

# Cauchy, Cosserat, Clausius, Einstein, Maxwell, Weyl Equations Revisited

J.-F. Pommaret 

CERMICS, Ecole des Ponts ParisTech, Paris, France

Email: jean-francois.pommaret@wanadoo.fr

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## Abstract

The Cauchy stress equations (1823), the Cosserat couple-stress equations (1909), the Clausius virial equation (1870) and the Maxwell/Weyl equations (1873, 1918) are among the most famous partial differential equations that can be found today in any textbook dealing with elasticity theory, continuum mechanics, thermodynamics or electromagnetism. Over a manifold of dimension  $n$ , their respective numbers are  $n, n(n-1)/2, 1, n$  with a total of  $N = (n+1)(n+2)/2$ , that is 15 when  $n = 4$  for space-time. This is also just the number of parameters of the Lie group of conformal transformations with  $n$  translations,  $n(n-1)/2$  rotations, 1 dilatation and  $n$  highly non-linear elations introduced by E. Cartan in 1922. The purpose of this paper is to prove that the form of these equations only depends on the structure of the conformal group for an arbitrary  $n \geq 1$  because they are described *as a whole* by the (formal) adjoint of the first Spencer operator existing in the Spencer differential sequence. Such a group theoretical implication is obtained by applying totally new differential geometric methods in field theory. In particular, when  $n = 4$ , the main idea is not to *shrink* the group from 10 down to 4 or 2 parameters by using the Schwarzschild or Kerr metrics instead of the Minkowski metric, but to *enlarge* the group from 10 up to 11 or 15 parameters by using the Weyl or conformal group instead of the Poincaré group of space-time. Contrary to the Einstein equations, these equations can be all parametrized by the adjoint of the second Spencer operator through  $Nn(n-1)/2$  potentials. These results bring the need to revisit the mathematical foundations of both General Relativity and Gauge Theory according to a clever but rarely quoted paper of H. Poincaré (1901). They strengthen the recent comments we already made about the dual confusions made by Einstein (1915) while following Beltrami (1892), both using the same Einstein operator but ignoring it is self-adjoint in the framework of differential double duality.

## Keywords

Janet Sequence, Spencer Sequence, Poincaré Sequence, Gauge Sequence, Lie Group of Transformations, Lie Pseudogroup of Transformations, Conformal Group of Transformations, Adjoint Representation

## 1. Introduction

Let  $G$  be a Lie group with coordinates  $(a^p) = (a^1, \dots, a^p)$  acting on a manifold  $X$  with a local action map  $y = f(x, a)$ . According to the *second fundamental theorem* of Lie, if  $\theta_1, \dots, \theta_p$  are the infinitesimal generators of the effective action of a lie group  $G$  on  $X$ , then  $[\theta_\rho, \theta_\sigma] = c_{\rho\sigma}^\tau \theta_\tau$  where the  $c = (c_{\rho\sigma}^\tau = -c_{\sigma\rho}^\tau)$  are the *structure constants* of a Lie algebra of vector fields which can be identified with  $\mathcal{G} = T_e(G)$  the tangent space to  $\mathcal{G}$  at the identity  $e \in G$  by using the action. Equivalently, introducing the non-degenerate inverse matrix  $\alpha = \omega^{-1}$  of right invariant vector fields on  $G$ , we obtain from crossed-derivatives the *compatibility conditions* (CC) for the previous system of partial differential (PD) equations called *Maurer-Cartan equations* or simply MC equations, namely:

$$\frac{\partial \omega_s^\tau}{\partial a^r} - \frac{\partial \omega_r^\tau}{\partial a^s} + c_{\rho\sigma}^\tau \omega_r^\rho \omega_s^\sigma = 0$$

(care to the sign used) or equivalently  $[\alpha_\rho, \alpha_\sigma] = c_{\rho\sigma}^\tau \alpha_\tau$  (See [1]-[5] for more details).

Using again crossed-derivatives, we obtain the corresponding *integrability conditions* (IC) on the structure constants and the Cauchy-Kowaleski theorem finally provides *the third fundamental theorem of Lie* saying that, for any Lie algebra  $\mathcal{G}$  defined by structure constants  $c = (c_{\rho\sigma}^\tau)$  satisfying:

$$c_{\rho\sigma}^\tau + c_{\sigma\rho}^\tau = 0, \quad c_{\mu\rho}^\lambda c_{\sigma\tau}^\mu + c_{\mu\sigma}^\lambda c_{\tau\rho}^\mu + c_{\mu\tau}^\lambda c_{\rho\sigma}^\mu = 0$$

one can construct an analytic group  $G$  such that  $\mathcal{G} = T_e(G)$  by recovering the MC forms from the MC equations.

**EXAMPLE 1.1:** Considering the affine group of transformations of the real line  $y = a^2 x + a^1$ , the orbits are defined by  $x = a^2 x_0 + a^1$ , a definition leading to  $dx = da^2 x_0 + da^1$  and thus  $dx = \left( \frac{1}{a^2} da^2 \right) x + \left( da^1 - \left( \frac{1}{a^2} da^2 \right) a^1 \right)$ . We obtain therefore  $\theta_1 = \partial_x$ ,  $\theta_2 = x \partial_x \Rightarrow [\theta_1, \theta_2] = \theta_1$  and  $\omega^1 = da^1 - \left( \frac{1}{a^2} da^2 \right) a^1$ ,  $\omega^2 = \left( \frac{1}{a^2} da^2 \right) a^1 \Rightarrow d\omega^1 + \omega^1 \wedge \omega^2 = 0$ ,  $d\omega^2 = 0 \Leftrightarrow [\alpha_1, \alpha_2] = \alpha_1$  with  $\alpha^1 = \partial_1$ ,  $\alpha_2 = a^1 \partial_1 + a^2 \partial_2$  in the following diagram:

$$\alpha \circ \omega = \begin{pmatrix} 1 & 0 \\ a^1 & a^2 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ -\frac{a^1}{a^2} & \frac{1}{a^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now, if  $x = a(t)x_0 + b(t)$  with  $a(t)$  a time depending orthogonal matrix (*rotation*) and  $b(t)$  a time depending vector (*translation*) describes the movement of a rigid body in  $\mathbb{R}^3$ , then the projection of the *absolute speed*  $v = \dot{a}(t)x_0 + \dot{b}(t)$  in

an orthogonal frame fixed in the body is the so-called *relative speed*  $a^{-1}v = (a^{-1}\dot{a})x_0 + (a^{-1}\dot{b})$  and the kinetic energy/Lagrangian is a quadratic function of the 1-forms  $A = (a^{-1}\dot{a}, a^{-1}\dot{b})$ . Meanwhile, taking into accounts the preceding example, the *Eulerian speed*  $v = v(x, t) = (\dot{a}a^{-1})x + (\dot{b} - \dot{a}a^{-1}b)$  is a quadratic function of the 1-forms  $B = (\dot{a}a^{-1}, \dot{b} - \dot{a}a^{-1}b)$ . We notice that  $a^{-1}\dot{a}$  and  $\dot{a}a^{-1}$  are both  $3 \times 3$  skew-symmetric time depending matrices that may be quite different.

**REMARK 1.2:** An easy computation in local coordinates for the case of the movement of a rigid body shows that the action of the  $3 \times 3$  skew-symmetric matrix  $\dot{a}a^{-1}$  on the position  $x$  at time  $t$  just amounts to the vector product by the *vortex vector*  $\omega = \frac{1}{2}\nabla \wedge v$ .

The above particular cases, well known by anybody studying the analytical mechanics of rigid bodies, can be generalized as follows.

If  $X$  is a manifold and  $G$  is a lie group (*not acting necessarily on X*), let us consider maps  $a: X \rightarrow G: (x) \rightarrow (a(x))$  or equivalently sections of the trivial (principal) bundle  $X \times G$  over  $X$ . If  $x + dx$  is a point of  $X$  close to  $x$ , then  $T(a)$  will provide a point  $a + da = a + \frac{\partial a}{\partial x}dx$  close to  $a$  on  $G$ . We may bring  $a$  back to  $e$  on  $G$  by acting on  $a$  with  $a^{-1}$ , *either on the left or on the right*, getting therefore a 1-form  $a^{-1}da = A$  or  $(da)a^{-1} = B$  with value in  $\mathcal{G}$ . As  $aa^{-1} = e$  we also get  $(da)a^{-1} = -ada^{-1} = -b^{-1}db$  if we set  $b = a^{-1}$  as a way to link  $A$  with  $B$ . When there is an action  $y = ax$ , we have  $x = a^{-1}y = by$  and thus  $dy = dax = (da)a^{-1}y$ , a result leading through the first fundamental theorem of Lie to the equivalent formulas:

$$a^{-1}da = A = (A_i^\tau(x)dx^i = -\omega_\sigma^\tau(b(x))\partial_i b^\sigma(x)dx^i)$$

$$(da)a^{-1} = B = (B_i^\tau(x)dx^i = \omega_\sigma^\tau(a(x))\partial_i a^\sigma(x)dx^i)$$

Introducing the induced bracket  $[A, A](\xi, \eta) = [A(\xi), A(\eta)] \in \mathcal{G}, \forall \xi, \eta \in T$ , we may define the *curvature* 2-form  $dA - [A, A] = F \in \wedge^2 T^* \otimes \mathcal{G}$  by the local formula (*care again to the sign*):

$$\partial_i A_j^\tau(x) - \partial_j A_i^\tau(x) - c_{\rho\sigma}^\tau A_i^\rho(x)A_j^\sigma(x) = F_{ij}^\tau(x)$$

This definition can also be adapted to  $B$  by using  $dB + [B, B]$  and we obtain from the second fundamental theorem of Lie:

**THEOREM 1.3:** There is a *nonlinear gauge sequence*.

$$\begin{array}{lcl} X \times G & \rightarrow & T^* \otimes \mathcal{G} \xrightarrow{MC} \wedge^2 T^* \otimes \mathcal{G} \\ a & \rightarrow & a^{-1}da = A \rightarrow dA - [A, A] = F \end{array} \tag{1}$$

In 1956, at the birth of GT, the above notations were coming from the EM potential  $A$  and EM field  $dA = F$  of relativistic Maxwell theory. Accordingly,  $G = U(1)$  (unit circle in the complex plane)  $\rightarrow \dim(\mathcal{G}) = 1$  was the *only possibility* to get a 1-form  $A$  and a 2-form  $F$  with vanishing structure constants  $c = 0$ .

Choosing now a “close” to  $e$ , that is  $a(x) = e + t\lambda(x) + \dots$  and linearizing as usual, we obtain the linear operator

$$d : \wedge^0 T^* \otimes \mathcal{G} \rightarrow \wedge^1 T^* \otimes \mathcal{G} : (\lambda^\tau(x)) \rightarrow (\partial_i \lambda^\tau(x)) \text{ leading to (See [2] for details).}$$

**COROLLARY 1.4:** There is a *linear gauge sequence*:

$$\boxed{\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} \dots \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \rightarrow 0} \tag{2}$$

which is the tensor product by  $\mathcal{G}$  of the Poincaré sequence for the exterior derivative.

In order to introduce the previous results into a variational framework, we may consider a Lagrangian on  $T^* \otimes \mathcal{G}$ , that is an *action*  $W = \int w(A) dx$  where  $dx = dx^1 \wedge \dots \wedge dx^n$  and to vary it. With  $A = a^{-1} da = -(db)b^{-1}$  we may introduce  $\lambda = a^{-1} \delta a = -(\delta b)b^{-1} \in \mathcal{G} = \wedge^0 T^* \otimes \mathcal{G}$  with local coordinates  $\lambda^\tau(x) = -\omega_\sigma^\tau(b(x)) \delta b^\sigma(x)$  and we obtain in local coordinates ([2], pp. 180-185):

$$\boxed{\delta A = d\lambda - [A, \lambda] \Leftrightarrow \delta A_i^\tau = \partial_i \lambda^\tau - c_{\rho\sigma}^\tau A_i^\rho \lambda^\sigma} \tag{3}$$

Then, setting  $\partial w / \partial A = \mathcal{A} = (\mathcal{A}_i^\tau) \in \wedge^{n-1} T^* \otimes \mathcal{G}^*$ , we get:

$$\delta W = \int \mathcal{A} \delta A dx = \int \mathcal{A} (d\lambda - [A, \lambda]) dx$$

and therefore, after integration by part, the Euler-Lagrange (EL) equations of Poincaré ([2] [6]):

$$\boxed{\partial_i \mathcal{A}_\tau^i + c_{\rho\tau}^\sigma A_i^\rho \mathcal{A}_\sigma^i = 0} \tag{4}$$

Such a linear operator for  $\mathcal{A}$  has non constant coefficients linearly depending on  $A$  and is the adjoint of the previous operator (up to sign).

However, setting now  $(\delta a)a^{-1} = \mu \in \mathcal{G}$ , we get  $\lambda = a^{-1} ((\delta a)a^{-1})a = Ad(a)\mu$  while, setting  $a' = ab$ , we get the *gauge transformation* for any  $b \in G$  (See [2], Proposition 14, p. 182):

$$\boxed{A \rightarrow A' = (ab)^{-1} d(ab) = b^{-1} a^{-1} ((da)b + adb) = Ad(b)A + b^{-1} db, \Rightarrow F' = Ad(b)F} \tag{5}$$

Setting  $b = e + t\lambda + \dots$  with  $t \ll 1$ , then  $\delta A$  becomes an infinitesimal gauge transformation. Finally,  $a' = ba$   
 $\Rightarrow A' = a^{-1} b^{-1} ((db)a + a(db)) = a^{-1} (b^{-1} db)a + A \Rightarrow \delta A = Ad(a)d\mu$  when  $b = e + t\mu + \dots$  with  $t \ll 1$ . Therefore, introducing  $\mathcal{B}$  such that  $\mathcal{B}\mu = \mathcal{A}\lambda$ , we get the divergence-like equations:

$$\boxed{\partial_i \mathcal{B}_\sigma^i = 0} \tag{6}$$

We provide some more details on the *Adjoint representation*  $Ad : G \rightarrow aut(\mathcal{G}) : a \rightarrow Ad(a)$ , which is defined by a linear map  $M(a) = (M_\rho^\tau(a)) \in aut(\mathcal{G})$  and we have the involutive system ([2], Proposition 10, p. 180):

$$\boxed{\frac{\partial M_\mu^\tau}{\partial a^\tau} + c_{\rho\sigma}^\tau \omega_r^\rho(a) M_\mu^\sigma = 0} \tag{7}$$

by using the fact that *any right invariant vector field on G commutes with any left invariant vector field on G* (See [7]-[9] for applications of (reciprocal distributions) to Differential Galois Theory). In addition, as  $a^{-1}\delta a = \lambda$ , we obtain therefore successively:

$$\begin{aligned} Ad(a^{-1})\lambda &= a\lambda a^{-1} = a(a^{-1}\delta a)a^{-1} = \delta a a^{-1} = \mu \\ \Rightarrow dAd(a^{-1})\lambda &= (d\delta a)a^{-1} - \delta a a^{-1} d a a^{-1} \\ Ad(a)dAd(a^{-1})\lambda &= a^{-1}(d\delta a a^{-1} - \delta a a^{-1} d a a^{-1})a \\ &= a^{-1}d\delta a - (a^{-1}\delta a a^{-1})da \\ &= a^{-1}\delta da + (\delta a^{-1})da \\ &= \delta A \end{aligned}$$

As a byproduct, the operator  $\nabla : \lambda \rightarrow d\lambda - [A, \lambda]$  only depends on  $A$  and is the “twist” of the derivative  $d$  under the action of  $Ad(a)$  on  $\mathcal{G}$  whenever  $A = a^{-1}da$  in the gauge sequence. Setting:

$$\nabla(\alpha \otimes \lambda) = d\alpha \otimes \lambda + (-1)^r \alpha \wedge \nabla \lambda, \forall \alpha \in \wedge^r T^*; \lambda \in \mathcal{G}$$

an easy computation shows that  $(\nabla \circ \nabla \lambda)_{ij}^r = c_{\rho\sigma}^r F_{ij}^{\rho} \lambda^{\sigma} = 0$  and we obtain:

**COROLLARY 1.5:** The following  $\nabla$ -sequence:

$$\boxed{\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{\nabla} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \wedge^n T^* \otimes \mathcal{G} \rightarrow 0} \tag{8}$$

is *another* locally exact linearization of the non-linear gauge sequence which is isomorphic to copies of the Poincaré sequence and describes infinitesimal gauge transformations.

We have the commutative diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ \mu & \in & \wedge^0 T^* \otimes \mathcal{G} & \xrightarrow[\text{grad}_1]{} & T^* \otimes \mathcal{G} \\ \downarrow Ad(a) & & \downarrow & & \downarrow \\ \lambda & \in & \wedge^0 T^* \otimes \mathcal{G} & \xrightarrow[\text{grad}_1]{} & T^* \otimes \mathcal{G} \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

both with the corresponding adjoint diagram that does not seem to be known:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \leftarrow & \wedge^n T^* \otimes \mathcal{G}^* & \xleftarrow[\text{div}_1]{} & \wedge^{n-1} T^* \otimes \mathcal{G}^* & \ni & B \\ & & \uparrow & & \uparrow & & Ad(a) \uparrow \\ 0 & \leftarrow & \wedge^n T^* \otimes \mathcal{G}^* & \xleftarrow[\text{ad}(\nabla)_1]{} & \wedge^{n-1} T^* \otimes \mathcal{G}^* & \ni & A \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

In a completely different local setting, if  $G$  acts on  $X$  and  $Y$  is a copy of  $X$  with an action graph  $X \times G \rightarrow X \times Y : (x, a) \rightarrow (x, y = ax = f(x, a))$ , we may use the theorems of  $S. Lie$  in order to find a basis  $\{\theta_\tau \mid 1 \leq \tau \leq p = \dim(G)\}$  of infinitesimal generators of the action. If  $\mu = (\mu_1, \dots, \mu_n)$  is a multi-index of length  $|\mu| = \mu_1 + \dots + \mu_n$  and  $\mu + 1_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$ , we may introduce the system of infinitesimal Lie equations or Lie algebroid  $R_q \subset J_q(T)$  with sections defined by  $\xi_\mu^k(x) = \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)$  for an arbitrary section  $\lambda \in \wedge^0 T^* \otimes \mathcal{G}$  and the trivially involutive operator  $j_q : T \rightarrow J_q(T) : \theta \rightarrow (\partial_\mu \theta, 0 \leq |\mu| \leq q)$  of order  $q$ . We finally obtain the *Spencer operator* through the chain rule for derivatives [1] [10]-[12]:

$$\boxed{\left( d \xi_{\mu,i}^k \right)_{\mu,i}^k(x) = \partial_i \xi_\mu^k(x) - \xi_{\mu+1_i}^k(x) = \partial_i \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)} \tag{9}$$

**THEOREM 1.6:** When  $q$  is large enough to have an isomorphism  $R_{q+1} \simeq R_q \simeq \wedge^0 T^* \otimes \mathcal{G}$  and the following *linear Spencer sequence* in which the operators  $D_r$  are induced by  $d$  as above:

$$\boxed{0 \rightarrow \Theta \xrightarrow{j_q} R_q \xrightarrow{D_1} T^* \otimes R_q \xrightarrow{D_2} \wedge^2 T^* \otimes R_q \xrightarrow{D_3} \dots \xrightarrow{D_n} \wedge^n T^* \otimes R_q \rightarrow 0} \tag{10}$$

is isomorphic to the linear gauge sequence but with a completely different meaning because  $G$  is now acting on  $X$  and  $\Theta \subset T$  is such that  $[\Theta, \Theta] \subset \Theta$ .

**EXAMPLE 1.7: (Weyl group):** For an arbitrary dimension  $n$ , the Lie pseudogroup of conformal transformations, considered as a Lie group of transformations, has  $n$  translations,  $n(n-1)/2$  rotations, 1 dilatation and  $n$  nonlinear elations, that is a total of  $(n+1)(n+2)/2$  parameters while the Weyl subgroup of transformations has only  $(n^2+n+2)/2$  parameters [13]. When  $n=2$  and the standard Euclidean metric, we may choose the infinitesimal generators  $\theta_1 = \partial_1, \theta_2 = \partial_2, \theta_3 = x^1 \partial_2 - x^2 \partial_1, \theta_4 = x^1 \partial_1 + x^2 \partial_2$  of the Weyl subgroup with  $2+1+1=4$  parameters by taking out the elations. Setting  $\xi_\mu^k = \lambda^\tau \partial_\mu \theta_\tau^k$  with  $\lambda = (\lambda^\tau(x)) \in \mathcal{G}$ , we have the  $4 \times 4$  full rank matrix allowing to describe the isomorphism  $R_2 \simeq R_1 \simeq \wedge^0 T^* \otimes \mathcal{G}$ :

$$\xi^1 = \lambda^1 - x^2 \lambda^3 + x^1 \lambda^4, \xi^2 = \lambda^2 + x^1 \lambda^3 + x^2 \lambda^4, \xi_1^2 = -\xi_2^1 = \lambda^3, \xi_1^1 = \xi_2^2 = \lambda^4$$

as follows:

$$\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi_1^2 = -\xi_2^1 \\ \xi_1^1 = \xi_2^2 = A \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x^2 & x^1 \\ 0 & 1 & x^1 & x^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{pmatrix}$$

Now, in order to determine  $ad(D_1)$ , we have to integrate by parts the duality summation:

$$\begin{aligned} &\sigma_1^1 (\partial_1 \xi^1 - \xi_1^1) + \sigma_2^1 (\partial_1 \xi^2 - \xi_1^2) + \sigma_1^2 (\partial_2 \xi^1 - \xi_2^1) + \sigma_2^2 (\partial_2 \xi^2 - \xi_2^2) \\ &+ \mu^{12,r} \partial_r \xi_1^2 + \nu^r \partial_r \xi_1^1 \end{aligned}$$

in which we have taken into account the Medolaghi equations  $\xi_2^2 - \xi_1^1 = 0, \xi_2^1 + \xi_1^2 = 0, \xi_{ij}^k = 0$  defining the Weyl algebroid. We get the 2 Cauchy + 1 Cosserat

+ 1 Clausius equations describing the adjoint of the first Spencer operator in which we may have  $\sigma^{1,2} \neq \sigma^{2,1}$ :

$$\partial_r \sigma^{i,r} = f^i, \quad \partial_r \mu^{12,r} + \sigma^{1,2} - \sigma^{2,1} = m^{12}, \quad \partial_r v^r + \sigma_r^r = u$$

that we can transform into the 4 *pure divergence* equations by comparing  $ad(D_1)$  with  $ad(d)$ :

$$\begin{aligned} \partial_r (\sigma^{i,r}) &= f^i, \\ \partial_r (\mu^{12,r} + x^1 \sigma^{2,r} - x^2 \sigma^{1,r}) &= m^{12} + x^1 f^2 - x^2 f^1, \\ \partial_r (v^r + x^i \sigma_i^r) &= u + x^i f_i \end{aligned}$$

a result not completely evident at first sight, both with the parametrization problem that we now study and solve. In the case of the Poincaré subgroup for the Euclidean metric, the Spencer sequence is isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a Lie algebra  $\mathcal{G}$  of dimension  $n + n(n-1)/2 = n(n+1)/2$ . The Spencer operator  $D_2 : T^* \otimes R_1 \rightarrow \wedge^2 T^* \otimes R_1$  is thus parametrized by the Spencer operator  $D_1$  and the operator  $ad(D_1)$  is thus parametrized by the operator  $ad(D_2)$  which is providing  $n^2(n^2-1)/4$  potentials, that is 3 when  $n=2$ . As the Spencer operator  $D_1$  may be defined by the 6 equations:

$$\partial_1 \xi_1 = A_{1,1}, \partial_1 \xi_2 - \xi_{1,2} = A_{2,1}, \partial_2 \xi_1 - \xi_{1,2} = A_{1,2}, \partial_2 \xi_2 = A_{2,2}, \partial_1 \xi_{1,2} = B_1, \partial_2 \xi_{1,2} = B_2$$

because  $R_1$  is defined by the Lie equations  $(\xi_{1,1} = 0, \xi_{1,2} + \xi_{2,1} = 0, \xi_{2,2} = 0)$ .

The three CC made by  $D_2$  are thus:

$$\partial_2 A_{1,1} - \partial_1 A_{2,1} + B_1 = 0, \quad \partial_2 A_{1,2} - \partial_1 A_{2,2} + B_2 = 0, \quad \partial_2 B_1 - \partial_1 B_2 = 0$$

Multiplying these three equations respectively by  $\phi^1, \phi^2, \phi^3$ , summing and integrating by parts, we obtain the first-order parametrization:

$$\begin{cases} A_{1,1} \rightarrow -\partial_2 \phi^1 \\ A_{1,2} \rightarrow -\partial_2 \phi^2 \\ A_{2,1} \rightarrow \partial_1 \phi^1 \\ A_{2,2} \rightarrow \partial_1 \phi^2 \end{cases} \quad \begin{cases} B_1 \rightarrow \phi^1 - \partial_2 \phi^3 \\ B_2 \rightarrow \phi^2 + \partial_1 \phi^3 \end{cases}$$

$$\boxed{\begin{aligned} \sigma^{1,1} &= -\partial_2 \phi^1, \quad \sigma^{1,2} = -\partial_2 \phi^2, \quad \sigma^{2,1} = \partial_1 \phi^1, \quad \sigma^{2,2} = \partial_1 \phi^2, \\ \mu^{12,1} &= -\partial_2 \phi^3 + \phi^1, \quad \mu^{12,2} = \partial_1 \phi^3 + \phi^2 \end{aligned}}$$

a result that could not have been even imagined by the Cosserat brothers.

## 2. Conformal Group of Transformations

The idea is to notice that the brothers are *always* dealing with the same group of rigid motions because the lines, surfaces or media they consider are all supposed to be in the same 3-dimensional background/surrounding space which is acted on by the group of rigid motions, namely a group with 6 parameters (3 *translations* + 3 *rotations*). In 1909 it should have been strictly impossible for the two brothers to extend their approach to bigger groups, in particular to include the only additional *dilatation* of the Weyl group that will provide the virial theorem and, a

*fortiori*, the *elations* of the conformal group considered later on by H. Weyl [14]. In order to emphasize the reason for using Lie equations, we provide the explicit form of the highly nonlinear  $n$  finite elations and their infinitesimal counterpart in a conformal Riemannian space of dimension  $n$ :

$$\boxed{y = \frac{x - x^2 b}{1 - 2(bx) + b^2 x^2} \Rightarrow \theta_s = -\frac{1}{2} x^2 \delta_s^r \partial_r + \omega_{sr} x^t x^r \partial_t}$$

$$\Rightarrow \partial_r \theta_s^r = n \omega_{sr} x^t, \forall 1 \leq r, s, t \leq n$$

where the non-degenerate metric  $\omega$  is used for the scalar products  $x^2, bx, b^2$  involved.

Our purpose is to exhibit *directly* the Cauchy [15], Cosserat [16], Clausius [4], Maxwell [17], Weyl [14] (CCCMW) equations by computing with full details the adjoint of the first Spencer operator  $D_1 : \hat{R}_3 \rightarrow T^* \otimes \hat{R}_3$  for the conformal involutive finite type third order system  $\hat{R}_3 \subset J_3(T)$  for any dimension  $n \geq 1$ , in particular for  $n \geq 1$  along with the results obtained in [13]. In general, one has  $n$  translations (t) +  $n(n-1)/2$  rotations (r) + 1 dilatation (d) +  $n$  nonlinear elations (e), that is a total of  $(n+1)(n+2)/2$  parameters, thus 15 when  $n=4$ . As a by-product, the Cosserat couple-stress equations will be obtained for the Killing involutive finite type second order system  $\hat{R}_2 \subset J_2(T)$ . It must be noticed that not even a single comma must be changed when  $n=3$  when our results are compared to the original formulas provided by the bothers Cosserat in 1909. We only need recall the specific features of the standard first-order Spencer operator  $d : \hat{R}_3 \rightarrow T^* \otimes \hat{R}_2$  as follows by considering the multi-indices for the  $n$  translations, the  $n(n-1)/2$  rotations, the only dilatation and the  $n$  elations separately as follows:

$$\begin{aligned} & (\xi^k(x), \xi_i^k(x), \xi_{ij}^k(x), \xi_{ijr}^k(x) = 0) \\ & \rightarrow (\partial_i \xi^k(x) - \xi_i^k(x), \partial_i \xi_j^k(x) - \xi_{ij}^k(x), \partial_i \xi_r^r(x) - \xi_{ri}^r(x), \partial_r \xi_{ij}^k(x)) \end{aligned}$$

in the *duality summation*:

$$\boxed{\sigma_k^i (\partial_i \xi^k(x) - \xi_i^k(x)) + \mu_k^{ij} (\partial_i \xi_j^k(x) - \xi_{ij}^k(x)) + \nu^i (\partial_i \xi_r^r(x) - \xi_{ri}^r(x)) + \pi_k^{ij,r} (\partial_r \xi_{ij}^k(x))}$$

We have obtained a first simplification by noticing that the third order jets vanish, that is to say  $\xi_{rij}^k = 0$ . Indeed, starting with the Euclidean or Minkowski metric  $\omega$  with vanishing Christoffel symbols  $\gamma = 0$ , the second-order conformal equations can be provided in the parametric form:

$$\boxed{\xi_{ij}^k = \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij} \omega^{kr} A_r(x) \Leftrightarrow \xi_{ri}^r = n A_i(x)}$$

The result follows from the fact that this system is homogeneous and it is well known that  $\hat{g}_3 = 0, \forall n \geq 3$  as in [4] but also in any case as in [13].

A second simplification may be obtained by using the (constant) metric in order to raise or lower the indices in the implicit summations considered. In particular, we have successively:

$$\omega_{ij}\xi_i^r + \omega_{ir}\xi_j^r - \frac{2}{n}\omega_{ij}\xi_r^r = 0 \Leftrightarrow \omega_{ij}\xi_i^r + \omega_{ir}\xi_i^r = 2A(x)\omega_{ij}$$

$$\Rightarrow A(x) = \xi_1^1(x) = \xi_2^2(x) = \dots = \xi_n^n(x) = \frac{1}{n}\xi_r^r(x) \Rightarrow \xi_{i,j} + \xi_{j,i} = 0, \forall i \neq j$$

Now, we recall that we have chosen the notations in such a way that the system is formally integrable if and only if we have:

**LEMMA 2.1:** The sum of the Spencer operators  $\partial_i A(x) - A_i(x)$  and  $\partial_i A_j(x) - 0$  is a linear combination of the original Spencer operators  $\partial_i \xi_r^r(x) - \xi_{ri}^r(x)$  and  $\partial_i \xi_{rj}^r(x) - 0$  and conversely.

*Proof:* Substituting, we obtain successively:

$$\partial_i A(x) - A_i(x) = \frac{1}{n}(\partial_i \xi_r^r - \xi_{ri}^r)$$

$$\partial_i A_j(x) - 0 = \frac{1}{n}(\partial_i (\xi_{rj}^r - 0)) = \frac{1}{n}(\partial_i \xi_{rj}^r - \xi_{rij}^r)$$

□

In this situation, it follows that  $\sigma^{i,j}\xi_{i,j} = \sum_{i < j} (\sigma^{i,j} - \sigma^{j,i})\xi_{i,j} + \frac{1}{n}\sigma_r^r \xi_r^r$  and we may set  $\mu_k^{ij}\xi_{ij}^k = -\mu^i A_i$  where  $\mu^i$  is a linear (tricky) function of the  $\mu_k^{ij}$  with constant coefficients only depending on  $\omega$ . The new equivalent duality summation becomes:

$$\sigma^{i,r}\partial_r \xi_i + \sum_{i < j} (\mu^{ij,r}\partial_r \xi_{i,j} - (\sigma^{i,j} - \sigma^{j,i})\xi_{i,j}) - \sigma_r^r A(x) - \mu^i A_i(x) + \nu^i (\partial_i A(x) - A_i(x)) + \pi^{i,r} (\partial_r A_i(x))$$

and is important to notice that we may have  $\sigma^{i,j} \neq \sigma^{j,i}$  when  $i < j$ . Integrating by parts and changing the signs, we finally obtain the Poincaré Euler-Lagrange equations in the following form that allows to avoid using the structure constants of the conformal Lie algebra:

$$\boxed{\xi_i \rightarrow \partial_r \sigma^{i,r} = f^i \text{ (Cauchy stress equations)}}$$

$$\boxed{\xi_{i,j}, i < j \rightarrow \partial_r \mu^{ij,r} + \sigma^{ij} - \sigma^{ji} = m^{ij} \text{ (Cosserat couple-stress equations)}}$$

$$\boxed{A(x) \rightarrow \partial_r \nu^r + \sigma_r^r = u \text{ (Clausius equation)}}$$

$$\boxed{A_i(x) \rightarrow \partial_r \pi^{i,r} + \mu^i + \nu^i = v^i \text{ (Maxwell/Weyl equations)}}$$

Transforming these equations into pure divergence-like equations as we already did for the Poincaré equations by using now the isomorphism  $R_q \simeq \wedge^0 T^* \otimes \mathcal{G}$  is much more difficult.

First of all, using the last example of the introduction dealing with the Weyl group, we obtain when the second-order jets vanish:

$$\boxed{\partial_r \sigma^{i,r} = f^i}$$

$$\boxed{\partial_r (\mu^{ij,r} + x^j \sigma^{i,r} - x^i \sigma^{j,r}) = m^{ij} + x^j f^i - x^i f^j, \forall 1 \leq i < j \leq n}$$

$$\boxed{\partial_r (\nu^r + x^i \sigma_i^r) = u + x^i f_i}$$

$$\partial_r (\pi^{i,r} + x^i v^r + \dots) = v^i + x^i u + \dots$$

the two first equations, introduced by the Cosserat brothers in ([16], p. 137, 167), are *exactly* the ones used in continuum mechanics in order to study the equilibrium of forces and couples bringing the symmetry of the stress-tensor when  $\mu = 0$  and  $m = 0$ , where the left member is the Stokes formula applied to the total surface density of momentum while the right member is the total volume density of momentum (See [2] [18] for details).

Accordingly, the only problem left is to fill in the details for the Maxwell/Weyl equations, but this will be the most difficult part of this paper because it will depend on the relations!

**REMARK 2.2:** As noticed by the Cosserat brothers themselves, a major difference existing between the Cauchy and the Cosserat elasticity theories is that the *compatibility conditions* (CC) for the respective *fields* are described by a second-order operator for the first but by a first-order operator for the second. When  $n = 2$ , we have indeed for the Killing operator  $\mathcal{D}$  [2] [19]:

$$\begin{aligned} 2\partial_1 \xi^1 &= \Omega_{11}, 2\partial_2 \xi^2 = \Omega_{22}, \partial_1 \xi^2 + \partial_2 \xi^1 = \Omega_{12} \\ \Rightarrow \partial_{22} \Omega_{11} + \partial_{11} \Omega_{22} - 2\partial_{12} \Omega_{12} &= 0 \end{aligned}$$

and for the corresponding Spencer operator  $D_1$  [1] [2] [20]:

$$\begin{aligned} \partial_i \xi^k - \xi_i^k &= X_i^k, \partial_i \xi_j^k = X_{j,i}^k \\ \Rightarrow \partial_i X_j^k - \partial_j X_i^k + X_{i,j}^k - X_{j,i}^k &= 0, \partial_i X_{l,j}^k - \partial_j X_{l,i}^k = 0 \end{aligned}$$

Though we are dealing in both cases with the Lie pseudogroup  $\Gamma$  of rigid transformations:

$$\omega_{kl} (f(x)) \partial_i f^k(x) \partial_l f^l(x) = \omega_{ij}(x)$$

for the Euclidean metric  $\omega$ , we discover that *the differential sequence used is not the same*. This fact has never been acknowledged by the people working on Cosserat media because this result only depends on the *fundamental diagram*  $I$  ([1], p. 183) [2] [10] [18]. Multiplying the Riemann operator  $\Omega \rightarrow \partial_{22} \Omega_{11} + \partial_{11} \Omega_{22} - 2\partial_{12} \Omega_{12}$  by a test function  $\phi$  and integrating by parts while taking into account that  $\sigma^{ij} \Omega_{ij} = \sigma^{11} \Omega_{11} + \sigma^{22} \Omega_{22} + 2\sigma^{12} \Omega_{12}$ , we obtain the well-known second order Airy parametrization [5] [21] [22]:

$$\sigma^{11} = \partial_{22} \phi, \sigma^{22} = \partial_{11} \phi, \sigma^{12} = \sigma^{21} = -\partial_{12} \phi$$

The same comment can be done on the possibility to enlarge the Lie pseudogroup  $\tilde{\Gamma}$  of Weyl transformations to the Lie pseudogroup  $\hat{\Gamma}$  of conformal transformations in arbitrary dimension [13]. The first difficulty is that the “*Vessiot structure equations*” are superseding the “*Cartan structure equations*”, a fact never acknowledged by Cartan and all followers up to now [18] [23] and the second is that the Spencer sequence is also largely superseding the Cartan structure equations which are using exterior calculus on jet bundles of order  $q$  without any possibility to quotient them down on the base manifold  $X$  as we saw.

**EXAMPLE 2.3:** (*Projective group of the real line*): With  $n = 1$  and  $K = \mathbb{Q}$ , let

us consider the Lie pseudogroup defined by the third-order Schwarzian OD equation with standard jet notations:

$$y = \frac{ax+b}{cx+d} \Leftrightarrow \frac{y_{xxx}}{y_x} - \frac{3}{2} \left( \frac{y_{xx}}{y_x} \right)^2 = 0 \Rightarrow \xi_{xxx} = 0$$

A basis of infinitesimal generators is  $\left\{ \theta_1 = \partial_x, \theta_2 = x\partial_x, \theta_3 = \frac{1}{2}x^2\partial_x \right\}$  and we have the following diagram in which the columns of the  $3 \times 3$  matrix describe the components of  $j_2(\theta)$ :

$$\begin{pmatrix} \xi \\ \xi_x = A \\ \xi_{xx} = A_x \end{pmatrix} = \begin{pmatrix} 1 & x & \frac{1}{2}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}$$

In order to construct the adjoint of the first Spencer operator  $D_1 : \hat{R}_3 \rightarrow T^* \otimes \hat{R}_3$  when there is only one translation, no rotation but only one dilatation and only one elation, we have to consider the duality sum with  $\xi_{xxx} = 0$ :

$$\sigma(\partial_x \xi - \xi_x) + \nu(\partial_x \xi_x - \xi_{xx}) + \pi(\partial_x \xi_{xx} - 0)$$

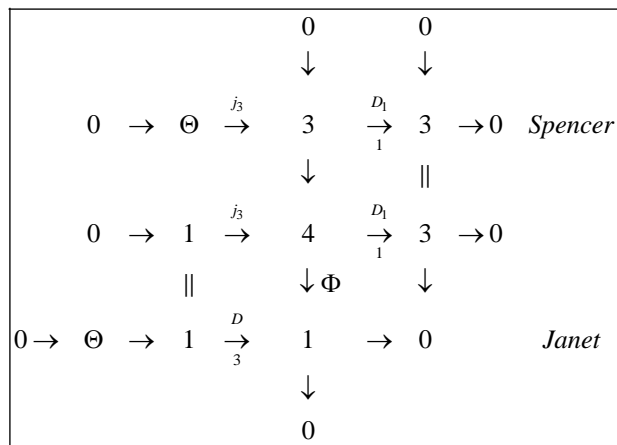
Integrating by parts and changing the sign, we get the board of first-order operators allowing to define  $ad(D_1)$ , namely the Cauchy/Cosserat stress equation, the Clausius virial equation and the Maxwell/Weyl equation successively:

$$\begin{cases} \xi & \rightarrow \partial_x \sigma = f \\ \xi_x & \rightarrow \partial_x \nu + \sigma = u \\ \xi_{xx} & \rightarrow \partial_x \pi + \nu = v \end{cases}$$

We may obtain therefore the pure divergence equations:

$$\begin{cases} \partial_x(\sigma) = f \\ \partial_x \pi + \nu = v \\ \partial_x \left( \pi + x\nu + \frac{1}{2}x^2\sigma \right) = \frac{1}{2}x^2 f + xu + v \end{cases}$$

Coming back now to the Janet sequence in the following *Fundamental Diagram* I of [13]:



we notice that the central row splits because  $j_3$  is an injective operator and the corresponding sequence of differential modules is the splitting sequence  $0 \rightarrow D^3 \rightarrow D^4 \rightarrow D \rightarrow 0$ . It follows that the adjoint sequence also splits in the following commutative and exact adjoint diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 & & & & ad(D_1) & & \\
 & 0 & \leftarrow & 3 & \xleftarrow{1} & 3 & \\
 & \uparrow & & \uparrow & & \parallel & \\
 0 & \leftarrow & 1 & \xleftarrow{ad(j_3)} & 4 & \xleftarrow{1} & 3 \leftarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 & & & & ad(D) & & \\
 0 & \leftarrow & 1 & \xleftarrow{3} & 1 & & 0 \\
 & \uparrow & & \uparrow & & & \\
 & 0 & & 0 & & & 
 \end{array}$$

In a more effective but quite surprising way, the kernel of  $ad(D_1)$  in the adjoint Spencer sequence is defined by the successive differential conditions:

$$(f = 0, u = 0, v = 0) \Rightarrow (\partial_x \sigma = 0, \partial_x v + \sigma = 0, \partial_x \pi + v = 0) \Rightarrow \boxed{\partial_{xxx} \pi = 0}$$

It is thus isomorphic to the kernel of  $ad(\mathcal{D})$  in the adjoint of the Janet sequence. But  $\mathcal{D}$  is a Lie operator in the sense that  $\mathcal{D}\xi = 0, \mathcal{D}\eta = 0 \Rightarrow \mathcal{D}[\xi, \eta] = 0$  or, equivalently,  $[\Theta, \Theta] \subset \Theta$ . It follows that the “stress” appearing in the Cauchy operator which is the adjoint of the Lie operator  $\mathcal{D}$  in the Janet sequence has *strictly nothing to do* with the “stress” appearing in the Cosserat couple-stress equations provided by the adjoint of  $D_1$  appearing in the Spencer sequence. This confusion, which is even worse than the ones we have described at length in the many recent references (See [1] [18] [23] [24] for details and letters on the controversy Cartan/Vessiot and [5] [21] [22] [25] for details on the controversy Beltrami/Einstein), leads to revisit the mathematical foundations of both continuum mechanics and general relativity.

It is important to point out that  $ad(j_3)$  is nothing else than the Euler-Lagrange equation in higher dimensional variational calculus (See chapter VIII part A of [2] and the last chapter of [26] for details on the variational sequence). More precisely, we have:

$$\begin{aligned}
 j_3 : (\xi = A, \partial_x \xi = B, \partial_{xx} \xi = C, \partial_{xxx} \xi = D) \\
 \Rightarrow D_1 : (\partial_x A - B = E, \partial_x B - C = F, \partial_x C - D = G)
 \end{aligned}$$

Multiplying by three test functions  $(\lambda^1, \lambda^2, \lambda^3)$  and integrating by parts while changing the sign as usual, we obtain the adjoint operator  $ad(D_1)$  with CC the adjoint operator  $ad(j_3)$ :

$$\begin{aligned}
 (\partial_{xxx} \mu^4 - \partial_{xx} \mu^3 + \partial_x \mu^2 - \mu^1 = \nu) \\
 \Leftarrow (\partial_x \lambda^1 = \mu^1, \partial_x \lambda^2 + \lambda^1 = \mu^2, \partial_x \lambda^3 + \lambda^2 = \mu^3, \lambda^3 = \mu^4)
 \end{aligned}$$

The central vertical short exact splitting sequence is described by:

$$\mu^4 \rightarrow (0, 0, 0, \mu^4), (\mu^1, \mu^2, \mu^3, \mu^4) \rightarrow (\mu^1, \mu^2, \mu^3) = (\sigma, \nu, \pi)$$

a result that could not be even imagined without these new methods.

**EXAMPLE 2.4:** (*Conformal Group*) When  $n = 2$ , the conformal group has 6 parameters and we should follow the same procedure after adding the two elations:

$$\theta_5 = \frac{1}{2} \left( (x^1)^2 - (x^2)^2 \right) \partial_1 + x^1 x^2 \partial_2, \theta_6 = x^1 x^2 \partial_1 + \frac{1}{2} \left( (x^2)^2 - (x^1)^2 \right) \partial_2$$

in such a way that  $\partial_r \theta_5^r = 2x^1, \partial_r \theta_6^r = 2x^2$  (See [27] for the relation with the Clausius virial theorem and [28] for the relation with the conformal group). The only difference is that we have now to deal with the 6 right members  $(f^1, f^2, u, v^1, v^2)$ . As we have been only using the Spencer bundles  $C_0$  and  $C_1$ , these results have strictly nothing to do with  $C_2$  involving 2-forms and the so-called Cartan curvature, *a result also proving that the mathematical foundations of Gauge theory must be totally revisited* as we have no longer any link with the unitary group  $U(1)$  [3].

With more details, when  $n = 2$ , the Lie equations are (See [13] for more details):

$$\begin{aligned} \xi_2^1 + \xi_1^2 &= 0, \xi_2^2 - \xi_1^1 = 0, \xi_{22}^1 - \xi_{11}^1 = 0, \xi_{22}^2 - \xi_{12}^1 = 0, \\ \xi_{12}^1 - \xi_{11}^2 &= 0, \xi_{12}^2 - \xi_{11}^1 = 0, \xi_{ijr}^k &= 0 \end{aligned}$$

and we may now add the two previous elations in order to obtain similarly the 6 parametric sections  $\xi_2(x)$  of the jet bundle  $\hat{R}_2 \subset J_2(T)$  from the arbitrary sections  $\lambda = \lambda(x)$ . We have the following diagram in which the columns of the  $6 \times 6$  matrix are the components of  $j_2(\theta)$ :

$$\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi_1^2 = -\xi_2^1 \\ \xi_1^1 = \xi_2^2 = A \\ \xi_{11}^1 = \xi_{12}^2 = -\xi_{22}^1 = A_1 \\ \xi_{12}^1 = \xi_{22}^2 = -\xi_{11}^2 = A_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x^2 & x^1 & \theta_5^1 & \theta_6^1 \\ 0 & 1 & x^1 & x^2 & \theta_5^2 & \theta_6^2 \\ 0 & 0 & 1 & 0 & x^2 & -x^1 \\ 0 & 0 & 0 & 1 & x^1 & x^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ \lambda^5 \\ \lambda^6 \end{pmatrix}$$

Using the fact that we have now  $\xi_{ijr}^k = 0$  because the system is homogeneous and we may assume that  $\hat{g}_3 = 0$  as in [13], we have the new duality summation:

$$\sigma_k^i \left( \partial_i \xi^k - \xi_i^k \right) + \mu^{12,r} \left( \partial_r \xi_1^2 - \xi_{1r}^2 \right) + \nu^r \left( \partial_r A - A_r \right) + \pi^{i,r} \partial_r A_i$$

Multiplying on the left by the row vector  $(f^1, f^2, m^{12}, u, v^1, v^2)$ , we obtain for the four columns on the left the same results that only involve  $(f^1, f^2, m^{12}, u)$  as in the preceding example and for the two columns on the right:

$$\theta_5^1 f^1 + \theta_5^2 f^2 + x^2 m^{12} + x^1 u + v^1, \theta_6^1 f^1 + \theta_6^2 f^2 - x^1 m^{12} + x^2 u + v^2$$

*a result which is totally new* because, though well known, *the explicit formulas for the infinitesimal generators of the elations have never been used in the literature up to our knowledge.*

Using the fact that  $\xi_{11}^2 = -\xi_{12}^1$  and  $\xi_{12}^2 = \xi_{11}^1$ , the integration by parts of the

summation brings the following *totally new terms* after changing sign:

$$A_1 = \xi_{11}^1 \rightarrow \partial_r \pi^{1,r} + \mu^{12,2} + \nu^1 = \nu^1, \quad A_2 = \xi_{12}^1 \rightarrow \partial_r \pi^{2,r} - \mu^{12,1} + \nu^2 = \nu^2$$

We obtain therefore for the divergence corresponding to the left term:

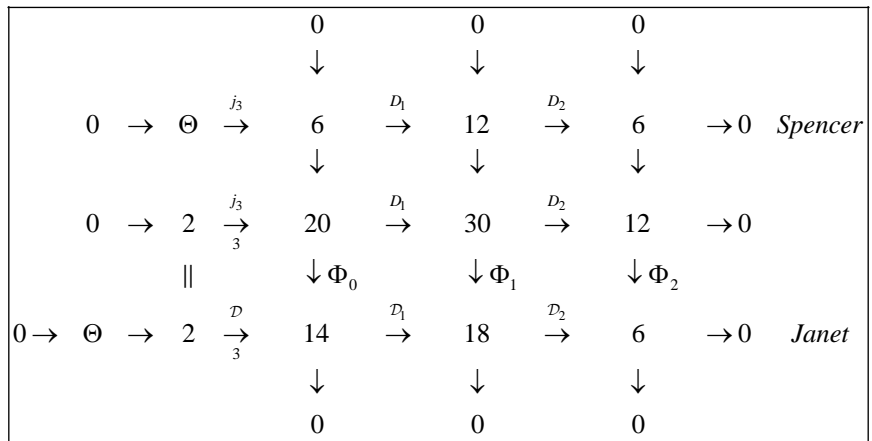
$$\begin{aligned} & \frac{1}{2} \left( (x^1)^2 - (x^2)^2 \right) f^1 + x^1 x^2 f^2 + x^2 m^{12} + x^1 u + \nu^1 \\ &= \frac{1}{2} \left( (x^1)^2 - (x^2)^2 \right) (\partial_r \sigma^{1,r}) + x^1 x^2 (\partial_r \sigma^{2,r}) \\ & \quad + x^2 (\partial_r \mu^{12,r} + \sigma^{2,1} - \sigma^{1,2}) + x^1 (\partial_r \nu^r + \sigma^{1,1} + \sigma^{2,2}) + \nu^1 \\ &= (\partial_r (\theta_5^1 \sigma^{1,r}) - x^1 \sigma^{1,1} + x^2 \sigma^{1,2}) + \partial_r (\theta_5^2 \sigma^{2,r}) - (x^2 \sigma^{2,1} + x^1 \sigma^{2,2}) \\ & \quad + (\partial_r (x^2 \mu^{12,r}) - \mu^{12,2} + x^2 \sigma^{2,1} - x^2 \sigma^{1,2}) \\ & \quad + (\partial_r (x^1 \nu^r) - \nu^1 + x^1 \sigma^{1,1} + x^1 \sigma^{2,2}) + \nu^1 \\ &= \partial_r (\theta_5^i \sigma_i^r + x^2 \mu^{12,r} + x^1 \nu^r + \pi^{1,r}) \end{aligned}$$

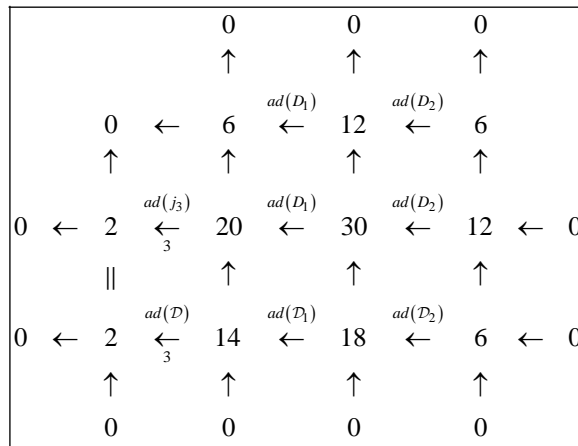
Similarly, we also obtain for the right term:

$$\begin{aligned} & x^1 x^2 f^1 + \frac{1}{2} \left( (x^2)^2 - (x^1)^2 \right) f^2 + x^1 m^{12} + x^2 u + \nu^2 \\ &= x^1 x^2 (\partial_r \sigma^{1,r}) + \frac{1}{2} \left( (x^2)^2 - (x^1)^2 \right) (\partial_r \sigma^{2,r}) \\ & \quad - x^1 (\partial_r \mu^{12,r} + \sigma^{2,1} - \sigma^{1,2}) + x^2 (\partial_r \nu^r + \sigma^{1,1} + \sigma^{2,2}) + \nu^2 \\ &= (\partial_r (\theta_6^1 \sigma^{1,r}) - x^2 \sigma^{1,1} - x^1 \sigma^{1,2}) + (\partial_r (\theta_6^2 \sigma^{2,r}) - x^2 \sigma^{2,2} + x^1 \sigma^{2,1}) \\ & \quad - \partial_r (x^1 \mu^{12,r}) + \mu^{12,1} - x^1 \sigma^{2,1} + x^1 \sigma^{1,2} \\ & \quad + (\partial_r (x^2 \nu^r) - \nu^2 + x^2 \sigma^{1,1} + x^2 \sigma^{2,2}) + \nu^2 \\ &= \partial_r (\theta_6^i \sigma_i^r - x^1 \mu^{12,r} + x^2 \nu^r + \pi^{2,r}) \end{aligned}$$

Once again, we notice that the Spencer operator has been used in a crucial way and that no other classical approach could have allowed to obtain these results.

We finally recall the fundamental diagram I and the corresponding adjoint diagram that has never been used in mathematical physics:



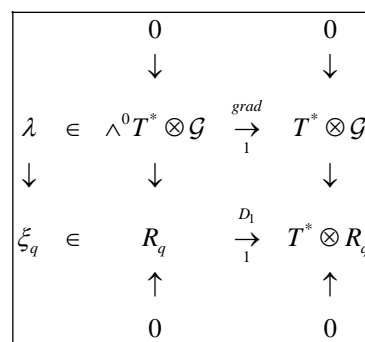


Three delicate snake chases are needed in order to prove that  $ad(\mathcal{D})$  is formally surjective, that the cohomology of the lower sequence at 18 is isomorphic to the kernel of  $ad(D_2)$  and finally that  $ad(\mathcal{D}_2)$  is formally injective, three facts that are not evident at first sight. As the second chase is not easy to work out, even for somebody familiar with homological algebra, we provide the details. For simplicity, starting with  $a \in 18$  means that  $a$  is a section of the central vector bundle of fiber dimension 18, that is locally a set of 18 functions. If such an  $a$  is killed by  $ad(\mathcal{D}_1)$ , we can get  $b \in 30$  killed by  $ad(D_1)$  because the lower central square is commutative. As the central sequence is known to be locally exact [1] [2], then  $b = ad(D_2)c$  for a certain  $c \in 12$ . Then we may project  $c$  to  $d \in 6$  which is killed by  $ad(D_2)$  by the commutativity of the upper right square. Now, if  $c \in 12$  should go to zero in 6, then  $c$  should come from  $e \in 6$  and, applying  $ad(D_2)$ , we should obtain a coboundary in 18. Finally,  $ad(\mathcal{D}_2)$  is injective because the central row of the diagram is a splitting sequence and  $ad(D_2)$  is injective. We have the long *ker/coker* exact sequence:

$$0 \rightarrow D^2 \rightarrow D^{20} \rightarrow D^{30} \rightarrow D^{12} \rightarrow 0$$

with the Euler-Poincaré formula:  $12 - 30 + 20 - 2 = 0$  in a coherent way.

In the case of an arbitrary dimension  $n \geq 1$ , we have the commutative diagrams:



and the corresponding adjoint diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \leftarrow & \wedge^n T^* \otimes \mathcal{G}^* & \xleftarrow[\downarrow]{\text{div}_1} & \wedge^{n-1} T^* \otimes \mathcal{G}^* & \ni & B \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \leftarrow & \wedge^n T^* \otimes R_q^* & \xleftarrow[\downarrow]{\text{ad}(D_1)_1} & \wedge^{n-1} T^* \otimes R_q^* & \ni & \mathcal{X}_q \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Accordingly, as “e” for “elation” is ranging from 1 to n, setting  $\omega_{e_s} x^s = x_e$ , we are sure that:

$$\partial_r (\theta_e^i \sigma_i^r + a_e(x) \mu + x_e \nu^r + \omega_{se} \pi^{s,r}) = \theta_e^i f_i + b_e(x) m + x_e u + \omega_{se} \nu^s$$

with  $a_e(x)$  and  $b_e(x)$  are (tricky) linear combinations of  $x$ . For example, we have successively:

$$\begin{aligned}
 \partial_r (\theta_e^i \sigma_i^r) &= \theta_e^i f_i + (\partial_r \theta_e^i) \sigma_i^r = \theta_e^i f_i + x_e \sigma_r^r + \dots \\
 \partial_r (x_e \nu^r + \omega_{se} \pi^{s,r}) &= (x_e u - x_e \sigma_r^r + \omega_{se} \nu^s) + \omega_{se} (\nu^s - \nu^s) + \dots \\
 &= x_e u - x_e \sigma_r^r + \omega_{se} \nu^s + \dots
 \end{aligned}$$

It follows that the trace of  $\sigma$  disappears. The explicit computation of the remaining terms should be awfully technical, a fact explaining why these results are not known, even for  $n = 2$ . The interest of using the Spencer sequence in the previous diagrams is to show out the interest of using “sections” of jet bundles instead of “solutions” but such an approach has never been done in mathematical physics. As a byproduct, we may quote:

**THEOREM 2.5:** The general Maxwell/Weyl equations that are depending on the second-order jets can be written as a pure divergence, totally independently of the fact that such a result is known in the framework discovered by Poincaré which is highly depending on the structure constants of the underlying Lie algebra as we already saw.

**REMARK 2.6:** (*Special relativity*): Though surprising it may look like at first sight, the above example with  $n = 2$  perfectly fits with the original presentation of Lorentz transformations if one uses the “hyperbolic” notations  $sh(\phi) = (e^\phi - e^{-\phi})/2$ ,  $ch(\phi) = (e^\phi + e^{-\phi})/2$ ,  $th(\phi) = sh(\phi)/ch(\phi)$  with  $ch^2(\phi) - sh^2(\phi) = 1$  instead of the classical  $\sin(\theta)$ ,  $\cos(\theta)$ ,  $\tan(\theta) = \sin(\theta)/\cos(\theta)$  with  $\cos^2(\theta) + \sin^2(\theta) = 1$ . Indeed, setting  $x^1 = x, x^2 = ct$  and using the well defined formula  $th(\phi) = u/c$  among dimensionless quantities, the Lorentz transformation can be written:

$$\begin{aligned}
 \bar{x}^1 &= \frac{x^1 - \frac{u}{c} x^2}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}, \quad \bar{x}^2 = \frac{-\frac{u}{c} x^1 + x^2}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} \\
 \Leftrightarrow \bar{x}^1 &= ch(\phi) x^1 - sh(\phi) x^2, \quad \bar{x}^2 = -sh(\phi) x^1 + ch(\phi) x^2
 \end{aligned}$$

Moreover, setting  $th(\psi) = v/c$ , we obtain easily for the composition of speeds:

$$th(\phi + \psi) = \frac{(th(\phi) + th(\psi))}{(1 + th(\phi)th(\psi))} \Leftrightarrow composition\left(\frac{u}{c}, \frac{v}{c}\right) = \frac{\left(\frac{u}{c} + \frac{v}{c}\right)}{\left(1 + \frac{u}{c} \frac{v}{c}\right)}$$

without the need of any “*gedanken experiment*” on light signals.

### 3. Electromagnetism and Gravitation

When  $n = 4$ , the comparison with the Maxwell equations of electromagnetism is easily obtained as follows. Indeed, writing a part of the dualizing summation in the form:

$$\begin{aligned} \mathcal{J}^i (\partial_i A - A_i) + \frac{1}{2} \mathcal{F}^{ij} (\partial_i A_j - \partial_j A_i) &= -\mathcal{J}^i A_i + \sum_{i \leq j} \mathcal{F}^{ij} (\partial_i A_j - \partial_j A_i) + \dots \\ &= -\mathcal{J}^1 A_1 + \dots + \mathcal{F}^{12} (\partial_1 A_2 - \partial_2 A_1) + \mathcal{F}^{13} (\partial_1 A_3 - \partial_3 A_1) + \mathcal{F}^{14} (\partial_1 A_4 - \partial_4 A_1) + \dots \\ &= -(\mathcal{J}^1 A_1 + \dots + (\mathcal{F}^{12} \partial_2 A_1 + \mathcal{F}^{13} \partial_3 A_1 + \mathcal{F}^{14} \partial_4 A_1) + \dots) \\ &= div(\dots) + (-\mathcal{J}^1 + \partial_2 \mathcal{F}^{12} + \partial_3 \mathcal{F}^{13} + \partial_4 \mathcal{F}^{14}) A_1 + \dots \end{aligned}$$

Integrating by parts and changing the sign as usual, we obtain as usual the second set of Maxwell equations for the induction  $\mathcal{F}$  :

$$\partial_r \mathcal{F}^{ir} - \mathcal{J}^i = 0 \Rightarrow \partial_i \mathcal{J}^i = 0$$

Such a result is coherent with the virial equation on the condition to have  $\sigma_r^r = 0$  in a coherent way with the classical Maxwell impulsion-energy stress tensor density:

$$\sigma_j^i = \mathcal{F}^{ir} F_{rj} + \frac{1}{4} \delta_j^i \mathcal{F}^{rs} F_{rs} \Rightarrow \sigma_r^r = 0$$

which is traceless with a divergence producing the Lorentz force (See [2], p. 444 and [8]). Hence, the mathematical foundations of EM entirely depend on the structure of the conformal group of space-time, a fact that can only be understood by using the Spencer operator as we saw.

As we have explained in the recent [22] [29], studying the mathematical structure of gravitation is much more delicate as it involves third-order jets. Our purpose at the end of this paper is to consider only the linearized framework. The crucial idea is to notice that the Poisson equation has only to do with the trace of the stress tensor density, contrary to the EM situation as we just saw.

Defining the vector bundle  $\hat{F}_0 = J_1(T)/\hat{R}_1 \simeq T^* \otimes \hat{g}_1$  when  $n \geq 4$ , another difficulty can be discovered in the following commutative and exact diagrams obtained by applying the Spencer  $\delta$ -map to the symbol sequence with

$$dim(\hat{g}_1) = dim(g_1) + 1 = (n(n-1)/2) + 1:$$

$$0 \rightarrow \hat{g}_1 \rightarrow T^* \otimes T \rightarrow \hat{F}_0 \rightarrow 0$$

then to its first prolongation with  $dim(\hat{g}_2) = n$  :

$$0 \rightarrow \hat{g}_2 \rightarrow S_2 T^* \otimes T \rightarrow T^* \otimes \hat{F}_0 \rightarrow 0$$

and finally to its second prolongation in which  $\hat{g}_3 = 0$ :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \hat{g}_3 & \rightarrow & S_3 T^* \otimes T & \rightarrow & S_2 T^* \otimes \hat{F}_0 \rightarrow \hat{F}_1 \rightarrow 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 0 & \rightarrow & T^* \otimes \hat{g}_2 & \rightarrow & T^* \otimes S_2 T^* \otimes T & \rightarrow & T^* \otimes T^* \otimes \hat{F}_0 \rightarrow 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 0 & \rightarrow & \wedge^2 T^* \otimes \hat{g}_1 & \rightarrow & \underline{\wedge^2 T^* \otimes T^* \otimes T} & \rightarrow & \wedge^2 T^* \otimes \hat{F}_0 \rightarrow 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \\
 0 & \rightarrow & \wedge^3 T^* \otimes T & = & \wedge^3 T^* \otimes T & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

A snake chase [30] allows to introduce the *Weyl* bundle  $\hat{F}_1$  defined by the short exact sequence:

$$0 \rightarrow T^* \otimes \hat{g}_2 \xrightarrow{\delta} Z_1^2(\hat{g}_1) \rightarrow \hat{F}_1 \rightarrow 0$$

in which the cocycle bundle  $Z_1^2(\hat{g}_1)$  is defined by the short exact sequence:

$$0 \rightarrow Z_1^2(\hat{g}_1) \rightarrow \wedge^2 T^* \otimes \hat{g}_1 \xrightarrow{\delta} \wedge^3 T^* \otimes T \rightarrow 0$$

We have of course  $\dim(\hat{F}_1) = 10$  when  $n = 4$  but more generally:

$$\begin{aligned}
 \dim(\hat{F}_1) &= (n(n+1)/2)(n(n+1)/2-1) - n^2(n+1)(n+2)/6 \\
 &= ((n(n-1)/2)(n(n-1)/2+1) - n^2(n-1)(n-2)/6) - n^2 \\
 &= n(n+1)(n+2)(n-3)/12
 \end{aligned}$$

In the purely Riemannian case, as  $g_2 = 0$ , we have  $F_1 \simeq Z_1^2(g_1)$  and thus:

$$\begin{aligned}
 \dim(F_1) &= (n(n+1)/2)(n(n+1)/2-1) - n^2(n+1)(n+2)/6 \\
 &= n^2((n-1)/2)^2 - n^2(n-1)(n-2)/6 \\
 &= n^2(n^2-1)/12
 \end{aligned}$$

a result leading to the unexpected formula  $\dim(F_1) - \dim(\hat{F}_1) = n(n+1)/2$ .

No classical method can produce such results which are summarized in the following crucial *Fundamental Diagram II* provided as early as in 1983 [18] but never acknowledged:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & Ricci \\
 & & & & & & \downarrow \\
 & & & & 0 & \rightarrow & Z_1^2(g_1) \rightarrow Riemann \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & \rightarrow & T^* \otimes \hat{g}_2 \xrightarrow{\delta} Z_1^2(\hat{g}_1) \rightarrow Weyl \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \rightarrow & S_2 T^* & \xrightarrow{\delta} & T^* \otimes T^* & \xrightarrow{\delta} & \wedge^2 T^* \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

**THEOREM 3.1:** This commutative and exact diagram splits and a diagonal snake chase proves that  $Ricci \approx S_2T^*$ . This FACT explains the confusion done by Einstein and Weyl in their tentatives to use the split lower sequence for linking gravitation with electromagnetism.

*Proof:* The monomorphism  $\delta : S_2T^* \rightarrow T^* \otimes T^*$  splits with  $\frac{1}{2}(A_{i,j} + A_{j,i}) \leftarrow A_{i,j}$  while the epimorphism  $\delta : T^* \otimes T^* \rightarrow \wedge^2 T^* : A_{i,j} \rightarrow A_{i,j} - A_{j,i}$  splits with  $\frac{1}{2}F_{ij} \leftarrow F_{ij} = -F_{ji}$ . We explain how the well-known result  $T^* \otimes T^* \approx S_2T^* \oplus \wedge^2 T^*$ , which is coming from the elementary formula  $n^2 = n(n+1)/2 + n(n-1)/2$  may be related to the Spencer  $\delta$ -cohomology interpretation of the Riemann and Weyl bundles. For this, we have to give details on the “snake” chase:

$$S_2T^* \rightarrow T^* \otimes T^* \rightarrow T^* \otimes \hat{g}_2 \rightarrow Z_1^2(\hat{g}_1) \rightarrow Z_1^2(g_1) \rightarrow Riemann \rightarrow Ricci$$

Starting with  $(A_{ij} = A_{i,j} = A_{j,i} = A_{ji}) \in S_2T^* \subset T^* \otimes T^*$ , we may define:

$$\begin{aligned} \zeta_{ri,j}^r &= nA_{i,j} = nA_{ij} = nA_{ji} = \zeta_{rj,i}^r \Rightarrow (\zeta_{lj,i}^k = \delta_l^k A_{j,i} + \delta_j^k A_{l,i} - \omega_{ij}^k \omega^{rs} A_{r,i}) \in T^* \otimes \hat{g}_2 \\ &\Rightarrow (R_{l,ij}^k = \zeta_{li,j}^k - \zeta_{lj,i}^k) \in Z_1^2(\hat{g}_1) \in \wedge^2 T^* \otimes \hat{g}_1 \in \wedge^2 T^* \otimes T^* \otimes T \\ &\Rightarrow R_{r,ij}^r = \zeta_{ri,j}^r - \zeta_{rj,i}^r = n(A_{i,j} - A_{j,i}) = 0 \Rightarrow (R_{l,ij}^k) \in Z_1^2(g_1) \\ &\Rightarrow nR_{l,ij}^k = (\delta_l^k \zeta_{ri,j}^r + \delta_i^k \zeta_{rl,j}^r - \omega_{li}^k \omega^{rs} \zeta_{rs,i}^r) - (\delta_l^k \zeta_{rj,i}^r + \delta_j^k \zeta_{rl,i}^r - \omega_{lj}^k \omega^{rs} \zeta_{rs,i}^r) \\ &\Rightarrow R_{l,ij}^k = (\delta_i^k A_{lj} - \delta_j^k A_{li}) - \omega^{ks} (\omega_{li} A_{sj} - \omega_{lj} A_{si}) \end{aligned}$$

Introducing  $tr(A) = \omega^{ij} A_{ij}$  and  $R_{ij} = R_{i,rj}^r = (nA_{ij} - A_{ij}) - (A_{ij} - \omega_{ij} tr(A))$ , we get:

$$R_{ij} = (n-2)A_{ij} + \omega_{ij} tr(A) = R_{ji} \Rightarrow tr(R) = \omega^{ij} R_{ij} = 2(n-1)tr(A)$$

Substituting, we finally obtain  $A_{ij} = \frac{1}{(n-2)}R_{ij} - \frac{1}{2(n-1)(n-2)}\omega_{ij}tr(R)$  and the tricky formula:

$$\begin{aligned} R_{l,ij}^k &= \frac{1}{(n-2)}(\delta_i^k R_{lj} - \delta_j^k R_{li} - \omega^{ks} (\omega_{li} R_{sj} - \omega_{lj} R_{si})) \\ &\quad - \frac{1}{(n-1)(n-2)}(\delta_i^k \omega_{lj} - \delta_j^k \omega_{li})tr(R) \end{aligned}$$

totally independently from the standard elimination of the derivatives of a conformal factor.

Contracting in  $k$  and  $i$ , we obtain indeed the lift:

$$Riemann = H_1^2(g_1) \rightarrow S_2T^* \approx Ricci : R_{l,ij}^k \rightarrow R_{i,rj}^r = R_{ij} = R_{ji}$$

in a coherent way. Using a standard result of homological algebra [26] [27] [30], we obtain therefore a splitting  $Weyl = H_1^2(\hat{g}_1) \rightarrow H_1^2(g_1) = Riemann :$

$$\begin{aligned} W_{l,ij}^k &= R_{l,ij}^k - \left( \frac{1}{(n-2)}(\delta_i^k R_{lj} - \delta_j^k R_{li} - \omega^{ks} (\omega_{li} R_{sj} - \omega_{lj} R_{si})) \right. \\ &\quad \left. - \frac{1}{(n-1)(n-2)}(\delta_i^k \omega_{lj} - \delta_j^k \omega_{li})tr(R) \right) \end{aligned}$$

in such a way that  $W_{i,rj}^r = 0$ , a result leading to the isomorphism  $Riemann \simeq Ricci \oplus Weyl$ .

□

Introducing the Christoffel symbols  $\gamma_{ij}^k$  of the metric  $\omega$ , we may define the (true) 1-form [2]:

$$\alpha_i = \chi_{r,i}^r + \gamma_{r,s}^s \chi_i^s = n(\partial_i a - A_i^r a_r) = n(\partial_i a - a_i) - n a_r g_s^r (\partial_i f^s - f_i^s)$$

because we have  $\gamma = 0$  in the conformal case and, taking the respective determinants:

$$\begin{aligned} \omega_{kl}(f(x)) f_i^k(x) f_j^l(x) &= e^{2a(x)} \omega_{ij}(x) \Rightarrow \det(f_j^l) = e^{na} \\ \Rightarrow g_i^r \partial_i f_r^l &= \frac{1}{\det(f_j^l)} \partial_i \det(f_j^l) = n \partial_i a \end{aligned}$$

It follows that:

$$\begin{cases} \Omega_{ij} \equiv (L(\hat{\xi}_1) \omega)_{ij} = 2A \omega_{ij} \\ \Gamma_{ij}^k \equiv (L(\hat{\xi}_2) \gamma)_{ij}^k = \delta_i^k A_j + \delta_j^k A_i - \omega_{ij} \omega^{kr} A_r \end{cases}$$

after linearization, in such a way that  $\partial_i a - a_i = 0 \Rightarrow \partial_i A - A_i = 0$  for a formally integrable system of defining finite Lie equations. It is important to notice that the geometric objects appearing in the Janet sequence, namely  $(\omega, \gamma) = j_1(\omega)$  according to the Levi-Civita isomorphism, are quite different from the  $\chi_q$  appearing in the Spencer sequence that has never been introduced in mathematical physics and has only been introduced for the first time in the study of Cosserat media [16] [20] [30].

We provide a short unusual motivating example in order to convince the reader that the construction of differential sequences is not an easy task indeed.

**EXAMPLE 3.2:** (Macaulay example)

With  $m = 1, n = 3, q = 2, K = \mathbb{Q}$ , let us consider the linear second-order system  $R_2 \subset J_2(E)$  with  $\dim(E) = 1$  provided by F. S. Macaulay in 1916 [24] [31] [32] while using jet notations:

$$y_{33} = 0, \quad y_{23} - y_{11} = 0, \quad y_{22} = 0$$

We let the reader check easily that  $\dim(g_2) = 3, \dim(g_3) = 1$  with only parametric jet  $y_{111}, g_4 = 0$  and thus  $\dim(R_2) = 8 = 2^3$ , a result leading to  $\dim(R_{3+r}) = 8$  that is  $R_{3+r} \simeq R_3, \forall r \geq 0$ . We recall the dimensions of the following jet bundles:

$$\begin{array}{rcccccccc} q & \rightarrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ S_q T^* & \rightarrow & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 \\ J_q(E) & \rightarrow & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 \end{array}$$

and the generic commutative and exact diagram allowing to construct the Spencer bundles  $C_r \subset C_r(E)$  and the Janet bundles  $F_r$  for  $r = 0, 1, \dots, n$  with  $F_0 = J_q(E)/R_q$  from the short exact sequence of vector bundles:

$$0 \rightarrow g_{q+1} \rightarrow S_{q+1} T^* \otimes E \rightarrow h_1 \rightarrow 0 \Rightarrow h_1 \subset T^* \otimes F_0$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \wedge^{r-1}T^* \otimes g_{q+1} & \xrightarrow{\delta} & \wedge^r T^* \otimes R_q & \rightarrow & C_r & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \wedge^{r-1}T^* \otimes S_{q+1}T^* \otimes E & \xrightarrow{\delta} & \wedge^r T^* \otimes J_q(E) & \rightarrow & C_r(E) & \rightarrow 0 \\
 \downarrow & & \downarrow & \searrow & \downarrow & & \\
 \wedge^{r-1}T^* \otimes h_1 & \xrightarrow{\delta} & \wedge^r T^* \otimes F_0 & \rightarrow & F_r & \rightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & & 
 \end{array}$$

showing that we have indeed:

$$\begin{aligned}
 C_r &= \wedge^r T^* \otimes R_q / \delta(\wedge^{r-1} T^* \otimes g_{q+1}) \\
 C_r(E) &= \wedge^r T^* \otimes J_q(E) / \delta(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E) \\
 F_r &= \wedge^r T^* \otimes J_q(E) / (\wedge^r T^* \otimes R_q + \delta(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E))
 \end{aligned}$$

When  $R_q \subset J_q(E)$  is involutive, that is formally integrable (FI) with an involutive symbol  $g_q$ , then these three differential sequences are formally exact on the jet level and, in the Spencer sequence:

$$0 \rightarrow \Theta \xrightarrow{j_q} C_0 \xrightarrow{D_1} C_1 \xrightarrow{D_2} \dots \xrightarrow{D_n} C_n \rightarrow 0$$

the first-order involutive operators  $D_1, D_2, \dots, D_n$  are induced by the standard Spencer operator  $d: R_{q+1} \rightarrow T^* \otimes R_q$  already defined that can be extended to  $d: \wedge^r T^* \otimes R_{q+1} \rightarrow \wedge^{r+1} T^* \otimes R_q$ . A similar condition is also valid for the Janet sequence:

$$0 \rightarrow \Theta \rightarrow E \xrightarrow{D} F_0 \xrightarrow{D_1} F_1 \xrightarrow{D_2} \dots \xrightarrow{D_n} F_n \rightarrow 0$$

which can be thus constructed “as a whole” from the previous extension of the Spencer operator (See [1] p. 183 + 185 + 391 for the main diagrams and [24] for other explicit computations on the Macaulay example). However, such a result is still not known and not even acknowledged today in mathematical physics, particularly in general relativity which is *never* using the Spencer  $\delta$ -cohomology in order to define the Riemann or Bianchi operators [25] [33] [34]. The study of the present Macaulay example will be sufficient in order to justify our comment.

First of all, as  $g_2$  is *not* 2-acyclic and the coefficients are constant, the CC are of order two as follows:

$$\begin{aligned}
 Qw - Rv = 0, Ru - Pw = 0, Pv - Qu = 0 \\
 \Rightarrow P(Qw - Rv) + Q(Ru - Pw) + R(Pv - Qu) \equiv 0
 \end{aligned}$$

The simplest formally exact resolution, *which is quite far from being a Janet sequence*, is thus:

$$0 \rightarrow \Theta \rightarrow 1 \xrightarrow{\frac{D}{2}} 3 \xrightarrow{\frac{D_1}{2}} 3 \xrightarrow{\frac{D_2}{2}} 1 \rightarrow 0$$

Secondly, as the first prolongation of  $R_2$  becoming involutive is  $R_4$ , an idea could be to start with the system  $R_3 \subset J_3(E)$  but we have proved in [24] that the simplest formally exact sequence that could be obtained, *which is also quite far from being a Janet sequence*, is:

$$0 \rightarrow \Theta \rightarrow 1 \xrightarrow{3} 12 \xrightarrow{1} 21 \xrightarrow{2} 46 \xrightarrow{1} 72 \xrightarrow{1} 48 \xrightarrow{1} 12 \rightarrow 0$$

Indeed, the Euler-Poincaré characteristic is  $1 - 12 + 21 - 46 + 72 - 48 + 12 = 0$  but we notice that the orders of the successive operators may vary up and down.

The reader discovers that the *Fundamental Diagram I* relating the upper Spencer sequence to the lower Janet sequence is (See [4], pp. 19-22 for details):

			0	0	0	0								
			↓	↓	↓	↓								
0	→	Θ	$\xrightarrow[4]{j_4}$	8	$\xrightarrow[1]{D_1}$	24	$\xrightarrow[1]{D_2}$	24	$\xrightarrow[1]{D_3}$	8	→	0		
			↓	↓	↓	↓	↓	↓	↓					
0	→	1	$\xrightarrow[4]{j_4}$	35	$\xrightarrow[1]{D_1}$	84	$\xrightarrow[1]{D_2}$	70	$\xrightarrow[1]{D_3}$	20	→	0		
				↓	↓	↓	↓	↓	↓					
0	→	Θ	→	1	$\xrightarrow[4]{D}$	27	$\xrightarrow[1]{D_1}$	60	$\xrightarrow[1]{D_2}$	46	$\xrightarrow[1]{D_3}$	12	→	0
					↓	↓	↓	↓	↓					
					0	0	0	0	0					

In the present example, the Spencer bundles are  $C_r = \wedge^r T^* \otimes R_4$  and their dimensions are quite lower than the dimensions of the Janet bundles. Among the long exact sequences that *must* be used the following involves a  $540 \times 600$  matrix and we wish good luck to anybody using computer algebra:

$$0 \rightarrow R_7 \rightarrow J_7(1) \rightarrow J_3(27) \rightarrow J_2(60) \rightarrow J_1(46) \rightarrow F_3 \rightarrow 0$$

$$\Rightarrow 8 - 120 + 540 - 600 + 184 = 12 = \dim(F_3)$$

We are now ready to apply the previous diagrams by proving the following crucial theorem:

**THEOREM 3.3:** When  $n = 4$ , the linear Spencer sequence for the Lie algebra  $\hat{\Theta}$  of infinitesimal conformal group of transformations projects onto a part of the Poincaré sequence for the exterior derivative *with a shift by one step* according to the following commutative and locally exact diagram:

0	→	$\hat{\Theta}$	$\xrightarrow{j_2}$	$\hat{R}_2$	$\xrightarrow{D_1}$	$T^* \otimes \hat{R}_2$	$\xrightarrow{D_2}$	$\wedge^2 T^* \otimes \hat{R}_2$
			↓	↙	↓	↓	↓	↓
				$T^*$	$\xrightarrow{d}$	$\wedge^2 T^*$	$\xrightarrow{d}$	$\wedge^3 T^*$
				A		$dA = F$		$dF = 0$

*This purely mathematical result also contradicts classical gauge theory* because it proves that EM only depends on the structure of the conformal group of space-time but not on  $U(1)$ .

*Proof:* Considering  $\omega$  and  $\gamma$  as geometric objects, we obtain at once the formulas:

$$\bar{\omega}_{ij} = e^{2a(x)} \omega_{ij} \Rightarrow \bar{\gamma}_{ri}^r = \gamma_{ri}^r + \partial_i a$$

With more details, on the infinitesimal level, we have successively:

$$L(\xi_1)\omega = 2A\omega \Rightarrow (\xi_r^r + \gamma_{ri}^r \xi^i) = nA, \forall \xi_1 \in \hat{R}_1$$

$$(L(\xi_2)\gamma)_{ij}^k = \delta^i A_j + \delta_j^k A_i - \omega_{ij} \omega^{kr} A_r \Rightarrow \xi_{ri}^r + \gamma_{rs}^r \xi^s + \xi^s \partial_s \gamma_{ri}^r = nA_i, \forall \xi_2 \in \hat{R}_2$$

We obtain therefore an isomorphism  $J_1(\wedge^0 T^*) = \wedge^0 T^* \times_X T^*$ , a result leading to the following commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & R_2 & \rightarrow & \hat{R}_2 & \rightarrow & J_1(\wedge^0 T^*) & \rightarrow 0 \\ & \downarrow d & & \downarrow d & & \downarrow d & \\ 0 \rightarrow & T^* \otimes R_1 & \rightarrow & T^* \otimes \hat{R}_1 & \rightarrow & T^* & \rightarrow 0 \end{array}$$

where the rows are exact by counting the dimensions. The operator  $d : (A, A_i) \rightarrow (\partial_i A - A_i)$  on the right is induced by the central Spencer operator, a result that could not have been imagined by Weyl and followers. *It provides a good transition towards the conformal origin of electromagnetism.*

We now restrict our study to the linear framework and introduce a new system  $\tilde{R}_1 \subset J_1(T)$  of infinitesimal Lie equations defined by  $L(\xi_1)\omega = 2A\omega$  with prolongation defined by  $L(\xi_2)\gamma = 0$  in such a way that  $R_1 \subset \tilde{R}_1 = \hat{R}_1$  with a strict inclusion and the strict inclusions  $R_2 \subset \tilde{R}_2 \subset \hat{R}_2$ .

**LEMMA 3.4:** One has an isomorphism  $\hat{R}_2 / \tilde{R}_2 \cong \hat{g}_2$ .

*Proof:* From the definitions, we obtain the following commutative and exact diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & 0 & \rightarrow & \hat{g}_2 \\ & & & & \downarrow & \searrow & \\ 0 & \rightarrow & \tilde{R}_2 & \rightarrow & \hat{R}_2 & \rightarrow & \hat{R}_2 / \tilde{R}_2 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \tilde{R}_1 & = & \hat{R}_1 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

The south-east arrow is an isomorphism as it is both a monomorphism and an epimorphism by using a snake chase showing that  $\hat{R}_2 = \tilde{R}_2 \oplus \hat{g}_2$ .

□

A first problem to solve is to construct vector bundles from the components of the image of  $D_1$ . Using the corresponding capital letter for denoting the linearization, let us introduce:

$$\begin{aligned} \partial_i \xi_\mu^k - \xi_{\mu+1_i}^k &= X_{\mu,i}^k \Rightarrow B_{\mu,i}^k \text{ (tensors)} \\ (B_{l,i}^k = X_{l,i}^k + \gamma_{ls}^k X_{,i}^s) &\in T^* \otimes T^* \otimes T \Rightarrow (B_{r,i}^r = B_i) \in T^* \end{aligned}$$

$$\begin{aligned} (B_{ij,i}^k &= X_{lj,i}^k + \gamma_{sj}^k X_{l,i}^s + \gamma_{ls}^k X_{j,i}^s - \gamma_{lj}^s X_{s,i}^k + X_{,i}^r \partial_r \gamma_{lj}^k) \in T^* \otimes S_2 T^* \otimes T \\ \Rightarrow (B_{ri,j}^r - B_{rj,i}^r) &\in \wedge^2 T^* \end{aligned}$$

We obtain from the relations  $\partial_i \gamma_{rj}^r = \partial_j \gamma_{ri}^r$  and the previous results:

$$\begin{aligned} F_{ij} &= B_{ri,j}^r - B_{rj,i}^r \\ &= X_{ri,j}^r - X_{rj,i}^r + \gamma_{rs}^r X_{i,j}^s - \gamma_{rs}^r X_{j,i}^s + X_{,j}^r \partial_r \gamma_{si}^s - X_{,i}^r \partial_r \gamma_{sj}^s \\ &= \partial_i X_{r,j}^r - \partial_j X_{r,i}^r + \gamma_{rs}^r (X_{i,j}^s - X_{j,i}^s) + X_{,j}^r \partial_i \gamma_{sr}^s - X_{,i}^r \partial_j \gamma_{sr}^s \\ &= \partial_i (X_{r,j}^r + \gamma_{rs}^r X_{,j}^s) - \partial_j (X_{r,i}^r + \gamma_{rs}^r X_{,i}^s) \\ &= \partial_i B_j - \partial_j B_i \end{aligned}$$

Now, using the contracted formula  $\xi_{ri}^r + \gamma_{rs}^r \xi_i^s + \xi^s \partial_s \gamma_{ri}^r = nA_i$ , we obtain:

$$\begin{aligned} B_i &= (\partial_i \xi_r^r - \xi_{ri}^r) + \gamma_{rs}^r (\partial_i \xi^s - \xi_i^s) \\ &= \partial_i \xi_r^r + \gamma_{rs}^r \partial_i \xi^s + \xi^s \partial_s \gamma_{ri}^r - nA_i \\ &= \partial_i (\xi_r^r + \gamma_{rs}^r \xi^s) - nA_i \\ &= n(\partial_i A - A_i) \end{aligned}$$

and we finally get  $F_{ij} = n(\partial_j A_i - \partial_i A_j)$  which is no longer depending on  $A$ , a result fully solving the dream of Weyl. Of course, when  $n=4$  and  $\omega$  is the Minkowski metric, then we have  $\gamma=0$  in actual practice and the previous formulas become particularly simple.

It follows that  $dB = F \Leftrightarrow -ndA = F$  in  $\wedge^2 T^*$  and thus  $dF = 0$ , that is  $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ , has an intrinsic meaning in  $\wedge^3 T^*$ . It is finally important to notice that the usual EM Lagrangian is defined on sections of  $\hat{C}_1$  killed by  $D_2$  but *not* on  $\hat{C}_2$ . Finally, the south-west arrow in the left square is the composition:

$$\xi_2 \in \hat{R}_2 \xrightarrow{D_1} X_2 \in T^* \otimes \hat{R}_2 \xrightarrow{\pi_1^2} X_1 \in T^* \otimes \hat{R}_1 \xrightarrow{(\gamma)} (B_i) \in T^* \Leftrightarrow \xi_2 \in \hat{R}_2 \rightarrow (nA_i) \in T^*$$

More generally, using the Lemma, we have the composition of epimorphisms:

$$\hat{C}_r \rightarrow \hat{C}_r / \tilde{C}_r = \wedge^r T^* \otimes (\hat{R}_2 / \tilde{R}_2) = \wedge^r T^* \otimes \hat{g}_2 = \wedge^r T^* \otimes T^* \xrightarrow{\delta} \wedge^{r+1} T^*$$

Accordingly, though  $A$  and  $B$  are potentials for  $F$ , then  $B$  can also be considered as a part of the *field* but the important fact is that the first set of (*linear*) Maxwell equations  $dF = 0$  is induced by the (*linear*) operator  $D_2$  because we are only dealing with involutive and thus formally integrable operators, a *fact* justifying the commutativity of the square on the left of the diagram.

□

**REMARK 3.5:** Similarly to Remark 2.6, we have now the identifications for  $k=1,2,3$ :

$\text{gauging} \left\{ \begin{array}{ll} \text{speed} & \leftrightarrow \text{rotations} \leftrightarrow \partial_4 \xi^k - \xi_4^k \\ \text{acceleration} & \leftrightarrow \text{elations} \leftrightarrow \partial_4 \xi_4^k - \xi_{44}^k \end{array} \right\} \text{.Spencer operator}$
--

We finally just sketch the nonlinear framework and a few among its consequences [2] [10] [18].

Taking the determinant of each term of the non-linear second order PD

equations defining  $\hat{\Gamma}$  while using jet notations, we obtain successively:

$$\omega_{kl}(y)y_i^k y_j^l = e^{2a(x)}\omega_{ij}(x) \Rightarrow \det(\omega)\left(\det(f_i^k)\right)^2 = e^{2na(x)}\det(\omega) \Rightarrow \det(y_i^k) = e^{na(x)}$$

in such a way that we may define  $a(x) = -b(f(x))$  over the target when  $\Delta(x) = \det(\partial_i f^k(x)) \neq 0$  while caring only about the connected component  $0 \rightarrow 1 \rightarrow \infty$  of the dilatation group. The problem is thus to pass from a (metric) tensor to a (metric) tensor density and to consider successively the two non-linear systems of finite defining Lie equations:

$$\omega_{kl}(y)y_i^k y_j^l = \omega_{ij}(x) \rightarrow \hat{\omega}_{kl}y_i^k y_j^l \left(\det(y_i^k)\right)^{-\frac{2}{n}} = \hat{\omega}_{ij}(x)$$

Now, with  $\gamma = 0$  we have  $\chi_{r,i}^s = g_k^s(\partial_i f_s^k - A_i^r f_{rs}^k)$  and:

$$g_k^s \partial_i f_s^k = \left(1/\det(f_i^k)\right) \partial_i \det(f_i^k) = n \partial_i a, \quad g_k^s f_{rs}^k = n a_r(x)$$

Finally, we have the jet compositions and contractions:

$$g_k^r f_i^k = \delta_i^r \Rightarrow g_k^r f_{ij}^k = -g_{kl}^r f_i^k f_j^l \\ \Rightarrow n a_i(x) = g_k^s f_{is}^k = -f_i^k f_r^l g_{kl}^r = -n f_i^k(x) b_k(f(x))$$

It follows that  $\alpha_i = n(\partial_i a(x) - A_i^r a_r(x))$  but we may also set  $a_i = -f_i^k b_k$  in order to obtain simply  $\alpha_i = -n\left(\frac{\partial b}{\partial y^k} - b_k\right) \partial_i f^k$  as a way to pass from source to target (Compare to [35]).

When  $X$  is a manifold of dimension  $n$  and  $Y$  is just a copy of  $X$ , let us consider the  $q$ -jet bundle  $\Pi_q = \Pi_q(X, Y)$  of invertible jets of order  $q$  with local coordinates  $(x, y_q) = (x^i, y^k, y_i^k, y_{ij}^k, \dots)$  such that  $\det(y_i^k) \neq 0$ . There is a canonical embedding  $\Pi_{q+1} \subset J_1(\Pi_q)$  defined by  $y_{\mu,i}^k = y_{\mu+1,i}^k$ . If  $\mathcal{R}_q \subset \Pi_q$  is a Lie groupoid of order  $q$  with first prolongation  $\mathcal{R}_{q+1} = J_1(\mathcal{R}_q) \cap \Pi_{q+1} \subset J_1(\Pi_q)$  is also a Lie groupoid and we may define the nonlinear Spencer operator:

$$\boxed{\bar{D}: \mathcal{R}_{q+1} \rightarrow T^* \otimes \mathcal{R}_q : f_{q+1} \rightarrow f_{q+1}^{-1} \circ J_1(f_q) - id_{q+1} = \chi_q}$$

Indeed,  $f_{q+1}^{-1} \circ J_1(f_q)$  is a well defined section of  $J_1(\mathcal{R}_q)$  over the section  $f_q^{-1} \circ f_q = id_q$  of  $\mathcal{R}_q$ , exactly like  $id_{q+1}$ , and  $id_q^{-1}(V(\mathcal{R}_q)) = \mathcal{R}_q \subset J_q(T = id_q^{-1}(V(\Pi_q)))$ . With a slight abuse of language we may say that  $f_{q+1}(\chi_q) = j_1(f_q) - f_{q+1}$  and we recall the inductive formula:

$$\boxed{f_r^k \chi_{\mu,i}^r + \dots + f_{\mu+1,r}^k \chi_{r,i}^r = \partial_i f_\mu^k - f_{\mu+1,i}^k}$$

We obtain in particular the following linear combinations of the Spencer operator:

$$\begin{cases} \chi_{r,i}^k = g_l^k \partial_i f^l - \delta_i^k = A_i^k - \delta_i^k = g_l^k (\partial_i f^r - f_i^l) \\ \chi_{j,i}^k = g_l^k (\partial_i f_j^l - A_i^r f_{rj}^l) = g_l^k (\partial_i f_j^l - f_{ij}^l) - g_l^k g_u^r f_{rj}^l (\partial_i f^u - f_i^u) \end{cases}$$

and the useful compatibility conditions [29]:

$$\partial_i A_j^k - \partial_j A_i^k - A_i^r \chi_{r,j}^k + A_j^r \chi_{r,i}^k = 0 \\ \partial_i \chi_{l,j}^k - \partial_j \chi_{l,i}^k - \chi_{l,i}^r \chi_{r,j}^k + \chi_{l,j}^r \chi_{r,i}^k - A_i^r \chi_{lr,j}^k + A_j^r \chi_{lr,i}^k = 0$$

**PROPOSITION 3.6:** When  $\det(A) \neq 0$ , that is when  $\Delta(x) = \det(\partial_i f^k(x)) \neq 0$  because we have  $\det(f_i^k(x)) \neq 0$  by assumption, there is a *nonlinear Spencer sequence* stabilized at order  $q$ :

$$0 \rightarrow \text{aut}(X) \xrightarrow{j_q} \Pi_q \xrightarrow{\bar{D}_1} C_1(T) \xrightarrow{\bar{D}_2} C_2(T)$$

where  $\bar{D}_1$  and  $\bar{D}_2$  are involutive and a *restricted second nonlinear Spencer sequence*:

$$0 \rightarrow \Gamma \xrightarrow{j_q} \mathcal{R}_q \xrightarrow{\bar{D}_1} C_1 \xrightarrow{\bar{D}_2} C_2$$

such that  $\bar{D}_1$  and  $\bar{D}_2$  are involutive whenever  $\mathcal{R}_q$  is involutive. In the case of Lie groups of transformations, the symbol of the involutive system  $R_q$  must be  $g_q = 0$  providing an isomorphism  $\mathcal{R}_{q+1} \simeq \mathcal{R}_q \Rightarrow R_{q+1} \simeq R_q$  and we have therefore  $C_r = \wedge^r T^* \otimes R_q$  for  $r = 1, \dots, n$ .

It remains to graft a variational procedure adapted to the previous results. Contrary to what happens in analytical mechanics or elasticity, for example, *the main idea is to vary sections but not points*. Hence, we may introduce the variation  $\delta f^k(x) = \eta^k(f(x))$  over the target  $y = f(x)$  and set  $\eta^k(f(x)) = \xi^i(x) \partial_i f^k(x) = f_i^k(x) \bar{\xi}^i(x)$  in order to bring it back over the source  $x$ .

As a major result first discovered in specific cases by the brothers Cosserat in 1909 [16] and by Weyl in 1918 [14], we shall prove and apply the following key result:

Such a procedure will only depend on the linear spencer operator and its formal adjoint.

In order to prove this result, if  $f_{q+1}, g_{q+1}, h_{q+1} \in \Pi_{q+1}$  can be composed in such a way that  $f'_{q+1} = g_{q+1} \circ f_{q+1} = f_{q+1} \circ h_{q+1}$ , we get:

$$\begin{aligned} \bar{D}'f'_{q+1} &= f_{q+1}^{-1} \circ g_{q+1}^{-1} \circ j_1(g_q) \circ j_1(f_q) - id_{q+1} = f_{q+1}^{-1} \circ \bar{D}g_{q+1} \circ j_1(f_q) + \bar{D}f_{q+1} \\ &= h_{q+1}^{-1} \circ f_{q+1}^{-1} \circ j_1(f_q) \circ j_1(h_q) - id_{q+1} = h_{q+1}^{-1} \circ \bar{D}f_{q+1} \circ j_1(h_q) + \bar{D}h_{q+1} \end{aligned}$$

Using the local exactness of the first nonlinear Spencer sequence or ([4], p. 219), we may state:

**LEMMA 3.7:** For any section  $f_{q+1} \in \mathcal{R}_{q+1}$ , the *finite gauge transformation*:

$$\chi_q \in T^* \otimes R_q \rightarrow \chi'_q = f_{q+1}^{-1} \circ \chi_q \circ j_1(f_q) + \bar{D}f_{q+1} \in T^* \otimes R_q$$

exchanges the solutions of the *field equations*  $\bar{D}'\chi_q = 0$  when  $f_{q+1}$  acts on  $R_q$  and  $j_1(f)$  on  $T^*$ .

We may introduce the *formal Lie derivative* on  $J_q(T)$  by linearity through the successive formulas:

$$\begin{aligned} \{j_{q+1}(\xi), j_{q+1}(\eta)\} &= j_q([\xi, \eta]) \\ L(\xi_{q+1})\eta_q &= \{\xi_{q+1}, \eta_{q+1}\} + i(\xi)d\eta_{q+1} = [\xi_q, \eta_q] + i(\eta)d\xi_{q+1} \\ L(j_1(\xi_{q+1}))\chi_q(\zeta) &= L(\xi_{q+1})(\chi_q(\zeta)) - \chi_q([\xi, \zeta]) \end{aligned}$$

**LEMMA 3.8:** Passing to the limit *over the source* with  $\chi_q = \bar{D}f_{q+1}$  and

$h_{q+1} = id_{q+1} + t\xi_{q+1} + \dots$  for  $t \rightarrow 0$ , we get an *infinitesimal gauge transformation* leading to the *infinitesimal variation*:

$$\begin{aligned} \delta\chi_q &= d\xi_{q+1} + L(j_1(\xi_{q+1}))\chi_q \\ \delta\chi_q(\zeta) &= i(\zeta)d\xi_{q+1} + \{\xi_{q+1}, \chi_{q+1}(\zeta)\} + i(\xi)d\chi_{q+1}(\zeta) - \chi_q([\xi, \zeta]) \\ &= i(\zeta)d\xi_{q+1} + L(\xi_{q+1})(\chi_q(\zeta)) - \chi_q([\xi, \zeta]) \end{aligned} \tag{4}$$

which *only depends on*  $\chi_q$  but *does not depend on* the parametrization of  $\chi_q$  by  $f_{q+1}$ .

**LEMMA 3.9:** Passing to the limit *over the target* with  $\chi_q = \bar{D}f_{q+1}$  and  $g_{q+1} = id_{q+1} + t\eta_{q+1} + \dots$  for  $t \rightarrow 0$  over the target  $Y$ , we get the other *infinitesimal variation*:

$$\delta\chi_q = f_{q+1}^{-1} \circ d_Y \eta_{q+1} \circ j_1(f_q) \tag{5}$$

which *highly depends on the parametrization of*  $\chi_q$ .

**EXAMPLE 3.10:** We obtain for  $q = 1$ :

$$\begin{cases} \delta\chi_{.i}^k = (\partial_i \xi^k - \xi_i^k) + (\xi^r \partial_r \chi_{.i}^k + \chi_{.i}^k \partial_i \xi^r - \chi_{.i}^r \xi_r^k) \\ \delta\chi_{j.i}^k = (\partial_i \xi_j^k - \xi_{ij}^k) + (\xi^r \partial_r \chi_{j.i}^k + \chi_{j,r}^k \partial_i \xi^r + \chi_{r,i}^k \xi_j^r - \chi_{r,i}^r \xi_j^k - \chi_{.i}^r \xi_{jr}^k) \end{cases}$$

Introducing the inverse matrix  $B = A^{-1}$ , we obtain therefore equivalently:

$$\begin{aligned} \delta A_i^k &= \xi^r \partial_r A_i^k + A_r^k \partial_i \xi^r - A_i^r \xi_r^k \Leftrightarrow \delta B_i^k = \xi^r \partial_r B_i^k - B_k^r \partial_r \xi^i + B_i^r \xi_r^k \\ \delta\chi_{j.i}^k &= (\partial_i \xi_j^k - A_i^r \xi_{jr}^k) + (\xi^r \partial_r \chi_{j.i}^k + \chi_{j,r}^k \partial_i \xi^r + \chi_{r,i}^k \xi_j^r - \chi_{r,i}^r \xi_j^k) \end{aligned}$$

The explicit general formulas of the three previous lemmas cannot be found somewhere else (The reader may compare them to the ones obtained in [12] by means of the so-called “diagonal” method that cannot be applied to the study of explicit examples). The following unusual difficult proposition generalizes well known variational techniques used in continuum mechanics and will be crucially used for applications like the photoelastic beam experiment in [35].

**PROPOSITION 3.11:** The same variation is obtained when

$\eta_q = f_{q+1}(\xi_q + \chi_q(\xi))$  in which we set  $\chi_q = \bar{D}f_{q+1}$ , a transformation which only depends on  $j_1(f_q)$  and is invertible if and only if  $\det(A) \neq 0$  or, equivalently,  $\Delta \neq 0$ .

*Proof:* First of all, we get  $\bar{\xi} = A(\xi)$  for  $q = 0$ , a transformation which is invertible if and only if  $\det(A) \neq 0$  and thus  $\Delta \neq 0$ . In the nonlinear framework, we have to keep in mind that there is no need to vary the object  $\omega$  which is given but only the need to vary the section  $f_{q+1}$  as we already saw, using  $\eta_q \in R_q(Y)$  over the target or  $\xi_q \in R_q$  over the source. With  $\eta_q = f_{q+1}(\bar{\xi}_q)$ , we obtain:

$$\begin{cases} \delta f^k = \eta^k = f_r^k \bar{\xi}^r = \xi^i \partial_i f^k \\ \delta f_r^k = \eta_u^k f_r^u = f_s^k \bar{\xi}^s + f_{rs}^k \bar{\xi}^s = \xi^i \partial_i f_r^k + f_s^k \xi_r^s \\ \delta f_{rs}^k = \eta_{uv}^k f_r^u f_s^v + \eta_u^k f_{rs}^u = f_u^k \bar{\xi}^u + f_{ur}^k \bar{\xi}^u + f_{us}^k \bar{\xi}^u + \bar{\xi}^i f_{irs}^k = f_u^k \xi_r^u + f_{ur}^k \xi_s^u + f_{us}^k \xi_r^u + \xi^i \partial_i f_{rs}^k \end{cases}$$

and so on, a reason for replacing  $\xi^i f_{\mu+1}^k$  by  $\xi^i \partial_i f_\mu^k$  in  $\eta_q = f_{q+1}(\xi_q)$  in order to obtain:

$$\delta f_\mu^k = \eta_r^k f_\mu^r + \dots = \xi^i (\partial_i f_\mu^k - f_{\mu+1_i}^k) + f_r^k \xi_\mu^r + \dots + f_{\mu+1_r}^k \xi^r$$

where the right member only depends on  $j_1(f_q)$  when  $|\mu| = q$ .

Using the inductive formula already found and multiplying it by  $\xi^i$ , we obtain:

$$f_r^k \chi_{\mu,i}^r \xi^i + \dots + f_{\mu+1_r}^k \chi_{\mu,i}^r \xi^i = \xi^i (\partial_i f_\mu^k - f_{\mu+1_i}^k)$$

Substituting in the previous formula provides  $\eta_q = f_{q+1}(\xi_q + \chi_q(\xi))$  and achieves the proof.

Checking directly the proposition is not evident even when  $q = 0$  as we have indeed:

$$\begin{aligned} \delta \chi_{,i}^s &= g_k^s \left( \frac{\partial \eta^k}{\partial y^u} - \eta_u^k \right) \partial_i f^u \\ &= (\partial_i \bar{\xi}^s - \bar{\xi}_i^s) + (\chi_{r,i}^s \bar{\xi}^r - \chi_{,i}^r \bar{\xi}_r^s) \\ &= \partial_i (A_r^s \xi^r) + (A_r^s \chi_{r,i}^s - A_i^r \chi_{r,i}^s) \xi^r - A_i^r \xi_r^s \\ &= (\partial_i \xi^s - \xi_i^s) + (\xi^r \partial_r \chi_{,i}^s + \chi_{,i}^r \partial_r \xi^r - \chi_{,i}^r \xi_r^s) \end{aligned}$$

□

For the Killing system  $R_1 \subset J_1(T)$  with  $g_2 = 0$ , these variations are *exactly* the ones that can be found in ([16], (49) + (50), p. 124 with a printing mistake corrected on p. 128) when replacing a  $3 \times 3$  skew-symmetric matrix by the corresponding vector. *The three last unavoidable Lemmas are thus essential in order to bring back the nonlinear framework of finite elasticity to the linear framework of infinitesimal elasticity that only depends on the linear Spencer operator.*

For the conformal Killing system  $\hat{R}_1 \subset J_1(T)$ , we obtain:

$$\begin{aligned} \alpha_i = \chi_{r,i}^r \Rightarrow \delta \alpha_i &= (\partial_i \xi_r^r - \xi_{ri}^r) + (\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r - \chi_{,i}^s \xi_{rs}^r) \\ &= (\partial_i \xi_r^r - A_i^s \xi_{rs}^r) + (\alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i) \end{aligned}$$

$$\varphi_{ij} = \partial_i \alpha_j - \partial_j \alpha_i \Rightarrow \delta \varphi_{ij} = (\partial_j (A_i^s \xi_{rs}^r) - \partial_i (A_j^s \xi_{rs}^r)) + (\varphi_{rj} \partial_i \xi^r + \varphi_{ir} \partial_j \xi^r + \xi^r \partial_r \varphi_{ij})$$

These are *exactly* the variations obtained by Weyl ([14], (76), p. 289) who was assuming implicitly  $A = 0$  when setting  $\bar{\xi}_r^r = 0 \Leftrightarrow \xi_r^r = -\alpha_i \xi^i$  by introducing a connection. Accordingly,  $\xi_{ri}^r$  is the variation of the EM potential itself, that is the  $\delta A_i$  of engineers used in order to exhibit the Maxwell equations from a variational principle ([14], § 26) but the introduction of the Spencer operator is new in this framework. If  $f_1 = id_1$ , we have  $\chi_0 = 0$  and

$$\delta \alpha_i = (\partial_i \xi_r^r - \xi_{ri}^r) + (\alpha_r \partial_i \xi^r + \xi^r \partial_r \alpha_i).$$

Then, using the definition of  $a$ , namely  $\det(f_i^k) = e^{na}$ , we have:

$$\begin{aligned} n \delta a &= (1/\det(f_i^k)) \delta \det(f_i^k) = g_k^i \delta f_i^k = \eta^s \\ &= g_k^i (\xi^r \partial_r f_i^k + f_r^k \xi_i^r) = n \xi^r \partial_r a + \xi_r^r \end{aligned}$$

Using the variation  $\delta A_i^s = \xi^r \partial_r A_i^s + A_r^s \partial_i \xi^r - A_i^r \xi_r^s = g_i^s \left( \frac{\partial \eta^l}{\partial y^k} - \eta_l^k \right) \partial_i f^k$  we finally get:

$$\begin{aligned}
 \delta\alpha_i &= (\partial_i \xi_r^r - \xi_{ri}^r) + (\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r) - \mathcal{X}_i^s \xi_{rs}^r \\
 &= (\partial_i \xi_r^r - A_i^s \xi_{rs}^r) + (\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r) \\
 &= \left[ \frac{\partial \eta_s^s}{\partial y^k} - n g_l^r \left( \frac{\partial \eta^l}{\partial y^k} - \eta_k^l \right) a_r \right] \partial_i f^k - A_i^r f_r^k \eta_{sk}^s \\
 &= \left[ \left( \frac{\partial \eta_s^s}{\partial y^k} - \eta_{sk}^s \right) - n b_l \left( \frac{\partial \eta^l}{\partial y^k} - \eta_k^l \right) \right] \partial_i f^k
 \end{aligned}$$

The terms  $\partial_i \xi_r^r + (\xi^r \partial_r \alpha_i + \alpha_r \partial_i \xi^r)$  of the variation, including the variation of  $\alpha = \alpha_i dx^i$  as a 1-form, are *exactly* the ones introduced by Weyl in ([14] formula (76), p. 289). We also recognize the variation  $\delta A_i$  of the 4-potential used by engineers now expressed by means of second-order jets.

The variation over the target is only depending on the components of the Spencer operator, in a coherent way with the general variational formulas that could have been used otherwise. We notice that these formulas, which have been obtained with difficulty for second-order jets, could not even be obtained by hand for third-order jets. They show the importance and usefulness of the general formulas providing the Spencer non-linear operators for an arbitrary order, in particular for the study of the conformal group which is defined by second-order lie equations with a 2-acyclic symbol when  $n = 4$ . It is also important to notice that, setting  $b(f(x)) = -a(x)$ ,  $a_i = -f_i^k b_k$  because of the inversion formula  $f_r^k g_u^r = \delta_u^k \Rightarrow g_{uv}^i = -g_u^r g_v^s g_k^i f_{rs}^k$ , we get:

$$\begin{aligned}
 \alpha_i &= n(\partial_i a - A_i^r a_r) = -n \left( \frac{\partial b}{\partial y^k} - b_k \right) \partial_i f^k = \beta_k \partial_i f^k \\
 \Rightarrow \varphi_{ij} &= \partial_i \alpha_j - \partial_j \alpha_i = \left( \frac{\partial \beta_l}{\partial y^k} - \frac{\partial \beta_k}{\partial y^l} \right) \partial_i f^k \partial_j f^l
 \end{aligned}$$

a formula showing that the EM field does not depend on the exchange of source and target.

Using the transformation  $\eta_1 = f_2(\xi_1)$  with  $f_2 \in \hat{\mathcal{R}}_2$  and  $\xi_1, \eta_1 \in \hat{\mathcal{R}}_1$  we obtain the contraction  $\eta^k = f_i^k \xi^i$ ,  $\eta_k^k = \xi_r^r + n a_i \xi^i$ . The inverse transformation allowing to describe  $\hat{\mathcal{R}}_1^*$  is thus  $\xi^i = g_i^k \eta^k$ ,  $\xi_r^r = \eta_k^k - n b_k \eta^k$ . Using the variational formula  $\delta \chi_1 = f_2^{-1} \circ d_y \eta_2 \circ j_1(f_1)$ , if the dual field of  $(\mathcal{X}_{,i}^k, \mathcal{X}_{r,i}^r)$  over the source is  $(\mathcal{X}_{,k}^i, \mathcal{X}^i)$ , we obtain for the dual field  $(\mathcal{Y}_i^k, \mathcal{Y}^k)$  over the target:

$$\Delta \mathcal{Y}_i^k = g_i^s \mathcal{X}_{,k}^i \partial_i f^k - n b_l \mathcal{X}^i \partial_i f^k, \quad \Delta \mathcal{Y}^k = \mathcal{X}^i \partial_i f^k$$

The specific action of the second-order jets is thus  $\mathcal{Y}_i^k \rightarrow \mathcal{Y}_i^k - n b_l \mathcal{Y}^k$ . If we set  $f(x) = x$ ,  $f_i^k = \delta_i^k$  and  $k = 4$ , we recognize the transformation  $p_i \rightarrow p_i - e A_i$  because  $\mathcal{Y}^4 = \mathcal{J}^4$  is the electric charge density. The factor  $a$  can also be used as the additional absolute temperature within the framework of the Weyl group and we can also use the full conformal group for introducing the transformation  $\Delta \mathcal{Y}^{rs} = \mathcal{X}^{ij} \partial_i f^k \partial_j f^l$  in order to obtain the extended action of the elations with  $\mathcal{Y}_i^k \rightarrow \mathcal{Y}_i^k - n b_l \mathcal{Y}^k + n \frac{\partial b_l}{\partial y^s} \mathcal{Y}^{ks}$  while taking into account the second set of Maxwell equations  $\frac{\partial \mathcal{Y}^{ks}}{\partial y^k} - \mathcal{Y}^s = 0$  in order to obtain the Lorentz force in the right member

of the divergence of the generalized Cauchy stress tensor density (See [4] for details).

#### 4. Conclusions

In the case of Special Relativity (1905), it is now known that Einstein was aware of the Michelson and Morley experiment (1887) but only a footnote in his paper “Electrodynamics of Moving Bodies” provides a reference to the *conformal group of space-time* for the Minkowski metric  $\omega$ . However, proving the local invariance of the Maxwell equations by such a group of transformations was not possible at that time because of the nonlinear *relations* introduced by E. Cartan quite later on (1922) and there is no proof that the conformal factor should be equal to 1. Considering a Lie group of transformations as a Lie pseudogroup of transformations, we have revisited in this new framework the mathematical foundations of both continuum mechanics, thermodynamics, elasticity and electromagnetism, thus general relativity and gauge theory, showing that the methods known for Lie groups cannot be adapted to Lie pseudogroups. One of the most striking results is that the analogue of the Weyl operator is a third-order operator with first-order CC when  $n=3$  but becomes a second-order operator with second-order CC when  $n=4$  [36].

In particular, the electromagnetic field, which is a 2-form with value in the Lie algebra of the unitary group  $U(1)$  which is not acting on  $X$  according to *classical gauge theory*, becomes part of a 1-form with value in a Lie algebroid in the new approach using the Lie pseudogroup of conformal transformations of  $X$ . More generally, *shifting by one step the interpretation of the differential sequences involved*, the “field” is no longer a 2-form with value in a Lie algebra but must be a 1-form with value in a Lie algebroid. Meanwhile, we have proved that the use of Lie equations allows to avoid any explicit description of the action of the underlying group, a fact particularly useful for the nonlinear relations of the conformal group. However, a main problem is that the formal methods developed by Spencer and coworkers around 1970 are still not acknowledged by physicists ([19] provides a fine example indeed!) and we don’t even speak about the Vessiot structure equations for Lie pseudogroups, not even acknowledged by mathematicians after more than a century [23]. Finally, as a very striking fact with deep roots in homological algebra, the Cauchy/Cosserat/Clausius/Maxwell/Weyl equations can be parametrized, contrary to Einstein equations [5] [30] [34]. We have finally proved in a few recent papers why gravitational waves cannot exist, not because a problem of detection but because a problem of equations that can only be understood through the *Fundamental Commutative Diagram II* [22] [25] [36]. We hope this paper will open new trends for future theoretical physics and field theory, based on the use of new differential geometric methods (Compare to [37]). Accordingly, paraphrasing W. Shakespeare, we may say as in [10]:

“TO ACT OR NOT TO ACT, THAT IS THE QUESTION”.

#### 5. Main Duality Notations and Recapitulating Formulas

$X$  manifold of dimension  $n$  with coordinates  $x$  and indices  $i, j, k, r, s$ ,  $Y$  copy of

$X$  with coordinates  $y$ .

$\mathcal{G}$  Lie group of dimension  $p$  with indices  $\rho, \sigma, \tau$ .

$T = T(X)$  tangent bundle,  $T^* = T^*(X)$  cotangent bundle,  $\wedge^r T^*$  bundle of  $r$ -forms.

$$a^{-1}da = A = (A_i^\tau) \in T^* \otimes \mathcal{G}$$

$$\Rightarrow dA - [A, A] = (\partial_i A_j^\tau - \partial_j A_i^\tau - c_{\rho\sigma}^\tau A_i^\rho A_j^\sigma) = (F_{ij}^\tau) = F \in \wedge^2 T^* \otimes \mathcal{G}$$

$a' = ab \Rightarrow A' = (a')^{-1} da' = Ad(b)A + b^{-1}db$  gauge transformation.

$a^{-1}\delta a = \lambda, (\delta a)a^{-1} = \mu \Rightarrow \lambda = a^{-1}((\delta a)a^{-1})a = Ad(a)\mu$  left versus right variations.

$\delta A = d\lambda - [A, \lambda] = \nabla\lambda \Leftrightarrow \delta A_i^\tau = \partial_i \lambda^\tau - c_{\rho\sigma}^\tau A_i^\rho \lambda^\sigma \Rightarrow \mathcal{A}_i^\tau \in \wedge^{n-1} T^* \otimes \mathcal{G}^*$  differential duality adjoint operator  $ad(\nabla) \Leftrightarrow \boxed{\partial_i \mathcal{A}_\tau^i + c_{\rho\tau}^\sigma A_i^\rho \mathcal{A}_\sigma^i = 0}$  Poincaré operator.

$\mathcal{B}\mu = \mathcal{A}\lambda \in \wedge^{n-1} T^* \Rightarrow \boxed{\partial_i \mathcal{B}_\sigma^i = 0}$  pure divergence.

$G$  acts on  $X$ :  $X \times G \rightarrow X \times Y : (x, a) \rightarrow (x, y = ax = f(x, a))$

Lie theorem  $\rightarrow$  infinitesimal generators  $\theta_\tau = \theta_\tau^k \partial_k \in T$

$\lambda \in \mathcal{G} \Rightarrow \xi_\mu^k(x) = \lambda^\tau(x) \partial_\mu \theta_\tau^k(x) \in R_{q+1} \simeq R_q \in J_q(T)$  jet bundle when  $q$  large enough.

$$d : R_{q+1} \rightarrow T^* \otimes R_q, \left( d \xi_{q+1}^\tau \right)_{\mu,i}^k(x) = \partial_i \xi_\mu^k(x) - \xi_{\mu+1,i}^k(x) = \partial_i \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)$$

Spencer operator.

### Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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