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Method for Fitting and Deriving the CKM and PMNS Matrices from Underlying Wavefunctions

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Abstract

The Cabibbo-Kobayashi-Maskawa (CKM) and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrices in the electroweak sector are now well-known experimentally. However, there have been only a few proposals to derive these matrices from an underlying theory. In this note, these matrices are derived from example scalar wavefunctions associated with a permutationally symmetric mass matrix with three states for each of the four fermion families. Such a mass matrix is consistent with an anomaly-free quantum field theory for the 4 fermion families. The derivation uses three-dimensional gaussian wavefunctions with specified widths and specified separations between the 3 wells implied by the theory. This approach first fits the diagonal elements of the matrices. A fourth element is then estimated from the properties of the scalar fields in the aforementioned model. Unitarity is then applied to compute the remaining matrix elements. The example calculations produce matrices that have a normalized root-mean-square error (RMSE) from the measured matrix of 6.09×10^{-4} and 8.34×10^{-3} for the CKM and PMNS matrices, respectively. The normalized RMSE for departure from unitarity is 9.18×10^{-4} and 8.95×10^{-3} for the two respective matrices. The results are within one standard deviation of almost all of the measured parameters for both matrices. The primary objective of this paper is to show that the matrices *can be fit accurately* in the context of at least one anomaly-free quantum field theory.

Keywords

Mixing Matrices, CKM Matrix, PMNS Matrix

1. Introduction

Measurements of both the CKM and PMNS matrices have stabilized in recent years [1]-[6]. There have been only a few proposals for calculation of these

matrices from more basic principles or theories (e.g., [7]-[9]). The primary objective of this paper is to show that fits of the CKM and PMNS matrices can be obtained from at least one quantum field theory within experimental error. It will be seen that this can be achieved with a relatively straightforward approach. With such fits, independent determinations of the parameters of the underlying wavefunctions of the theory then imply a predictive capability.

The approach is to compute these matrices based on the hypothesis that the fundamental fermions, e.g., quarks, electrons, and neutrinos, are composite particles. One class of models that treats the fundamental fermions as composite particles are preon models. Preons models are known to present a more basic structure for the fundamental fermions of the up, down, electron, and neutrino families [10]-[17]. Preons have not been particularly productive for quantitative explanations of observed phenomena, but they do explain the structure of the fermion families (up, down electron, neutrino), and some progress has been made in the recent past using preons to explain primordial matter-antimatter asymmetry [18]-[20]. In this note, scalar fields similar to the original Harari form of preons [10] are appended to an anomaly-free quantum field theory [17] that uses permutationally symmetric mass matrices. This theory has the property that the mass matrices can be interpreted as an interaction involving particles occupying three potential wells ("knots") in a closed-ring topology. Hence, it is natural to ask about the nature of such particles in the context of this theory.

This theory, involving permutational symmetry without underlying preon-like wavefunctions, provides an explanation for the three generations of particles and their masses, and also derives SU(3) color as a property of quarks ([17], Ch. 2). The quark families involve the full triplet of color states and have a broken SU(3) symmetry in this theory. The electron family of fermions are found to occupy singlet SU(3) states that emerge from the theory, and for which the usual SU(3) symmetry and SU(3) force do not apply ([17], Chs. 3 and 9), as is required by observations. The $SU(2)_L \times U(1)$ symmetry emerges by changing from the native circulant basis to the more familiar mass basis with handed states, in which all fermions, including neutrinos, are Dirac states (not Majorana). The additional assumption of "preon-like" wavefunctions occupying the aforementioned potential wells allows an accurate fit and explanation of the electroweak sector parameters. These fits include the masses of the electroweak bosons and the Weinberg angle (and a less accurate fit of the Higgs particle mass as a composite state) ([17], Ch. 11). These are among a number of useful predictions and/or explanations of what would otherwise be inputs into the standard model.

Assuming such preon-like wavefunctions in the context of these potential wells, the charged electroweak current can be interpreted as a transfer of a single charged preon from one set of potential wells to another ([17], Ch. 12). It will be seen below that these assumptions allow a derivation of the CKM and PMNS matrices. This paper thus begs the question, "Do such preons exist and have they been observed?" The results of this paper support the hypothesis that such particles might exist,

but not that they have been observed in any direct way (much as in the case of quarks). That said, some considerations suggest a link between such scalars and the Higgs sector, prior to spontaneous symmetry breaking [20]. One might also ask whether this fit is related to the mass matrices, since the CKM and PMNS matrices are required to be consistent with the Yukawa mass couplings. Although this derivation does indeed draw upon the mass matrices of the extended-color theory as will be seen below, this consistency requirement does not provide a significant constraint on the matrices as measured, as discussed in ([5] [17] Ch. 13).

Section 2 outlines the calculational approach and justifications. Section 3 presents results. Means are discussed that might improve the accuracy of the resulting fits. The fits are not unique as will be seen below, so other related solutions are possible that may provide better fits. Once the parameters of the underlying wavefunctions are specified, one then has a derivation of the CKM and PMNS matrices from a well-defined physical system in the conventional sense, *i.e.*, from particles occupying 3 equally-spaced potential wells in a ring. Section 4 summarizes the results.

2. Calculational Approach

The approach assumes that the hypothesized three scalar fields in a fermion satisfy a scalar equation in the three corresponding parabolic potential wells. With this assumption, the time-stationary ground-state spatial wavefunction in well j, $\Psi\left(\boldsymbol{x},\boldsymbol{x}_j,\sigma\right)$, is a gaussian in a rest frame in which the particle is instantaneously at rest:

$$\Psi\left(\boldsymbol{x},\boldsymbol{x}_j,\sigma\right)=\exp\left[-\left|\boldsymbol{x}-\boldsymbol{x}_j\right|^2/2\sigma^2\right]\Big/\left(2\pi\sigma^2\right)^{3/2}, \tag{1}$$

where $\boldsymbol{x}_j$ is the offset of well j relative to the center of the ring, and σ is the root-mean-square width of the gaussian wavefunction. It should be noted that the parabolic potential well is often used to approximate other potential wells, e.g., a gaussian potential well or a diatomic potential well [21]-[23]. It is also well-known that a gaussian wavefunction applies for the ground state in a parabolic potential in the context of either the Schrodinger equation or the Klein-Gordon equation. It will be seen that the results are not sensitive to the exact assumed form of the wavefunctions.

Next, the approach assumes that the three potential wells have a separation r_{sep} around a circular ring. The value of r_{sep} can be estimated using electrostatic repulsion. One may use the relationship $(e/3)^2/r_{\text{sep}}$ for the electrostatic potential energy of two particles of charge $e/3$ separated by r_{sep} (e is the charge of the electron). From this and using the electrostatic energy from the "preon" bonds in electrons and quarks ([17], Ch. 11), one obtains a value of r_{sep} equal to 1.88×10^{-21} meters. This separation is well below the current experimental upper bound on the size of the electron, which is of the order of 10^{-18} meters [24]. The relationship between r_{sep} and the radius of the ring r_0 when there are three equally-spaced potential wells around the ring is $r_0=r_{\text{sep}}/\sqrt{3}$. This yields a radius of about 1.08×10^{-21} m. This

small size seems to violate Heisenberg's uncertainty principle, implying very large momenta and energy. However, there are reasons why it does not, as detailed in a companion paper. The radius of this ring is allowed to vary in a limited way from one generation to the next, e.g., the radius r_0 of the down quark and the strange quark can be slightly different.

It is easy to show that there are three permutationally-equivalent sets of 3 eigenstates for 3 × 3 permutationally-symmetric mass matrices for a given set of 3 eigenvalues ([17], Ch. 2). One such set of 3 eigenstates is given by the columns of the unitary matrix $\boldsymbol{U}_1$:

$$\boldsymbol{U}_1 = 1/\sqrt{3}\begin{bmatrix} 1 & \mathrm{e}^{-\pi i/3} & -\mathrm{e}^{\pi i/3} \\ 1 & \mathrm{e}^{\pi i/3} & -\mathrm{e}^{-\pi i/3} \\ 1 & -1 & 1 \end{bmatrix}, \tag{2}$$

So, for example, the first column is the electron eigenstate vector, the second column is the muon eigenstate, and the third column is the tauon eigenstate. The eigenvalues are the corresponding fermion rest masses. The three elements of the column are the amplitudes to be in the three respective potential wells. The above three paragraphs delineate the basic assumptions used here to calculate the matrix elements of the CKM and PMNS matrices. The process is next detailed.

The matrix elements of the CKM and PMNS matrices are then computed according to an inner-product formula familiar to any student of linear algebra or lower-division chemistry. For the $(i, j)^{\text{th}}$ matrix element of a matrix $\boldsymbol{M}$,

$$\boldsymbol{M}_{i,j} = \int \mathrm{d}x^3 \boldsymbol{\varphi}_i^H(\boldsymbol{x})\boldsymbol{\varphi}_j(\boldsymbol{x}), \tag{3}$$

where $\boldsymbol{M}$ is either the CKM or PMNS matrix. For the CKM matrix, the index i refers to an up-quark family member, *i.e.*, an up, charm, or top fermion. The index j refers to a down-quark family member, *i.e.*, a down, strange, or bottom fermion. Similarly, for the PMNS matrix, i refers to an electron family generation and j refers to the neutrino family generation. The wavefunctions $\boldsymbol{\varphi}_j(\boldsymbol{x})$ are vectors since they are proportional to the column vectors in Equation (2), so the Hermitian conjugate notation "H" is used. The integrals at each potential well are performed in the instantaneous reference frame in which the particles are at rest. In the context of this analysis, this integral becomes an integral over the wavefunctions given by Equation (1) in each of the three potential wells and therefore is a sum over the three potential wells for each fermion. Using Equations (1) to (3) one then has

$$\boldsymbol{M}_{i,j} = \sum_m \boldsymbol{U}_1^*(m,i)\sum_n \boldsymbol{U}_1(n,j)\int \mathrm{d}x^3 \Psi^*(\boldsymbol{x}, \boldsymbol{x}_{mi}, \sigma_i)\Psi(\boldsymbol{x}, \boldsymbol{x}_{nj}, \sigma_j). \tag{4}$$

Here the sums over rows (potential wells) m and n obviously run from 1 to 3. The $\boldsymbol{x}_{mi}$ denote the location of well m for fermion i and similarly for $\boldsymbol{x}_{nj}$. The matrices computed according to the above yield the 3 × 3 identity matrix as expected when the same well locations are used for the rings of the respective families and the gaussian spreads are much smaller than the separations. In such cases this follows since the integrals are equal to $\delta_{m,n}$ and the matrix $\boldsymbol{U}_1$ is unitary.

The relative clocking of the wavefunctions in the rings is assumed to be the same, so that each of the three wavefunctions of $\varphi_i(\boldsymbol{x})$ and $\varphi_j(\boldsymbol{x})$ is in radial lines from the common center of the rings. This is in accord with an analysis of the decay of the pion in this context ([17], Ch. 12). In this context, the charged-current electroweak interaction corresponds to wavefunctions that are localized to potential wells that match in azimuthal position between the particles in the two different families.

In order to obtain diagonal elements that differ from unity for the CKM matrix, the relative radii of the rings of the up and down families are allowed to vary from one generation to the next. Similarly, the relative radii of the rings of the electron and neutrino families are allowed to vary for the PMNS matrix. There is no reason why small variations of radius between fermions of different generations should be disallowed in the theory.

The expression (4) allows for overlaps between nearest-neighbor wavefunctions in one potential well and a similar wavefunction in a neighboring well. This effect was initially included and has a non-zero contribution. It is found that such overlaps do not change the off-diagonal elements of $\boldsymbol{M}_{i,j}$, which are zero within roundoff error with the above approach, whether or not such nearest-neighbor overlaps are included. Such overlaps do alter the diagonal elements in proportion to the overlap between neighboring potential wells. In keeping with the initial hypothesis that the charged-current electroweak interaction is localized to specific matching potential wells ([17], Ch. 12), these nearest-neighbor overlap contributions are set to zero. These overlaps could be made zero by other means as well, such as a smaller width of the wavefunctions. This would then impact the other parameters, such as smaller relative radii of the rings, but it is found that this does not affect the final fit agreements after such adjustments.

It is found that by small variations of the radius between generations one can obtain exact fits of the magnitudes of 3 diagonal elements of the CKM and PMNS matrices. As is well-known, 4 parameters are required to specify such unitary matrices, so at least one more matrix parameter is required. This is done by estimating at least one off-diagonal matrix element from parameters of the mass matrix. The process outlined above gives zero for the off-diagonal matrix elements, due to the symmetries of the $\boldsymbol{U}_1$ matrix and the integrals. To obtain non-zero off-diagonal elements, the above equation is amended with the following term:

$$\boldsymbol{M}'_{i,j} = \boldsymbol{M}_{i,j} + \eta \sum_{i,j=1,\cdots,3} \left|\varepsilon_{ijk}\right| \sum_k \boldsymbol{U}_1^*(k,i)\boldsymbol{U}_1(k,j) \int \mathrm{d}x^3 \Psi^*(\boldsymbol{x}, \boldsymbol{x}_{ki}, \sigma_i) \Psi(\boldsymbol{x}, \boldsymbol{x}_{kj}, \sigma_j). \quad (5)$$

Here, ε_{ijk} is the Levi-Civita antisymmetry tensor and η is an antisymmetry factor for the matrix $\boldsymbol{M}'_{i,j}$, (CKM or PMNS). There are several items to note. First, $|\varepsilon_{ijk}|$ has cyclic permutational symmetry in accord with the theory. Second, the sum in Equation (5) is zero for diagonal matrix elements, $i = j$, and only has one term for off-diagonal elements when $i \neq j$. Note also that there is an overall sign that is not determined in the contributions of Equation (5), *i.e.*, $\pm\left|\varepsilon_{ijk}\right|$ are both possible, and this degree of freedom is utilized below. Finally, the form of the

above term leads to phases for the off-diagonal elements which are integer multiples of π/3 for real-valued Ψ. These phases fortuitously are a close match with the current best estimates of the *CP*-violating phases for both the CKM and PMNS matrices, as seen below. Future efforts might allow for additional corrections in Equation (5) in order to obtain better fits; the matrix elements of $\boldsymbol{U}_1$ in principle span any possible *CP*-violating phase. That is, the asymmetry term in Equation (5), might be an oversimplification. Equation (5) could also conceivably use a complex value of η (preferably with a phase which is a small fraction of a radian). The latter approach can trivially give an exact fit to the *CP*-violating phase. Physically, Equation (5) allows for a slight asymmetric electroweak interaction between the three aligned potential wells.

The antisymmetry factor is estimated approximately from the parameters of the mass matrices for the respective families. This might be expected because of the relationship between mass matrices and the CKM and PMNS matrices in the standard model, or in any other self-consistent model of a similar form. For the PMNS matrix, the normal mass hierarchy is used for the purposes of this paper. The antisymmetry factor η for the off-diagonal terms is estimated by using the probability to stay in a potential well using the mass matrices of the respective family mass matrices ([17], Ch. 2). The **Appendix** shows that this probability $P(\Delta t)$ to stay in a potential well *j* in a *W*-boson interaction time, Δt, can be approximated by $1-\alpha\left|d_{hop}\right|^2$. So, for example, when the normalized amplitude d_{hop} has the minimum value of zero, the particles are not hopping at all. Physically a particle then resides entirely in a single potential well, so the electroweak interaction must compete with the preon bonds holding it there, and hence the electroweak interaction is less localized. The interaction is more like a "tear" than a "transfer" of the scalar particle in such cases with smaller hop amplitudes. The magnitude of $\left|d_{hop}\right|$ is much less for the lepton families than for the quark families, which provides an explanation for why the PMNS matrix is so far from diagonal compared to the CKM matrix—the preon particle transfer is more like a tear for the PMNS matrix. Following this perspective, one has the following approximate bounds on the antisymmetry factor from the **Appendix**. Using $\alpha = 1$, one obtains

$$\begin{aligned} &0.022 \approx P_u(\Delta t) \lesssim \eta_{CKM} \lesssim P_d(\Delta t) = 0.069 \text{ , for the CKM matrix, and} \\ &0.157 \approx P_e(\Delta t) \lesssim \eta_{PMNS} \lesssim P_\nu(\Delta t) = 0.572 \text{ , for the PMNS matrix.} \end{aligned} \quad (6)$$

Here $P_{u,d,e,\nu}(\Delta t)$ denotes one minus the probability of a transition in time Δt for the up, down, electron, and neutrino families, respectively. The chosen values are η_{CKM} = 0.011 and η_{PMNS} = 0.54. The use of Equations (5) with these values of η produces 6 off-diagonal terms. The resulting CKM matrix is approximately unitary, with the non-diagonal elements of $\boldsymbol{CKM}^H\boldsymbol{CKM}$ less than 0.01. The resulting PMNS matrix is not particularly close to a unitary matrix. Since only one of these off-diagonal elements is needed to complete the specification of a unitary matrix, one element is selected, which is the (1, 3) element (1st row, 3rd

column). This seems rather arbitrary but is done for two reasons. First, it is the only element of the "standard" Euler parameterization that directly contains the non-zero complex *CP*-violating phase, and the only element in the first row. Second, and more fundamentally, the theory of [17] indicates that the singlet representation, a left-handed free Dirac (or Weyl) state, which is the (1, 1, 1) eigenvector in $\boldsymbol{U}_1$, should be used for the up- and electron-family states for precise consistency with observations and the corresponding standard model electroweak calculations. This corresponds to column 1 for the unitary matrix $\boldsymbol{U}_1$ shown above. This leads to a required choice of elements only in row 1, with these conventions (for the other 2 color solutions, these columns are permuted). Combining both these justifications, Equation (5) should be applied to the (1, 3) element as the needed 4th matrix element. This completes the definition of the unitary matrix, as mentioned above. With these 4 matrix elements, the rest of the unitary CKM and PMNS matrices are computed using unitarity. The unitarity calculations are very straightforward and these are summarized here for completeness, for the CKM matrix:

$$\begin{aligned}
&\left|\boldsymbol{CKM}_e(1,2)\right| = \left[1-\left|\boldsymbol{CKM}_{e0}(1,3)\right|^2-\left|\boldsymbol{CKM}_{e0}(1,1)\right|^2\right]^{1/2},\\
&\left|\boldsymbol{CKM}_e(2,3)\right| = \left[1-\left|\boldsymbol{CKM}_{e0}(1,3)\right|^2-\left|\boldsymbol{CKM}_{e0}(3,3)\right|^2\right]^{1/2},\\
&\left|\boldsymbol{CKM}_e(2,1)\right| = \left[1-\left|\boldsymbol{CKM}_{e0}(2,3)\right|^2-\left|\boldsymbol{CKM}_{e0}(2,2)\right|^2\right]^{1/2},\\
&\left|\boldsymbol{CKM}_e(3,2)\right| = \left[1-\left|\boldsymbol{CKM}_{e0}(1,2)\right|^2-\left|\boldsymbol{CKM}_{e0}(2,2)\right|^2\right]^{1/2},\\
&\boldsymbol{CKM}_e(3,1)_a = \left[1-\left|\boldsymbol{CKM}_{e0}(2,1)\right|^2-\left|\boldsymbol{CKM}_{e0}(1,1)\right|^2\right]^{1/2},\\
&\boldsymbol{CKM}_e(3,1)_b = \left[1-\left|\boldsymbol{CKM}_{e0}(3,2)\right|^2-\left|\boldsymbol{CKM}_{e0}(3,3)\right|^2\right]^{1/2},\\
&\left|\boldsymbol{CKM}_e(3,1)\right| = 0.5\left[\boldsymbol{CKM}_e(3,1)_a+\boldsymbol{CKM}_e(3,1)_b\right].
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\Phi\left[\boldsymbol{CKM}_{e0}(2,2)\right] = \text{(see text)},\\
&\Phi\left[\boldsymbol{CKM}_e(2,1)\right]\\
&= -\Phi\left\{-\left[\boldsymbol{CKM}_{e0}(1,2)\boldsymbol{CKM}_{e0}(2,2)^*+\boldsymbol{CKM}_{e0}(1,3)\boldsymbol{CKM}_{e0}(2,3)^*\right]\Big/\boldsymbol{CKM}_{e0}(1,1)\right\},\\
&\Phi\left[\boldsymbol{CKM}_e(3,2)\right]\\
&= -\Phi\left\{-\left[\boldsymbol{CKM}_{e0}(1,3)\boldsymbol{CKM}_{e0}(1,2)^*+\boldsymbol{CKM}_{e0}(2,3)\boldsymbol{CKM}_{e0}(2,2)^*\right]\Big/\boldsymbol{CKM}_{e0}(3,3)\right\},\\
&\Phi\left[\boldsymbol{CKM}_e(3,1)\right]_a\\
&= -\Phi\left\{-\left[\boldsymbol{CKM}_{e0}(1,2)\boldsymbol{CKM}_e(3,2)^*+\boldsymbol{CKM}_{e0}(1,3)\boldsymbol{CKM}_{e0}(3,3)^*\right]\Big/\boldsymbol{CKM}_{e0}(1,1)\right\},\\
&\Phi\left[\boldsymbol{CKM}_e(3,1)\right]_b\\
&= -\Phi\left\{-\left[\boldsymbol{CKM}_{e0}(1,3)\boldsymbol{CKM}_{e0}(1,1)^*+\boldsymbol{CKM}_{e0}(2,3)\boldsymbol{CKM}_e(2,1)^*\right]\Big/\boldsymbol{CKM}_{e0}(3,3)\right\},\\
&\Phi\left[\boldsymbol{CKM}_e(3,1)\right] = \Phi\left[\exp\left(i0.5\left\{\Phi\left[\boldsymbol{CKM}_e(3,1)\right]_b+\Phi\left[\boldsymbol{CKM}_e(3,1)\right]_b\right\}\right)\right].
\end{aligned} \tag{8}$$

The same pattern of calculations is used for the PMNS matrix. Here "$\Phi[z]$" denotes the angle of the complex number z. Equations (7) and (8) are computed in the order shown. Here, $\boldsymbol{CKM}_{e0}$ is the matrix obtained from the process associated with Equations (5) and (6), which sets the diagonal elements and also the (1, 3) element. Note that the phases of the first row and the last column are set to zero in the standard Euler parameterization, except the (1, 3) element. Hence the phases of those elements need not be computed. Note also that $\Phi\left[\boldsymbol{CKM}_e(3,2)\right]$ is used in the calculation of $\Phi\left[\boldsymbol{CKM}_e(3,1)\right]_a$ and $\Phi\left[\boldsymbol{CKM}_e(2,1)\right]$ is used in the calculation of $\Phi\left[\boldsymbol{CKM}_e(3,1)\right]_b$. The phase of the (2,2) element cannot be computed in a straightforward manner using the above approach. The phase of the (2, 2) element is obtained by solving for the Euler angles using Equation (7) and the phase from the (1, 3) element from Equation (5). The Euler angles are solved from the magnitudes of the (1, 3), (1, 1), and (3, 3) elements obtained from Equation (7). Recall that the Euler parameterization is possible because of unitarity.

The input values are summarized in Table 1. There are seven distinct parameters used here for each matrix, along with an overall size scale parameter r_0, which is set to 10^{-21} meters. The most important parameters are the ratios of the ring radii in the complementary families, e.g., 1.129/1 for $r_{\text{sep,d}}/r_{\text{sep,u}}$ and 1.5565/1 for $r_{\text{sep},\nu}/r_{\text{sep,e}}$. Since only 4 parameters are needed to specify a unitary matrix, it seems that the seven parameters are more than enough parameters to achieve the objective of fitting the unitary CKM and PMNS matrices.

Table 1. Input parameters of scalar wavefunction used to fit the CKM and PMNS matrices. Lengths are given in units of r_0, the nominal ring radius, which is set to 10^{-21} m. For the ring radii in the d and ν families, each generation is assigned a distinct radius.

Parameter	CKM value (u or d)	PMNS value (e or ν)
Particle gaussian radii, σ, u or e	0.4	0.8
Particle gaussian radii, σ, d or ν	0.4	1.2
Ring radii, u or e	1.0	1.0
Ring radii, d or ν	1.129, 1.1312, 1.024	1.5565, 2.303, 2.089
Antisymmetry factor η and sign	(−) 0.011	(+) 0.54
Unitary matrix	U_1	U_1

The result of these calculations produces matrices that are both good approximations to the CKM and PMNS matrices, respectively, and are also approximately unitary. The metric for comparison between the observationally-inferred CKM matrix, $\boldsymbol{CKM}$, and the estimated matrix, $\boldsymbol{CKM}_e$, is

$$err_{meas} = \left[\sum_{i,j=1\text{ to }3}\left|\boldsymbol{CKM}(i,j)-\boldsymbol{CKM}_e(i,j)\right|^2\right]^{\frac{1}{2}} \Big/ \left[\sum_{i,j=1\text{ to }3}\left|\boldsymbol{CKM}(i,j)\right|^2\right]^{\frac{1}{2}}. \quad (9)$$

In addition to this error-with-measured metric, there is also a unitarity error metric that is computed as follows:

$$err_{un} = \left[\sum_{i,j=1\text{ to }3} \left|\boldsymbol{I}(i,j) - \boldsymbol{U}_e(i,j)\right|^2\right]^{\frac{1}{2}} \Big/ \left[\sum_{i,j=1\text{ to }3} \left|\boldsymbol{I}(i,j)\right|^2\right]^{\frac{1}{2}}, \quad (10)$$

where $\boldsymbol{I}$ is the 3×3 identity matrix and $\boldsymbol{U}_e = \boldsymbol{CKM}_e^H \boldsymbol{CKM}_e$. Note that unitarity need not hold exactly because the complex value of the (1, 3) element is used instead of just the phase, and this supplies an unnecessary fifth parameter into the calculations. This then can cause small deviations from unitarity. Similar expressions are used for the PMNS matrix. These error metrics are presented and discussed in the next section.

3. Results

The resulting theoretically estimated CKM matrix in accord with the standard Euler parameterization is

$$\boldsymbol{CKM}_e = \begin{bmatrix} 0.97433 & 0.22508 & 0.001786 - i0.003094 \\ -0.22491 - i0.000127 & 0.97346 - i0.292 \times 10^{-4} & 0.042258 \\ 0.008792 - i0.003420 & -0.04134 - i0.000692 & 0.9991 \end{bmatrix}$$

The resulting theoretically estimated PMNS matrix in accord with the standard Euler parameterization is

$$\boldsymbol{PMNS}_e = \begin{bmatrix} 0.82324 & 0.54801 & -0.074091 + i0.12833 \\ -0.33529 + i0.08099 & 0.58728 + i0.052999 & 0.73028 \\ 0.44555 + i0.06918 & -0.59131 + i0.04816 & 0.66688 \end{bmatrix}$$

The recent experimentally-derived values are from the 2022 PDG values, given in [1] and [2], respectively. The corresponding fits to the Euler angles and *CP*-violating phase δ are shown in Table 2.

Table 2. Comparison of fits to Euler angles and *CP*-violating phase for the CKM and PMNS matrices. All values are given in degrees.

Parameter/Matrix	CKM measured	CKM fit	PMNS measured	PMNS fit
θ_{12}	13.00 ± 0.04	13.01	33.64 ± 0.80	33.65
θ_{13}	0.2114 ± 0.006	0.205	8.53 ± 0.14	8.52
θ_{23}	$2.397^{+0.049}_{-0.042}$	2.42	47.64 ± 1.20	47.60
δ	65.5 ± 1.55	60	-115^{+36}_{-29}	−120

A brief discussion of the sources used for the measured values in Table 2 may be appropriate. The CKM numbers all come from [1], and the error bars are transformed from the sine of the angles to the angles for the Euler angles. This transformation is not needed for δ. The PMNS Euler angles all come from [2] and are given in terms of the square of the sine of the angles, and so the same process is used as for the CKM matrix after taking the square root. The only parameter in Table 2 that is not approximately within 1-sigma error bars is the CP-violating phase δ_{CKM} for the CKM matrix. The deviation in this case is 3.5 sigma, which is considerable. There are other current estimates for the CP-violating phase for the CKM matrix

in the literature, e.g., [4] from 2020. In this reference, $\delta_{CKM} = 68.5^{\circ} \pm 2.5^{\circ}$. As mentioned above, it is possible that Equation (5) might be generalized to reduce this error. On the other hand, the agreement for the PMNS matrix is quite good, within about 5/29 = 0.17 sigma.

As stated in the abstract, the resulting CKM matrix has an RMS error from measured of $err_{meas} = 6.09 \times 10^{-4}$ and an RMS departure from unitarity $err_{un} = 9.18 \times 10^{-4}$. The resulting PMNS matrix has an RMS error from measured of $err_{meas} = 0.00834$ and an RMS departure from unitarity $err_{un} = 0.00895$. As discussed earlier, these relatively good matches are based on simple matrix-element calculations involving inner products of gaussian wavefunctions for scalar fields as well as the properties of the mass matrices of [17] and their eigenstates. These fits are also significantly better than those given by alternative proposals [7] [9], as shown in Table 3. Reference 7 uses a Jordan algebra, and only computes the CKM matrix. Reference 9 uses the Monster group and provides an estimate of both matrices. The CKM Euler angles for Reference 9 are computed from the magnitudes of the CKM matrix elements that are provided in that reference; the angles are not explicitly provided.

Table 3. Comparison of fits to Euler angles and CP-violating phase for the CKM and PMNS matrices from other approaches, references (7) and (9). All values are given in degrees.

Parameter/Matrix	CKM measured	CKM fit, Ref 7/9	PMNS measured	PMNS fit, Ref. 9
θ_{12}	13.00 ± 0.04	11.09/12.98	33.64 ± 0.80	34.29
θ_{13}	0.2114 ± 0.006	0.172/0.562	± 8.53 ± 0.14	(−)8.56
θ_{23}	$2.397^{+0.49}_{-0.42}$	4.054/12.98	± 47.64 ± 1.20	(−)42.85
δ	65.5 ± 1.55	NA/NA	-115^{+36}_{-29}	$\lvert\delta\rvert < 14.8$

4. Summary

The CKM and PMNS matrices are computed from inner products that depend on the dimensionless ratios of the separations of potential wells relative to the spatial spread of the gaussian wavefunctions. These are varied in a limited way for each fermion generation. The results also depend on the 3-vector representations of the fermions in the circulant basis. These would not add a complex part to the calculation but for the insight that in weak interactions, terms may arise that are related to the propensity for the wavefunction to be localized to a specific potential well, based on properties of the mass matrix. This leads to a complex off-diagonal contribution which has a magnitude based on the 3-vector representations in $\boldsymbol{U}_1$ of the particles as well as the properties of the mass matrices. Unitarity is then applied to complete the calculations of the matrices.

For the PMNS matrix, the fit values of Euler angles and the *CP*-violating phase are well within 1 standard deviation of the experimental values. Accordingly, it is found that the magnitudes of all the matrix elements are well within published 3-sigma error bars [25], and also comfortably within 1-sigma error bars for the

magnitudes of all elements, if a gaussian distribution of measurement errors is assumed. The largest error in terms of the measured error bars for the magnitudes is only 0.13-sigma! The *CP*-violating phase is within 0.2 sigma. For the CKM matrix, all the Euler angles are within approximately 1-sigma of the experimental values. Inspection indicates that the magnitudes of the computed matrix elements are within 1-sigma error bars for 7 of the 9 elements [2]. Of the remaining two elements, the (2, 3) element is off by about 2.7 standard deviations and the (3, 1) element is off by about 4.4 standard deviations. The latter large error is probably due to propagation of errors in the multi-step process from the approximate initial values used. The *CP*-violating phase is off by about 3 standard deviations. A future effort might obtain a better fit to this parameter using the $\boldsymbol{U}_1$ matrix elements that span any possible *CP*-violating phase. Yet another alternative to obtain a better fit for this phase is simply to allow the asymmetry factor to be complex with a phase of about 0.1 radians. This would trivially allow an exact fit for the CKM *CP* phase.

One might argue that the above fits are trivial because there are more inputs (7) than unitary matrix parameters (4). Of the seven input parameters for the CKM matrix, the only critical parameters are the three ring radii of the down family members relative to the corresponding up-family ring radii, and also the antisymmetry factor. This amounts to 4 input parameters for the 4 matrix parameters for the CKM matrix. This is also true for the PMNS matrix. Hence the quality of these fits may not be viewed as so trivial, but they are still not unique.

This paper shows that the assumed properties of underlying gaussian wavefunctions that occupy potential wells are consistent with both the measured CKM and PMNS matrices, and also with properties of the underlying theory. These properties include 1) the mass eigenstate vectors ([17], Ch. 2), 2) the mass matrices ([17], Ch. 2), 3) the requirement for left-handed Dirac states (or Weyl states) for up and electron families ([17], Chs. 4-6, 13), and 4) the localization of wavefunctions to individual wells during the brief interaction lifetimes of the massive electroweak bosons ([17], Ch. 12). These properties of the theory also result in calculations that match all known charged-current electroweak standard model calculations for leptons, and certainly are also a match for quarks at tree level, at least. The resulting CKM matrix is nearly diagonal because of features of the quark mass matrices that lead to tight bounds on the antisymmetry factor. For a similar reason, the PMNS matrix is quite different from a diagonal matrix due to the significantly smaller normalized hop amplitudes of the mass matrices. This results in an exchange of a resident charge that is more of a "tear" than a simple transfer, and this is manifested by larger off-diagonal terms for the PMNS matrix. The CKM and PMNS matrices are also consistent with the Yukawa mass couplings as presented in ([5] [17] Ch. 13). There is also the choice of which non-diagonal matrix element to estimate first, the (1, 3) element, which is not determined except by several qualitative but justified rationales. This choice could be viewed as a drawback of this method.

In summary, if the particular set of wavefunction length scales shown above were somehow determined by independent means, they would "predict" the measured

CKM and PMNS matrices to the error levels stated above, when combined with the mass-eigenstate vectors of the underlying theory. Hence, the above calculations not only offer a process for obtaining relatively accurate fits of the CKM and PMNS, but in principle are a method to predict the CKM and PMNS matrices. This paper demonstrates that *at least one* anomaly-free quantum field theory can fit the CKM and PMNS matrices within experimental error for almost all of the matrix parameters.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Probability to Stay in a Potential Well over a Short Duration

The quantum field theory of [17] utilizes 3 × 3 Hermitian mass matrices of the form:

$$\boldsymbol{H} = \boldsymbol{H}\left(m_f, d, \phi\right) = m_f c^2 \left[\boldsymbol{I} + d_{hop} e^{i\phi} \boldsymbol{P} + d_{hop} e^{-i\phi} \boldsymbol{P}^{-1}\right], \tag{A1}$$

where m_f is "the family mass parameter" which has units of MeV/c^2, $\boldsymbol{I}$ is the 3 × 3 identity matrix, and $d_{hop} e^{i\phi}$ is the normalized complex "hop" amplitude. The matrix $\boldsymbol{P}$ is the 3 × 3 positive permutation matrix, given by

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{A2}$$

Note that $\boldsymbol{P}^{-1} = \boldsymbol{P}^2$ is also a permutation matrix in the reverse direction. It can be shown that m_f is equal to the average of the experimentally-derived or otherwise inferred three masses of the respective family, e.g., $m_{f,d}$ = average of down, strange, and bottom masses for the down family. The other two parameters, d_{hop} and ϕ, are used to match the mass eigenvalues of the matrices to the measured masses in a family. One finds $m_{f,u}$ = 58127.4 MeV, $m_{f,d}$ = 1426.9 MeV/c^2, $m_{f,e}$ = 627.676 MeV/c^2, and $m_{f,\nu}$ = 0.02273 × 10^{-6} MeV/c^2, for the up, down, electron, and neutrino families, respectively, using the 2016 PDG values of the masses. For the neutrinos, masses of 0.0054, 0.0102, and 0.0515 eV/c^2 are assumed.

The probability $P\left(\Delta t\right)$ to stay in a potential well *j* in a time Δt can be approximated by $1 - \sum_{k \neq j} \left|\left\langle j \left| \boldsymbol{H} \right| k \right\rangle\right|^2 \left(\Delta t/\hbar\right)^2$ for small values of Δt, where $\boldsymbol{H}$ is the 3 × 3 mass matrix, treated as a Hamiltonian matrix. The off-diagonal matrix elements have magnitude $\left|\left\langle j \left| \boldsymbol{H} \right| k \right\rangle\right| = m_f c^2 d_{hop}$. The time duration Δt is approximately 3 × 10^{-25} sec or less, using the W-boson lifetime. Hence, the probability to stay in a potential well *j* can be written as $1 - \alpha \left|d_{hop}\right|^2$. One may compute $\alpha = 2\left(m_f c^2 \Delta t/\hbar\right)^2$, using the mass parameter for a specific particle family *f*. The family-specific values of the hop amplitude d_{hop} are $d_{hop,u}$ = 0.989, $d_{hop,d}$ = 0.965, $d_{hop,e}$ = 0.917, and $d_{hop,\nu}$ = 0.654 for the respective families. One then obtains the following transition probabilities for the four families:

$$P_d\left(\Delta t\right) \approx 0, \tag{A3}$$

$$P_d\left(\Delta t\right) = 0.60, \tag{A4}$$

$$P_e\left(\Delta t\right) = 0.93, \text{ and} \tag{A5}$$

$$P_\nu\left(\Delta t\right) = 1. \tag{A6}$$

The probability to stay in the potential well for the up family is approximated as zero because the probability would be less than zero if the full lifetime of the W-boson were used in the above approximation. If an alternative approximation is used, in which α is set to one for all of the families, one obtains

$$P_d\left(\Delta t\right) = 0.022, \tag{A7}$$

$$P_d\left(\Delta t\right) = 0.069, \tag{A8}$$

$$P_e\left(\Delta t\right) = 0.159, \text{ and} \tag{A9}$$

$$P_\nu\left(\Delta t\right) = 0.572. \tag{A10}$$

This latter approximation is valid if the hop amplitude is interpreted as the square root of the probability of a hop in a duration equal to $\hbar \Big/ \left(\sqrt{2} m_f c^2\right)$.