

Analytic algorithms for Some Models of Nonlinear Age-Structured Population Dynamics and Epidemiology

Vipul K. Baranwal, Ram K. Pandey, Manoj P. Tripathi, Om. P. Singh
Department of Applied Mathematics, Institute of Technology, Banaras Hindu University,
Varanasi, India

E-mail: singhom@gmail.com, opsingh.apm@itbhu.ac.in

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Abstract

Three analytic algorithms based on Adomian decomposition, homotopy perturbation and homotopy analysis methods are proposed to solve some models of nonlinear age-structured population dynamics and epidemiology. Truncating the resulting convergent infinite series, we obtain numerical solutions of high accuracy for these models. Three numerical examples are given to illustrate the simplicity and accuracy of the methods.

Keywords: Age-Structured Population Models, Population Dynamics, SIS Epidemic Models, Adomian Decomposition Method, Homotopy Perturbation and Homotopy Analysis Methods.

1. Introduction

Individuals in a structured population are distinguished by age, size, maturity and some other individual physical characteristics. The basic assumption when modelling the evaluation of such a population is that the structure of the population with respect to these individual physical characteristics at a given time, and possibly some environmental inputs as time evolves, completely determines the dynamical behaviours of the population. Mathematical models describing this evolution have attracted a considerable amount of interest among scientists as a tool for modelling the interaction of different population communities in such diverse fields as demography, epidemiology, ecology, cell kinetics, tumor growth etc.

For a long time, there has been an interest in modelling population dynamics. The first discrete population model appears in Liber Abaci by Leonardo Pisano in 1228 [1], which gives rise to the celebrated Fibonacci sequences. The simplest continuous model is due to Malthus in 1798 [2]. His model is an unstructured one and it leads to an exponential growth of the population which is usually invalid for large time. Forty years later, in 1838, Verhulst proposed a logistic model which impose a maximum size for the population by considering the effects of crowding and the limitation of resources.

In order to build adequate models for population dynamics, some detail concerning individual behaviour and its effects on vital rates of growth, production, and death

must be included. Perhaps the most natural way to consider such effects is to introduce the age variable into the model describing the population dynamics.

Among the first continuous population models incorporating age effects were those of Sharpe and Lotka [3] and McKendrick [4]. Basically, the Sharpe-Lotka-McKendrick models assume that birth and mortality processes are linear functions of population density. In 1974, Gurtin and MacCamy [5] proposed a nonlinear age structured model. The allowed the mortality rate and the fertility rate to be affected by the total population, which is true for the most real cases. Their model generalizes Verhulst's, and under reasonable assumptions on the vital rates of the population, results in a logistic model with bounded growth.

We consider the following nonlinear age-structured population model [5] and name it as model (A):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mu(x, P(t))u = 0, \quad 0 \leq x < A, t > 0, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x < A, \quad (1.2)$$

$$u(0, t) = \int_0^A \beta(x, P(t))u(x, t)dx, \quad t > 0, \quad (1.3)$$

$$P(t) = \int_0^A u(x, t)dx, \quad t \geq 0, \quad (1.4)$$

and some related epidemic models introduced latter, where $u(x, t)$ is the population density (or age-density)

with respect to age x of a population at time t . The units of $u(x, t)$ are given in units of population divided by units of time. Hence, the total number of individuals between ages a and $a + \Delta a$ is given by $\int_a^{a+\Delta a} u(x, t) dx$,

where $u(x, t)$ is taken to be smooth function of (x, t) . The model (A) describes the evolution of the age-density $u(x, t)$ of a population, with maximum age $A < \infty$, whose growth is regulated by the vital rates β and μ [5-7]. To be specific, β and μ denote fertility and death rate respectively. If $\mu = d_1(x) + d_2(x)P(t)$ and $\beta = b_1(x) - b_2(x)P(t)$, then the model (A) reduces to what we call model (B):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -[d_1(x) + d_2(x)P(t)]u(x, t), \quad (1.5)$$

$$0 \leq x < A, t \geq 0,$$

$$u(x, 0) = u_0(x), \quad 0 \leq x < A, \quad (1.6)$$

$$u(0, t) = \int_0^A [b_1(\zeta) - b_2(\zeta)P(t)]u(\zeta, t) d\zeta, \quad t \geq 0, \quad (1.7)$$

$$P(t) = \int_0^A u(x, t) dx, \quad t \geq 0, \quad (1.8)$$

where $d_1(x)$ is the natural death rate (without considering competition), $d_2(x)P(t)$ is the increase of death rate due to competition, $b_1(x)$ is the natural fertility rate (without considering competition), $b_2(x)P(t)$ is the decrease of fertility rate considering competition. Equation (1.3) or (1.7), also called the renewal equation, gives the number of new born individuals introduced into the population. The value of $u(0, t)$ at any time t depends on the age distribution of the population at that time.

The nonlocal boundary condition (1.7) complicates the application of standard numerical techniques such as finite difference, finite elements, spectral methods and so on [8]. So, to avoid the complexity involved in applying those numerical methods to the population model, it is important to convert the nonlocal boundary value problem into more desirable forms. However, it turns out to be a hard work in many cases. Therefore, for nonlinear age-structured population models, there are only few methods for solving them. In recent years, the numerical approximation of the model (A) has been studied by several authors like Abia and Lopez-Marcos, they applied difference schemes based on Runge-Kutta method and other numerical integration techniques to solve it [9-11]. In [12], Kim and Park developed an upwind scheme for the model (A). Iannelli *et al.* [13] solved it by using splitting methods. Reproducing kernel method was successfully applied by Cui and Chen [14] and Krzyzanowski et

al. gave a discontinuous Galerkin method for non linear age structured population model [15]. Norhayati and Wake [16] used Laplace transform technique to solve and analysed the existence of steady age distribution and its stability.

Recently methods like Adomian decomposition method (ADM), homotopy perturbation method (HPM), homotopy analysis method (HAM) have been used successfully to solve a variety of non linear problems [17-20]. Dehghan and Salehi [21] used VIM and ADM to solve the delay logistic equation which has been extensively used as models in biology with particular emphasis on population dynamics. In 2009, Li [8] applied VIM to solve the model (B) with great success, but ADM, HPM and HAM have yet not been used for the purpose.

The aim of the present paper is to apply these techniques for the numerical evaluation of the non linear age-structured population model (B). The basic ideas of these methods apply to other problems related to (B). In fact the same approaches are used for approximation of the age-structured SIS model.

The paper is organized as follows. In sections 2-4, we introduce the algorithms based on ADM, HPM and HAM respectively. In sec. 5, we apply these algorithms on some numerical experiments and finally conclusions are given in sec. 6.

2. Adomian Decomposition Method (ADM)

In this section we give a brief outline of ADM for solving nonlinear age-structured population model (NASPM). Equation (1.5) may be written in the operator form as

$$L_t(u(x, t)) + L_x(u(x, t)) + d_1(x)u(x, t) + d_2(x)N(u(x, t)) = 0, \quad (2.1)$$

where the notations $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial}{\partial x}$ symbolize the linear differential operators and the notation $N(u(x, t)) = u(x, t) \int_0^A u(x, t) dx$ symbolizes the nonlinear operator.

The inverse operator L_t^{-1} , is defined by $L_t^{-1}(\cdot) = \int_0^t (\cdot) d\xi$.

Thus, applying the inverse operator L_t^{-1} to Equation (2.1), we get

$$u(x, t) = u(x, 0) - L_t^{-1} \left[L_x(u(x, t)) + d_1(x)u(x, t) + d_2(x)N(u(x, t)) \right]. \quad (2.2)$$

The ADM [17,18] assumes an infinite series solution for the unknown functions $u(x, t)$, given by

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \tag{2.3}$$

The nonlinear operator $N(u(x, t))$ is decomposed as

$$N(u(x, t)) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n), \tag{2.4}$$

where A_n is an approximate Adomian's polynomial which can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian

$$A_0 = N(u_0), A_1 = N'(u_0)u_1, A_2 = N'(u_0)u_2 + N''(u_0)\frac{u_1^2}{2!}, A_3 = N'(u_0)u_3 + N''(u_0)u_1u_2 + N'''(u_0)\frac{u_1^3}{3!}, \dots$$

Substituting (2.3) and (2.4) in (2.2), we get

$$\sum_{n=0}^{\infty} u_n = u(x, 0) - L_t^{-1} \left[L_x \sum_{n=0}^{\infty} u_n + d_1(x) \sum_{n=0}^{\infty} u_n + d_2(x) \sum_{n=0}^{\infty} A_n \right]. \tag{2.6}$$

Identifying the zeroth component $u_0(x, t)$ by the initial condition $u(x, 0)$, we obtain the subsequent com-

$$u_0(x, t) = u(x, 0), \quad u_{n+1}(x, t) = -L_t^{-1} \left[L_x(u_n(x, t)) + d_1(x)u_n(x, t) + d_2(x)A_n \right]. \tag{2.7}$$

3. Homotopy Perturbation Method (HPM)

In this method, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a ‘‘small parameter’’. This method became very popular amongst the scientists and engineers, even though it involves continuous deformation of a simple problem into a more difficult problem under consideration. Most of the perturbation methods depend on the existence of a small perturbation parameter but many nonlinear problems have no small perturbation parameter at all. Many new methods have been proposed in the late nineties to solve such nonlinear equation devoid of such small parameters. Late 1990s saw a surge in applications of homotopy theory in the scientific and engineering computations [19]. When the homotopy theory is coupled with perturbation theory it provides a powerful mathematical tool. To illustrate the basic concept of HPM, consider the following nonlinear functional equation

$$A(u) = f(r), \quad r \in \Omega,$$

with the boundary conditions $B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \partial\Omega,$ (3.1)

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and $\partial\Omega$ is the boundary of the domain Ω . The operator A is decomposed as $A = L + N$, where L is the linear and N is the nonlinear operator. Hence Equation (3.1) can be written as

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega.$$

[17,18]. For a given nonlinear operator $N(u(x, t))$, these polynomials are calculated using the basic formula:

$$A_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k u_k\right) \right]_{\lambda=0}, \quad n \geq 0. \tag{2.5}$$

The above formula is used to set a computer code to compute the various Adomian polynomials A_n . The first few polynomials are given as follows:

ponents by the following recursive formula

We construct a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow R$ satisfying

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega. \tag{3.2}$$

Hence,

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{3.3}$$

where u_0 is an initial approximation for the solution of (3.1). As

$$H(v, 0) = L(v) - L(u_0) \text{ and } H(v, 1) = A(v) - f(r), \tag{3.4}$$

it shows that $H(v, p)$ continuously traces an implicitly defined curve from a starting point $H(u_0, 0)$ to a solution $H(v, 1)$. The embedding parameter p increases monotonously from zero to one as the trivial linear part $L(u) = 0$ deforms continuously to the original problem $A(u) = f(r)$. The embedding parameter $p \in [0, 1]$ can be considered as an expanding parameter [19] to obtain

$$v = v_0 + pv_1 + p^2v_2 + \dots. \tag{3.5}$$

The solution is obtained by taking the limit as p tends to 1 in equation.(3.5). Hence

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots. \tag{3.6}$$

The series (3.6) converges for most cases and the rate of convergence depends on $A(u) - f(r)$.

For nonlinear age-structured population model, we choose the initial approximation $u_0(x, t) = u_0(x)$, and construct the following homotopy:

$$(1-p)\left(\frac{\partial v(x,t)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t}\right) + p\left(\frac{\partial v(x,t)}{\partial t} + \frac{\partial v(x,t)}{\partial x} + \left(d_1(x) + d_2(x) \int_0^A v(x,t) dx\right)v(x,t)\right) = 0, \tag{3.7}$$

which is equivalent to

$$\frac{\partial v(x,t)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t} + p \frac{\partial u_0(x,t)}{\partial t} + p\left(\frac{\partial v(x,t)}{\partial x} + \left(d_1(x) + d_2(x) \int_0^A v(x,t) dx\right)v(x,t)\right) = 0, \tag{3.8}$$

where $p \in [0,1]$ is an embedding parameter. Using the parameter p , we expand the solution in the following form

$$v(x,t) = v_0(x,t) + pv_1(x,t) + p^2v_2(x,t) + p^3v_3(x,t) + \dots \tag{3.9}$$

Substituting Equation (3.9) into Equation (3.8), and equating the terms with the identical powers of p , we obtain:

$$\begin{aligned} p^0 : \frac{\partial v(x,t)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t} &= 0, \quad v_0(x,0) = u(x,0), \\ p^1 : \frac{\partial u_1(x,t)}{\partial t} + \frac{\partial u_0(x,t)}{\partial t} + \frac{\partial v_0(x,t)}{\partial x} &+ d_1(x)v_0(x,t) + d_2(x)v_0(x,t) \int_0^A v_0(x,t) dx = 0, \quad v_1(x,0) = 0, \\ p^2 : \frac{\partial v_2(x,t)}{\partial t} + \frac{\partial v_1(x,t)}{\partial x} &+ d_1(x)v_1(x,t) + d_2(x)v_1(x,t) \int_0^A v_0(x,t) dx + d_2(x)v_0(x,t) \int_0^A v_1(x,t) dx = 0, \\ v_2(x,0) &= 0, \\ p^3 : \frac{\partial v_3(x,t)}{\partial t} + \frac{\partial v_2(x,t)}{\partial x} &+ d_1(x)v_2(x,t) + d_2(x)v_2(x,t) \int_0^A v_0(x,t) dx + d_2(x)v_1(x,t) \int_0^A v_1(x,t) dx \\ &+ d_2(x)v_0(x,t) \int_0^A v_2(x,t) dx = 0, \quad v_3(x,0) = 0, \dots \end{aligned} \tag{3.10}$$

We use the iterative scheme (3.10) to compute the various v_i 's. Hence the solution of Equation (1.5) is given by,

$$u(x,t) = \lim_{p \rightarrow 1} v(x,t) = \sum_{m=0}^{\infty} v_m(x,t). \tag{3.11}$$

4. Homotopy Analysis Method (HAM)

Homotopy analysis method (HAM) was first proposed by Liao [20] based on homotopy, a fundamental concept in topology and differential geometry. The HAM is based on construction of homotopy which continuously deforms an initial guess approximation to the exact solution of the given problem. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary linear parameter is used to control the region of convergence of the solution series, which is not possible in the other methods like perturbation techniques, homotopy perturbation methods, decomposition methods. The HAM provides the greater flexibility in choosing initial approximations and auxiliary linear operators and hence a complicated nonlinear problem can be transformed into infinite number simpler, linear sub problems as shown by Liao and Tan [22].

Here we give a brief description of HAM [20] to han-

dle the general non linear problem,

$$N[u(x,t)] = 0, \quad t > 0, \tag{4.1}$$

where N is a nonlinear operator and $u(x,t)$ is unknown function of the independent variables x,t . Liao [20] constructed the zero order deformation equation

$$\begin{aligned} (1-q)L[\varphi(x,t;q) - u_0(x,t)] \\ = q\hbar H(x,t)N[\varphi(x,t;q)], \end{aligned} \tag{4.2}$$

where $q \in [0,1]$ is the homotopy or embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ an initial guess of $u(x,t)$ and $\varphi(x,t;q)$ is an unknown function.

Putting $q = 0$, and $q = 1$, in Equation (4.2), we see that

$$\varphi(x,t;0) = u_0(x,t), \tag{4.3}$$

$$\varphi(x,t;1) = u(x,t), \tag{4.4}$$

Therefore, according to Equations (4.3) & (4.4), $\varphi(x,t;q)$ deforms continuously from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$ as the embedding parameter q increases from 0 to 1. Liao [20] expanded $\varphi(x,t;q)$ in a Taylor series as follows

$$\varphi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m, \tag{4.5}$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \varphi(x, t; q)}{\partial q^m} \right|_{q=0}. \tag{4.6}$$

The convergence of the series (4.5) is controlled by \hbar . Assume that the auxiliary parameter \hbar , the auxiliary function H , the initial approximation $u_0(x, t)$, and the auxiliary linear operator L are so properly chosen that the series (4.5) converges at $q=1$. Then, at $q=1$ and using (4.4) the series (4.5) gives the exact solution $u(x, t)$ as

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t). \tag{4.7}$$

The above expression provides us with a relationship between the initial guess $u_0(x, t)$ and the exact solution $u(x, t)$ by means of the terms $u_m(x, t) (m=1, 2, 3, \dots)$, which are still to be determined. The process of their evaluations is given as follows :

Differentiating the zero order deformation Equation (4.2) m times with respect to embedding parameter q , then setting $q=0$ and dividing by $m!$, we get the following m th-order deformation equation,

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar H(t) R_m(u_{m-1}(x, t)), \tag{4.8}$$

where

$$R_m(u_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\varphi(x, t; q)]}{\partial q^{m-1}} \right|_{q=0}, \tag{4.9}$$

$$u_m = \{u_0(x, t), u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}, \tag{4.10}$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & \text{otherwise} \end{cases}. \tag{4.11}$$

For any given operators L and N we get the m th order deformation Equation (4.8) and solving it we get different $u_m(x, t)$. The solution of problem (4.1) is obtained by putting these $u_m(x, t)$'s in (4.7) and choosing a suitable value of \hbar for the convergence of the series. The symbolic computation software like Maple and Mathematica can solve (4.8) easily.

5. Numerical Applications

In this paper, we apply ADM, HPM and HAM to solve the nonlinear age-structured population models. In the following examples $\tilde{u}_n(x, t)$ will denote an approximate solution of the problem under consideration, obtained by truncating the solution series (4.7) at level

$m = n$. Also $E_n(\hbar) = |u_{exact}(x, t) - \tilde{u}_n(x, t)|$ denotes the error between exact and approximate solution at \hbar . In **Table 1**, E_1 denote the error between exact solution and approximate solution obtained by reproducing kernel method [14].

Example 5.1 Consider the following nonlinear age-structured population model [8, 14].

$$\frac{\partial u(x, t)}{\partial t} + \frac{\partial u(x, t)}{\partial x} = -P(t)u(x, t), \quad t \geq 0, 0 \leq x < A, \tag{5.1}$$

$$u(x, 0) = \frac{e^{-x}}{2}, \quad 0 \leq x < A, \tag{5.2}$$

$$u(0, t) = P(t), \quad t \geq 0, \tag{5.3}$$

$$P(t) = \int_0^A u(x, t) dx, \quad t \geq 0, \tag{5.4}$$

where, $A = +\infty$, with $u(x, t) = \frac{e^{-x}}{1+e^{-t}}, t \geq 0, x \geq 0$, as the exact solution of (5.1).

Case (a) Solution by ADM

Rewriting Equation (5.1) in the operator form

$$L_t u(x, t) = -L_x u(x, t) - u(x, t) \int_0^A u(x, t) dx, \tag{5.5}$$

taking the initial approximation $u_0(x, t) = \frac{e^{-x}}{2}$, and using the recursive formula (2.7), we find that all the even iterates $u_{2n} = 0, n = 0, 1, 2, 3, \dots$ and

$$u_1(x, t) = t \frac{e^{-x}}{4}, \quad u_3(x, t) = -t^3 \frac{e^{-x}}{48},$$

$$u_5(x, t) = t^5 \frac{e^{-x}}{480}, \quad u_7(x, t) = -t^7 \frac{e^{-x}}{80640}, \dots$$

Hence, the solution is given by

$$\begin{aligned} u(x, t) &= \lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x, t) \\ &= e^{-x} \left(\frac{1}{2} + \frac{t}{4} - \frac{t^3}{48} + \frac{t^5}{480} - \frac{t^7}{80640} + \dots \right) \\ &= \frac{e^{-x}}{1+e^{-t}}, \end{aligned}$$

which is the exact solution.

Case (b) Solution by HPM

Taking the same initial approximation $u_0(x, t) = \frac{e^{-x}}{2}$, and using the equations (3.10), we find that all the even iterates $v_{2n} = 0, n = 0, 1, 2, 3, \dots$ and

Table 1. Comparison between HAM and reproducing kernel method solutions.

Nodes		Exact Solution	Approximate Solution	Error	Error [14]
t	x	$u(x,t)$	$\tilde{u}_7(x,t), \hbar = -1$	$E_7(-1) = u(x,t) - \tilde{u}_7(x,t) $	E_1
0.00	0.00	0.5	0.5	0	0
0.20	0.00	0.549834	0.549834	1.08906E - 11	2.6162E - 02
0.40	0.00	0.598688	0.598688	5.50928E - 09	4.4321E - 02
0.60	0.00	0.645656	0.645656	2.07654E - 07	4.4703E - 02
0.80	0.00	0.689974	0.689972	2.69192E - 06	3.6536E - 02
1.00	0.00	0.731059	0.731039	1.93921E - 05	1.1234E - 02
0.00	1.00	0.18394	0.18394	0	0
0.20	1.00	0.202273	0.202273	4.00643E - 12	6.467E - 03
0.40	1.00	0.220245	0.220245	2.02675E - 09	1.2146E - 02
0.60	1.00	0.237524	0.237524	7.63918E - 08	1.4046E - 02
0.80	1.00	0.253827	0.253826	9.90302E - 07	7.887E - 03
1.00	1.00	0.268941	0.268934	7.13396E - 06	-5.552E - 03
0.00	2.00	0.0676676	0.0676676	0	0
0.20	2.00	0.0744119	0.0744119	1.47389E - 12	5.5869E - 03
0.40	2.00	0.0810236	0.0810236	7.456E - 10	9.6181E - 03
0.60	2.00	0.0873801	0.0873801	2.8103E - 08	1.03579E - 02
0.80	2.00	0.0933779	0.0933775	3.64312E - 07	5.4159E - 03
1.00	2.00	0.098938	0.0989354	2.62444E - 06	6.5159E - 03
0.00	3.00	0.0248935	0.0248935	0	0
0.20	3.00	0.0273746	0.0273746	5.42216E - 13	4.4722E - 03
0.40	3.00	0.0298069	0.0298069	2.74291E - 10	7.3509E - 03
0.60	3.00	0.0321453	0.0321453	1.03385E - 08	8.2104E - 03
0.80	3.00	0.0343518	0.0343517	1.34023E - 07	6.4072E - 03
1.00	3.00	0.0363973	0.0363963	9.65477E - 07	2.9965E - 03
0.00	4.00	0.00915782	0.00915782	0	0
0.20	4.00	0.0100706	0.0100706	1.99471E - 13	1.8242E - 03
0.40	4.00	0.0109653	0.0109653	1.00906E - 10	3.2864E - 03
0.60	4.00	0.0118256	0.0118256	3.80332E - 09	3.7749E - 03
0.80	4.00	0.0126373	0.0126373	4.93043E - 08	3.1609E - 03
1.00	4.00	0.0133898	0.0133894	3.55179E - 07	1.8526E - 03
0.00	5.00	0.00336897	0.00336897	0	0
0.20	5.00	0.00370475	0.00370475	7.33805E - 14	-3.3172E - 04
0.40	5.00	0.00403393	0.00403393	3.71212E - 11	-2.5414E - 04
0.60	5.00	0.0043504	0.0043504	1.39916E - 09	-1.3464E - 04
0.80	5.00	0.00464901	0.00464899	1.8138E - 08	-3.5880E - 04
1.00	5.00	0.00492583	0.0049257	1.30663E - 07	-8.72E - 04
0.00	6.00	0.00123938	0.00123938	0	0
0.20	6.00	0.0013629	0.0013629	2.69953E - 14	8.9154E - 04
0.40	6.00	0.001484	0.001484	1.36561E - 11	2.3233E - 03
0.60	6.00	0.00160042	0.00160042	5.14724E - 10	3.85194E - 03
0.80	6.00	0.00171028	0.00171027	6.67261E - 09	5.38069E - 03
1.00	6.00	0.00181211	0.00181206	4.80683E - 08	6.49366E - 03
0.00	8.00	0.000167731	0.000167731	0	0
0.20	8.00	0.000184449	0.000184449	3.65338E - 15	4.58849E - 03
0.40	8.00	0.000200837	0.000200837	1.84816E - 12	6.31876E - 03
0.60	8.00	0.000216594	0.000216593	6.96603E - 11	9.81151E - 03
0.80	8.00	0.000231461	0.00023146	9.03039E - 10	1.30453E - 02
1.00	8.00	0.000245243	0.000245236	6.50533E - 09	1.6041E - 02
0.00	10.00	0.0000227	0.0000227	0	0
0.20	10.00	0.0000249624	0.0000249624	4.94433E - 16	2.14306E - 04
0.40	10.00	0.0000271804	0.0000271804	2.50121E - 13	1.14205E - 03
0.60	10.00	0.0000293128	0.0000293127	9.42749E - 12	2.52654E - 03
0.80	10.00	0.0000313248	0.0000313247	1.22213E - 10	3.89648E - 03
1.00	10.00	0.00003319	0.0000331891	8.80401E - 10	5.20734E - 03

$$v_1(x,t) = t \frac{e^{-x}}{4}, v_3(x,t) = -t^3 \frac{e^{-x}}{48},$$

$$v_5(x,t) = t^5 \frac{e^{-x}}{480}, v_7(x,t) = -t^7 \frac{e^{-x}}{80640}, \dots$$

Thus we see that the various terms obtained by using HPM are same as those obtained by using ADM. In general, the ADM solution is a part of HPM solution.

Substituting these values in Equation (3.11), the solution is given by

$$u(x,t) = \lim_{p \rightarrow 1} v(x,t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N v_n(x,t)$$

$$= e^{-x} \left(\frac{1}{2} + \frac{t}{4} - \frac{t^3}{48} + \frac{t^5}{480} - \frac{t^7}{80640} + \dots \right)$$

$$= \frac{e^{-x}}{1 + e^{-t}},$$

which is the exact solution.

Case (c) Solution by HAM

Choosing the linear operator as $L = \frac{\partial}{\partial t}$ and using the Equations (4.8-4.10), we obtain the following m th order deformation equation as

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \int_0^t \left[\frac{\partial u_{m-1}(x,\tau)}{\partial \tau} + \frac{\partial u_{m-1}(x,\tau)}{\partial x} \right. \\ \left. + \sum_{i=0}^{m-1} u_i(x,\tau) \int_0^A u_{m-1-i}(x,\tau) dx \right] d\tau,$$

$$m = 1, 2, 3, \dots$$

(5.6)

Taking the initial guess as $u_0(x,t) = u(x,0) = \frac{e^{-x}}{2}$, and solving the m th Equation (5.6), we get the following

$$u_1(x,t) = -\hbar t \frac{e^{-x}}{4},$$

$$u_2(x,t) = -(1 + \hbar) \hbar t \frac{e^{-x}}{4},$$

$$u_3(x,t) = -(1 + \hbar)^2 \hbar t \frac{e^{-x}}{4} + \hbar^3 t^3 \frac{e^{-x}}{48}, \dots$$

Truncating the series (4.7) at level $m = 7$, we obtain an approximate solution of (5.1) as

$$\tilde{u}_7(x,t) = u_0(x,t) + \sum_{m=1}^7 u_m(x,t),$$

the accuracy of approximation is controlled by the auxiliary parameter \hbar as illustrated by **Figures 3** and **4**.

The approximate solution $\tilde{u}_m(x,t)$ converges to the

exact solution $\frac{e^{-x}}{1 + e^{-t}}$, as $m \rightarrow \infty$ for $\hbar = -1$.

The ADM and HPM solutions are obtained by taking $\hbar = -1$ in HAM solution.

Table 1 shows that the approximate solution $\tilde{u}_7(x,t)$ obtained by HAM for $\hbar = -1$ is more accurate compared to that obtain by reproducing kernel method [14].

Figure 1 shows the approximate solution for $\hbar = -1$, whereas **Figures 2, 3** show the errors $E_7(\hbar)$ for different values of \hbar . It is observed that $E_7(-0.99)$ is smaller than $E_7(-1)$.

Example 5.2 Consider the following nonlinear age-structured population model [8,13].

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = -(P(t)+1)u(x,t), \quad (5.7)$$

$$t \geq 0, 0 \leq x < A,$$

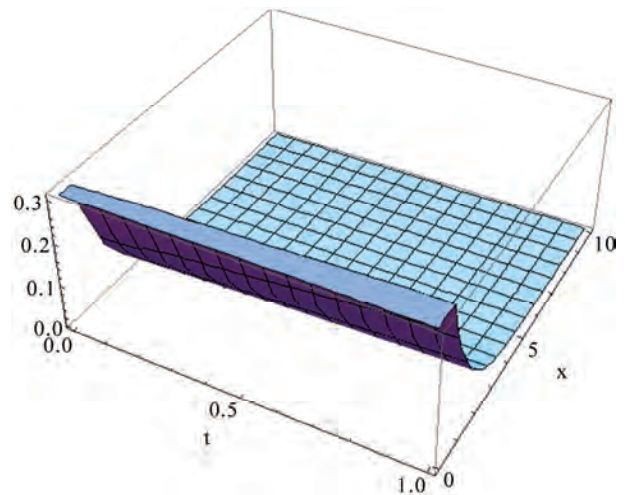


Figure 1. Approximate solution $\tilde{u}_7(x,t)$, $\hbar = -1$.

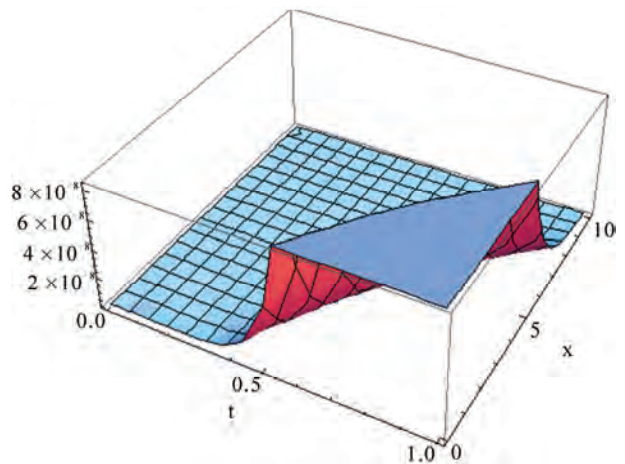


Figure 2. Error $E_7(-1)$.

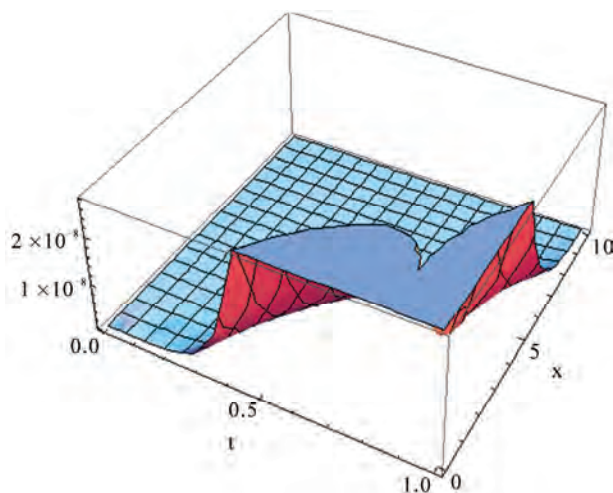


Figure 3. Error $E_7(-0.99)$.

$$u(x, 0) = \frac{e^{-x}}{2}, \quad 0 \leq x < A, \tag{5.8}$$

$$u(0, t) = P(t), \quad t \geq 0, \tag{5.9}$$

$$P(t) = \int_0^A u(x, t) dx, \quad t \geq 0, \tag{5.10}$$

where $A = +\infty$. It is easy to verify that

$$u(x, t) = \frac{e^{-x}}{2+t}, \quad t \geq 0, x \geq 0, \text{ is the exact solution of (5.7).}$$

Case (a) Solution by ADM

Rewriting Equation (5.7) in operator form

$$L_x u(x, t) = -L_x u(x, t) - u(x, t) - u(x, t) \int_0^A u(x, t) dx, \tag{5.11}$$

taking initial approximation $u_0(x, t) = \frac{e^{-x}}{2}$, and using the recursive formula (2.7), we get

$$u_1(x, t) = -t \frac{e^{-x}}{4}, u_2(x, t) = t^2 \frac{e^{-x}}{8}, u_3(x, t) = -t^3 \frac{e^{-x}}{16},$$

$$u_4(x, t) = t^4 \frac{e^{-x}}{32}, u_5(x, t) = -t^5 \frac{e^{-x}}{64}, \dots$$

The solution is given by

$$u(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x, t)$$

$$= e^{-x} \left(\frac{1}{2} - \frac{t}{4} + \frac{t^2}{8} - \frac{t^3}{16} + \frac{t^4}{32} - \frac{t^5}{64} + \frac{t^6}{128} - \frac{t^7}{256} + \dots \right)$$

$$= \frac{e^{-x}}{2+t},$$

which is the exact solution.

Case (b) Solution by HPM

Taking initial approximation $u_0(x, t) = \frac{e^{-x}}{2}$, and using the Equations (3.10), we get

$$v_1(x, t) = -t \frac{e^{-x}}{4}, v_2(x, t) = t^2 \frac{e^{-x}}{8}, v_3(x, t) = -t^3 \frac{e^{-x}}{16},$$

$$v_4(x, t) = t^4 \frac{e^{-x}}{32}, v_5(x, t) = -t^5 \frac{e^{-x}}{64}, \dots$$

Substituting these values in Equation (3.11), the solution is given by

$$u(x, t) = \lim_{p \rightarrow 1} v(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N v_n(x, t)$$

$$= e^{-x} \left(\frac{1}{2} - \frac{t}{4} + \frac{t^2}{8} - \frac{t^3}{16} + \frac{t^4}{32} - \frac{t^5}{64} + \frac{t^6}{128} - \frac{t^7}{256} + \dots \right)$$

$$= \frac{e^{-x}}{2+t},$$

which is the exact solution.

Case (c) Solution by HAM

Choosing the linear operator as $L = \frac{\partial}{\partial t}$ and using the Equations (4.8-4.10), we obtain the following m th order deformation equation as

$$u_m(x, t) = \mathcal{H}_m u_{m-1}(x, t) + \hbar \int_0^t \left[\frac{\partial u_{m-1}(x, \tau)}{\partial \tau} + \frac{\partial u_{m-1}(x, \tau)}{\partial x} \right. \\ \left. + u_{m-1}(x, \tau) + \sum_{i=0}^{m-1} u_i(x, \tau) \int_0^A u_{m-1-i}(x, \tau) dx \right] d\tau,$$

$$m = 1, 2, 3, \dots \tag{5.12}$$

Taking the initial guess as $u_0(x, t) = u(x, 0) = \frac{e^{-x}}{2}$, and solving the m th Equation (5.12), we get the following

$$u_1(x, t) = \hbar t \frac{e^{-x}}{4},$$

$$u_2(x, t) = (1 + \hbar) \hbar t \frac{e^{-x}}{4} + \hbar^2 t^2 \frac{e^{-x}}{8},$$

$$u_3(x, t) = (1 + \hbar)^2 \hbar t \frac{e^{-x}}{4} + 2(1 + \hbar) \hbar^2 t^2 \frac{e^{-x}}{8} + \hbar^3 t^3 \frac{e^{-x}}{16}, \dots$$

Truncating the series (4.7) at level $m = 8$, we obtain an approximate solution of (5.7) as

$$\tilde{u}_8(x, t) = u_0(x, t) + \sum_{m=1}^8 u_m(x, t).$$

The approximate solution $\tilde{u}_m(x,t)$ converges to the exact solution $\frac{e^{-x}}{2+t}$, as $m \rightarrow \infty$ for $\hbar = -1$.

Figure 4 shows the approximate solution for $\hbar = -1$, whereas Figures 5, 6 show the errors $E_8(\hbar)$ for different values of \hbar . It is observed that $E_8(-0.7905)$ is smaller than $E_8(-1)$.

In the next example, we consider a SIS model describing the evolution of a human nonlethal disease which does not import immunity [23-25]. Some infections, for example the group of those responsible for the common cold, do not confer any long lasting immunity. Such infections do not have a recovered state and individuals become susceptible again after infection. Maybe, the most specific parameter of biological system is the age, and, especially for some infectious diseases, it has a deep influence on the dynamics of its spreading in a population. Many of the parameters may depend on age, especially the contact rate, which summarizes the ‘infectious effectiveness’ of contacts between susceptible and infectious subjects. This effectiveness has, thus, to take into account both the age of the infectious and the age of the susceptible. Epidemic models modelling the age structure of a population are very complex. Example (5.3) illustrates the utility of our algorithm on such types of complex models.

Example 5.3 We consider the following age-structured SIS model [13]:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} + \frac{1}{1-x}u(x,t) = (i(x,t) - u(x,t))P(t),$$

$$t \geq 0, 0 \leq x < A, \tag{5.13}$$

$$u(x,0) = 2(1-x), \quad 0 \leq x < A, \tag{5.14}$$

$$u(0,t) = \int_0^A \beta(x,i)u(x,t)dx, \quad t \geq 0, \tag{5.15}$$

$$P(t) = \int_0^A u(x,t)dx, \quad t \geq 0. \tag{5.16}$$

where $A=1, \beta = 4(1-x)^2$ or $\beta = \pi \sin \pi x$.

For the total population, we take a steady state distribution such that $i(x,t) = 4(1-x)$. Then the exact solution of (5.13) is $u(x,t) = \frac{4(1-x)}{1+e^{-2t}}$.

Following the procedures adopted in the previous examples, we apply the three methods and obtain the various iterates of the solution as follows:

As ADM and HPM iterates are identical for $p=1$, we list them once.

1) ADM and HPM iterates

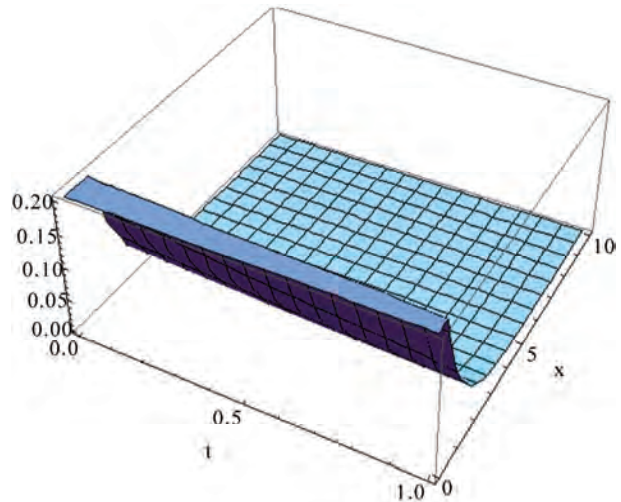


Figure 4. Approximate solution $\tilde{u}_8(x,t)$, $\hbar = -1$.

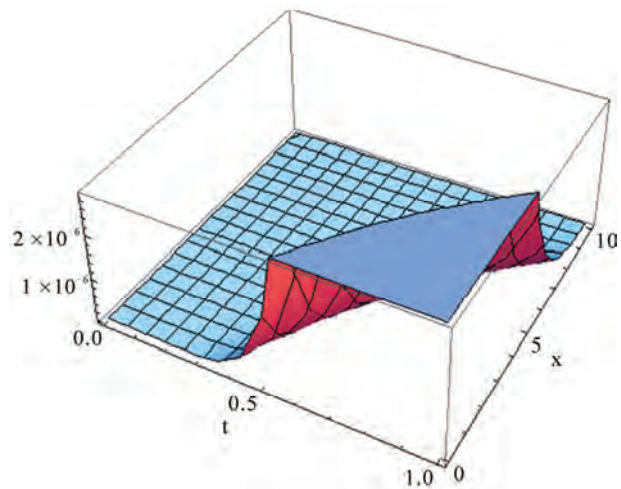


Figure 5. Error $E_8(-1)$.

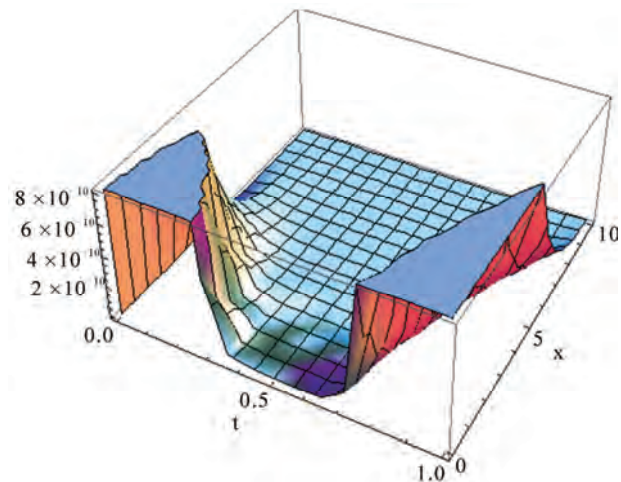


Figure 6. Error $E_8(-0.7905)$.

$$\begin{aligned}
 u_0(x,t) &= 2(1-x), u_1(x,t) = 2t(1-x), \\
 u_3(x,t) &= -\frac{2}{3}t^3(1-x), u_5(x,t) = \frac{4}{15}t^5(1-x), \\
 u_7(x,t) &= -\frac{34}{315}t^7(1-x), u_9(x,t) = \frac{124}{2835}t^9(1-x), \\
 u_{11}(x,t) &= -\frac{3268}{155925}t^{11}(1-x), u_{2n}(x,t) = 0, n = 1, 2, \dots
 \end{aligned}$$

The solution is given by

$$\begin{aligned}
 u(x,t) &= \lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x,t) \\
 &= 2(1-x) \left(1+t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835} - \frac{1634t^{11}}{155925} + \dots \right) \\
 &= \frac{4(1-x)}{1+e^{-2t}},
 \end{aligned}$$

which is the exact solution.

2) HAM iterates

$$\begin{aligned}
 u_0(x,t) &= 2(1-x), \\
 u_1(x,t) &= -2\hbar t(1-x), \\
 u_2(x,t) &= -2(1+\hbar)\hbar t(1-x), \\
 u_3(x,t) &= -2(1+\hbar)^2\hbar t(1-x) + \frac{2}{3}\hbar^3 t^3(1-x), \\
 u_4(x,t) &= -2(1+\hbar)^3\hbar t(1-x) + 2(1+\hbar)\hbar^3 t^3(1-x), \dots
 \end{aligned}$$

Truncating the series (4.7) at the level $m = 11$, we obtain an approximate solution of (5.13) as

$$\tilde{u}_{11}(x,t) = u_0(x,t) + \sum_{m=1}^{11} u_m(x,t).$$

The approximate solution $\tilde{u}_m(x,t)$ converges to the exact solution $\frac{4(1-x)}{1+e^{-2t}}$, as $m \rightarrow \infty$ for $\hbar = -1$.

Figure 7 shows the approximate solution for $\hbar = -1$, whereas **Figures 8, 9** show the errors $E_{11}(\hbar)$ for different values of \hbar . It is observed that $E_{11}(-0.95312)$ is smaller than $E_{11}(-1)$.

6. Conclusion

In this paper, we have given three simple, easy to implement analytic algorithms based on ADM, HPM and HAM for nonlinear models of age-structured population dynamics and epidemiology. These algorithms involve infinite convergent series and in many cases, the closed form solutions are obtained. Where the closed form analytical solutions are not readily available, it is established through the three numerical experiments that truncating the solution series at relatively lower level of truncation give fairly accurate solutions. Moreover, the accuracy of

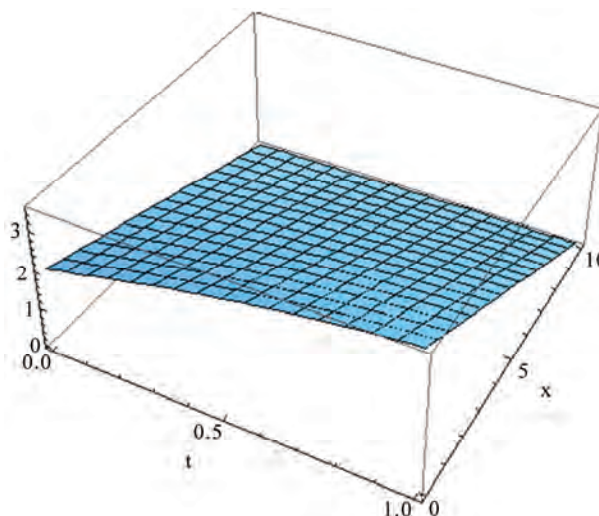


Figure 7. Approximate solution $\tilde{u}_{11}(x,t)$, $\hbar = -1$.

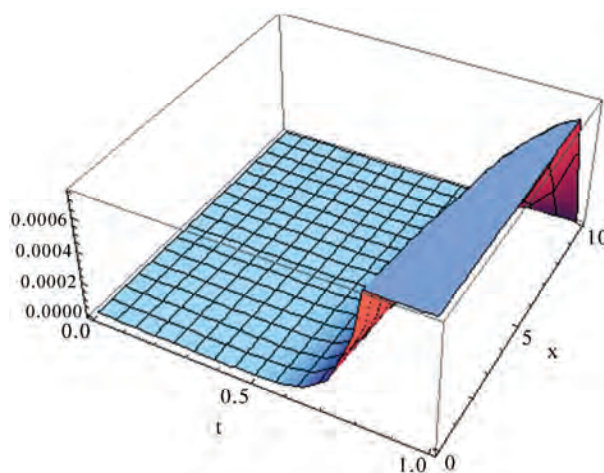


Figure 8. Error $E_{11}(-1)$.

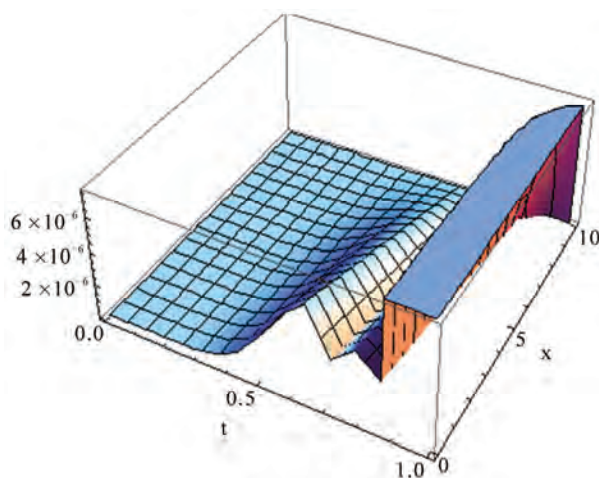


Figure 9. Error $E_{11}(-0.95312)$.

the solution may be increased by the suitable choice of the auxiliary parameter \hbar in HAM. **Table 1** clearly establishes the accuracy of our method compared to that of the reproducing kernel method proposed by Cui and Chen [14].

7. Acknowledgements

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