}{\varepsilon_\nu}\left[\boldsymbol{n}_\mu^r\left[\boldsymbol{n}_\nu^i\right]\right]\left[\boldsymbol{n}_\nu^i,[\boldsymbol{N},\boldsymbol{n}]\right]\\ &=-\frac{1}{\varepsilon_\nu}\left[\boldsymbol{n}_\mu^r,\left[\boldsymbol{n}_\nu^i,\left[\boldsymbol{n}_\nu^i,[\boldsymbol{N},\boldsymbol{n}]\right]\right]\right]\\ &=\frac{\boldsymbol{n}_\nu^i\boldsymbol{n}_\nu^i}{\varepsilon_\nu}\left[\boldsymbol{n}_\mu^r,[\boldsymbol{N},\boldsymbol{n}]\right]\\ &=\left[\boldsymbol{n}_\mu^r,[\boldsymbol{N},\boldsymbol{n}]\right], \quad (\mu,\nu=(1,2)).\end{aligned} \tag{7.3}$$

The antisymmetric operator $\left[\left[\boldsymbol{n}_\mu^r,\boldsymbol{n}_\nu^i\right]\right]\equiv\left[\boldsymbol{n}_\mu^r\left[\boldsymbol{n}_\nu^i\right]\right]=\left[\left[\boldsymbol{n}_\mu^r\right]\boldsymbol{n}_\nu^i\right]$ transforms vectors $\left[\boldsymbol{n}_\nu^i,[\boldsymbol{N},\boldsymbol{n}]\right]$ into the new vectors $\boldsymbol{n}_\nu^i\boldsymbol{n}_\nu^i\cdot\left[\boldsymbol{n}_\mu^r,[\boldsymbol{N},\boldsymbol{n}]\right]$[6]. From this together with (2) follows for $\boldsymbol{E}_\nu^i$ polarized in the incident plane

$$-\frac{1}{\varepsilon_\nu}\left[\boldsymbol{n}_\mu^r,\left[\boldsymbol{n}_\nu^i,\boldsymbol{E}_\nu^i\right]\right]=\left[\boldsymbol{n}_\mu^r,[\boldsymbol{N},\boldsymbol{n}]\right]\boldsymbol{N}\boldsymbol{E}_\nu^i, \quad (\mu,\nu=(1,2)). \tag{7.4}$$

7.1. Electric Field Polarized Perpendicular to the Incident Plane

All vector amplitudes of the electric field are polarized according to (7.2) in this

[6]In our notation of vector products $\boldsymbol{x}[\boldsymbol{y}]=[\boldsymbol{x},\boldsymbol{y}]\equiv[\boldsymbol{x}]\boldsymbol{y}$ the object $[\boldsymbol{x}]$ may be considered as antisymmetric operator which acts onto the vector $\boldsymbol{y}$ from the left to the right and the written identity can be obtained by simple displacement of a quadratic bracket. The vector $\boldsymbol{x}$ is here substituted by a vector product, say $\boldsymbol{x}\to[\boldsymbol{a},\boldsymbol{b}]$. Then $\left[[\boldsymbol{a},\boldsymbol{b}]\right]\boldsymbol{y}=\left[[\boldsymbol{a},\boldsymbol{b}],\boldsymbol{y}\right]$.

case in direction of the vector $\left[\boldsymbol{N},\boldsymbol{n}\right]$ (or $\boldsymbol{E}_{\nu\perp}^{i} \propto \left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{\nu}^{i}\right]\cdot\left[\boldsymbol{N},\boldsymbol{n}\right]$) and there is no difficulty to represent (6.15) in the scalar form

$$\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{1}^{r}\right]=\frac{\boldsymbol{n}_{1}^{i}\boldsymbol{N}-\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{1}^{i}\right]+\frac{2\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{2}^{i}\right],$$

$$\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{2}^{r}\right]=\frac{2\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{1}^{i}\right]+\frac{\boldsymbol{n}_{2}^{i}\boldsymbol{N}-\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{E}_{2}^{i}\right]. \tag{7.5}$$

The transition from the electric field amplitudes to the magnetic field amplitudes transforms the polarization perpendicular to the incidence plane to polarizations parallel to the incidence plane and according to (7.2) one has the proportionalities $\boldsymbol{B}_{\nu,\parallel}^{i} \propto \boldsymbol{N}\boldsymbol{B}_{\nu}^{i}\cdot\left[\boldsymbol{n}_{\nu}^{i},\left[\boldsymbol{N},\boldsymbol{n}\right]\right]$. Inserting this into (6.16) and using the auxiliary formula (7.4), it leads to the scalar representation

$$\boldsymbol{N}\boldsymbol{B}_{1}^{r}=\frac{\boldsymbol{n}_{1}^{i}\boldsymbol{N}-\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{B}_{1}^{i}+\frac{2\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{B}_{2}^{i},$$

$$\boldsymbol{N}\boldsymbol{B}_{2}^{r}=\frac{2\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{B}_{1}^{i}+\frac{\boldsymbol{n}_{2}^{i}\boldsymbol{N}-\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{B}_{2}^{i}, \tag{7.6}$$

or using (7.1) equivalently to the scalar representation

$$\bar{\boldsymbol{n}}\boldsymbol{B}_{1}^{r}=-\frac{\boldsymbol{n}_{1}^{i}\boldsymbol{N}-\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\bar{\boldsymbol{n}}\boldsymbol{B}_{1}^{i}+\frac{2\boldsymbol{n}_{1}^{i}\boldsymbol{N}}{\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\bar{\boldsymbol{n}}\boldsymbol{B}_{2}^{i},$$

$$\bar{\boldsymbol{n}}\boldsymbol{B}_{2}^{r}=+\frac{2\boldsymbol{n}_{2}^{i}\boldsymbol{N}}{\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\bar{\boldsymbol{n}}\boldsymbol{B}_{1}^{i}-\frac{\boldsymbol{n}_{2}^{i}\boldsymbol{N}-\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\bar{\boldsymbol{n}}\boldsymbol{B}_{2}^{i}. \tag{7.7}$$

If the involved refraction vectors and amplitude vectors are real vectors it is not difficult to represent these formulae in coordinates.

7.2. Magnetic Field Polarized Perpendicular to the Incident Plane

All vector amplitudes of the magnetic field are polarized according to (7.2) in this case in direction of the vector $\left[\boldsymbol{N},\boldsymbol{n}\right]$ (or $\boldsymbol{B}_{\nu\perp}^{i} \propto \left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{\nu}^{i}\right]\cdot\left[\boldsymbol{N},\boldsymbol{n}\right]$) Therefore, we may present (6.25) in the scalar form

$$\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{1}^{r}\right]=\frac{\varepsilon_{2}\boldsymbol{n}_{1}^{i}\boldsymbol{N}-\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\varepsilon_{2}\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{1}^{i}\right]+\frac{2\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\varepsilon_{2}\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{2}^{i}\right],$$

$$\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{2}^{r}\right]=\frac{2\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\varepsilon_{1}\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{1}^{i}\right]+\frac{\varepsilon_{1}\boldsymbol{n}_{2}^{i}\boldsymbol{N}-\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\varepsilon_{1}\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\left[\boldsymbol{N},\boldsymbol{n},\boldsymbol{B}_{2}^{i}\right]. \tag{7.8}$$

The transition from the magnetic field amplitudes to the electric field amplitudes transforms the polarization perpendicular to the incidence plane to polarizations parallel to the incidence plane and according to (7.2) one has the proportionalities $\boldsymbol{E}_{\nu,\parallel}^{i} \propto \boldsymbol{N}\boldsymbol{E}_{\nu}^{i}\cdot\left[\boldsymbol{n}^{i},\left[\boldsymbol{N},\boldsymbol{n}\right]\right]$. Inserting this into (7.10) and using the auxiliary formula (7.4) it leads to the scalar representation

$$\boldsymbol{N}\boldsymbol{E}_{1}^{r}=\frac{\varepsilon_{2}\boldsymbol{n}_{1}^{i}\boldsymbol{N}-\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\varepsilon_{2}\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{E}_{1}^{i}+\frac{2\varepsilon_{2}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}{\varepsilon_{2}\boldsymbol{n}_{1}^{i}\boldsymbol{N}+\varepsilon_{1}\boldsymbol{n}_{2}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{E}_{2}^{i},$$

$$\boldsymbol{N}\boldsymbol{E}_{2}^{r}=\frac{2\varepsilon_{1}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\varepsilon_{1}\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{E}_{1}^{i}+\frac{\varepsilon_{1}\boldsymbol{n}_{2}^{i}\boldsymbol{N}-\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}{\varepsilon_{1}\boldsymbol{n}_{2}^{i}\boldsymbol{N}+\varepsilon_{2}\boldsymbol{n}_{1}^{r}\boldsymbol{N}}\boldsymbol{N}\boldsymbol{E}_{2}^{i}, \tag{7.9}$$

or using (7.1) equivalently to the scalar representation

$$\bar{\boldsymbol{n}}\boldsymbol{E}_1^r = -\frac{\varepsilon_2 \boldsymbol{n}_1^i \boldsymbol{N} - \varepsilon_1 \boldsymbol{n}_2^r \boldsymbol{N}}{\varepsilon_2 \boldsymbol{n}_1^i \boldsymbol{N} + \varepsilon_1 \boldsymbol{n}_2^r \boldsymbol{N}} \bar{\boldsymbol{n}}\boldsymbol{E}_1^i + \frac{2\varepsilon_2 \boldsymbol{n}_1^i \boldsymbol{N}}{\varepsilon_2 \boldsymbol{n}_1^i \boldsymbol{N} + \varepsilon_1 \boldsymbol{n}_2^r \boldsymbol{N}} \bar{\boldsymbol{n}}\boldsymbol{E}_2^i,$$

$$\bar{\boldsymbol{n}}\boldsymbol{E}_2^r = +\frac{2\varepsilon_1 \boldsymbol{n}_2^i \boldsymbol{N}}{\varepsilon_1 \boldsymbol{n}_2^i \boldsymbol{N} + \varepsilon_2 \boldsymbol{n}_1^r \boldsymbol{N}} \bar{\boldsymbol{n}}\boldsymbol{E}_1^i - \frac{\varepsilon_1 \boldsymbol{n}_2^i \boldsymbol{N} - \varepsilon_2 \boldsymbol{n}_1^r \boldsymbol{N}}{\varepsilon_1 \boldsymbol{n}_2^i \boldsymbol{N} + \varepsilon_2 \boldsymbol{n}_1^r \boldsymbol{N}} \bar{\boldsymbol{n}}\boldsymbol{E}_2^i. \quad (7.10)$$

In the coordinate-invariant representation, it is not necessary to discuss the direction of the normal unit vector $\boldsymbol{N}$ (invariance $\boldsymbol{N} \to -\boldsymbol{N}$) or the direction of the vectors $\boldsymbol{E}_1^r$ and $\boldsymbol{E}_2^r$ in dependence on the directions of the vectors $\boldsymbol{E}_1^i$ and $\boldsymbol{E}_2^i$. The new aspect is that the relations calculated as vectorial relations in a coordinate-invariant form and are then specialized to the restrictions of the field amplitudes and expressed by scalar components. We did not express the formulae by introduction of the moduli of the involved vectors and the angles between them that is not difficult to make if the refraction vectors and the vectorial amplitudes are real vectors but it requires also much place for discussion. It seems that the scalar reflection and refraction proportionality factors are in agreement with the known factors which are mostly given in coordinate representations (e.g., [6] [7]). Our coordinate-invariant approach to the amplitude relations in reflection and refraction problems at isotropic media distinguishes from that of Fyodorov [1] who makes more propositions for scalar components and determines these then but also that he takes into account a magnetic permeability $\mu(\omega)$ and gives then representations by moduli of vectors and angles between them.

8. Reflection and Refraction of Plane Monochromatic Waves at the Boundary between Two Isotropic Media in Compact Form

The boundary conditions (6.9) after elimination of the vectorial amplitude of the reflected or refracted wave in medium 2 possess as mentioned there still a general form independent on the polarization of the involved waves. We take them now as the starting point for the derivation of a more compact form of the reflection and refraction formulae for isotropic media and write them in the form

$$\left[\boldsymbol{N}, \mathsf{A}_{21}^{rr}\boldsymbol{E}_1^r\right] = \left[\boldsymbol{N}, \mathsf{A}_{22}^{ri}\boldsymbol{E}_2^i - \mathsf{A}_{21}^{ri}\boldsymbol{E}_1^i\right], \quad (8.1)$$

with the following abbreviations for three-dimensional operators

$$\mathsf{A}_{21}^{rr} \equiv \boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_1^r\right)\mathsf{I} + \boldsymbol{n}_1^r \cdot \boldsymbol{n}_2^r,$$

$$\mathsf{A}_{21}^{ri} \equiv \boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_1^i\right)\mathsf{I} + \boldsymbol{n}_1^i \cdot \boldsymbol{n}_2^r,$$

$$\mathsf{A}_{22}^{ri} \equiv \boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_2^i\right)\mathsf{I} + \boldsymbol{n}_2^i \cdot \boldsymbol{n}_2^r. \quad (8.2)$$

Using now the orthogonality $\boldsymbol{n}_1^r\boldsymbol{E}_1^r = 0$ one may express $\boldsymbol{E}_1^r$ by vectorial multiplication of (8.1) with the vector $\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}$ and finds in a first step

$$\mathsf{A}_{21}^{rr}\boldsymbol{E}_1^r = \frac{\left[\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}, \left[\boldsymbol{N}, \mathsf{A}_{21}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{22}^{ri}\boldsymbol{E}_2^i\right]\right]}{\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}\boldsymbol{N}}. \quad (8.3)$$

By multiplication of this equation from the left with the operator $\overline{\mathsf{A}_{21}^{rr}}$ one may resolve this equation with the result

$$\boldsymbol{E}_1^r = \frac{\overline{\mathsf{A}_{21}^{rr}}\left[\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}},\left[\boldsymbol{N},\mathsf{A}_{21}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{22}^{ri}\boldsymbol{E}_2^i\right]\right]}{\left|\mathsf{A}_{21}^{rr}\right|\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}\boldsymbol{N}}, \quad \Rightarrow \quad \boldsymbol{n}_1^r\boldsymbol{E}_1^r = 0. \tag{8.4}$$

This is already a coordinate-invariant form of the refracted (or reflected) vectorial amplitude in medium 2 in dependence on the incident waves in medium 1 and medium 2. Using now the well-known identity for double vectorial products $\left[\boldsymbol{x},\left[\boldsymbol{y},\boldsymbol{z}\right]\right] = \left(\tilde{\boldsymbol{x}}\boldsymbol{z}\right)\boldsymbol{y} - \left(\boldsymbol{x}\boldsymbol{y}\right)\boldsymbol{z}$ one may represent (8.4) in the form

$$\boldsymbol{E}_1^r = \frac{\left(\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}\left(\mathsf{A}_{21}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{22}^{ri}\boldsymbol{E}_2^i\right)\right)\overline{\mathsf{A}_{21}^{rr}}\boldsymbol{N}}{\left|\mathsf{A}_{21}^{rr}\right|\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}\boldsymbol{N}} - \frac{\overline{\mathsf{A}_{21}^{rr}}\left(\mathsf{A}_{21}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{22}^{ri}\boldsymbol{E}_2^i\right)}{\left|\mathsf{A}_{21}^{rr}\right|}. \tag{8.5}$$

The complementary operator $\overline{\mathsf{A}_{21}^{rr}}$ to operator A_{21}^{rr} and the determinant of A_{21}^{rr} are

$$\overline{\mathsf{A}_{21}^{rr}} = \boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_1^r\right)\left(\left(\boldsymbol{n}_2^r\boldsymbol{n}_2^r\right)\mathsf{I} - \boldsymbol{n}_1^r\cdot\boldsymbol{n}_2^r\right), \qquad \left|\mathsf{A}_{21}^{rr}\right| = \left(\boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_1^r\right)\right)^2\boldsymbol{n}_2^r\boldsymbol{n}_2^r, \tag{8.6}$$

that can be calculated from relations (A.16) of Appendix A. From this follows

$$\begin{aligned}\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}\boldsymbol{N} &= \boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_1^r\right)\left(\boldsymbol{n}_2^r\boldsymbol{n}_2^r\cdot\boldsymbol{n}_1^r\boldsymbol{N} - \boldsymbol{n}_1^r\boldsymbol{n}_1^r\cdot\boldsymbol{n}_2^r\boldsymbol{N}\right)\\ &= \boldsymbol{n}_2^r\left(\boldsymbol{n}_2^r - \boldsymbol{n}_1^r\right)\left(\varepsilon_2\boldsymbol{n}_1^r\boldsymbol{N} - \varepsilon_1\boldsymbol{n}_2^r\boldsymbol{N}\right).\end{aligned} \tag{8.7}$$

This expression is in the denominator of the reflection and refraction formula (8.4) and its different representations. We recognize here the two different denominators in the reflection and refraction formulae of Section 6 separated in the electric field amplitude perpendicular and parallel to the incident plane. The two sum terms in the split form (8.6) taken alone do not satisfy the orthogonality $\boldsymbol{n}_1^r\boldsymbol{E}_1^r = 0$ but only taken together.

One may give (8.4) still another form using the identity (B.7) ($\overline{\mathsf{A}}\left[\tilde{\boldsymbol{x}}\overline{\mathsf{A}},\tilde{\boldsymbol{y}}\right] = \left|\mathsf{A}\right|\left[\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}\mathsf{A}\right]$, second line, first identity)

$$\boldsymbol{E}_1^r = \frac{\left[\boldsymbol{n}_1^r,\left[\boldsymbol{N},\mathsf{A}_{21}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{22}^{ri}\boldsymbol{E}_2^i\right]\mathsf{A}_{21}^{rr}\right]}{\boldsymbol{n}_1^r\overline{\mathsf{A}_{21}^{rr}}\boldsymbol{N}}. \tag{8.8}$$

The solution for the reflected (or refracted) vector amplitude $\boldsymbol{E}_2^r$ in medium 2 can be obtained due to the similarity of both media by simple permutations of the lower indices "1" and "2" in all formulae, for example, instead of (8.8)

$$\boldsymbol{E}_2^r = -\frac{\left[\boldsymbol{n}_2^r,\left[\boldsymbol{N},\mathsf{A}_{11}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{12}^{ri}\boldsymbol{E}_2^i\right]\mathsf{A}_{12}^{rr}\right]}{\boldsymbol{n}_2^r\overline{\mathsf{A}_{12}^{rr}}\boldsymbol{N}}, \tag{8.9}$$

and in analogous way also (8.4) and (8.5) with the abbreviations for operators

$$\begin{aligned}\mathsf{A}_{12}^{rr} &\equiv \boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r - \boldsymbol{n}_2^r\right)\mathsf{I} + \boldsymbol{n}_2^r\cdot\boldsymbol{n}_1^r,\\ \mathsf{A}_{11}^{ri} &\equiv \boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r - \boldsymbol{n}_1^i\right)\mathsf{I} + \boldsymbol{n}_1^i\cdot\boldsymbol{n}_1^r,\\ \mathsf{A}_{12}^{ri} &\equiv \boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r - \boldsymbol{n}_2^i\right)\mathsf{I} + \boldsymbol{n}_2^i\cdot\boldsymbol{n}_1^r.\end{aligned} \tag{8.10}$$

The incident vectorial amplitudes which one can insert into the above formulae taking into account $\boldsymbol{n}_\nu^i \boldsymbol{E}_\nu^i = 0$ underly again the restrictions (8.6) and are $\boldsymbol{E}_\nu^s \to \frac{1}{\varepsilon_\nu}\left[\boldsymbol{n}_\nu^s,\left[\boldsymbol{E}_\nu^s,\boldsymbol{n}_\nu^s\right]\right]$. For the reflected and refracted vectorial amplitudes $\boldsymbol{E}_\nu^r$ they follow then automatically. We do not make a more detailed consideration of the above formulae since they should only lead to the formulae of previous Section. However, they are important for us to derive in compact form corresponding formulae for reflection and refraction at the boundary from an isotropic to an anisotropic medium in next section.

9. Reflection and Refraction at the Plane Boundary between an Isotropic and an Anisotropic Medium

We now suppose that we have two homogeneous media without spatial dispersion but with frequency dispersion which appears widely only as parameter of the material properties that we do not explicitly show in the notations. The first medium with number 1 should be isotropic and the second medium with number 2 is supposed to be generally anisotropic. Their permittivity tensors are

$$\boldsymbol{\varepsilon}_1(\omega) = \varepsilon_1(\omega)\mathsf{I}, \quad \varepsilon_1(\omega) \equiv \varepsilon_1, \qquad \boldsymbol{\varepsilon}_2(\omega) \equiv \boldsymbol{\varepsilon}_2. \tag{9.1}$$

In medium 1 we have two coupled plane monochromatic waves where one is the incident wave (upper index "i") with vector amplitude $\boldsymbol{E}_1^i$ of the electric field and the other a reflected or refracted wave (upper index "r") with vector amplitude $\boldsymbol{E}_1^r$ each with two possible independent directions of polarizations and with refraction vectors which we denote by $\boldsymbol{n}_1^i$ and $\boldsymbol{n}_1^r$, correspondingly

$$\boldsymbol{n}_1^i = \overline{\boldsymbol{n}} + \boldsymbol{n}_1^i \boldsymbol{N} \cdot \boldsymbol{N}, \quad \boldsymbol{n}_1^r = \overline{\boldsymbol{n}} + \boldsymbol{n}_1^r \boldsymbol{N} \cdot \boldsymbol{N}, \quad \boldsymbol{n}_1^r \boldsymbol{N} = -\boldsymbol{n}_1^i \boldsymbol{N}. \tag{9.2}$$

In medium 2 we have, in general, four coupled plane monochromatic waves corresponding to the degree four of the dispersion equation from which two can be considered as incident waves (upper indices "i_1" and "i_2") with known polarization vectors $\boldsymbol{e}_2^{i_1}$ and $\boldsymbol{e}_2^{i_2}$ and two reflected or refracted waves (upper indices "r_1" and "r_2") with known polarization vectors $\boldsymbol{e}_2^{r_1}$ and $\boldsymbol{e}_2^{r_2}$ and with amplitudes $E_2^{i_1}, E_2^{i_2}$ and $E_2^{r_1}, E_2^{r_2}$, correspondingly, and with refraction vectors

$$\boldsymbol{n}_2^{i_\mu} = \overline{\boldsymbol{n}} + \boldsymbol{n}_2^{i_\mu} \boldsymbol{N} \cdot \boldsymbol{N}, \quad \boldsymbol{n}_2^{r_\mu} = \overline{\boldsymbol{n}} + \boldsymbol{n}_2^{r_\mu} \boldsymbol{N} \cdot \boldsymbol{N}, \quad (\mu = 1, 2). \tag{9.3}$$

Between the coefficients $\boldsymbol{n}_2^{i_\mu}\boldsymbol{N}$ and $\boldsymbol{n}_2^{r_\mu}\boldsymbol{N}$, in general, do not exist simple relations. The refraction vectors are illustrated for a uniaxial medium in special axis position within the incidence plane in **Figure 2**.

The vector $\overline{\boldsymbol{n}}$ is the common tangential part of the refraction vectors of all coupled waves in a reflection and refraction problem. The amplitudes of the incident waves are supposed to be given and the amplitudes of the refracted or reflected waves are then to determine from them. Furthermore, we suppose that via the dispersion relations for the media we have successfully determined from the fixed tangential component $\overline{\boldsymbol{n}}$ of the given incident waves all refraction vectors $\boldsymbol{n}$ with their normal components $\boldsymbol{n}\boldsymbol{N} \cdot \boldsymbol{N}$ and will now come to the calculation of the amplitude relations. From the determined refraction vectors in medium 2 follow then also the corresponding polarization vectors of the electric fields of the waves in medium 2. The basis for the solution of the problem of the amplitude relations is formed again by the boundary conditions of the electric and magnetic field to both sides at the plane boundary between the two media.

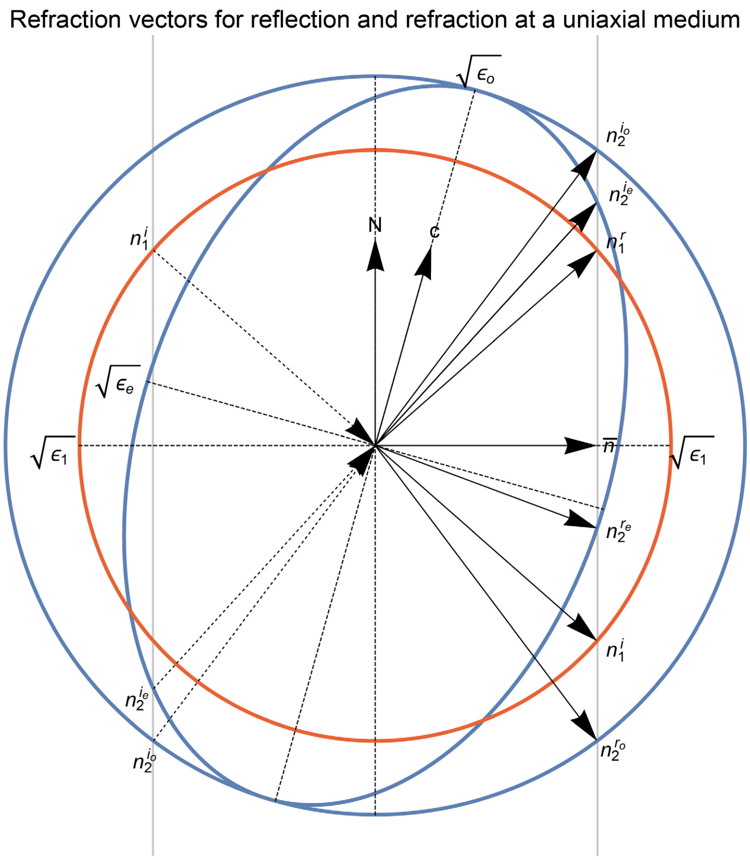

Figure 2. Refraction vectors of incident and reflected and refracted waves at the plane boundary from an isotropic medium 1 to an anisotropic medium 2. The paper plane is the incidence plane. In the Figure is drawn a special case of a uniaxial medium with position of the optic axis in the incident plane but not in normal direction to the boundary plane. In general case the optic axis is somewhere outside the incident plane. The red circle is the intersection of the dispersion surface with the incident plane in isotropic medium 1 and the blue curves the same in uniaxial medium 2. The normal unit vector to the boundary plane is $\boldsymbol{N}$ and the unit vector to the optic axis is $\boldsymbol{c}$. The tangential part to the boundary plane which all refraction vectors have in common is denoted by $\overline{\boldsymbol{n}}$. Instead of the general notations of the refraction vectors for the incident and reflected or refracted waves $\left(\boldsymbol{n}_2^{i_1}, \boldsymbol{n}_2^{i_2}, \boldsymbol{n}_2^{r_1}, \boldsymbol{n}_2^{r_2}\right)$ there are used here the notations $\left(\boldsymbol{n}_2^{i_o}, \boldsymbol{n}_2^{i_e}, \boldsymbol{n}_2^{r_o}, \boldsymbol{n}_2^{r_e}\right)$ with "o" for ordinary waves and "e" for extraordinary waves. We may see here that for the reflection of extraordinary waves the usual law of incident angle equal to reflection angle is no more true (look to $\boldsymbol{n}_2^{i_e}$ and $\boldsymbol{n}_2^{r_e}$) whereas for ordinary waves it remains to be true (look to $\boldsymbol{n}_2^{i_o}$ and $\boldsymbol{n}_2^{r_o}$). In biaxial media with general position of the symmetry elements in comparison to the boundary and the incidence plane all coupled waves are extraordinary waves.

Thus the vectorial fields $\boldsymbol{E}_1\left(\overline{\boldsymbol{r}}, t\right)$ and $\boldsymbol{E}_2\left(\overline{\boldsymbol{r}}, t\right)$ at the two sides of the boundary plane $\boldsymbol{Nr} = 0$ of two-dimensional vectors ($\overline{\boldsymbol{r}} \equiv \left[\left[\boldsymbol{N}, \boldsymbol{r}\right], \boldsymbol{N}\right]$) are

$$\boldsymbol{E}_1(\overline{\boldsymbol{r}},t)=\exp\left(\mathrm{i}\frac{\omega}{c}(\overline{\boldsymbol{n}}\overline{\boldsymbol{r}}-ct)\right)\left(\boldsymbol{E}_1^i+\boldsymbol{E}_1^r\right),$$

$$\boldsymbol{E}_2(\overline{\boldsymbol{r}},t)=\exp\left(\mathrm{i}\frac{\omega}{c}(\overline{\boldsymbol{n}}\overline{\boldsymbol{r}}-ct)\right)\left(\boldsymbol{e}_2^{i_1}E_2^{i_1}+\boldsymbol{e}_2^{i_2}E_2^{i_2}+\boldsymbol{e}_2^{r_1}E_2^{r_1}+\boldsymbol{e}_2^{r_2}E_2^{r_2}\right), \quad (9.4)$$

with the same phase factor at the boundary plane. In the bulks of the two media, one has in addition the phase factor $\mathrm{e}^{\mathrm{i}\frac{\omega}{c}(\boldsymbol{nN}\cdot\boldsymbol{Nr})}$ with $\boldsymbol{n}$ specified for each of the coupled waves.

The boundary conditions which have to be satisfied at a plane boundary $\boldsymbol{Nr}=0$ are again the continuity of the tangential components of the electric and of the magnetic field. For the electric field, this leads to the following relation (in this form Fyodorov [1] [2])[7]

$$\left[\boldsymbol{N},\boldsymbol{E}_1^i+\boldsymbol{E}_1^r\right]=\left[\boldsymbol{N},\boldsymbol{e}_2^{i_1}E_2^{i_1}+\boldsymbol{e}_2^{i_2}E_2^{i_2}+\boldsymbol{e}_2^{r_1}E_2^{r_1}+\boldsymbol{e}_2^{r_2}E_2^{r_2}\right]. \quad (9.5)$$

With neglect of spatial dispersion or of a magnetic permeability of the medium magnetic field and magnetic induction are the same and apart from the normal component of the magnetic field which in each case is continuous also its tangential component is continuous and thus the whole magnetic field [1] [2][8]. Expressed by the electric field this leads to the conditions

$$\left[\boldsymbol{n}_1^i,\boldsymbol{E}_1^i\right]+\left[\boldsymbol{n}_1^r,\boldsymbol{E}_1^r\right]=\left[\boldsymbol{n}_2^{i_1},\boldsymbol{e}_2^{i_1}\right]E_2^{i_1}+\left[\boldsymbol{n}_2^{i_2},\boldsymbol{e}_2^{i_2}\right]E_2^{i_2}+\left[\boldsymbol{n}_2^{r_1},\boldsymbol{e}_2^{r_1}\right]E_2^{r_1}+\left[\boldsymbol{n}_2^{r_2},\boldsymbol{e}_2^{r_2}\right]E_2^{r_2}. \quad (9.6)$$

As already said in case of two isotropic media there is a redundancy in this equation (3 instead of 2 necessary independent coefficients) but this cannot lead to contradictions.

The problem of the amplitude relations is to determine from (9.5) and (9.6) the amplitudes of the reflected and refracted waves (upper index r) in dependence of the amplitudes of the incident waves (upper index i). This problem was tried to solve in our old articles [9] [10] similar to present one. We repeat this derivation in parts but with some modifications. The first step is the same as in case of two isotropic media.

By vectorial multiplication of Equation (9.6) with the vector $\boldsymbol{n}_1^r$ using $\boldsymbol{n}_1^r\boldsymbol{E}_1^r=0$ one obtains

$$\boldsymbol{E}_1^r=\frac{1}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\left\{\left[\boldsymbol{n}_1^r,\left[\boldsymbol{n}_1^i,\boldsymbol{E}_1^i\right]-\left[\boldsymbol{n}_2^{i_1},\boldsymbol{e}_2^{i_1}\right]E_2^{i_1}-\left[\boldsymbol{n}_2^{i_2},\boldsymbol{e}_2^{i_2}\right]E_2^{i_2}-\left[\boldsymbol{n}_2^{r_1},\boldsymbol{e}_2^{r_1}\right]E_2^{r_1}-\left[\boldsymbol{n}_2^{r_2},\boldsymbol{e}_2^{r_2}\right]E_2^{r_2}\right]\right\}. \quad (9.7)$$

Inserting $\boldsymbol{E}_1^r$ into Equation (9.5) and multiplying it by $\boldsymbol{n}_1^r\boldsymbol{n}_1^r$ this vector amplitude is eliminated from the boundary condition and one finds the vectorial relations

$$\left[\boldsymbol{N},\mathrm{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}\right]E_2^{r_1}+\left[\boldsymbol{N},\mathrm{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}\right]E_2^{r_2}=\left[\boldsymbol{N},\mathrm{A}_{11}^{ri}\boldsymbol{E}_1^i\right]-\left[\boldsymbol{N},\mathrm{A}_{12}^{ri_1}\boldsymbol{e}_2^{i_1}\right]E_2^{i_1}-\left[\boldsymbol{N},\mathrm{A}_{12}^{ri_2}\boldsymbol{e}_2^{i_2}\right]E_2^{i_2}, \quad (9.8)$$

[7]The genuine tangential part of an electric field $\boldsymbol{E}$ at the boundary plane is $\left[\left[\boldsymbol{N},\boldsymbol{E}\right],\boldsymbol{N}\right]$ but $\left[\boldsymbol{N},\boldsymbol{E}\right]$ is equivalent since it is only rotated on the boundary plane by 90° and this is made with all parts at the same time.

[8]In more general cases of additional tensorial magnetic permeabilities the components of $\boldsymbol{H}$ are continuous or in still more general cases of spatial dispersion (e.g., natural optical activity) it has to be established which are the modified boundary conditions.

with the following abbreviations for three-dimensional operators

$$\mathsf{A}_{11}^{ri} \equiv \boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r - \boldsymbol{n}_1^i\right)\mathsf{I} + \boldsymbol{n}_1^i \cdot \boldsymbol{n}_1^r, \quad \Rightarrow \quad \mathsf{A}_{11}^{ri}\boldsymbol{E}_1^i = \varepsilon_1\boldsymbol{E}_1^i + \left[\boldsymbol{n}_1^r, \left[\boldsymbol{n}_1^i, \boldsymbol{E}_1^i\right]\right],$$

$$\mathsf{A}_{12}^{ri_\mu} \equiv \boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r - \boldsymbol{n}_2^{i_\mu}\right)\mathsf{I} + \boldsymbol{n}_2^{i_\mu} \cdot \boldsymbol{n}_1^r, \quad \Rightarrow \quad \mathsf{A}_{12}^{ri_\mu}\boldsymbol{e}_2^{i_\mu} = \varepsilon_1\boldsymbol{e}_2^{i_\mu} + \left[\boldsymbol{n}_1^r, \left[\boldsymbol{n}_2^{i_\mu}, \boldsymbol{e}_2^{i_\mu}\right]\right],$$

$$\mathsf{A}_{12}^{rr_\mu} \equiv \boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r - \boldsymbol{n}_2^{r_\mu}\right)\mathsf{I} + \boldsymbol{n}_2^{r_\mu} \cdot \boldsymbol{n}_1^r, \quad \Rightarrow \quad \mathsf{A}_{12}^{rr_\mu}\boldsymbol{e}_2^{r_\mu} = \varepsilon_1\boldsymbol{e}_2^{r_\mu} + \left[\boldsymbol{n}_1^r, \left[\boldsymbol{n}_2^{r_\mu}, \boldsymbol{e}_2^{r_\mu}\right]\right]. \quad (9.9)$$

By scalar multiplication of the vectorial Equation (9.8) with the vector $\mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}$ or $\mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}$ one obtains two scalar equations for $E_2^{r_1}$ and $E_2^{r_2}$ where one of the two sum terms on the left-hand side of (9.8) vanishes and the other has a volume product as factor of $E_2^{r_1}$ and $E_2^{r_2}$. This leads to

$$E_2^{r_1} = \frac{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}, \mathsf{A}_{11}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{12}^{ri_1}\boldsymbol{e}_2^{i_1}E_2^{i_1} - \mathsf{A}_{12}^{ri_2}\boldsymbol{e}_2^{i_2}E_2^{i_2}\right]}{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}\right]},$$

$$E_2^{r_2} = \frac{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}, \mathsf{A}_{11}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{12}^{ri_1}\boldsymbol{e}_2^{i_1}E_2^{i_1} - \mathsf{A}_{12}^{ri_2}\boldsymbol{e}_2^{i_2}E_2^{i_2}\right]}{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}\right]}. \quad (9.10)$$

The dedenominators' in these formulae being volume products due to $\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}\right] = -\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}\right]$ possess only a different sign but otherwise are the same. With these formulae, the problem of the determination of the refracted (or reflected) waves in the anisotropic medium 2 in dependence on the incident waves is already solved, at least, principally. Inserting these solutions into (9.7) it is also solved, at least principally, the problem of determination of the vector amplitude of the reflected (or refracted) wave in isotropic medium 1. Using the solutions (9.10) this leads to

$$\begin{aligned}\boldsymbol{E}_1^r = \frac{1}{\varepsilon_1}\Bigg\{&\left[\boldsymbol{n}_1^r, \left[\boldsymbol{n}_1^i, \boldsymbol{E}_1^i\right] - \left[\boldsymbol{n}_2^{i_1}, \boldsymbol{e}_2^{i_1}\right]E_2^{i_1} - \left[\boldsymbol{n}_2^{i_2}, \boldsymbol{e}_2^{i_2}\right]E_2^{i_2}\right] \\ &- \left[\boldsymbol{n}_1^r, \left[\boldsymbol{n}_2^{r_1}, \boldsymbol{e}_2^{r_1}\right]\right]\frac{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}, \mathsf{A}_{11}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{12}^{ri_1}\boldsymbol{e}_2^{i_1}E_2^{i_1} - \mathsf{A}_{12}^{ri_2}\boldsymbol{e}_2^{i_2}E_2^{i_2}\right]}{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}\right]} \\ &- \left[\boldsymbol{n}_1^r, \left[\boldsymbol{n}_2^{r_2}, \boldsymbol{e}_2^{r_2}\right]\right]\frac{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}, \mathsf{A}_{11}^{ri}\boldsymbol{E}_1^i - \mathsf{A}_{12}^{ri_1}\boldsymbol{e}_2^{i_1}E_2^{i_1} - \mathsf{A}_{12}^{ri_2}\boldsymbol{e}_2^{i_2}E_2^{i_2}\right]}{\left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}\right]}\Bigg\}.\end{aligned} \quad (9.11)$$

These are vectorial relations between the incident amplitudes $\boldsymbol{E}_1^i$ and $\boldsymbol{e}_2^{i_1}E_2^{i_1}, \boldsymbol{e}_2^{i_2}E_2^{i_2}$ and the reflected or refracted amplitudes $\boldsymbol{E}_1^r$ and $\boldsymbol{e}_2^{r_1}E_2^{r_1}, \boldsymbol{e}_2^{r_2}E_2^{r_2}$ since, in general, they do not possess the same direction of polarization.

With the formula (9.11) is, at least principally, also solved the problem of determination of the vectorial amplitude of the reflected or refracted wave in medium 1 in dependence on the incident waves in medium 1 and in medium 2. One has for the incident waves $\boldsymbol{E}_1^i$ as well as for $\boldsymbol{e}_2^{i_1}E_2^{i_1}$ and $\boldsymbol{e}_2^{i_2}E_2^{i_2}$ in each case 3 partial sum terms but desirable would be a representation in form of one more compact sum term only but we could not find up to now such a form. In Section 11 we specialize these formulae for the case of perpendicular incidence.

If the incident waves with amplitudes $\boldsymbol{E}_1^i, E_2^{i_1}, E_2^{i_2}$ are vanishing in the formulae (9.10) and (9.11) then, nevertheless, solutions are possible for the "reflected" or "refracted" amplitudes $\boldsymbol{E}_1^r$ and $\boldsymbol{e}_2^{r_1}E_2^{r_1}, \boldsymbol{e}_2^{r_2}E_2^{r_2}$ if the denominator in these formulae vanishes that means if

$$\begin{aligned} 0 &= \left[\boldsymbol{N}, \mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1}, \mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2} \right] \\ &= [\boldsymbol{N}, \left(\varepsilon_1 - \boldsymbol{n}_1^r\boldsymbol{n}_2^{r_1}\right)\boldsymbol{e}_2^{r_1} + \boldsymbol{n}_1^r\boldsymbol{e}_2^{r_1}\cdot\boldsymbol{n}_2^{r_1}, \left(\varepsilon_1 - \boldsymbol{n}_1^r\boldsymbol{n}_2^{r_2}\right)\boldsymbol{e}_2^{r_2} + \boldsymbol{n}_1^r\boldsymbol{e}_2^{r_2}\cdot\boldsymbol{n}_2^{r_2}] \\ &= \left(\varepsilon_1 - \boldsymbol{n}_1^r\boldsymbol{n}_2^{r_1}\right)\left(\varepsilon_1 - \boldsymbol{n}_2^r\boldsymbol{n}_2^{r_2}\right)\left[\boldsymbol{N}, \boldsymbol{e}_2^{r_1}, \boldsymbol{e}_2^{r_2}\right] - \left(\varepsilon_1 - \boldsymbol{n}_1^r\boldsymbol{n}_2^{r_1}\right)\boldsymbol{n}_1^r\boldsymbol{e}_2^{r_2}\left[\boldsymbol{N}, \boldsymbol{n}, \boldsymbol{e}_2^{r_1}\right] \\ &\quad + \left(\varepsilon_1 - \boldsymbol{n}_1^r\boldsymbol{n}_2^{r_2}\right)\boldsymbol{n}_1^r\boldsymbol{e}_2^{r_1}\left[\boldsymbol{N}, \boldsymbol{n}, \boldsymbol{e}_2^{r_2}\right], \end{aligned} \tag{9.12}$$

where we took into account $\left[\boldsymbol{N}, \boldsymbol{n}_2^{r_1}, \boldsymbol{n}_2^{r_2}\right] = 0$ because all refraction vectors are in the incidence plane and therefore this volume product vanishes. For undetermined $\boldsymbol{n}$ an arbitrary of the coupled refraction vectors can be inserted since their normal components are suppressed in the corresponding volume products or, equivalently, also their common tangential component $\overline{\boldsymbol{n}}$ can be inserted. The expression on the right-hand side of (9.12) gives also an impression how the widely analogous expressions in the numerators of the reflection and refraction formulae look like if one writes them more in detail.

The main purpose, however, why we wrote down (9.12) is that this equation determines the possible surface waves and Brewster cases. In analogous case of two isotropic media from the special case of polarization of the coupled amplitudes of the electric fields parallel to the incidence plane follows the dispersion equation for surface waves and at once for the Brewster case where only two vector amplitudes are coupled at the boundary in both cases, one in medium 1 and one in medium 2 instead of 3 or 4 [20]-[22]. The dispersion equation in this case is an equation alone for the tangential part $\overline{\boldsymbol{n}}$ of the two coupled refraction vectors which in case of surface waves are complex vectors with an imaginary part which describes exponential decay in normal direction to both sides of the boundary plane. The notions "reflected" or "refracted" become inappropriate in this case. It is not clear whether it is possible and how by elimination processes of the normal components $\boldsymbol{n}_1^r\boldsymbol{N}, \boldsymbol{n}_2^{r_1}\boldsymbol{N}, \boldsymbol{n}_2^{r_2}\boldsymbol{N}$ of the involved refraction vectors a scalar algebraic equation (*i.e.*, without roots) for $\overline{\boldsymbol{n}}$ can be obtained in the general anisotropic case and of which polynomial degree it is (see next section). Such an equation would describe only their possible existence but not the injection of these waves to the surface.

10. Determination of Refraction Vectors in an Anisotropic Medium

All coupled refraction vectors in a reflection and refraction problem possess the same tangential component $\overline{\boldsymbol{n}}$ to the boundary plane but different normal components $\boldsymbol{nN}\cdot\boldsymbol{N}$ to this plane. To determine the normal components $\boldsymbol{nN}$ of the waves in the anisotropic medium 2 one has to go with the proposition $\boldsymbol{n} = \boldsymbol{nN}\cdot\boldsymbol{N} + \overline{\boldsymbol{n}}$ into the dispersion Equation (3.1) (we omit here the indices $s = \left(i_\mu, r_\mu\right)$, $\left(\mu, \nu\right) = \left(1, 2\right)$) with the determinant $\left|\mathsf{L}\left(\boldsymbol{n}\right)\right|$ explicitly given in (2.17) that leads to the following equation for $\boldsymbol{nN}$ (supposed $\boldsymbol{\varepsilon}$ is in general asymmetric)

$$
\begin{aligned}
0 &= \mathrm{L}\left(\boldsymbol{nN}\cdot\boldsymbol{N}+\overline{\boldsymbol{n}}\right)\\
&= \left(\boldsymbol{nN}\right)^4 \boldsymbol{N\varepsilon N} + \left(\boldsymbol{nN}\right)^3\left(\boldsymbol{N\varepsilon}\overline{\boldsymbol{n}} + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon N}\right)\\
&\quad + \left(\boldsymbol{nN}\right)^2\left(\overline{\boldsymbol{n}}\boldsymbol{\varepsilon}\overline{\boldsymbol{n}} + \boldsymbol{N\varepsilon N}\left(\overline{\boldsymbol{n}}^2 - \langle\varepsilon\rangle\right) + \boldsymbol{N\varepsilon}^2\boldsymbol{N}\right)\\
&\quad + \boldsymbol{nN}\left(\left(\boldsymbol{N\varepsilon}\overline{\boldsymbol{n}} + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon N}\right)\left(\overline{\boldsymbol{n}}^2 - \langle\varepsilon\rangle\right) + \left(\boldsymbol{N\varepsilon}^2\overline{\boldsymbol{n}} + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon}^2\boldsymbol{N}\right)\right)\\
&\quad + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon}\overline{\boldsymbol{n}}\left(\overline{\boldsymbol{n}}^2 - \langle\varepsilon\rangle\right) + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon}^2\overline{\boldsymbol{n}} + |\boldsymbol{\varepsilon}|.
\end{aligned}
\tag{10.1}
$$

This is a polynomial equation of 4-th degree for the normal components $\boldsymbol{nN}$ of the refraction vectors $\boldsymbol{n}$ in dependence on the common tangential component $\overline{\boldsymbol{n}}$ to all involved refraction vectors with, in general, 4 different solutions. Although it may be solved in radicals this, however, is hardly to manage without a computer.

From Equation (10.1) one obtains 4 relations (from them 3 nonlinear ones) between the roots $\boldsymbol{nN}$ and the coefficients to the powers of $\left(\boldsymbol{nN}\right)^\mu$, $\left(\mu = 0,1,2,3\right)$ in form of their invariants as follows (lower index "2" omitted)

$$
\boldsymbol{n}^{i_1}\boldsymbol{N} + \boldsymbol{n}^{i_2}\boldsymbol{N} + \boldsymbol{n}^{r_1}\boldsymbol{N} + \boldsymbol{n}^{r_2}\boldsymbol{N} = -\frac{\boldsymbol{N\varepsilon}\overline{\boldsymbol{n}} + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon N}}{\boldsymbol{N\varepsilon N}},
$$

$$
\begin{aligned}
&\boldsymbol{n}^{i_1}\boldsymbol{N}\cdot\left(\boldsymbol{n}^{i_2} + \boldsymbol{n}^{r_1} + \boldsymbol{n}^{r_2}\right)\boldsymbol{N} + \boldsymbol{n}^{i_2}\boldsymbol{N}\cdot\left(\boldsymbol{n}^{r_1} + \boldsymbol{n}^{r_2}\right)\boldsymbol{N} + \boldsymbol{n}^{r_1}\boldsymbol{N}\cdot\boldsymbol{n}^{r_2}\boldsymbol{N}\\
&= \frac{\overline{\boldsymbol{n}}\boldsymbol{\varepsilon}\overline{\boldsymbol{n}} + \boldsymbol{N\varepsilon N}\left(\overline{\boldsymbol{n}}^2 - \langle\varepsilon\rangle\right) + \boldsymbol{N\varepsilon}^2\boldsymbol{N}}{\boldsymbol{N\varepsilon N}},
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{n}^{i_1}\boldsymbol{N}\cdot\boldsymbol{n}^{i_2}\boldsymbol{N}\cdot\left(\boldsymbol{n}^{r_1} + \boldsymbol{n}^{r_2}\right)\boldsymbol{N} + \left(\boldsymbol{n}^{i_1} + \boldsymbol{n}^{i_2}\right)\boldsymbol{N}\cdot\boldsymbol{n}^{r_1}\boldsymbol{N}\cdot\boldsymbol{n}^{r_2}\boldsymbol{N}\\
&= -\frac{\left(\boldsymbol{N\varepsilon}\overline{\boldsymbol{n}} + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon N}\right)\left(\overline{\boldsymbol{n}}^2 - \langle\varepsilon\rangle\right) + \left(\boldsymbol{N\varepsilon}^2\overline{\boldsymbol{n}} + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon}^2\boldsymbol{N}\right)}{\boldsymbol{N\varepsilon N}},
\end{aligned}
$$

$$
\boldsymbol{n}^{i_1}\boldsymbol{N}\cdot\boldsymbol{n}^{i_2}\boldsymbol{N}\cdot\boldsymbol{n}^{r_1}\boldsymbol{N}\cdot\boldsymbol{n}^{r_2}\boldsymbol{N} = \frac{\overline{\boldsymbol{n}}\boldsymbol{\varepsilon}\overline{\boldsymbol{n}}\left(\overline{\boldsymbol{n}}^2 - \langle\varepsilon\rangle\right) + \overline{\boldsymbol{n}}\boldsymbol{\varepsilon}^2\overline{\boldsymbol{n}} + |\boldsymbol{\varepsilon}|}{\boldsymbol{N\varepsilon N}}.
\tag{10.2}
$$

In the special case of perpendicular incidence

$$
\overline{\boldsymbol{n}} = \boldsymbol{0},
\tag{10.3}
$$

the general Equation (10.1) for biaxial media simplifies to the biquadratic equation for $\boldsymbol{nN}$

$$
0 = \left(\boldsymbol{nN}\right)^4 \boldsymbol{N\varepsilon N} - \left(\boldsymbol{nN}\right)^2\left(\langle\varepsilon\rangle\boldsymbol{N\varepsilon N} - \boldsymbol{N\varepsilon}^2\boldsymbol{N}\right) + |\boldsymbol{\varepsilon}|,
\tag{10.4}
$$

with, in general, 4 different solutions

$$
\begin{aligned}
\boldsymbol{nN} &= (\pm)\sqrt{\frac{1}{2}\left(\langle\varepsilon\rangle - \frac{\boldsymbol{N\varepsilon}^2\boldsymbol{N}}{\boldsymbol{N\varepsilon N}} \pm \sqrt{\left(\langle\varepsilon\rangle - \frac{\boldsymbol{N\varepsilon}^2\boldsymbol{N}}{\boldsymbol{N\varepsilon N}}\right)^2 - \frac{4|\boldsymbol{\varepsilon}|}{\boldsymbol{N\varepsilon N}}}\right)}\\
&= (\pm)\frac{1}{2}\left\{\sqrt{\langle\varepsilon\rangle - \frac{\boldsymbol{N\varepsilon}^2\boldsymbol{N}}{\boldsymbol{N\varepsilon N}} + 2\sqrt{\frac{|\boldsymbol{\varepsilon}|}{\boldsymbol{N\varepsilon N}}}} \pm \sqrt{\langle\varepsilon\rangle - \frac{\boldsymbol{N\varepsilon}^2\boldsymbol{N}}{\boldsymbol{N\varepsilon N}} - 2\sqrt{\frac{|\boldsymbol{\varepsilon}|}{\boldsymbol{N\varepsilon N}}}}\right\}, \quad \boldsymbol{n} \to \boldsymbol{n}^s.
\end{aligned}
\tag{10.5}
$$

The correspondence of the signs $(\pm)$ to $s = (i, r)$ depends on the chosen direction of the normal unit vector $\boldsymbol{N}$ to the boundary plane.

In case of uniaxial media Equation (10.1) simplifies essentially. For ordinary waves with the dispersion Equation (4.10) it leads to

$$0=(\boldsymbol{nN})^2+\overline{\boldsymbol{n}}^2-\varepsilon_o, \tag{10.6}$$

with the solutions

$$\boldsymbol{nN}=\pm\sqrt{\varepsilon_o-\overline{\boldsymbol{n}}^2},\quad \boldsymbol{nN}\to\boldsymbol{n}^s\boldsymbol{N},\quad (s=i_o,r_o) \tag{10.7}$$

that is the same as in case of an isotropic medium with permittivity ε. For extraordinary waves with the dispersion Equation (4.11) inserting $\boldsymbol{n}=\boldsymbol{nN}\cdot\boldsymbol{N}+\overline{\boldsymbol{n}}$ it leads to the quadratic equation for the normal components $\boldsymbol{nN}$ of the refraction vectors

$$0=(\boldsymbol{nN})^2\left(\varepsilon_e(\boldsymbol{Nc})^2+\varepsilon_o[\boldsymbol{N},\boldsymbol{c}]^2\right)+2\boldsymbol{nN}(\varepsilon_e-\varepsilon_o)\boldsymbol{Nc}\cdot\overline{\boldsymbol{n}}\boldsymbol{c}+\varepsilon_e(\overline{\boldsymbol{n}}\boldsymbol{c})^2+\varepsilon_o[\overline{\boldsymbol{n}},\boldsymbol{c}]^2-\varepsilon_e\varepsilon_o, \tag{10.8}$$

with the solutions

$$\boldsymbol{nN}=\frac{(\varepsilon_o-\varepsilon_e)\boldsymbol{Nc}\cdot\overline{\boldsymbol{n}}\boldsymbol{c}\pm\sqrt{\varepsilon_e\varepsilon_o\left(\varepsilon_e(\boldsymbol{Nc})^2+\varepsilon_o[\boldsymbol{N},\boldsymbol{c}]^2-(\boldsymbol{Nc})^2\overline{\boldsymbol{n}}^2-(\overline{\boldsymbol{n}}\boldsymbol{c})^2\right)-\varepsilon_o^2\left([\boldsymbol{N},\boldsymbol{c}]^2\overline{\boldsymbol{n}}^2-(\overline{\boldsymbol{n}}\boldsymbol{c})^2\right)}}{\varepsilon_e(\boldsymbol{Nc})^2+\varepsilon_o[\boldsymbol{N},\boldsymbol{c}]^2}. \tag{10.9}$$

One may check that in the isotropic case of $\varepsilon_e=\varepsilon_o\equiv\varepsilon$ this formula becomes identical with (10.7). The determination of the normal components $\boldsymbol{nN}$ in dependence on the tangential component $\overline{\boldsymbol{n}}$ with respect to the boundary plane by general position of the optic axis is for extraordinary waves already quite complicated. For special positions of the optic axis perpendicular to the boundary plane $[\boldsymbol{N},\boldsymbol{c}]=\boldsymbol{0}$, $\overline{\boldsymbol{n}}\boldsymbol{c}=0$ or parallel to the boundary plane $\boldsymbol{Nc}=0$ formula (10.9) simplifies considerably.

11. Special Case of Perpendicular Incidence in Reflection and Refraction at an Anisotropic Medium

We specialize from the general formulae of Section 9 and 10 the special case of a perpendicular incident wave in isotropic medium 1. Since it possesses then a vanishing tangential component $\overline{\boldsymbol{n}}=\boldsymbol{0}$ all other waves which may be coupled in medium 1 and medium 2 (incident ones in medium 2 and reflected and refracted) possess also a vanishing tangential component. Thus we investigate now the special case

$$\overline{\boldsymbol{n}}=\boldsymbol{0},\quad\Rightarrow\quad \boldsymbol{n}_1^i=\boldsymbol{n}_1^i\boldsymbol{N}\cdot\boldsymbol{N}=-\boldsymbol{n}_1^r,\quad \boldsymbol{n}_2^{i_\mu}=\boldsymbol{n}_2^{i_\mu}\boldsymbol{N}\cdot\boldsymbol{N}=-\boldsymbol{n}_2^{r_\mu},\quad (\mu=(1,2)). \tag{11.1}$$

The vectorial amplitudes in isotropic medium 1 are

$$\boldsymbol{n}_1^s\boldsymbol{N}=\pm\sqrt{\varepsilon_1},\quad (s=(i,r)). \tag{11.2}$$

In medium 2 one has to assign from the 4 solutions corresponding to inner fixed signs of "$\pm$" in (10.5) 2 solutions to incident waves and 2 to reflected or refracted waves. From fixed outer signs "$(\pm)$" belong in each case one to an incident wave and the other to a refracted or reflected wave. Due to symmetry of the wave-equation operator $\mathsf{L}(\boldsymbol{n})=\mathsf{L}(-\boldsymbol{n})$ (see (2.16)) the polarization vectors to refraction vectors $\boldsymbol{n}$ and $-\boldsymbol{n}$ can be chosen as the same that means

$$\boldsymbol{e}_2^{i_\mu}=\boldsymbol{e}_2^{r_\mu},\quad (\mu=(1,2)). \tag{11.3}$$

We consider now the amplitude relations. The scalar amplitudes $E_2^{r_1}$ and $E_2^{r_2}$ to fixed polarization vectors $\boldsymbol{e}_2^{r_1}$ and $\boldsymbol{e}_2^{r_2}$ are calculated in (9.10). First we consider how the denominator $\left[\boldsymbol{N},\mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1},\mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}\right]$ simplifies in case of perpendicular incidence when all refraction vectors are in direction of the unit vector $\boldsymbol{N}$ to the incidence plane (see also (9.9))

$$\begin{aligned}&\left[\boldsymbol{N},\mathsf{A}_{12}^{rr_1}\boldsymbol{e}_2^{r_1},\mathsf{A}_{12}^{rr_2}\boldsymbol{e}_2^{r_2}\right]\\&=\left[\boldsymbol{N},\left(\boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r-\boldsymbol{n}_2^{r_1}\right)\mathsf{I}+\left(\boldsymbol{n}_1^r\boldsymbol{n}_2^{r_1}\right)\boldsymbol{N}\cdot\boldsymbol{N}\right)\boldsymbol{e}_2^{r_1},\left(\boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r-\boldsymbol{n}_2^{r_2}\right)\mathsf{I}+\left(\boldsymbol{n}_1^r\boldsymbol{n}_2^{r_2}\right)\boldsymbol{N}\cdot\boldsymbol{N}\right)\boldsymbol{e}_2^{r_2}\right]\\&=\left(\boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r-\boldsymbol{n}_2^{r_1}\right)\right)\left(\boldsymbol{n}_1^r\left(\boldsymbol{n}_1^r-\boldsymbol{n}_2^{r_2}\right)\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{e}_2^{r_2}\right].\end{aligned}\tag{11.4}$$

After the same scheme simplify the three sum terms in numerator of (9.10). In this way, one finds from (9.10) the scalar amplitudes of the reflected or refracted wave in medium 2 in dependence on the incident waves

$$\begin{aligned}E_2^{r_1}&=\frac{\left(\boldsymbol{n}_1^r\boldsymbol{N}-\boldsymbol{n}_1^i\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{E}_1^i\right]-\left(\boldsymbol{n}_1^r\boldsymbol{N}-\boldsymbol{n}_2^{i_1}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{e}_2^{i_1}\right]E_2^{i_1}}{\left(\boldsymbol{n}_1^r\boldsymbol{N}-\boldsymbol{n}_2^{r_1}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{e}_2^{r_1}\right]}\\&=\frac{2\boldsymbol{n}_1^i\boldsymbol{N}\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{E}_1^i\right]}{\left(\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_1}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{e}_2^{r_1}\right]}-\frac{\left(\boldsymbol{n}_1^i\boldsymbol{N}-\boldsymbol{n}_2^{r_1}\boldsymbol{N}\right)E_2^{i_1}}{\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_1}\boldsymbol{N}},\\E_2^{r_2}&=\frac{\left(\boldsymbol{n}_1^r\boldsymbol{N}-\boldsymbol{n}_1^i\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{E}_1^i\right]-\left(\boldsymbol{n}_1^r\boldsymbol{N}-\boldsymbol{n}_2^{i_2}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{e}_2^{i_2}\right]E_2^{i_2}}{\left(\boldsymbol{n}_1^r\boldsymbol{N}-\boldsymbol{n}_2^{r_2}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{e}_2^{r_2}\right]}\\&=\frac{2\boldsymbol{n}_1^i\boldsymbol{N}\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{E}_1^i\right]}{\left(\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_2}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{e}_2^{r_2}\right]}-\frac{\left(\boldsymbol{n}_1^i\boldsymbol{N}-\boldsymbol{n}_2^{r_2}\boldsymbol{N}\right)E_2^{i_2}}{\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_2}\boldsymbol{N}}.\end{aligned}\tag{11.5}$$

In case of perpendicular incidence, the incident and reflected waves in medium 2 with vectorial amplitudes $\boldsymbol{e}_2^{i_1}E_2^{i_1}$ and $\boldsymbol{e}_2^{r_1}E_2^{r_1}$ on one side and $\boldsymbol{e}_2^{i_2}E_2^{i_2}$ and $\boldsymbol{e}_2^{r_2}E_2^{r_2}$ on the other side are fully decoupled. The reason from mathematical side is that the polarization vectors (11.3) can be chosen as the same or are, at least, parallel to each other. This is no more the case for oblique incidence.

The vectorial amplitude of the reflected or refracted wave $\boldsymbol{E}_1^r$ in medium 1 can be obtained from (9.7) using (11.5) with the intermediate result

$$\begin{aligned}\boldsymbol{E}_1^r=&\frac{\boldsymbol{n}_1^r\boldsymbol{n}_1^i}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{E}_1^i\right]\right]-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{i_1}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{e}_2^{i_1}\right]\right]E_2^{i_1}-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{i_2}}{\boldsymbol{n}_1^r\boldsymbol{n}_2^{r_1}}\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{e}_2^{i_2}\right]\right]E_2^{i_2}\\&-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{r_1}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\left(\frac{2\boldsymbol{n}_1^i\boldsymbol{N}\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{E}_1^i\right]}{\left(\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_1}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{e}_2^{r_1}\right]}-\frac{\left(\boldsymbol{n}_1^i\boldsymbol{N}-\boldsymbol{n}_2^{r_1}\boldsymbol{N}\right)E_2^{i_1}}{\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_1}\boldsymbol{N}}\right)\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1}\right]\right]\\&-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{r_2}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\left(\frac{2\boldsymbol{n}_1^i\boldsymbol{N}\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{E}_1^i\right]}{\left(\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_2}\boldsymbol{N}\right)\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{e}_2^{r_2}\right]}-\frac{\left(\boldsymbol{n}_1^i\boldsymbol{N}-\boldsymbol{n}_2^{r_2}\boldsymbol{N}\right)E_2^{i_2}}{\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_2}\boldsymbol{N}}\right)\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2}\right]\right].\end{aligned}\tag{11.6}$$

Using in addition (11.1) and (11.3) and the relation $0=\boldsymbol{n}_1^i\boldsymbol{E}_1^i=\boldsymbol{n}_1^i\boldsymbol{N}\cdot\boldsymbol{N}\boldsymbol{E}_1^i$ specialized to perpendicular incidence one finds the simplified expression

$$\begin{aligned}\boldsymbol{E}_1^r=&\boldsymbol{E}_1^i-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{i_1}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\frac{2\boldsymbol{n}_1^i\boldsymbol{N}}{\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_1}\boldsymbol{N}}\frac{\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{E}_1^i\right]}{\left[\boldsymbol{N},\boldsymbol{e}_2^{r_2},\boldsymbol{e}_2^{r_1}\right]}-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{i_2}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\frac{2\boldsymbol{n}_1^i\boldsymbol{N}}{\boldsymbol{n}_1^i\boldsymbol{N}+\boldsymbol{n}_2^{r_2}\boldsymbol{N}}\frac{\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{E}_1^i\right]}{\left[\boldsymbol{N},\boldsymbol{e}_2^{r_1},\boldsymbol{e}_2^{r_2}\right]}\\&-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{i_1}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\frac{2\boldsymbol{n}_1^i\boldsymbol{N}}{\boldsymbol{n}_1^i+\boldsymbol{n}_2^{r_1}\boldsymbol{N}}\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{e}_2^{i_1}\right]\right]E_2^{i_1}-\frac{\boldsymbol{n}_1^r\boldsymbol{n}_2^{i_2}}{\boldsymbol{n}_1^r\boldsymbol{n}_1^r}\frac{2\boldsymbol{n}_1^i\boldsymbol{N}}{\boldsymbol{n}_1^i+\boldsymbol{n}_2^{r_2}\boldsymbol{N}}\left[\boldsymbol{N},\left[\boldsymbol{N},\boldsymbol{e}_2^{i_2}\right]\right]E_2^{i_2}.\end{aligned}\tag{11.7}$$

A more compact formula for the three sum terms proportional to $\boldsymbol{E}_1^i$ was up to now not found.

Thus, there are obtained formulae for the amplitudes of the reflected or refracted waves in medium 1 and in medium 2 in case of perpendicular incidence.

12. About the Impossibility of Negative Refraction

The notion of negative refraction was introduced by J.B. Pendry [11] for a behavior of a light beam at the boundary plane between two isotropic media with both negative electric permittivity $\varepsilon(\omega)<0$ and negative magnetic permeability $\mu(\omega)<0$ and therefore $\varepsilon(\omega)\mu(\omega)>0$. It is stated that a beam in such a medium is refracted to the opposite side of the normal vector $\boldsymbol{N}$ to the boundary in the incidence plane in comparison to a normally broken beam at two isotropic media. I was appalled to read in a recent journal article [12] that it is also now believed that such an effect may exist more than twenty years after its invention and very intensive discussion. There is a genuinely looking illustration of this effect as if it is already realized. Judging by this article and by a given lecture years ago U. Leonhardt appears as someone as a protagonist of this phenomenon. However, it is impossible by well-established laws of refraction of light beams.

We show that in isotropic media also in such with the dispersion equation

$$\boldsymbol{k}^2=\frac{\omega^2}{c^2}\varepsilon(\omega)\mu(\omega),\quad\Rightarrow\quad|\boldsymbol{k}|\equiv(+)\sqrt{\boldsymbol{k}^2}=(+)\frac{\omega}{c}\sqrt{\varepsilon(\omega)\mu(\omega)},\tag{12.1}$$

the group velocity $\boldsymbol{v}\equiv\frac{\partial\omega}{\partial\boldsymbol{k}}$ is in the same direction as the wave vector $\boldsymbol{k}$. The group velocity determines the direction of beam propagation and the modulus of its velocity but it does not take into account the diffraction of the beam during propagation due to further terms in the equation for beam propagation. Equation (12.1) can be resolved, at least in principle, to the form $\omega=\omega(\boldsymbol{k})$. By differentiation of (12.1) with respect to wave vector $\boldsymbol{k}$ follows

$$\begin{aligned}2\boldsymbol{k}&=\left(\frac{\partial}{\partial\omega}\frac{\omega^2\varepsilon(\omega)\mu(\omega)}{c^2}\right)\frac{\partial\omega}{\partial\boldsymbol{k}}\\&=\frac{1}{c^2}\left(2\omega\varepsilon(\omega)\mu(\omega)+\omega^2\frac{\partial}{\partial\omega}\big(\varepsilon(\omega)\mu(\omega)\big)\right)\boldsymbol{v},\end{aligned}\tag{12.2}$$

and the group velocity becomes

$$\begin{aligned}\boldsymbol{v}&=\frac{c^2\boldsymbol{k}}{\omega\left(\varepsilon(\omega)\mu(\omega)+\frac{\omega}{2}\frac{\partial}{\partial\omega}\big(\varepsilon(\omega)\mu(\omega)\big)\right)}\\&=\boldsymbol{c}\frac{\sqrt{\varepsilon(\omega)\mu(\omega)}}{\varepsilon(\omega)\mu(\omega)+\frac{\omega}{2}\frac{\partial}{\partial\omega}\big(\varepsilon(\omega)\mu(\omega)\big)}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}.\end{aligned}\tag{12.3}$$

It shows that the group velocity is in direction of the wave vector $\boldsymbol{k}$ that is even the case for arbitrary dispersion equations of the form $c^2\boldsymbol{k}^2=\omega^2 f(\omega)$. Equation (12.3) can be also represented by the logarithmic derivative

$$\boldsymbol{v}=\frac{c}{\sqrt{\varepsilon(\omega)\mu(\omega)}}\frac{1}{1+\frac{\omega}{2}\frac{\partial}{\partial\omega}\log\left(\left|\varepsilon(\omega)\mu(\omega)\right|\right)}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \tag{12.4}$$

Neglecting the frequency dispersion means the omission of terms with derivatives of $\varepsilon(\omega)\mu(\omega)$ with respect to ω and the above expressions for the group velocities $\boldsymbol{v}$ simplify that does not basically change the situation of proportionality to $\frac{\boldsymbol{k}}{|\boldsymbol{k}|}=\frac{\boldsymbol{n}}{|\boldsymbol{n}|}$. The wave vector of the refracted wave $\boldsymbol{k}$ or its refraction vector $\boldsymbol{n}\equiv\frac{c}{\omega}\boldsymbol{k}$ possess the same tangential components $\boldsymbol{k}$ or $\boldsymbol{n}$ as the corresponding quantities of the incident wave and reflected wave (see Figure 1). If a normally refracted wave possesses the refraction vector $\boldsymbol{n}^r=\overline{\boldsymbol{n}}+\boldsymbol{n}^r\boldsymbol{N}\cdot\boldsymbol{N}$ a wave with negative refraction would possess the refraction vector $\boldsymbol{n}'^r=\boldsymbol{n}^r-2\overline{\boldsymbol{n}}=-\overline{\boldsymbol{n}}+\boldsymbol{n}^r\boldsymbol{N}\cdot\boldsymbol{N}$ that means a tangential part $-\overline{\boldsymbol{n}}$ which cannot be coupled with an incident wave with refraction vector $\boldsymbol{n}^i=\overline{\boldsymbol{n}}+\boldsymbol{n}^i\boldsymbol{N}\cdot\boldsymbol{N}$. Therefore, such a medium (liquid or solid) with negative refraction is impossible and can never be realized in described way.

13. Conclusions

We derived in coordinate-invariant form the vectorial amplitude relations for the refraction and refraction of plane monochromatic waves at the plane boundary from an isotropic to an anisotropic medium with corresponding permittivity tensors. The representation of the same problem for two isotropic media was thought to give an orientation what is to make and what is to generalize. We see also some possibilities of variation of the applied method. For example, one could start from the equations for the magnetic field of the coupled waves at the boundary but we do not expect that this is less difficult than for the electric field. One may also interchange the order of the elimination processes of the reflected or refracted waves and may begin with this already in the boundary conditions. Nevertheless it seems that the resulting formulae for the reflection and refraction at anisotropic media become very complicated and calculations with numerical values of the parameters are only managed in steps by a computer (first involved refraction vectors, then polarization vectors of electric fields and last vectorial amplitude relations writing them in detail and inserting there the possible incident fields). To pack all this into one formula is practically impossible for their lengths and complexity. The calculation of polarization vectors of the electric field amplitudes by projection operators was discussed in general case and then illustrated for the special case of uniaxial media. In the Appendices, we present some topics which are important for coordinate-invariant calculations in three-dimensional case.

Coordinate-invariant methods possess their main advantage compared with coordinate methods for reflection and refraction at anisotropic media whereas for isotropic media, the differences of both methods are not very important. Mostly, applying coordinate-invariant methods, one has also within these methods possibilities of variations. In present paper, they are applied in the three-dimensional case. They can be extended also to the four-dimensional case where, however, many necessary and useful identities have to be still derived and are not written down up to now (some I have in my files). Lorentz transformations [3] and Special Relativity Theory offer themselves as typical topics for such an approach.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Fyodorov, F.I. (1958) Optika anisotropnykh sred (Optics of anisotropic media). Isdatel'stvo AN BSSR.

[2] Fyodorov, F.I. (1976) Teoriya of girotropii (Theory of Gyrotropy). Isdatel'stvo Naúka i tekhnika.

[3] Fyodorov, F.I. (1979) Gruppa lorentsa (Lorentz group). Naúka.

[4] Szivessy, G. (1928) Kristalloptik. In: Grebe, L., Herzfeld, K.F., König, W., Landé, A., Ley, H., Mecke, R., *et al.*, Eds., *Licht Als Wellenbewegung*, Springer, 635-904. https://doi.org/10.1007/978-3-642-90780-7_11

[5] Szivessy, G. (1929) Besondere Fälle von Doppelbrechung (sog. künstliche oder akzidentelle Doppelbrechung). In: Dreisch, T., Frerichs, R., Grebe, L., Hopmann, J., Jentzsch, F., Jordan, P., *et al.*, Eds., *Licht und Materie*, Springer, 724-884. https://doi.org/10.1007/978-3-642-90785-2_19

[6] Landau, L.D. and Lifshitz, E.M. (1982) Electrodynamics of Continuous Media. 2nd Edition, Addison-Wesley.

[7] Born, M. and Wolf, E. (1964) Principles of Optics. 2nd Edition, Pergamon Press.

[8] Lekner, J. (2016) Theory of Reflection: Reflection and Transmission of Electromagnetic, Particle and Acoustic Waves. 2nd Edition, Springer. https://doi.org/10.1007/978-3-319-23627-8

[9] Wünsche, A. (1970) Neue Formeln für die Reflexion und Brechung des Lichtes an anisotropen Medien. *Annalen der Physik*, **480**, 201-214. https://doi.org/10.1002/andp.19704800207

[10] Wünsche, A. (1978) Amplitudenverhältnisse bei der Reflexion und Brechung des Lichtes an der Grenzebene zwischen isotropen und anisotropen Medien. *Annalen der Physik*, **490**, 99-115. https://doi.org/10.1002/andp.19784900203

[11] Pendry, J.B. (2000) Negative Refraction Makes a Perfect Lens. *Physical Review Letters*, **85**, 3966-3969. https://doi.org/10.1103/physrevlett.85.3966

[12] Alú, A. (2023) Tarnkappen für Licht und Schall. Spektrum Edition 4, (Quantenphysik), 122-126. (German Translation from Scientific American)

[13] Agranovich, V.M. and Ginzburg, V.L. (1968) Spatial Dispersion in Crystal Optics and the Theory of Excitons. Wiley.

[14] Ginzburg, V.L. (1981) Teoreticheckaya fizika i astrofizika, (Theoretical Physics and Astrophysics). Naúka.

[15] Jackson, J.D. (1962) Classical Electrodynamics. John Wiley & Sons.

[16] Maxwell, J.C. (1873) A Treatise on Electricity and Magnetism, vol. 2. 3rd Edition, Clarendon Press.

[17] Wünsche, A. (2022) Two Concepts in Optics of Anisotropic Dispersive Media and Polariton Case in Coordinate-Invariant Way. *Journal of Modern Physics*, **13**, 574-619. https://doi.org/10.4236/jmp.2022.134035

[18] Wünsche, A. (1980) Grenzbedingungen der Elektrodynamik für Medien mit räumlicher Dispersion und Übergangsschichten. *Annalen der Physik*, **492**, 121-142. https://doi.org/10.1002/andp.19804920205

[19] Wünsche, A. (2021) Boundary Conditions for Sturm-Liouville Equation with Transition Regions and Barriers or Wells. *Advances in Pure Mathematics*, **11**, 254-295. https://doi.org/10.4236/apm.2021.114018

[20] Agranovich, V.M. (1975) Crystal Optics of Surface Polaritons and the Properties of Surfaces. *Uspekhi Fizicheskih Nauk*, **115**, 199-237. https://doi.org/10.3367/ufnr.0115.197502b.0199

[21] Borstel, G. and Falge, H.J. (1977) Surface Phonon-Polaritons at Semi-Infinite Crystals. *Physica Status Solidi (b)*, **83**, 11-45. https://doi.org/10.1002/pssb.2220830102

[22] Wünsche, A. (1979) Dispersionsgleichungen für Oberflächenwellen und verallgemeinerte Brewster-Fälle an der Grenze zu optisch einachsigen Medien. *Annalen der Physik*, **491**, 281-303. https://doi.org/10.1002/andp.19794910405

Appendix A. Hamilton-Cayley Identity and Invariants of Three-Dimensional Operators

Hamilton-Cayley identities are operator identities which are specific for linear operators acting in N-dimensional vector spaces. We consider the general non-degenerate case of a linear operator A acting in an N-dimensional vector space which is determined by the condition that the operator possesses a complete set of N mutually different eigenvalues α_i to N eigenvectors $\boldsymbol{a}_i$ (I is N-dimensional identity operator)

$$\left(\mathsf{A}-\alpha_i\mathsf{I}\right)a_i=\mathbf{0},\quad\left(\alpha_i\neq\alpha_j,(i,j)=1,2,\cdots,N\right). \tag{A.1}$$

The different operators $\mathsf{A}-\alpha_i\mathsf{I}, i=1,2,\cdots,N$ are commutative

$$\left[\mathsf{A}-\alpha_i\mathsf{I},\mathsf{A}-\alpha_j\mathsf{I}\right]\equiv\left(\mathsf{A}-\alpha_i\mathsf{I}\right)\left(\mathsf{A}-\alpha_i\mathsf{I}\right)-\left(\mathsf{A}-\alpha_j\mathsf{I}\right)\left(\mathsf{A}-\alpha_i\mathsf{I}\right)=0. \tag{A.2}$$

Therefore, such operators in a product can be ordered in arbitrary way and each factor can be moved to the end of a product. From this follows that for arbitrary linear combinations $\sum_{j=1}^{N}\lambda^i\boldsymbol{a}_i$ of the N eigenvectors $\boldsymbol{a}_i,(i=1,2,\cdots,N)$ holds

$$\prod_{i=1}^{N}\left(\mathsf{A}-\alpha_i\mathsf{I}\right)\sum_{j=1}^{N}\lambda^j\boldsymbol{a}_j=\mathbf{0}. \tag{A.3}$$

Due to the supposed completeness of the eigenvectors, this is equivalent to

$$\prod_{i=1}^{N}\left(\mathsf{A}-\alpha_i\mathsf{I}\right)\equiv\sum_{k=0}^{N}A_k\mathsf{A}^{N-k}=0,\quad\left(\mathsf{A}^0\equiv\mathsf{I}\right). \tag{A.4}$$

The vanishing product of N operator equations (A.1) is transformed to a vanishing polynomial of degree N of the operator A with coefficients $A_k,(k=0,1,2,\cdots,N)$ built from the eigenvalues α_i, in particular

$$A_0=1,\quad A_1\equiv\left\langle\mathsf{A}\right\rangle=\sum_{i=1}^{N}\alpha_i,\quad\cdots,\quad A_N=\prod_{i=1}^{N}\alpha_i, \tag{A.5}$$

where A_1 is the trace and A_N the determinant of the operator A. The sum structure of the determinant corresponds exactly to the cycle structure of the permutations as elements of the symmetric group S_N which is described in illustrative way by the Young diagrams. Relation (A.4) is the Hamilton-Cayley identity for linear operators A in N-dimensional case. The N-dimensional Hamilton-Cayley identity is the minimal vanishing polynomial for any N-dimensional linear operator.

In three-dimensional case, besides the trace $\left\langle\mathsf{A}\right\rangle$, we add two further specific notations, $\left[\mathsf{A}\right]$ for the second invariant and $\left|\mathsf{A}\right|$ for the determinant of A according to

$$\left\langle\mathsf{A}\right\rangle=\sum_{i=1}^{3}A_i^i=\alpha_1+\alpha_2+\alpha_3,$$

$$\left[\mathsf{A}\right]=\frac{1}{2}\left(\left\langle\mathsf{A}\right\rangle^2-\left\langle\mathsf{A}^2\right\rangle\right)=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3,$$

$$\left|\mathsf{A}\right|=\frac{1}{6}\left(\left\langle\mathsf{A}\right\rangle^3-3\left\langle\mathsf{A}\right\rangle\left\langle\mathsf{A}^2\right\rangle+2\left\langle\mathsf{A}^3\right\rangle\right)=\alpha_1\alpha_2\alpha_3. \tag{A.6}$$

The Hamilton-Cayley identity is then

$$\mathsf{A}^3 - \langle \mathsf{A} \rangle \mathsf{A}^2 + [\mathsf{A}]\mathsf{A} - |\mathsf{A}|\mathsf{I} = 0. \tag{A.7}$$

The complementary operator $\bar{\mathsf{A}}$ to the operator A is defined by

$$\bar{\mathsf{A}} \equiv [\mathsf{A}]\mathsf{I} - \langle \mathsf{A} \rangle \mathsf{A} + \mathsf{A}^2, \tag{A.8}$$

with the properties due to the Hamilton-Cayley identity (A.7)

$$\mathsf{A}\bar{\mathsf{A}} = \bar{\mathsf{A}}\mathsf{A} = |\mathsf{A}|\mathsf{I}, \tag{A.9}$$

and with the invariants

$$\langle \bar{\mathsf{A}} \rangle = [\mathsf{A}], \quad [\bar{\mathsf{A}}] = |\mathsf{A}|\langle \mathsf{A} \rangle, \quad |\bar{\mathsf{A}}| = |\mathsf{A}|^2. \tag{A.10}$$

The main purpose of the introduction of the complementary operator $\bar{\mathsf{A}}$ is the possible determination of the inverse operator A^{-1} by

$$\mathsf{A}^{-1} = \frac{\bar{\mathsf{A}}}{|\mathsf{A}|}, \tag{A.11}$$

that is important for coordinate-invariant calculations. Furthermore, one calculates

$$\bar{\mathsf{A}}^2 \equiv (\bar{\mathsf{A}})^2 = [\mathsf{A}]\mathsf{A} - |\mathsf{A}|(\langle \mathsf{A} \rangle \mathsf{I} - \mathsf{A}), \tag{A.12}$$

but in contrast to

$$\bar{\bar{\mathsf{A}}} \equiv \overline{(\bar{\mathsf{A}})} = |\mathsf{A}|\mathsf{A}. \tag{A.13}$$

The operator $\langle \mathsf{A} \rangle \mathsf{I} - \mathsf{A}$ in (A.12) corresponds to the complementary operator for two-dimensional operators.

If the determinant $|\mathsf{A}| = 0$ and taking into account (A.10) one finds (A.12)

$$|\mathsf{A}| = 0 :\Rightarrow \left(\frac{\bar{\mathsf{A}}}{[\mathsf{A}]} \right)^2 = \frac{\bar{\mathsf{A}}}{[\mathsf{A}]} = \frac{\bar{\mathsf{A}}}{\langle \bar{\mathsf{A}} \rangle}, \quad \left\langle \frac{\bar{\mathsf{A}}}{\langle \bar{\mathsf{A}} \rangle} \right\rangle = 1. \tag{A.14}$$

Then the operator $\Pi \equiv \frac{\bar{\mathsf{A}}}{\langle \bar{\mathsf{A}} \rangle}$ is projection operator of operator A to simple eigenvalue 0. For eigenvalues $\alpha \neq 0$ of the operator A one may apply analogous formulae as discussed only with substitution of the operator A by the operator $\mathsf{A} - \alpha \mathsf{I}$. The cases of degeneration of the eigenvalues of the operator A expressible by additional relations for the invariants and the complementary operator we do not discuss here.

As examples, we consider the following operator

$$\mathsf{A} = \lambda \mathsf{I} + \boldsymbol{a} \cdot \tilde{\boldsymbol{a}}, \tag{A.15}$$

where λ is an arbitrary (complex) number and $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ are two arbitrary (complex), in general, different vectors. From this we calculate

$$\mathsf{A}^2 = \lambda^2 \mathsf{I} + 2\lambda \boldsymbol{a} \cdot \tilde{\boldsymbol{a}} + (\tilde{\boldsymbol{a}}\boldsymbol{a}) \boldsymbol{a} \cdot \tilde{\boldsymbol{a}}, \quad \langle \mathsf{A} \rangle = 3\lambda + \tilde{\boldsymbol{a}}\boldsymbol{a}, \quad [\mathsf{A}] = \lambda(3\lambda + 2\tilde{\boldsymbol{a}}\boldsymbol{a}),$$
$$\bar{\mathsf{A}} = \lambda((\lambda + \tilde{\boldsymbol{a}}\boldsymbol{a})\mathsf{I} - \boldsymbol{a} \cdot \tilde{\boldsymbol{a}}), \quad |\mathsf{A}| = \lambda^2(\lambda + \tilde{\boldsymbol{a}}\boldsymbol{a}). \tag{A.16}$$

We use this example in our main text.

Appendix B. Identities for Vector Products in Connection with Operators

Starting from the identities for volume products with application of linear operators A to the vectors $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$

$$[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}],\quad [\boldsymbol{x}\mathsf{A},\boldsymbol{y}\mathsf{A},\boldsymbol{z}\mathsf{A}]=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}],\tag{B.1}$$

which can be considered as definition of the determinant $|\mathsf{A}|$ we derive in this Appendix identities for vector products in combination with operators. We consider two variants, first the left-hand variant where the operators act from the left onto vectors and second the right-hand variant where the operators act from the right onto vectors but make proofs only for the left-hand variant.

From the relation of volume products as scalar products of a vector product with a vector using (B.1) follows

$$[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]\mathsf{A}\boldsymbol{z}=[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]=\left[\boldsymbol{x},\boldsymbol{y},\overline{\mathsf{A}}\mathsf{A}\boldsymbol{z}\right]=[\boldsymbol{x},\boldsymbol{y}]\overline{\mathsf{A}}\mathsf{A}\boldsymbol{z}.\tag{B.2}$$

Since $\boldsymbol{z}$ is an arbitrary vector we may omit $\mathsf{A}\boldsymbol{z}$ in this identity and find the following identities (we add also the right-hand variant of the operator acting on vectors to left side)

$$[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]=[\boldsymbol{x},\boldsymbol{y}]\overline{\mathsf{A}},\quad [\boldsymbol{x}\mathsf{A},\boldsymbol{y}\mathsf{A}]=\overline{\mathsf{A}}[\boldsymbol{x},\boldsymbol{y}].\tag{B.3}$$

In analogous way to (B.2) from (B.1) follows

$$[\mathsf{A}\boldsymbol{x},\boldsymbol{y}]\mathsf{A}\boldsymbol{z}=[\mathsf{A}\boldsymbol{x},\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=\frac{\left[\mathsf{A}\boldsymbol{x},\mathsf{A}\overline{\mathsf{A}}\boldsymbol{y},\mathsf{A}\boldsymbol{z}\right]}{|\mathsf{A}|}=\left[\boldsymbol{x},\overline{\mathsf{A}}\boldsymbol{y},\boldsymbol{z}\right]=\left[\boldsymbol{x},\overline{\mathsf{A}}\boldsymbol{y}\right]\boldsymbol{z}.\tag{B.4}$$

By omission of the arbitrary vector $\boldsymbol{z}$ from this identity follows (adding also the right-hand variant)

$$\begin{aligned}[\mathsf{A}\boldsymbol{x},\boldsymbol{y}]\mathsf{A}&=\left[\boldsymbol{x},\overline{\mathsf{A}}\boldsymbol{y}\right],\quad\Leftrightarrow\quad [\boldsymbol{x},\mathsf{A}\boldsymbol{y}]\mathsf{A}=\left[\overline{\mathsf{A}}\boldsymbol{x},\boldsymbol{y}\right],\\ \mathsf{A}[\boldsymbol{x}\mathsf{A},\boldsymbol{y}]&=\left[\boldsymbol{x},\boldsymbol{y}\overline{\mathsf{A}}\right],\quad\Leftrightarrow\quad \mathsf{A}[\boldsymbol{x},\boldsymbol{y}\mathsf{A}]=\left[\boldsymbol{x}\overline{\mathsf{A}},\boldsymbol{y}\right].\end{aligned}\tag{B.5}$$

If we substitute the operator A in the previous identities by $\overline{\mathsf{A}}$ then due to $\overline{\left(\overline{\mathsf{A}}\right)}=|\mathsf{A}|\mathsf{A}$ (see (A.13)) from (B.3) follows (plus right-hand variant)

$$\left[\overline{\mathsf{A}}\boldsymbol{x},\overline{\mathsf{A}}\boldsymbol{y}\right]=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y}]\mathsf{A},\quad \left[\boldsymbol{x}\overline{\mathsf{A}},\boldsymbol{y}\overline{\mathsf{A}}\right]=|\mathsf{A}|\mathsf{A}[\boldsymbol{x},\boldsymbol{y}].\tag{B.6}$$

and from (B.5)

$$\begin{aligned}\left[\overline{\mathsf{A}}\boldsymbol{x},\boldsymbol{y}\right]\overline{\mathsf{A}}&=|\mathsf{A}|[\boldsymbol{x},\mathsf{A}\boldsymbol{y}],\quad\Leftrightarrow\quad \left[\boldsymbol{x},\overline{\mathsf{A}}\boldsymbol{y}\right]\overline{\mathsf{A}}=|\mathsf{A}|[\mathsf{A}\boldsymbol{x},\boldsymbol{y}],\\ \overline{\mathsf{A}}\left[\boldsymbol{x}\overline{\mathsf{A}},\boldsymbol{y}\right]&=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y}\mathsf{A}],\quad\Leftrightarrow\quad \overline{\mathsf{A}}\left[\boldsymbol{x},\boldsymbol{y}\overline{\mathsf{A}}\right]=|\mathsf{A}|[\boldsymbol{x}\mathsf{A},\boldsymbol{y}].\end{aligned}\tag{B.7}$$

Further identities for double vectorial products can be derived from the above identities, for example

$$\begin{aligned}\overline{\mathsf{A}}[\boldsymbol{x},[\boldsymbol{y},\mathsf{A}\boldsymbol{z}]]&=[\boldsymbol{x}\mathsf{A},[\boldsymbol{y},\mathsf{A}\boldsymbol{z}]\mathsf{A}]=\frac{1}{|\mathsf{A}|}\left[\boldsymbol{x}\mathsf{A},\left[\mathsf{A}\overline{\mathsf{A}}\boldsymbol{y},\mathsf{A}\boldsymbol{z}\right]\mathsf{A}\right]\\ &=\frac{1}{|\mathsf{A}|}\left[\boldsymbol{x}\mathsf{A},\left[\overline{\mathsf{A}}\boldsymbol{y},\boldsymbol{z}\right]\overline{\mathsf{A}}\mathsf{A}\right]=\left[\boldsymbol{x}\mathsf{A},\left[\overline{\mathsf{A}}\boldsymbol{y},\boldsymbol{z}\right]\right],\\ \left[[\boldsymbol{x},\boldsymbol{y}\mathsf{A}],\boldsymbol{z}\right]\overline{\mathsf{A}}&=\left[\left[\boldsymbol{x}\overline{\mathsf{A}},\boldsymbol{y}\right],\mathsf{A}\boldsymbol{z}\right],\end{aligned}\tag{8}$$

which afterwards can be easily checked by the decomposition rule of double vector products[10].

Appendix C. Operator of Vectorial Wave Equation of Magnetic Field

We derive in this Appendix the general vectorial wave equation for the magnetic field $\boldsymbol{B}(\boldsymbol{k},\omega)$ in the concept of spatial dispersion $\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$ of the permittivity tensor and calculate then the invariants of the operator of this equation and the complementary operator under neglect of spatial dispersion.

Starting from the second of the Fourier-transformed vectorial Maxwell equations (2.7) one finds the equation ($\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$)

$$\mathbf{0}=\frac{c}{\omega}\boldsymbol{\varepsilon}^{-1}\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right]+\boldsymbol{E}(\boldsymbol{k},\omega). \tag{C.1}$$

Forming the vector product of this equation with the wave vector $\boldsymbol{k}$ and using the first of the vectorial Maxwell Equation (2.7) one obtains

$$\begin{aligned}\mathbf{0}&=\frac{c^2}{\omega^2|\boldsymbol{\varepsilon}|}\left[\boldsymbol{k},\overline{\boldsymbol{\varepsilon}}\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right]\right]+\boldsymbol{B}(\boldsymbol{k},\omega)\\&=\frac{c^2}{\omega^2|\boldsymbol{\varepsilon}|^2}\left[\overline{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}\boldsymbol{k},\overline{\boldsymbol{\varepsilon}}\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right]\right]+\boldsymbol{B}(\boldsymbol{k},\omega)\\&=\frac{c^2}{\omega^2|\boldsymbol{\varepsilon}|^2}\left[\boldsymbol{\varepsilon}\boldsymbol{k},\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right]\right]\overline{\overline{\boldsymbol{\varepsilon}}}+\boldsymbol{B}(\boldsymbol{k},\omega)\\&=\frac{c^2}{\omega^2|\boldsymbol{\varepsilon}|}\left[\boldsymbol{\varepsilon}\boldsymbol{k},\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right]\right]\boldsymbol{\varepsilon}+\boldsymbol{B}(\boldsymbol{k},\omega)\\&=\boldsymbol{B}(\boldsymbol{k},\omega)\left\{\frac{c^2}{\omega^2}\frac{\boldsymbol{\varepsilon}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{\varepsilon}-(\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k})\boldsymbol{\varepsilon}}{|\boldsymbol{\varepsilon}|}+\mathsf{I}\right\}.\end{aligned} \tag{C.2}$$

It was here used that for general linear operators A and vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ holds $[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]=[\boldsymbol{x},\boldsymbol{y}]\overline{\mathsf{A}}$ and $\overline{\overline{\mathsf{A}}}=|\mathsf{A}|\mathsf{A}$ (see (A.13)). Since we consider here the general asymmetric case $\boldsymbol{\varepsilon}\neq\boldsymbol{\varepsilon}^{\mathrm{T}}$ (e.g., optical activity) of the permittivity tensor it is here rigorous that the magnetic field $\boldsymbol{B}(\boldsymbol{k},\omega)$ stands on the left-hand side from the corresponding operator for this field according to the rule for the resolution of the double vector product in (C.2). If one wants to bring it to the right-hand side one has to use the transposed permittivity tensor $\boldsymbol{\varepsilon}^{\mathrm{T}}$ and finds

$$\left\{\frac{c^2}{\omega^2}\frac{\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{\varepsilon}^{\mathrm{T}}-(\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k})\boldsymbol{\varepsilon}^{\mathrm{T}}}{|\boldsymbol{\varepsilon}|}+\mathsf{I}\right\}\boldsymbol{B}(\boldsymbol{k},\omega)=\mathbf{0}. \tag{C.3}$$

For symmetric permittivity tensors $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{\mathrm{T}}$ it doesn't matter on which sides of the operator of the wave equation $\boldsymbol{B}(\boldsymbol{k},\omega)$ stands.

The vectorial wave Equation (C.2) for the magnetic field can now be written

$$\mathbf{0}=\boldsymbol{B}(\boldsymbol{k},\omega)\mathsf{L}'(\boldsymbol{k},\omega), \tag{C.4}$$

[10]To see the advantages of my notations I propose to try write down these last identities in the mostly applied vectorial notations in connection with operators preserving the notations for the letters $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\mathsf{A}$.

with the operator of the equation ($\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$)

$$\mathsf{L}'(\boldsymbol{k},\omega) \equiv \frac{c^2}{\omega^2}\frac{\boldsymbol{\varepsilon k}\cdot\boldsymbol{k\varepsilon}-(\boldsymbol{k\varepsilon k})\boldsymbol{\varepsilon}}{|\boldsymbol{\varepsilon}|}+\mathsf{I} \tag{C.5}$$

Under neglect of the spatial dispersion $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}(\omega)$ one may use the introduction of refraction vectors $\boldsymbol{n}$ by definition (2.13) and the wave Equation (C.4) for the magnetic field $\boldsymbol{B}$ can be written

$$\mathbf{0}=\boldsymbol{B}\mathsf{L}'(\boldsymbol{n}), \tag{C.6}$$

with the operator

$$\mathsf{L}'(\boldsymbol{n}) \equiv \frac{\boldsymbol{\varepsilon n}\cdot\boldsymbol{n\varepsilon}-(\boldsymbol{n\varepsilon n})\boldsymbol{\varepsilon}}{|\boldsymbol{\varepsilon}|}+\mathsf{I}. \tag{C.7}$$

For the calculation of the invariants of the operator $\mathsf{L}'(\boldsymbol{n})$ one has in preparation first to calculate its second and third power (or instead of the third power the complementary operator) and has then to apply the formulae (A.11) with the result

$$\begin{aligned}
\langle\mathsf{L}'(\boldsymbol{n})\rangle &= -\frac{\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\boldsymbol{\varepsilon}|}+3,\\
[\mathsf{L}'(\boldsymbol{n})] &= \frac{\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}}{|\boldsymbol{\varepsilon}|}-2\frac{\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\boldsymbol{\varepsilon}|}+3=2\frac{|\mathsf{L}(\boldsymbol{n})|}{|\boldsymbol{\varepsilon}|}-\frac{\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}-|\boldsymbol{\varepsilon}|}{|\boldsymbol{\varepsilon}|},\\
|\mathsf{L}'(\boldsymbol{n})| &= \frac{\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}}{|\boldsymbol{\varepsilon}|}-\frac{\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\boldsymbol{\varepsilon}|}+1=\frac{|\mathsf{L}(\boldsymbol{n})|}{|\boldsymbol{\varepsilon}|}.
\end{aligned} \tag{C.8}$$

Necessary but not sufficient condition in case of $|\mathsf{L}(\boldsymbol{n})|=0$ for the presence of a degeneration case is

$$\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}-|\boldsymbol{\varepsilon}|=0. \tag{C.9}$$

This condition is satisfied, e.g., for uniaxial media in direction of optic axes.

For the complementary operator $\overline{\mathsf{L}}'(\boldsymbol{n})$ to $\mathsf{L}'(\boldsymbol{n})$ one finds by applying (A.8)

$$\begin{aligned}
\overline{\mathsf{L}}'(\boldsymbol{n}) &= \frac{(\boldsymbol{n\varepsilon n})^2\left(\boldsymbol{\varepsilon}^2-\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{\varepsilon}\right)-\boldsymbol{n\varepsilon n}\left(\boldsymbol{\varepsilon}^2\boldsymbol{n}\cdot\boldsymbol{n\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{\varepsilon n}\cdot\boldsymbol{n\varepsilon}+\boldsymbol{\varepsilon n}\cdot\boldsymbol{n\varepsilon}^2-(\boldsymbol{n\varepsilon}^2\boldsymbol{n})\boldsymbol{\varepsilon}\right)}{|\boldsymbol{\varepsilon}|^2}\\
&\quad-\frac{\boldsymbol{\varepsilon n}\cdot\boldsymbol{n\varepsilon}-(\boldsymbol{n\varepsilon n})\boldsymbol{\varepsilon}}{|\boldsymbol{\varepsilon}|}+\underbrace{\left(\frac{\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}}{|\boldsymbol{\varepsilon}|}-\frac{\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\boldsymbol{\varepsilon}|}+1\right)}_{=|\mathsf{L}'(\boldsymbol{n})|}\mathsf{I}.
\end{aligned} \tag{C.10}$$

One may check the general identity (A.10) and the transversality condition of solutions (left-hand eigenvectors) to vector $\boldsymbol{n}$ in case of $|\mathsf{L}'(\boldsymbol{n})|=0$

$$\langle\overline{\mathsf{L}}'(\boldsymbol{n})\rangle=[\mathsf{L}'(\boldsymbol{n})],\qquad \overline{\mathsf{L}}'(\boldsymbol{n})\boldsymbol{n}=|\overline{\mathsf{L}}'|\boldsymbol{n}=0. \tag{C.11}$$

In case of $|\mathsf{L}'(\boldsymbol{n})|=0$ the operator $\dfrac{\overline{\mathsf{L}}'(\boldsymbol{n})}{\langle\overline{\mathsf{L}}'(\boldsymbol{n})\rangle}$ is projection operator for the direct determination of solutions of the vectorial wave Equation (C.6) for the magnetic field. It is clear that having a corresponding solution for the electric field one may determine a corresponding solution for the magnetic field also by the first of the vectorial Maxwell Equation (2.14).

Appendix D. Operator of Vectorial Wave Equation of Electric Induction

The vectorial wave equation for the electric induction $\boldsymbol{D}(\boldsymbol{k},\omega)$ can be immediately obtained from the vectorial wave equation for the electric field $\boldsymbol{E}(\boldsymbol{k},\omega)$ in (2.9) together with (2.10) by setting $\boldsymbol{E}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}^{-1}(\boldsymbol{k},\omega)\boldsymbol{D}(\boldsymbol{k},\omega)$ and $\boldsymbol{\varepsilon}^{-1}=\frac{\overline{\boldsymbol{\varepsilon}}}{|\boldsymbol{\varepsilon}|}$

$$\mathbf{0}=\mathsf{L}''(\boldsymbol{k},\omega)\boldsymbol{D}(\boldsymbol{k},\omega), \tag{D.1}$$

with the operator of this equation defined by ($\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$)

$$L''(k,\omega)=\frac{c^2}{\omega^2}\frac{\boldsymbol{k}\cdot\boldsymbol{k}\overline{\varepsilon}-(\boldsymbol{k}^2)\overline{\varepsilon}}{|\varepsilon|}+\boldsymbol{I}. \tag{D.2}$$

In case of neglect of the spatial dispersion $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}(\omega)$ this equation simplifies to

$$\mathbf{0}=\mathsf{L}''(\boldsymbol{n})\boldsymbol{D}, \tag{D.3}$$

with the operator ($\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\omega)$)

$$\mathsf{L}''(\boldsymbol{n})\equiv\frac{\boldsymbol{n}\cdot\boldsymbol{n}\overline{\boldsymbol{\varepsilon}}-(\boldsymbol{n}^2)\overline{\boldsymbol{\varepsilon}}}{|\boldsymbol{\varepsilon}|}+\mathsf{I}. \tag{D.4}$$

The invariants of the operator $\mathsf{L}''(\boldsymbol{n})$ are

$$\begin{aligned}
\langle\mathsf{L}''(\boldsymbol{n})\rangle&=-\frac{\langle\varepsilon\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\varepsilon|}+3=\langle\mathsf{L}'(\boldsymbol{n})\rangle,\\
[\mathsf{L}''(\boldsymbol{n})]&=\frac{\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}}{|\boldsymbol{\varepsilon}|}-2\frac{\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\varepsilon|}+3=[\mathsf{L}'(\boldsymbol{n})],\\
|\mathsf{L}''(\boldsymbol{n})|&=\frac{\boldsymbol{n}^2\cdot\boldsymbol{n\varepsilon n}}{|\boldsymbol{\varepsilon}|}-\frac{\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n\varepsilon}^2\boldsymbol{n}}{|\boldsymbol{\varepsilon}|}+1=|\mathsf{L}'(\boldsymbol{n})|=\frac{|\mathsf{L}(\boldsymbol{n})|}{|\boldsymbol{\varepsilon}|}.
\end{aligned} \tag{D.5}$$

The operators to the vectorial wave equations of the magnetic field $\mathsf{L}'(\boldsymbol{n})$ and of the electric induction $\mathsf{L}''(\boldsymbol{n})$ possess the same invariants.

The operators to the vectorial wave equations for the electric field $\mathsf{L}(\boldsymbol{n})$ and for the electric induction $\mathsf{L}''(\boldsymbol{n})$ possess a simple connection

$$\mathsf{L}''(\boldsymbol{n})=\mathsf{L}(\boldsymbol{n})\boldsymbol{\varepsilon}^{-1}=\frac{\mathsf{L}(\boldsymbol{n})\overline{\boldsymbol{\varepsilon}}}{|\boldsymbol{\varepsilon}|},\quad\Rightarrow\quad|\mathsf{L}''(\boldsymbol{n})|=\frac{|\mathsf{L}(\boldsymbol{n})|}{|\boldsymbol{\varepsilon}|}. \tag{D.6}$$

From this follows for the complementary operator $\overline{\mathsf{L}}''(\boldsymbol{n})$ to $\mathsf{L}''(\boldsymbol{n})$

$$\overline{\mathsf{L}}''(\boldsymbol{n})=\frac{|\mathsf{L}''(\boldsymbol{n})|}{|\mathsf{L}(\boldsymbol{n})|}\boldsymbol{\varepsilon}\overline{\mathsf{L}}(\boldsymbol{n})=\frac{\boldsymbol{\varepsilon}\overline{\mathsf{L}}(\boldsymbol{n})}{|\boldsymbol{\varepsilon}|}, \tag{D.7}$$

and explicitly if we use the representation (2.20)

$$\overline{\mathsf{L}}''(\boldsymbol{n})=\frac{(\boldsymbol{n}^2)\boldsymbol{\varepsilon n}\cdot\boldsymbol{n}-(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{\varepsilon n}\cdot\boldsymbol{n}-\boldsymbol{\varepsilon}^2\boldsymbol{n}\cdot\boldsymbol{n}-\boldsymbol{\varepsilon n}\cdot\boldsymbol{n\varepsilon}+(\boldsymbol{n\varepsilon n})\boldsymbol{\varepsilon})+|\boldsymbol{\varepsilon}|\mathsf{I}}{|\boldsymbol{\varepsilon}|}. \tag{D.8}$$

One may check the general identity (A.10) and the transversality condition of solutions (right-hand eigenvectors) to vector n in case of $\left|\mathsf{L}''(\boldsymbol{n})\right| = 0$

$$\left\langle \overline{\mathsf{L}}''(\boldsymbol{n}) \right\rangle = \left[\mathsf{L}''(\boldsymbol{n})\right], \qquad \boldsymbol{n}\overline{\mathsf{L}}''(\boldsymbol{n}) = \left|\overline{\mathsf{L}}''(\boldsymbol{n})\right|\boldsymbol{n}. \tag{D.9}$$

In case of $\left|\mathsf{L}''(\boldsymbol{n})\right| = 0$ the operator $\dfrac{\overline{\mathsf{L}}''(\boldsymbol{n})}{\left\langle \overline{\mathsf{L}}''(\boldsymbol{n}) \right\rangle}$ is projection operator for the direct determination of solutions of the vectorial wave Equation (D.3) for the electric induction.

Appendix E. Correspondences of Notations between Fyodorov and Ours in Three-Dimensional Case

For an easy comparison we collect the most important notations of Fyodorov and of ours in the following Table 1.

Table 1. Comparison of some notations of F.I. Fyodorov [1] [2] with ours in three-dimensional case and to reflection and refraction.

notation	Fyodorov	We
scalars	$\alpha, \beta, \ldots, A, B, \ldots$	$\alpha, \beta, \ldots, A, B, \ldots$
vectors	$\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{A}, \boldsymbol{B}, \ldots$	$\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{A}, \boldsymbol{B}, \ldots$
linear operators	$\alpha, \beta, \ldots$	$\mathsf{A}, \mathsf{B}, \ldots, \mathsf{L}, \ldots$
identical operator	1	$\mathsf{I} \equiv \mathsf{A}^0$
complementary operators	$\overline{\alpha}, \overline{\beta}, \ldots$	$\overline{\mathsf{A}}, \overline{\mathsf{B}}, \ldots$
trace	$\alpha_c, \beta_c, \ldots$	$\langle \mathsf{A} \rangle, \langle \mathsf{B} \rangle, \ldots$
second invariant	$\overline{\alpha}_c, \overline{\beta}_c, \ldots$	$[\mathsf{A}], [\mathsf{B}], \ldots$
determinant	$\lvert\alpha\rvert, \lvert\beta\rvert, \ldots$	$\lvert\mathsf{A}\rvert, \lvert\mathsf{B}\rvert, \ldots$
scalar product	$\boldsymbol{xy}$	$\boldsymbol{xy}$
dyadic product	$\boldsymbol{x} \cdot \boldsymbol{y}$	$\boldsymbol{x} \cdot \boldsymbol{y}$
vector product	$[\boldsymbol{xy}], [\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z}]$	$[\boldsymbol{x}, \boldsymbol{y}]$
volume product	$\boldsymbol{x}[\boldsymbol{yz}], [\boldsymbol{xy}]\boldsymbol{z}$	$[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$
special antisymmetric operator	$\boldsymbol{x}^\times; \ \boldsymbol{x}^\times \boldsymbol{y} = [\boldsymbol{xy}]$	$[\boldsymbol{x}]; \ [\boldsymbol{x}]\boldsymbol{y} = [\boldsymbol{x}, \boldsymbol{y}]$
refraction vectors	$\boldsymbol{m} = n\boldsymbol{n}, \ \boldsymbol{n}^2 = 1$	$\boldsymbol{n}, \ \boldsymbol{n}^2 = \lvert\boldsymbol{n}\rvert^2 \neq 1$
incident wave medium 1	$\boldsymbol{m}^{(0)}, \ \boldsymbol{E}^{(0)}$	$\boldsymbol{n}_1^i, \ \boldsymbol{E}_1^i$
reflected wave medium 1	$\boldsymbol{m}^{(1)}, \ \boldsymbol{E}^{(1)}$	$\boldsymbol{n}_1^r, \ \boldsymbol{E}_1^r$
refracted wave medium 2	$\boldsymbol{m}^{(2)}, \ \boldsymbol{E}^{(2)}$	$\boldsymbol{n}_2^r, \ \boldsymbol{E}_2^r$
incident wave medium 2	$-$	$\boldsymbol{n}_2^i, \ \boldsymbol{E}_2^i$
normal vector to boundary plane	$\boldsymbol{q}, \quad \boldsymbol{q}^2 = 1$	$\boldsymbol{N}, \quad \boldsymbol{N}^2 = 1$
normal vector to incident plane	$\boldsymbol{a} = [\boldsymbol{mq}],$	$[\boldsymbol{N}, \boldsymbol{n}]$
tangential component of refraction vectors	$\boldsymbol{b} = [\boldsymbol{qa}],$	$\overline{\boldsymbol{n}} = [[\boldsymbol{N}, \boldsymbol{n}], \boldsymbol{N}]$
normal component of refraction vectors	η	$\boldsymbol{nN}$
permittivities in uniaxial media	$\varepsilon_e, \ \varepsilon_o$	$\varepsilon_e, \ \varepsilon_o$
unit vector in axis direction of uniaxial media	$\boldsymbol{c}, \quad \boldsymbol{c}^2 = 1$	$\boldsymbol{c}, \quad \boldsymbol{c}^2 = 1$

(E.1)

There are a lot of notations connected with angles in the work of Fyodorov which is omitted in the table together with some other very special notations. Furthermore, the identical operator is mostly omitted in formulae of Fyodorov if this is possible at all. Wave equation operators are mostly written down in its full length that impedes to write down their invariants. The invariants of three-dimensional operators, in our notation $\langle \mathsf{A} \rangle, [\mathsf{A}], \lvert\mathsf{A}\rvert$, are inconspicuously in Fyodorov's notation and are easy to be confused with other notations. The symbol $\boldsymbol{x}^\times$ for an antisymmetric operator built from vector $\boldsymbol{x}$ was some progress but possesses the disadvantage that it acts only to the right onto vectors. Our symbol $[\boldsymbol{x}]$

may act also to the left onto vectors. With the notation $[\boldsymbol{x}]$ we did not come up to now in ambiguities if one only knows that $\boldsymbol{x}$ is a vector. Concerning the used letters our notations are more similar to that of Landau and Lifshits [6].