Journal of Mathematical Finance, 2023, 13, 271-283
https://www.scirp.org/journal/jmf
ISSN Online: 2162-2442
ISSN Print: 2162-2434

Estimating the Gerber-Shiu Function by Fourier Cosine Series Expansion in the Wiener-Poisson Risk Model

Marcelin Romeo Noumegni Kenmoe[1*], Jane Akinyi Aduda[2], Mbele Bidima Martin Le Doux[3]

[1]Department of Mathematics, Pan African University Institute for Basic Sciences Technology and Innovation (PAUSTI), Nairobi, Kenya
[2]Department of Statistics and Actuarial Sciences, Jomo Kenyatta University of Agriculture and Technology (JKUAT), Nairobi, Kenya
[3]Department of Mathematics, University of Yaounde 1, Yaounde, Cameroon
Email: *mrnounoumegni@gmail.com, jaduda@jkuat.ac.ke, mbelebidima@gmail.com

How to cite this paper: Kenmoe, M.R.N., Aduda, J.A. and Le Doux, M.B.M. (2023) Estimating the Gerber-Shiu Function by Fourier Cosine Series Expansion in the Wiener-Poisson Risk Model. *Journal of Mathematical Finance*, **13**, 271-283.
https://doi.org/10.4236/jmf.2023.133017

Received: May 27, 2023
Accepted: July 28, 2023
Published: July 31, 2023

Abstract

Gerber-Shiu function is the joint distribution of the time to ruin, the surplus before ruin and the deficit at ruin. In this paper, we propose a non-parametric estimator of the expected discounted penalty function; the Gerber-Shiu function for a compound Poisson risk model perturbed by diffusion is also called the Wiener-Poisson risk model. The estimator is based on the Fourier cosine series expansion method. It shows that our estimator has a fast convergence rate. We also derived some simulation examples to show the effectiveness of the estimator under a finite sample.

Keywords

COS Method, Perturbed Compound Poisson Risk Model, Surplus, Claim Sizes, Claim Number, Gerber-Shiu, Ruin, Poisson Process, Expected Discounted Penalty Function

1. Introduction

In this paper, we consider that the financial surplus of an insurance company evolves as a compound Poisson risk model perturbed by a Wiener process

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i + \sigma W(t), \quad t \geq 0 \tag{1}$$

where $u \geq 0$ is the initial capital and $c > 0$ is the constant premium per time. The aggregate claims $\sum_{i=1}^{N_t} X_i$ follows a compound Poisson process, where the number of claims $\{N_t\}_{t\geq 0}$ is an homogeneous Poisson process with intensity $\lambda > 0$, and the individual claim sizes $\{X_i\}_{i\geq 1}$ is a sequence of positive i.i.d. random variables generated by a generic variable X with density f_X and mean μ. Finally, $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion with $W(0)=0$, and $\sigma > 0$ is the diffusion volatility parameter. We suppose that $\{N_t\}$, $\{X_i\}$ and $\{W(t)\}$ are mutually independent. In this paper, we shall assume throughout the safety loading condition $c > \lambda\mu$, so that ruin is an uncertain event.

The ruin time is defined by $\tau = \inf\{t \geq 0 : U(t) < 0\}$ with the convention $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. Problems related to ruin are a hot topic in risk theory literature. Starting with Cramér and Lundberg's fundamental collective insurance risk model which was established at the turn of the century [1] [2], actuarial scholars are still investigating into specific examples of it or tweaking some of its aspects to make it more useful. [3] proposed three equivalent equations for ruin probability in a Cramér Lundberg model, and the solutions were means of inverse Laplace transform. To extend the ruin problem, [4] was the first to propose the joint distribution of the time to ruin, the surplus before ruin, and the deficit at ruin, named the Gerber-Shiu function.

$$\phi(u) = \mathbb{E}\left[\mathrm{e}^{-\delta\tau} w\left(U(\tau_-), |U(\tau)|\right) 1_{\{\tau<\infty\}} | U_0 = u\right], \quad u \geq 0 \tag{2}$$

where $w : [0,\infty) * [0,\infty) \mapsto [0,\infty)$ is a measurable penalty function of the surplus prior to ruin and the deficit at ruin, $\delta \geq 0$ represents the force of interest and $1_{\{A\}}$ is the indicator function of the event *A*. Set $w_0 = w(0,0)$. The Gerber-Shiu function is a powerful tool for studying ruin related problems. To make it simple, when $w \equiv 1$, ϕ becomes the ruin probability when $\delta = 0$ or the Laplace transform of the time to ruin when $\delta > 0$. For $w(x,y) \equiv y$, ϕ becomes the expected discounted deficit at ruin when ruin is caused by a claim. Finally, for $w(x,y) \equiv x + y$, ϕ becomes the expected discounted claim size causing ruin. Over the last decades, Gerber-Shiu has drawn the attention of numerous actuarial scholars in various risk models. [5] [6] [7] [8] [9]

The Wiener-Poisson risk model was initially proposed in the literature of actuarial science by [10] to extend the classical compound Poisson risk model, where the diffusion perturbation is used to describe the uncertainty of the premium income and aggregate claims. Since then, many scholars have made substantial contributions to this model. For instance, [11] showed that the probability of ruin satisfied certain renewal equations; [12] studied the ruin problems; [13] derived and solved boundary conditions for the n-th moment of the discounted dividend payments; [14] investigated some explicit solutions of the expected discounted penalty function; [15] considered a model with two-sided jumps. For further investigation, we refer the interested readers to [16] [17] and [18]. The explicit formula of the Gerber-Shiu function can be determined when it is assumed that the Poisson intensity, claim size density and diffusion volatility parameter are known. However, insurance companies do not have these probability distributions, instead they have the data information on the surplus flow levels, individual claim sizes and claim numbers. So the big challenge is to figure out how to estimate the Gerber-Shiu function based on the data information.

Recently, non-parametric estimation of risk measures has emerged as a hot topic in risk theory. For the classical model, [19] proposed a semi parametric estimator of the ruin probability, [20] constructed a non parametric estimator for ruin probability by Fourier inversion and kernel density estimation method, [21] estimated the finite time ruin by double Fourier transform. For the Wiener-Poisson risk model, [22] presented the estimator of the Gerber-Shiu function based on a regularized inversion of the Laplace transform, [23] presented an alternative method, the Fourier sinc series expansion to ameliorate the regularized inversion of Laplace transform, [24] proposed a new method for the estimation of the Gerber-Shiu function, Laguerre series expansion method which is not based on the Fourier transform and the Laplace transform.

In this paper, we implement the Fourier cosine series expansion to estimate the Gerber-Shiu function in the risk model (1). The Fourier cosine series expansion has some applications in pricing financial products and it is also named the COS method in the literature. In the field of risk theory, [25] approximated the Gerber-Shiu function under the Levy subordinator model, [26] studied the density of the time to ruin in the classical compound Poisson risk model and [27] estimated the expected discounted penalty function in the Levy risk model based on the Fourier cosine series expansion.

The remainder of this paper is organized as follows. In Section 2, we introduce the Fourier cosine series expansion. In Section 3, we show how to construct our estimator based on the COS method. The convergence rate of the estimator is derived in Section 4. Finally, some numerical simulations are presented in Section 5 to show that the estimator performs well when the sample size is finite and Section 6 is the conclusion.

2. Fourier Cosine Series Expansion

This section presents some known results of the COS method. For an integrable function g with a finite support $[a_1, a_2]$, we have the following Fourier Cosine series expansion,

$$g(x) = {\sum_{k=0}^{\infty}}' A_{g,k} \cos\left(k\pi \frac{x - a_1}{a_2 - a_1}\right) \tag{3}$$

where $\sum'$ indicates that the first term in the summation is weighted by one-half, and the cosine coefficients are given by

$$A_{g,k} = \frac{2}{a_2 - a_1} Re\left\{\int_{a_1}^{a_2} g(x) \exp\left(ik\pi \frac{x - a_1}{a_2 - a_1}\right) \mathrm{d}x\right\} \tag{4}$$

Let define $\mathcal{F}g(s) = \int \mathrm{e}^{isx} g(x)\mathrm{d}x$ as the Fourier transform of g. Assume that

$$\int_{a_1}^{a_2} \mathrm{e}^{isx} g(x)\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{isx} g(x)\mathrm{d}x = \mathcal{F}g(s) \tag{5}$$

Then

$$A_{g,k} \approx B_{g,k} := \frac{2}{a_2 - a_1} Re\left\{ \mathcal{F}g\left(\frac{k\pi}{a_2 - a_1}\right) \exp\left(\frac{-ik\pi a_1}{a_2 - a_1}\right) \right\} \tag{6}$$

Finally, the approximation of our function is given by

$$g(x) \approx \tilde{g}(x) := \sum_{k=0}^{K}{}' B_{g,k} \cos\left(k\pi \frac{x - a_1}{a_2 - a_1}\right) \tag{7}$$

The approximation error for the COS method is provided in [28] by the following Lemma.

Lemma 1. For the real-value integrable function g supported on $(0,\infty]$, suppose that $\left|g'(0^+)\right| < \infty$, $\left|g'(a)\right| < \infty$ and $\int_0^\infty \left|g''(y)\mathrm{d}y\right| < \infty$. Then, for some positive constants C_1 and C_2, we have

$$\sup_{x \in [0,a]} \left|g(x) - \tilde{g}(x)\right| \le C_1 a K^{-1} + C_2 K a^{-1} \int_a^\infty \left|g(y)\right| \mathrm{d}y. \tag{8}$$

Remark. Lemma 8 shows that COS method has two types of approximation errors depending on the parameter a which is in fact an integration domain truncation parameter in the COS method and the parameter K which is used in the approximation of the of the COS coefficients by the Fourier transform $\mathcal{F}g$. The first error term aK^{-1} means that larger K yields better approximation, but large a may slowdown the convergence rate. The second error term $C_2 K a^{-1} \int_a^\infty \left|g(y)\right| \mathrm{d}y$ means that larger a can result in more accurate approximation of the COS coefficient. However, this error term is increasing with respect to K since larger K means more COS coefficients have to be approximated.

3. Estimation Procedure

In this section, we shall first approximate the expected discounted penalty function by the COS method. Afterward, we shall replace the estimate values in the approximate function to have our estimator.

3.1. The COS Approximation Method

We consider how to use the COS method to approximate the expected discounted penalty function $\phi(u)$. The Laplace transform of the Gerber-Shiu function is given as follow, see [29]

$$\mathcal{L}\phi(s) = \frac{\frac{\sigma^2}{2} w_0(s - \rho) + \lambda\left[\mathcal{L}\omega(\rho) - \mathcal{L}\omega(s)\right]}{\psi_U(s) - \delta} \tag{9}$$

where $w_0 = w(0,0)$ and $\omega(u) = \int_u^\infty w(u, x-u) f_X(x)\mathrm{d}x, u \ge 0$.

From the relation between Laplace transform and Fourier transform, it follows that

$$\mathcal{F}\phi(s) = \mathcal{L}\phi(-is) = \frac{\frac{\sigma^2}{2} w_0(-is - \rho) + \lambda\left[\mathcal{L}\omega(\rho) - \mathcal{F}\omega(s)\right]}{\psi_U(-is) - \delta} \tag{10}$$

By Formula 7, we use the COS method approximation as follow

$$\phi(u) \approx \tilde{\phi}(u) := \sum_{k=0}^{K}{}' B_{\phi,k} \cos\left(k\pi \frac{u}{a}\right), \quad 0 \le u \le a \tag{11}$$

where the COS coefficients are given by:

$$B_{\phi,k} := \frac{2}{a} Re\left\{\mathcal{F}\phi\left(\frac{k\pi}{a}\right)\right\}$$

Remark. The approximation error can also be obtained Lemma 8. Furthermore, suppose that $\left|\phi'(0^+)\right| < \infty$, $\left|\phi''(0^+)\right| < \infty$, $\left|\phi'(a)\right| < \infty$, $\left|\phi''(a)\right| < \infty$ and $\int_0^\infty \left|\phi'(u)\right| \mathrm{d}u < \infty$, $\int_0^\infty \left|\phi''(u)\right| \mathrm{d}u < \infty$ and let C be a positive constant, then from Lemma 8 and Remark 2, we can obtain

$$\left|\tilde{\phi}(u) - \phi(u)\right| \le C \cdot \xi_\phi(a, K) \tag{12}$$

where

$$\xi_\phi(a, K) = aK^{-1} + Ka^{-1} \int_a^\infty \phi(u) \mathrm{d}u + Ka^{-1} \int_a^\infty \left|\phi'(u)\right| \mathrm{d}u \tag{13}$$

3.2. The COS Estimation Method

Since we have approximated the expected discounted penalty function, it can now be estimated based of the dataset of the surplus flow level, a random sample on individual claim sizes and claim numbers.

Let us assume that the premium rate c is constant, but the Poisson parameter λ and the claim size density function f_X are both unknown, but as in [23], we assume that the surplus process can be observed over a long time interval $[0, T]$. Let $\Delta \ge 0$ be a sampling interval. Without loss of generality, we assume that T/Δ is an integer and let $n = T/\Delta$.

Suppose that the insurer can get the following data-set.

- Data-set of surplus level: $\{U_{j\Delta} : j = 0,1,2,\cdots,n\}$ Where $U_{j\Delta}$ is the observed surplus level at time $t = j\Delta$
- Data-set of claim numbers and claim sizes:

$\{N_{j\Delta}, X_1, X_2, \cdots, X_{N_{j\Delta}}\}, j = 0,1,2,\cdots,N_T$ where $N_{j\Delta}$ is the total claim number up to time $t = j\Delta$.

We shall propose an estimator for the expected discounted penalty function $\phi(u)$. Obviously, we need to estimate the following quantities
$\sigma^2, \lambda, \rho, \psi_U'(\rho), \mathcal{F}f_X(s), \tilde{h}_+(u)$ and $\tilde{g}_+(u)$. As in [23], we can estimate
$\lambda, \rho, \mathcal{F}f_X(s)$ and $\mathcal{L}f_X(s)$ by

$$\hat{\sigma}^2 = \frac{1}{n\Delta} \sum_{j=1}^{N_T} \left[U_{j\Delta} - U_{(j-1)\Delta} - c\Delta + \sum_{k=N_{(j-1)\Delta+1}}^{N_{j\Delta}} X_k \right]^2$$

$$\hat{\lambda} = \frac{N_T}{T}$$

$$\widehat{\mathcal{F}f}_X(s) = \frac{1}{N_T} \sum_{j=1}^{N_T} \mathrm{e}^{isX_j}$$

$$\widehat{\mathcal{L}f}_X(s) = \frac{1}{N_T}\sum_{j=1}^{N_T} e^{-sX_j}$$

It is known that $\hat{\rho} \in \left[\left(-c+\sqrt{c^2+2\hat{\sigma}^2\delta}\right)\Big/\hat{\sigma}^2, \frac{\hat{\lambda}+\delta}{c}\right]$ and $\hat{\rho} \geq \delta/c$.

Finally, we propose an estimator for the expected discounted penalty function $\phi(u)$. It is easily seen that

$$\begin{aligned}\mathcal{L}\omega(s) &= \int_0^\infty \int_0^x e^{-su} w(u, x-u) \mathrm{d}u f_X(x) \mathrm{d}x \\ &= \mathbb{E}\left(\int_0^X e^{-su} w(u, X-u) \mathrm{d}u\right)\end{aligned}$$

$$\begin{aligned}\mathcal{F}\omega(s) &= \int_0^\infty \int_0^x e^{isu} w(u, x-u) \mathrm{d}u f_X(x) \mathrm{d}x \\ &= \mathbb{E}\left(\int_0^X e^{isu} w(u, X-u) \mathrm{d}u\right)\end{aligned}$$

see [23]. From which we obtain the estimators for $\mathcal{L}\omega(s)$ and $\mathcal{F}\omega(s)$

$$\widehat{\mathcal{L}\omega}(\hat{\rho}) = \frac{1}{N_T}\sum_{j=1}^{N_T}\int_0^{X_j} e^{-\hat{\rho}u} w(u, X_j - u) \mathrm{d}u$$

$$\widehat{\mathcal{F}\omega}(s) = \frac{1}{N_T}\sum_{j=1}^{N_T}\int_0^{X_j} e^{isu} w(u, X_j - u) \mathrm{d}u$$

Now the Fourier transforms $\mathcal{F}\phi(s)$ can be estimated as follows,

$$\widehat{\mathcal{F}\phi}(s) = \frac{\frac{\hat{\sigma}^2}{2} w_0(-is-\hat{\rho}) + \lambda\left[\widehat{\mathcal{L}\omega}(\hat{\rho}) - \widehat{\mathcal{F}\omega}(s)\right]}{\hat{\psi}_U(-is) - \delta} \tag{14}$$

Then $\phi(u)$ is estimated as follows,

$$\hat{\phi}(u) := \sum_{k=0}^{K}{}' \hat{B}_{\phi,k} \cos\left(k\pi\frac{u}{a}\right), \quad 0 \leq u \leq a \tag{15}$$

where the COS coefficients are given by:

$$\hat{B}_{\phi,k} := \frac{2}{a} Re\left\{\widehat{\mathcal{F}\phi}\left(\frac{k\pi}{a}\right)\right\}$$

4. Consistency Properties

In this section, we derive the consistency properties of our estimators when the observation interval $[0, T]$ is very large. First, we know that $\hat{\lambda} - \lambda = O_P\left(T^{-\frac{1}{2}}\right)$, $\hat{\sigma}^2 - \sigma^2 = O_P\left(T^{-\frac{1}{2}}\right)$

Lemma 2. Suppose that $c \geq \lambda\mathbb{E}(X)$, $\mathbb{E}(X^2) < \infty$ and $a = o(K)$ and $\mathbb{E}\left(\int_0^X w(X-x)\mathrm{d}x\right) \leq \infty$, $\mathbb{E}\left(\int_0^X xw(X-x)\mathrm{d}x\right)^2 \leq \infty$, then we have $\hat{\rho} - \rho = O_p\left(T^{-1/2}\right)$ and

$$\sup_{0\leq u\leq a}\left|\hat{\phi}(u) - \tilde{\phi}(u)\right| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right)$$

Proof. The convergence of $\hat{\rho}-\rho=O_p\left(T^{-1/2}\right)$ is well known, see [30].

For the Gerber-Shiu function, we have that

$$\begin{aligned}
\sup_{0\le u\le a}\left|\hat{\phi}(u)-\tilde{\phi}(u)\right| &= \sup_{0\le u\le a}\left|\sum_{k=0}^{K}{}'\hat{B}_{\phi,k}\cos\left(k\pi\frac{u}{a}\right)-\sum_{k=0}^{K}{}'B_{\phi,k}\cos\left(k\pi\frac{u}{a}\right)\right| \\
&\le \sup_{0\le u\le a}\left|\sum_{k=0}^{K}{}'\left(\hat{B}_{\phi,k}-B_{\phi,k}\right)\right| \\
&\le \sup_{0\le u\le a}\left|\sum_{k=0}^{K}{}'\frac{2}{a}\left[Re\left\{\widehat{\mathcal{F}\phi}\left(\frac{k\pi}{a}\right)\right\}-Re\left\{\mathcal{F}\phi\left(\frac{k\pi}{a}\right)\right\}\right]\right| \qquad (16) \\
&\le \sup_{s\in S}\frac{2}{a}\sum_{k=0}^{K}{}'1\left|\widehat{\mathcal{F}\phi}(s)-\mathcal{F}\phi(s)\right| \\
&= \frac{2(K+1)}{a}\sup_{s\in S}\left|\widehat{\mathcal{F}\phi}(s)-\mathcal{F}\phi(s)\right|
\end{aligned}$$

where

$$\begin{aligned}
\sup_{s\in S}\left|\widehat{\mathcal{F}\phi}(s)-\mathcal{F}\phi(s)\right| &\le \sup_{s\in S}\left|\frac{\frac{\hat{\sigma}^2}{2}w_0(-is-\hat{\rho})}{\hat{\psi}_U(-is)-\delta}-\frac{\frac{\sigma^2}{2}w_0(-is-\rho)}{\psi_U(-is)-\delta}\right| \\
&\quad+\sup_{s\in S}\left|\frac{\lambda\left[\widehat{\mathcal{L}\omega}(\hat{\rho})-\widehat{\mathcal{F}\omega}(s)\right]}{\hat{\psi}_U(-is)-\delta}-\frac{\lambda\left[\mathcal{L}\omega(\rho)-\mathcal{F}\omega(s)\right]}{\psi_U(-is)-\delta}\right| \\
&= M_1(s)+M_2(s)
\end{aligned}$$

It is shown in [31] that $M_1(s)=M_2(s)=O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right)$ Hence,

$$\sup_{s\in S}\left|\widehat{\mathcal{F}\phi}(s)-\mathcal{F}\phi(s)\right|=O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right) \qquad (17)$$

Then by formula 16 and 17, we can conclude that

$$\sup_{0\le u\le a}\left|\hat{\phi}(u)-\tilde{\phi}(u)\right|=O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right)$$

5. Numerical Simulation

In this section, we present some numerical results to show the effectiveness of our method. All computations are done in MATLAB on a EliteBook, with Intel(R) Core(TM) i5-6300U CPU@2.40GHz 2.50GHz and a RAM of 8GB. Throughout this section, we set $c=8$, $\lambda=5$, $\delta=0.1$ and consider claim size density functions, the exponential EXP(1): $f_X(x)=e^{-x}, x>0$ and the Erlang (2, 2): $f_X(x)=4xe^{-2x}, x>0$. Then the true value of the Gerber-Shiu function can be found in [23]. Now for these two claim size density functions, closed form of Fourier transforms exist so that the COS method approximation of $\tilde{\phi}(u)$ can be computed. When using Fourier Cosine Series expansion method to approximate $\phi(u)$, we apply the cumulant method given in [32] to determine the parameter *a*. We take $L=10$, $K=2^7$ to provide the benchmark.

First of all, we test the effectiveness of the Fourier Cosine method for approximating $\phi(u)$. In **Table 1** and **Table 2**, we present some average relative errors and average absolute errors for $\tilde{\phi}(u)$ respectively for exponential and Erlang distributions which are calculated by the following

$$\text{Average relative errors}: \frac{1}{\#\mathcal{U}} \sum_{u \in \mathcal{U}} \frac{\left|\tilde{\phi}(u) - \phi(u)\right|}{\phi(u)}$$

$$\text{Average absolute errors}: \frac{1}{\#\mathcal{U}} \sum_{u \in \mathcal{U}} \left|\tilde{\phi}(u) - \phi(u)\right|$$

Here, we take $\mathcal{U} = \{1, 2, \cdots, 15\}$, since when $\mathcal{U} > 15$, $\phi(u)$ is very small. We consider the truncation parameter $K = 2^q$, $q = 4, 5, 6, 7$. It can be observed that in each column of **Table 1** and **Table 2**, both average relative errors and the average absolute errors are decreasing w.r.t. *q*, which implies that large truncation parameter can reduce the bias under the model setting. In **Table 1** and **Table 2**, we find that the average relative errors are larger than the average absolute errors, which is due to that $\phi(u)$ is always smaller than 1, and in particular, it is close to zero for large initial surplus.

Next we test the performance of our estimator $\hat{\phi}(u)$. We fixe $K = 2^7$. As for the observation interval $[0, T]$, We shall take $T = 2^p$ for $p = 0, 1, 2, 3, 4, 5$. For the COS parameter *a*, we also apply the cumulant method given in [32]. We take $\hat{a} = \widehat{k}_1 + L\sqrt{\widehat{k}_2 + \sqrt{\widehat{k}_4}}$ where $\widehat{k}_j = \frac{d^j}{dj} \log \widehat{\mathcal{F}f}(-is)\Big|_{s=0}$. We repeat 300 simulations, and compute empirical average relative errors and average absolute errors for $\hat{\phi}(u)$. Which are defined by

$$\text{Average relative errors}: \frac{1}{\#\mathcal{U}} \sum_{u \in \mathcal{U}} \frac{1}{300} \sum_{j=1}^{300} \frac{\left|\hat{\phi}_j(u) - \phi(u)\right|}{\phi(u)}$$

$$\text{Average absolute errors}: \frac{1}{\#\mathcal{U}} \sum_{u \in \mathcal{U}} \frac{1}{300} \sum_{j=1}^{300} \left|\hat{\phi}_j(u) - \phi(u)\right|$$

where $\hat{\phi}_j(u)$ denote the j^{th} simulation values of $\hat{\phi}(u)$. In **Table 3** and **Table 4**, we present the empirical estimation errors for $\hat{\phi}(u)$ respectively for exponential and Erlang distributions. As expected, both the empirical average errors and empirical absolute errors are decreasing w.r.t. p which is due to that, as p increases (or equivalently T increases), more sample can be used to estimate $\hat{\phi}(u)$. Again, we observe that the average relative errors are larger than the average absolute errors.

Finally, we plot 300 consecutive estimators (green curves) on the same picture together with the true curve (red curve) to illustrate variability bands and show the stability of the procedures. In **Figure 1**, we observe that the beams of estimators are much closed to the true curves. In particular, it follows that the variances for estimating $\phi(u)$ are very small for *a* large observation interval. We find that $\phi(u)$ is a decreasing function of the initial surplus *u*, which means that the ruin is more likely to happen when u is small. At the same time, we can also observe that as T increases, the estimator tends to be stable and converges to $\phi(u)$. We plot the corresponding curves. Again, we can observe that the estimator becomes better as T becomes larger.

Table 1. Approximation Errors for Gerber-Shiu function $\tilde{\phi}(u)$ based on EXP (1).

q	Empirical average relative errors	Empirical relative absolute errors
4	0.08763	0.001577
5	0.01456	0.08598
6	0.01423	0.0006745
7	0.01367	0.0005797

Table 2. Approximation Errors for the Gerber-Shiu function $\tilde{\phi}(u)$ based on Erlang (2, 2).

q	Empirical average relative errors	Empirical relative absolute errors
4	0.7763	0.05377
5	0.6556	0.05198
6	0.4223	0.009745
7	0.2367	0.005797

Table 3. Estimation Errors Errors for the Gerber-Shiu function $\tilde{\phi}(u)$ based on EXP (1).

T	Empirical average relative errors	Empirical relative absolute errors
1000	0.13347	0.0811
2000	0.13050	0.0811
3000	0.120248	0.0749
4000	0.114248	0.0646

Table 4. Estimation Errors for the Gerber-Shiu function $\tilde{\phi}(u)$ based on Erlang (2, 2).

T	Empirical average relative errors	Empirical relative absolute errors
1000	0.15347	0.0861
2000	0.14550	0.0831
3000	0.141248	0.0799
4000	0.112248	0.00726

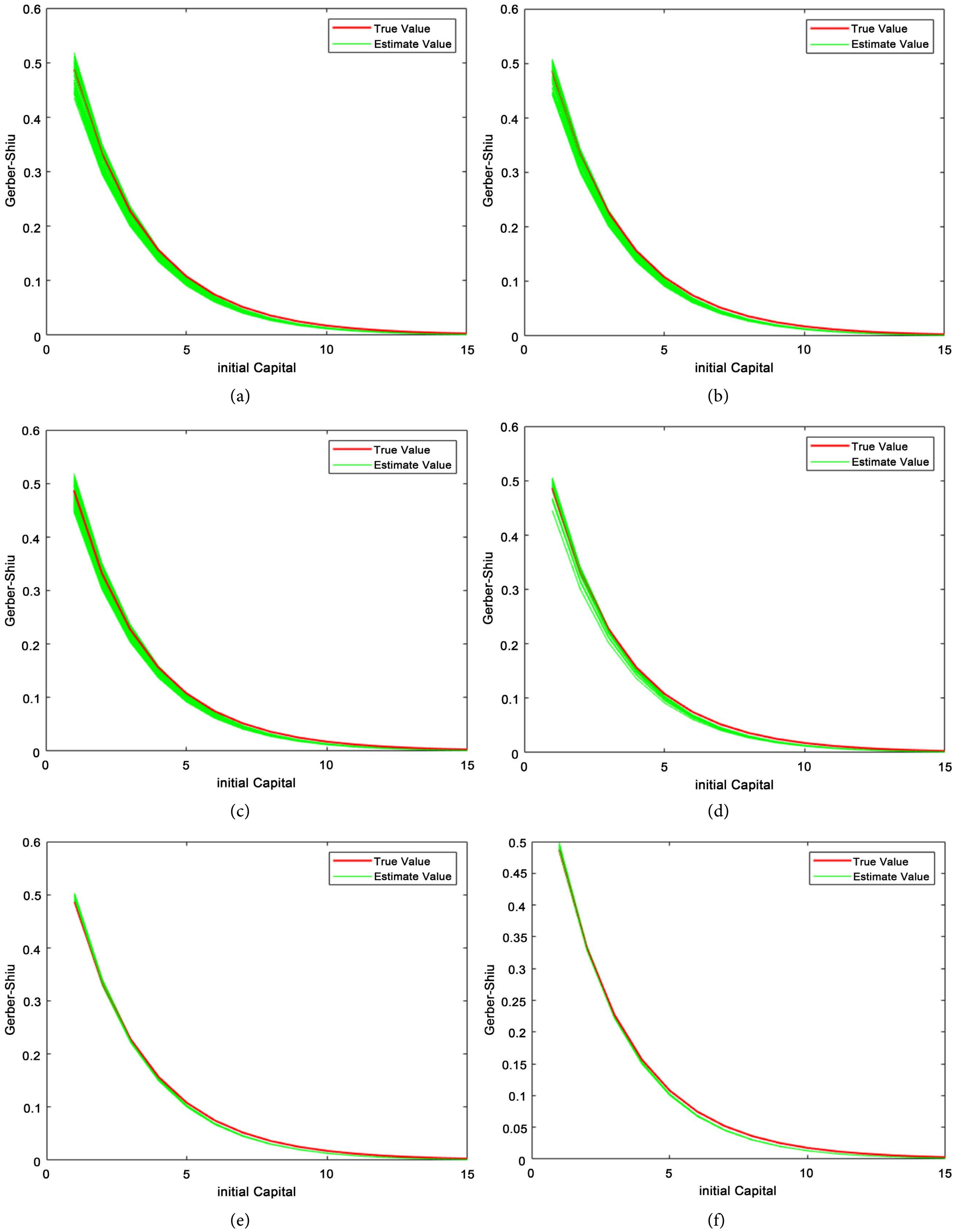

Figure 1. Beams for estimating the Gerber-Shiu function $\phi(u)$: 300 estimators in green, and the true value in bold red. (a) q = 1000; (b) q = 2000; (c) q = 3000; (d) q = 4000; (e) q = 5000; (f) q = 6000.

6. Conclusion

In this paper, we have proposed a new estimator of the expected discounted penalty function in the perturbed compound Poisson risk model. Our estimator is based on the COS method. The COS coefficient is easily derived based on the Fourier transform of the Gerber-Shiu function which is useful when approximating the function by the COS method. Suppose we have the surplus flow level, the claim number, and the claim sizes over a long-term interval. Then, we construct our estimator based on the COS approximation we have derived, by replacing the different functions with their estimates. We have derived theoretical errors and presented some simulation results to show the effectiveness of our estimator. We have shown that our estimator has an accurate convergence rate. Since the COS method does not utilize the Fast Fourier transform algorithm, our estimator is easier to compute compared to the Fourier sinc method. Since this method and the Fourier sinc method use the Fourier transform of the Gerber-Shiu, the comparison of the two methods could be future research.

Acknowledgements

The authors would like to thank the Pan African University Institute of Basic Sciences Technology and Innovation (PAUSTI) for their financial assistance.

Conflicts of Interest

The authors declare that there is no conflict of interests.

References

[1] Lundberg, F. (1903) 1. Approximerad framställning af sannolikhetsfunktionen: 2. Återförsäkring af kollektivrisker. Almqvist &Wiksells, Stockholm.

[2] Lundberg, F. and Riskutjämning, F. (1926) I: Teori. II: Statistik (Insurance Technical Smoothing of Risks) F. Englunds Boktryckeri AB, Stockholm.

[3] Constantinescu, C., Samorodnitsky, G. and Zhu, W. (2018) Ruin Probabilities in Classical Risk Models with γ Claims. *Scandinavian Actuarial Journal Taylor & Francis*, **2018**, 555-575. https://doi.org/10.1080/03461238.2017.1402817

[4] Gerber, H.U. and Shiu, E.S.W. (1998) On the Time Value of Ruin. *North American Actuarial Journal*, **2**, 48-72. https://doi.org/10.1080/10920277.1998.10595671

[5] Mnatsakanov, R., Ruymgaart, L.L. and Ruymgaart, F.H. (2008) Nonparametric Estimation of Ruin Probabilities Given a Random Sample of Claims. *Mathematical Methods of Statistics*, **17**, 35-43. https://doi.org/10.3103/S1066530708010031

[6] He, Y., Kawai, R., Shimizu, Y. and Yamazaki, K. (2022) The Gerber-Shiu Discounted Penalty Function: A Review from Practical Perspectives. *Insurance: Mathematics and Economics*, **109**, 1-28. https://doi.org/10.1016/j.insmatheco.2022.12.003

[7] Wang, W.Y. and Zhang, Z.M. (2019) Computing the Gerber-Shiu Function by Frame Duality Projection. *Scandinavian Actuarial Journal*, **2019**, 291-307. https://doi.org/10.1080/03461238.2018.1557739

[8] Xie, J.Y. and Zhang, Z.M. (2022) Recursive Approximating to the Finite-Time Gerber-Shiu Function in Lévy Risk Models under Periodic Observation. *Journal of*

Computational and Applied Mathematics, **399**, Article ID: 113703. https://doi.org/10.1016/j.cam.2021.113703

[9] Li, X.L., Shi, Y.F., Phillip, Y., Sheung, C. and Yang, H.L. (2021) Fourier-Cosine Method for Finite-Time Gerber-Shiu Functions. *Scandinavian Actuarial Journal*, **43**, B650-B677. https://doi.org/10.1137/20M1328580

[10] Gerber, H.U. (1970) An Extension of the Renewal Equation and Its Application in the Collective Theory of Risk. *Scandinavian Actuarial Journal*, **1970**, 205-210. https://doi.org/10.1080/03461238.1970.10405664

[11] Dufresne, F. and Gerber, H.U. (1991) Risk Theory for the Compound Poisson Process That Is Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **10**, 51-59. https://doi.org/10.1016/0167-6687(91)90023-Q

[12] Cai, J. and Yang, H.L. (2005) Ruin in the Perturbed Compound Poisson Risk Process under Interest Force. *Advances in Applied Probability*, **37**, 819-835. https://doi.org/10.1239/aap/1127483749

[13] Li, S.M. (2006) The Distribution of the Dividend Payments in the Compound Poisson Risk Model Perturbed by Diffusion. *Scandinavian Actuarial Journal*, **2006**, 73-85. https://doi.org/10.1080/03461230600589237

[14] Yin, C.C. and Wang, C.W. (2010) The Perturbed Compound Poisson Risk Process with Investment and Debit Interest. *Methodology and Computing in Applied Probability*, **12**, 391-413. https://doi.org/10.1007/s11009-008-9109-z

[15] Zhang, Z.M., Yang, H. and Li, S.M. (2010) The Perturbed Compound Poisson Risk Model with Two-Sided Jumps. *Journal of Computational and Applied Mathematics*, **233**, 1773-1784. https://doi.org/10.1016/j.cam.2009.09.014

[16] Tsai, C.C.L. (2001) On the Discounted Distribution Functions of the Surplus Process Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **28**, 401-419. https://doi.org/10.1016/S0167-6687(01)00067-1

[17] Tsai, C.C.L. and Willmot, G.E. (2002) A Generalized Defective Renewal Equation for the Surplus Process Perturbed by Diffusion. *Insurance: Mathematics and Economics*, **30**, 51-66. https://doi.org/10.1016/S0167-6687(01)00096-8

[18] Zhang, Z.M. and Yang, H. (2011) Gerber-Shiu Analysis in a Perturbed Risk Model with Dependence between Claim Sizes and Interclaim Times. *Journal of Computational and Applied Mathematics*, **235**, 1189-1204. https://doi.org/10.1016/j.cam.2010.08.003

[19] Masiello, E. (2014) On Semiparametric Estimation of Ruin Probabilities in the Classical Risk Model. *Scandinavian Actuarial Journal*, **2014**, 283-308. https://doi.org/10.1080/03461238.2012.690247

[20] Zhang, Z.M., Yang, H.L. and Yang, H. (2014) On a Nonparametric Estimator for Ruin Probability in the Classical Risk Model. *Scandinavian Actuarial Journal*, **2014**, 309-338. https://doi.org/10.1080/03461238.2012.691427

[21] Zhang, Z.M. (2017) Nonparametric Estimation of the Finite Time Ruin Probability in the Classical Risk Model. *Scandinavian Actuarial Journal*, **2017**, 452-469. https://doi.org/10.1080/03461238.2016.1174876

[22] Shimizu, Y. (2012) Non-Parametric Estimation of the Gerber-Shiu Function for the Wiener—Poisson Risk Model. *Scandinavian Actuarial Journal*, **2012**, 56-69. https://doi.org/10.1080/03461238.2010.523515

[23] Zhang, Z.M. (2017) Estimating the Gerber-Shiu function by Fourier-Sinc Series Expansion. *Scandinavian Actuarial Journal*, **2017**, 898-919. https://doi.org/10.1080/03461238.2016.1268541

[24] Su, W., Yong, Y.D. and Zhang, Z.M. (2019) Estimating the Gerber-Shiu Function in the Perturbed Compound Poisson Model by Laguerre Series Expansion. *Journal of Mathematical Analysis and Applications*, **469**, 705-729. https://doi.org/10.1016/j.jmaa.2018.09.033

[25] Su, W. and Wang, Y.Y. (2021) Estimating the Gerber-Shiu Function in Lévy Insurance Risk Model by Fourier-Cosine Series Expansion. *Mathematics*, **9**, Article 1402. https://doi.org/10.3390/math9121402

[26] Zhang, Z.M. (2017) Approximating the Density of the Time to Ruin via Fourier-Cosine Series Expansion. *ASTIN Bulletin*: *The Journal of the IAA*, **47**, 169-198. https://doi.org/10.1017/asb.2016.27

[27] Huang, Y.J., Yu, W.G., Pan, Y. and Cui, C.R. (2019) Estimating the Gerber-Shiu Expected Discounted Penalty Function for Lévy Risk Model. *Discrete Dynamics in Nature and Society*, **2019**, Article ID: 3607201. https://doi.org/10.1155/2019/3607201

[28] Xie, J.Y. and Zhang, Z.M. (2020) Statistical Estimation for Some Dividend Problems under the Compound Poisson Risk Model. *Insurance*: *Mathematics and Economics*, **95**, 101-115. https://doi.org/10.1016/j.insmatheco.2020.09.002

[29] Morales, M. (2007) On the Expected Discounted Penalty Function for a Perturbed Risk Process Driven by a Subordinator. *Insurance*: *Mathematics and Economics*, **40**, 293-301. https://doi.org/10.1016/j.insmatheco.2006.04.008

[30] Shimizu, Y. (2011) Estimation of the Expected Discounted Penalty Function for Lévy Insurance Risks. *Mathematical Methods of Statistics*, **20**, 125-149. https://doi.org/10.3103/S1066530711020037

[31] Yang, Y., Xie, J.Y. and Zhang, Z.M. (2023) Nonparametric Estimation of Some Dividend Problems in the Perturbed Compound Poisson Model. *Probability in the Engineering and Informational Sciences*, **37**, 418-441. https://doi.org/10.1017/S0269964822000298

[32] Fang, F. and Oosterlee, C.W. (2009) A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions. *SIAM Journal on Scientific Computing*, **31**, 826-848. https://doi.org/10.1137/080718061