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Portfolio Management Problem with Stochastic Wage Income and Inflation-Adjusted Wealth

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Abstract

In this paper, the portfolio management problem with stochastic wage income and inflation risk for CRRA investors is solved. In real life, investors experience stochastic wage income and inflation risk. This could be due to events such as COVID-19, fiscal policy, financial policy adjustments, and climate change. We consider an agent who invests in the financial market with one risk-free security (e.g. a money market account or bond) and one risky security (e.g. a stock or stock index). Our goal is to choose the optimal controls that maximize the objective function in order to obtain the value function. By applying Dynamic Programming Principle, we determine the HJB PDE. Solving the HJB PDE, we establish the value function and optimal controls.

Keywords

Portfolio Management, Inflation Risk, Dynamic Programming Principle, Stochastic Wage, CRRA Investor

1. Introduction

In modern finance, stochastic optimal control problems are key in practice. Such problems are a major concern to individual and institutional investors who seek to allocate wealth among various assets over a certain or uncertain lifetime. Our study was motivated by the need to adequately address challenges emanating from randomness and uncertainty in the portfolio management of modern finance. So far, different researchers have explicitly solved stochastic optimal control problems via methods, such as the dynamic programming principle, the maximum principle, the viscosity solution concept, and Backward Stochastic Differential Equations (BSDEs). Viscosity solutions and BSDE have a strong link through their PDE representation. This research work builds on the celebrated work of [1] [2] who solved the optimal control problem for an agent who invests in one risk-free asset and one risky asset but under constant interest rate and volatility.

The celebrated work of [3] was the first to introduce Mean-Variance (MV) optimal portfolio selection problems in discrete time. Markowitz defined an optimal portfolio as an efficient portfolio (frontier) for any investor. The Mean-Variance (MV) model, formulated the portfolio choice problem as an optimization problem without accounting for the consumption of an investor. According to this study, the MV model minimizes the variance of the terminal wealth for a desired level of expectation. This study also showed that there are possibilities for different portfolios to have a different combination of return and risk. The second pioneer work of [1] [2] investigated portfolio optimization based on the Utility functions. The theory of portfolio optimization is based on the preferences of an investor as described by a utility function. Merton solved a stochastic control problem in continuous time for a financial market consisting of one risk-free asset and one risky asset. The Hamilton-Jacobi-Bellman equation for the value function was determined by applying the dynamic programming principle. Merton showed that closed-form solutions exist for such non-linear partial differential equations. In the academic literature, there is a wide range of portfolio selection problems. Most common are problems formulated either in the Mean-Variance framework pioneered by [3] or problems of expected utility maximization type pioneered by [1] [2] for a diffusion-type model. In recent years, jump models have become increasingly popular in academic and financial research to explain randomness. This is due to the shortcomings of the classical Brownian motion model developed in [4]. Studies so far have shown that stock market returns have higher peaks and heavier tails see [5] [6] [7]. Often, jumps occur in the prices of stocks that cannot be explained by a Brownian motion model. These jumps also have large down movements in stock prices, but not equally large up movements. Another feature often observed in stock price distributions is that large changes in prices are often followed by large changes and small changes tend to be followed by small changes. So far, researchers have solved these observations using the jump models which capture many of the empirical features of stock price returns. However, because of the independent increment property (*i.e.* the Markov property), jump processes may not model the effect of volatility clustering. On the other hand, jump models are generally assumed to have finite jumps during a finite time interval which represent rare events in real life. This is our motivation to study diffusion-type models.

This study builds on the celebrated work of Merton to include inflation risk and stochastic wage income. The inflation risk is modeled as a Consumer Price Index (CPI). In real life, a financial market experience inflation risk and stochastic wage due to uncertain events such as COVID-19, climate change, wars, inflation, natural disasters, fiscal policy, and financial policy adjustments. Inflation risk parameters as well as stochastic wage affect optimal investment decisions. We consider the stochastic control problem of a single investor with a portfolio consisting of one risky-free security (e.g. a money market account or bond) and one risky security (e.g. a stock or stock index). Our goal is to choose the optimal investment policy that maximizes terminal wealth. The investor preferences are modeled as a Constant Relative Risk Aversion (CRRA) function and trading takes place in a finite horizon.

The study of [1] [2] considered constant interest rate and constant volatility rate. However, such assumptions are not practical in modern finance. In this study, we extend Merton's work to include stochastic wage income and inflation risk simultaneously. This results in sophisticated HJB PDEs. This paper outlines some new results in the field of Mathematics of Finance. The development of these new ideas was motivated by the need to adequately address challenges emanating from the interaction effects of randomness and uncertainty in the portfolio management of modern finance. The most important contribution of this study is that we have extended Merton's problems with a unique mixture of stochastic wage income and inflation risk simultaneously.

2. Links to the Literature

The problem of optimal investment has attracted a number of extensions. For instance, [8] investigated portfolio selection problems in a stochastic environment including inflation risk, and also apply Dynamic Programming Principle (DPP) to determine the value function and optimal policies. [9] applied a duality approach in solving a stochastic control problem. [10] used the duality approach to portfolio optimization problems with borrowing and short-sale constraints. [11] considered an optimal investment for a pension fund under inflation risk by applying the Martingale method for a financial market consisting of a money account, a stock, and an inflation-linked bond. In the paper by [12], dynamic asset allocation under inflation was investigated by applying DPP to determine the value function and optimal policies. [13] investigated a stochastic control problem with stochastic volatility and constant interest rate. [14] considered optimal Investment-Consumption Strategy under Inflation in a Markovian Regime-Switching Market. [15] researched optimal portfolio selection with life insurance under Inflation Risk for CRRA investors. This paper analyzed how risk aversion, the correlation coefficient between inflation and the stock price, the inflation parameters, and the coefficient of utility affect the optimal investment and consumption strategy. [16] investigated optimal investment, consumption and insurance problems. They considered a market with a real zero coupon bond, the inflation-linked real money account and a risky share following a jump-diffusion process. They applied a Backward Stochastic Differential Equation (BSDE) with jumps to derive the explicit solutions. A paper by [17] was the first to extend Merton's work to include life insurance in a study titled optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. [18] extended Merton's work by adding life insurance but with constant labor income in the study titled optimal life insurance purchase and consumption/investment under uncertain lifetime. In their study, the agent has initial wealth but also receives an income continuously which can be terminated upon premature death. In this study, Merton's work is extended in a unique way by studying the stochastic control problem for an agent who faces inflation risk and stochastic wage income. Such assumptions are realistic and practical in the real financial world. Our goal is to allocate initial wealth between a risk-free asset account and a risky asset account in order to maximize the discounted expected utility of terminal wealth.

The outline of this paper is as follows. Section 2 is introduction. Section 3 is literature review. Section 4 is description of the financial market model. In Section 5, the wealth model is determined. Section 6 is optimization criterion description. In Section 7, the Hamilton-Jacobi-Bellman (HJB) equation for the value function is derived. In Section 8, we investigate the value function and optimal policy. In Section 9, numerical examples and simulations are provided. Here, the effect of market parameters on the optimal investment policy is illustrated. In Section 10, the conclusion and suggested possible future research work are stated.

3. Financial Market Model

Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space with filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions such as $(\mathcal{F}_t)_{0\le t\le T}$ being right continuous complete filtration and $\mathbb{P}$-complete. Let all stochastic processes be well-defined and adapted in the filtered complete probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Consider a stochastic control problem of a single investor with a portfolio consisting of one risky-free security (e.g. a money market account or bond) B and one risky security (e.g. a stock or stock index) $S(t)$. Let the price dynamics of the risk-free security B evolve as follows:

$$\begin{cases} \mathrm{d}B(t) = r(t)B(t)\mathrm{d}t, \\ B(0) = 1, \end{cases} \tag{1}$$

with a constant risk-free interest rate r.

Let the dynamics of the stochastic wage income $\eta(t)$ evolve as follows:

$$\begin{cases} \mathrm{d}\eta(t) = \mu_\eta(t)\eta(t)\mathrm{d}t + \sigma_\eta(t)\eta(t)\mathrm{d}W^\eta, \\ \eta(0) = \eta. \end{cases} \tag{2}$$

Let the inflation risk be controlled by price index (e.g. CPI) denoted by $I(t)$ representing a fixed basket of goods and evolving as follows:

$$\begin{cases} \mathrm{d}I(t) = \mu_I(t)I(t)\mathrm{d}t + \sigma_I I(t)\mathrm{d}W^I(t), \\ I(0) = i, \end{cases} \tag{3}$$

Let the price dynamics of the risky security stock (or share), $S(t)$, follow a Geometric Levy process which is essentially a Geometric Brownian Motion (GBM) model with an added integral for the discontinuous part given by

$$\begin{cases} \mathrm{d}S(t) = S(t)\mu_s\mathrm{d}t + \sigma_s S(t)\mathrm{d}W^S(t), \\ S(0) > 0, \end{cases} \tag{4}$$

where $W^S(t)$ is a Wiener process modeling the random market risk factor, μ_s is the mean rate of return and σ_s is the volatility.

4. The Wealth Model

Consider an investor with initial amount of money $\mathcal{X}(0)>0$ and a time horizon of interest T. Over the time interval, $[0,T]$, the investor changes his portfolio dynamically. Let $\mathcal{C}(t)$ denote the rate of continuous consumption, $1-\pi(t)$ be the amount invested in the risk-free security and $\pi(t)$ denote the wealth to be invested in the risky asset, S. Note that the pair $(\mathcal{C}(t),\pi(t))$ is a trading strategy.

Lemma 1. The net wealth $X(t)$ for an investor who faces stochastic wage income evolves as follows:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{X}(t)}{\mathcal{X}(t)} = (1-\pi(t))\dfrac{\mathrm{d}B(t)}{B(t)} + \pi(t)\dfrac{\mathrm{d}S(t)}{S(t)} - C(t)\mathrm{d}t + \eta(t)\mathrm{d}t \\ \qquad\qquad = \left[\pi\mu_s + (1-\pi)r\right]\mathrm{d}t + \pi\sigma_s\mathrm{d}W^S(t) - \mathcal{C}(t)\mathrm{d}t + \eta(t)\mathrm{d}t, \\ \mathcal{X}(0) > 0. \end{cases}$$

Note that the inflation-adjusted real wealth process at time t denoted by $\bar{X}(t)$ is calculated as follows:

$$\bar{\mathcal{X}}(t) = \frac{\mathcal{X}(t)}{I(t)} \tag{5}$$

Applying Itô lemme on 5, we obtain the following real net wealth model

$$\begin{aligned} \mathrm{d}\bar{\mathcal{X}}(t) &= \frac{1}{I(t)}\mathrm{d}\mathcal{X}(t) + \frac{-\mathcal{X}(t)}{[I(t)]^2}\mathrm{d}I(t) - \frac{1}{[I(t)]^2}\mathrm{d}I(t)\mathrm{d}\mathcal{X}(t) \\ &\quad + \frac{-\mathcal{X}(t)}{[I(t)]^3}\mathrm{d}I(t)\mathrm{d}I(t) - C(t)\mathrm{d}t + \eta(t)\mathrm{d}t \\ &= \frac{\mathcal{X}}{I}\left[\frac{\mathrm{d}\mathcal{X}(t)}{\mathcal{X}(t)} - \frac{\mathrm{d}I(t)}{I(t)} + \frac{\mathrm{d}I^2(t)}{I^2(t)} - \frac{\mathrm{d}\mathcal{X}(t)\mathrm{d}I(t)}{\mathcal{X}(t)I(t)}\right] - \mathcal{C}(t)\mathrm{d}t + \eta(t)\mathrm{d}t. \end{aligned}$$

Simplifying further gives:

$$\begin{cases} \dfrac{\mathrm{d}\bar{\mathcal{X}}(t)}{\bar{\mathcal{X}}(t)} = \left[\pi\mu_s + (1-\pi)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 - C(t) + \eta(t)\right]\mathrm{d}t \\ \qquad\qquad - \sigma_I\mathrm{d}W^I(t) + \pi(t)\sigma_s\mathrm{d}W^s(t) \\ \bar{\mathcal{X}}(0) > 0, \end{cases} \tag{6}$$

where $\bar{\mathcal{X}}(t)$ is inflation adjusted real wealth, $C(t)$ is consumption and $\eta(t)$ is stochastic wage income.

Remark 1. The correlation coefficient $\rho\in[-1,1]$ of W^S and W^η in 4 and 2 is such that $W^S W^\eta = \rho\mathrm{d}t$.

5. The Optimization Criterion

Suppose the set of all admissible strategies is denoted by $\mathcal{A}$.

Definition 5.1. An investment and consumption strategy $\mathcal{A} = \left(\pi(t), \mathcal{C}(t)\right)$ is said to be admissible if the following conditions are satisfied.

1) The pair $\left(\pi(t), \mathcal{C}(t)\right)$ is progressively $\mathcal{F}_t$*-measurable*.

2) $\int_0^T \pi(t)^2 \, \mathrm{d}t < \infty$, $\int_0^T \mathcal{C}(t) \mathrm{d}t < \infty$, for all $T > 0$.

3) If $\left(\pi(t), \mathcal{C}\right)$ is the strategy, the wealth process 6 with $\bar{\mathcal{X}}(0) > 0$ has a path wise unique solution.

Remark 2. *The investor's objective is to maximize the net expected discounted utility of terminal real wealth plus consumption.*

The objective function for this stochastic control problem is then formulated mathematically as follows:

$$J\left(t, x; \pi(t), \mathcal{C}(t)\right) = \mathbb{E}\left[\int_0^T \phi \mathrm{e}^{-\lambda t} U_1\left(\mathcal{C}(t)\right) \mathrm{d}t + (1-\phi)\mathrm{e}^{-\lambda T} U_2\left(\bar{\mathcal{X}}(T)\right)\right]. \tag{7}$$

Definition 5.2. *The value function is defined as*

$$V(t, x) = \sup_{\left(\pi(t), \mathcal{C}(t)\right) \in \mathcal{A}} \mathbb{E}\left[\int_0^T \phi \mathrm{e}^{-\lambda t} U_1\left(\mathcal{C}(t)\right) \mathrm{d}t + (1-\phi)\mathrm{e}^{-\lambda T} U_2\left(\bar{\mathcal{X}}(T)\right)\right], \tag{8}$$

with boundary conditions $V(T, x) = (1-\phi)\mathrm{e}^{-\lambda T} U_2\left(\bar{\mathcal{X}}(T)\right)$.

Here, $\bar{\mathcal{X}}(t) \geq 0$ for all *t*, with T being the date of death, $\bar{\mathcal{X}}(T)$ is the value at time T of a trading strategy. The parameter λ is the subjective discount rate and ϕ determines the relative importance of the intermediate consumption and the bequest. $\mathbb{E}$ denotes the conditional expectation operator. $U_1\left(\mathcal{C}(t)\right)$ and $U_2\left(\bar{\mathcal{X}}(T)\right)$ are consumption and bequest functions respectively.

6. The Hamilton-Jacobi-Bellman Equation

By applying Dynamic Programming Principle, we obtain the Hamilton-Jacobi-Bellman equation (HJB equation) for the value function. The fully HJB PDE associated with the stochastic control problem 8 is the second-order nonlinear PDE given as follows:

$$\begin{aligned} V_t + \sup_{(\pi, \mathcal{C}) \in \mathcal{A}} \Bigg[& \left[\pi\mu_s + (1-\pi)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 - C(t) + \eta(t)\right]\bar{\mathcal{X}} V_x \\ & + \left[\frac{1}{2}\pi^2\sigma_s^2 + \sigma_I^2 - 2\pi\sigma_s\sigma_I\rho\right]\bar{\mathcal{X}}^2 V_{xx} + \mu_\eta \eta(t) V_\eta + \frac{1}{2}\sigma_\eta^2\eta^2(t) V_{\eta\eta} \\ & + \pi\sigma_s\sigma_I\eta(t)\bar{\mathcal{X}} V_{x\eta} + \phi\mathrm{e}^{-\lambda t} U_1(\mathcal{C}) \Bigg] = 0, \end{aligned} \tag{9}$$

where V_t, V_x, V_{xx}, V_η, $V_{\eta\eta}$ and $V_{x\eta}$ denote partial derivatives.

During the phase of retirement, the investor consumes from the accumulated surplus. The HJB PDE in the retirement phase is as follows:

$$\begin{aligned} V_t + \sup_{(\pi, \mathcal{C}) \in \mathcal{A}} \Bigg[& \left[\pi\mu_s + (1-\pi)r - \mu_I - \sigma_s\sigma_I\rho + \sigma_I^2 - C(t)\right]\bar{\mathcal{X}} V_x \\ & + \left[\frac{1}{2}\pi^2\sigma_s^2 + \sigma_I^2 - 2\pi\sigma_s\sigma_I\rho\right]\bar{\mathcal{X}}^2 V_{xx} + \phi\mathrm{e}^{-\lambda t} U_1(\mathcal{C}) \Bigg] = 0. \end{aligned} \tag{10}$$

Thus, the candidate optimal controls for 10 are as follows:

$$\pi^*(t) = -\frac{-(\mu_s - r)\bar{\mathcal{X}}V_x - 2\sigma_s\sigma_I\rho\bar{\mathcal{X}}^2V_{xx}}{\sigma_s^2\bar{\mathcal{X}}V_x} \tag{11}$$

and

$$\mathcal{C}^* = \left[\frac{\bar{\mathcal{X}}V_x}{\phi\mathrm{e}^{-\lambda t}}\right]^{\frac{-1}{\delta}}, \tag{12}$$

where the utility function is defined as

$$U_1(x) = U_2(x) = \frac{x^{1-\delta}}{1-\delta},\ \delta > 0,\ \delta \neq 1, \tag{13}$$

with δ being the risk aversion factor.

7. The Value Function and Optimal Policies

Assume the solution V for 10 take the form

$$V(t,x) = \frac{x^{1-\delta}}{1-\delta}G(t),\ g(T) = 1. \tag{14}$$

Partial derivatives for V are as follows:

$$V_t = \frac{x^{1-\delta}}{1-\delta}G_t,\ V_x = x^{-\delta}G,\ V_{xx} = -\delta x^{-\delta-1}G. \tag{15}$$

Note that 11 and 12 can be simplified further having known 14 and 15 as follows:

$$\pi^*(t) = \frac{(\mu_s - r) + 2\sigma_s\sigma_I\rho\delta}{\delta} \tag{16}$$

and

$$\mathcal{C}^* = \left[\frac{\bar{\mathcal{X}}^{-\delta+1}G}{\phi\mathrm{e}^{-\lambda t}}\right]^{\frac{-1}{\delta}}. \tag{17}$$

Substituting 15, 16 and 17 into 10 gives:

$$\begin{aligned} &\frac{x^{1-\delta}}{1-\delta}G_t + \sup_{\pi\in\mathcal{A}}\Bigg(\Bigg[\Bigg(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Bigg)\mu_s + \Bigg(1 - \frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Bigg)r \\ &-\mu_I - \sigma_s\sigma_I\rho + \sigma_I^2\Bigg]xx^{-\delta}G + \frac{1}{2}\Bigg[\Bigg(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Bigg)^2\sigma_s^2 + \sigma_I^2 \\ &-2\Bigg(\frac{\mu_s - r + \sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\Bigg)\sigma_s\sigma_I\rho\Bigg]x^2\left(-\delta x^{-\delta-1}G\right)\Bigg) + \Theta(t) = 0, \end{aligned} \tag{18}$$

where,

$$\Theta(t) = -C(t)\bar{\mathcal{X}}V_x + \phi\mathrm{e}^{-\lambda t}U_1(\mathcal{C}). \tag{19}$$

Equation (18) is still a Differential Equation (DE) which is difficult to solve. Inspired by [19] [20], we assume G is given by:

$$G = \int_t^T \hat{G}\mathrm{d}u + \hat{G}. \tag{20}$$

This implies18 reduces to the following DE with well-defined solutions:

$$\begin{aligned}&\frac{x^{1-\delta}}{1-\delta}\hat{G}_t+\sup_{\pi\in\mathcal{A}}\left(\left[\left[\left(\frac{\mu_s-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)\mu_s+\left(1-\frac{\mu_s-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)r\right.\right.\right.\\&\left.-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2\right]xx^{-\delta}\hat{G}+\frac{1}{2}\left[\left(\frac{\mu_s-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)^2\sigma_s^2+\sigma_I^2\right.\\&\left.\left.-2\left(\frac{\mu_s-r+\sigma_s\sigma_I\rho\delta}{\sigma_s^2\delta}\right)\sigma_s\sigma_I\rho\right]x^2\left(-\delta x^{-\delta-1}\hat{G}\right)\right)=0.\end{aligned}\tag{21}$$

Simplifying further gives the following DE which has a well-defined solution:

$$\frac{\hat{G}_t}{\hat{G}}=\left(1-\delta\right)\left[-\frac{\left[\mu_s-r+\sigma_s\sigma_I\rho\right]^2}{2\sigma_s^2\delta}-\left(r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right)\right].\tag{22}$$

Solving 22, we obtain the following solution:

$$\hat{G}(t)=\exp\left[\left(1-\delta\right)\left(-\frac{\left[\mu_s-r+\sigma_s\sigma_I\rho\right]^2}{2\sigma_s^2\delta}-\left[r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right]\right)\left(T-t\right)\right].\tag{23}$$

Therefore, the value function V having solved 22 is given by:

$$\begin{aligned}V\left(t,x\right)=\frac{x^{1-\delta}}{1-\delta}\exp&\left[\left(1-\delta\right)\left(-\frac{\left[\mu_s-r+\sigma_s\sigma_I\rho\right]^2}{2\sigma_s^2\delta}\right.\right.\\&\left.\left.-\left[r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right]\right)\left(T-t\right)\right].\end{aligned}\tag{24}$$

In addition, the optimal controls are given as follows:

$$\pi^*\left(t\right)=\frac{\left(\mu_s-r\right)+2\sigma_s\sigma_I\rho\delta}{\delta}\tag{25}$$

and

$$\mathcal{C}^*=\left[\frac{\bar{\mathcal{X}}^{-\delta+1}\hat{G}}{\phi\mathrm{e}^{-\lambda t}}\right]^{\frac{-1}{\delta}},\tag{26}$$

where,

$$\hat{G}(t)=\exp\left[\left(1-\delta\right)\left(-\frac{\left[\mu_s-r+\sigma_s\sigma_I\rho\right]^2}{2\sigma_s^2\delta}-\left[r-\mu_I-\sigma_s\sigma_I\rho+\sigma_I^2+\frac{\sigma_I^2\delta}{2}\right]\right)\left(T-t\right)\right].\tag{27}$$

8. Numerical Examples and Simulations

In this section, we determine how parameters affect optimal investment $\pi^*\left(t\right)$ and optimal consumption $\mathcal{C}^*\left(t\right)$ controls. We first assess the effects of correlation coefficient ρ on optimal investment $\pi^*\left(t\right)$ and optimal consumption $\mathcal{C}^*\left(t\right)$ over time through surface analysis.

Figure 1 shows the effect of the correlation coefficient ρ on the investment $\pi^*\left(t\right)$. The curve results indicate that the correlation coefficient ρ affects investment in a positive way over time. In summary, optimal investment $\pi^*\left(t\right)$ increases as the correlation coefficient ρ heads to a perfect positive correlation over time.

In **Figure 2**, the optimal investment $\pi^*(t)$ increases with larger values of risk aversion factor δ as this lead to smaller relative risk aversion $1-\delta$ for the investor. The investor becomes vigorous in investing resulting in more investment in risky assets.

Figure 3 shows the effect of interest rate r on optimal investment $\pi^*(t)$. The curve results indicate that interest r affects optimal investment in a positive way over time.

In **Figure 4**, the optimal investment strategy $\pi^*(t)$ increases with increases in growth rate μ_s of the risky asset over time. When the growth rate of the stock is high, an agent invests more in the stock for more wealth and consumption. This agrees with practical investments and our intuition.

In **Figure 5**, the optimal investment strategy $\pi^*(t)$ increase with an increase in inflation volatility σ_I. When the inflation volatility is increasing, an agent becomes aggressive and invests more in the risky asset for more wealth.

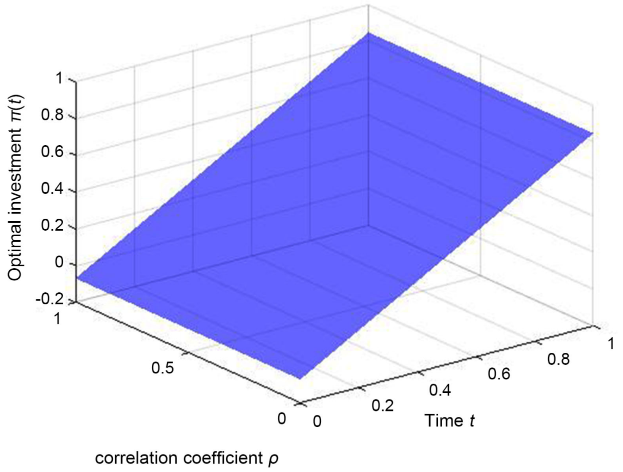

Figure 1. The effects of the correlation coefficient ρ on optimal investment $\pi^*(t)$ over time *t*.

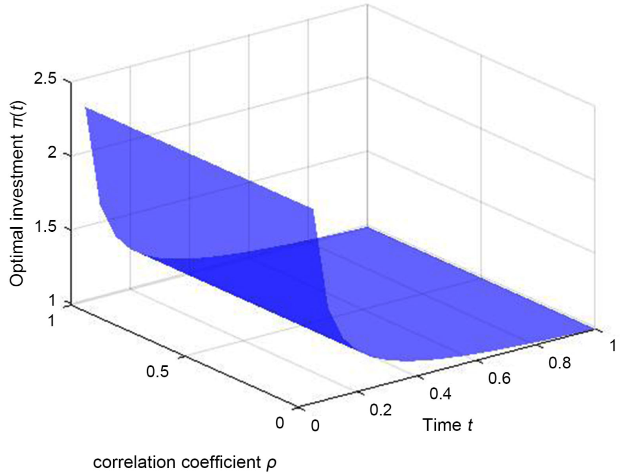

Figure 2. The effects of risk aversion factor δ on optimal investment $\pi^*(t)$ over time *t*.

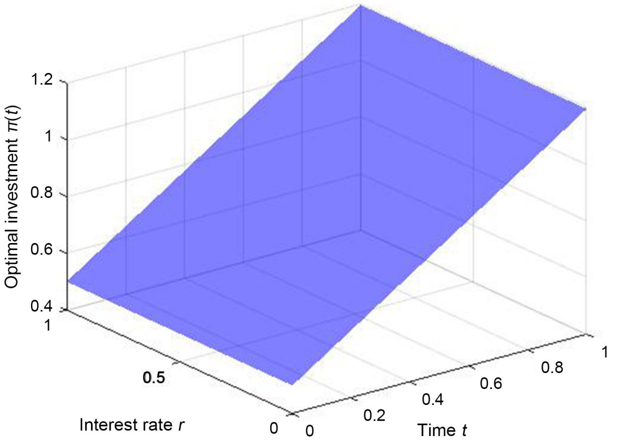

Figure 3. The effects of interest rate *r* on optimal investment $\pi^*(t)$ over time *t*.

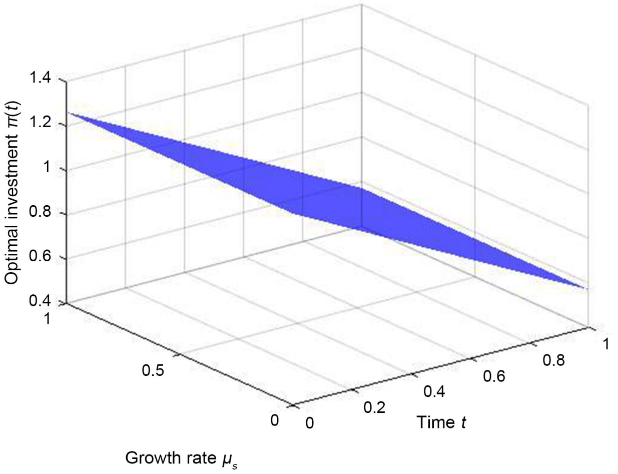

Figure 4. The effects of growth rate μ_s of the risky asset on optimal investment $\pi^*(t)$ over time *t*.

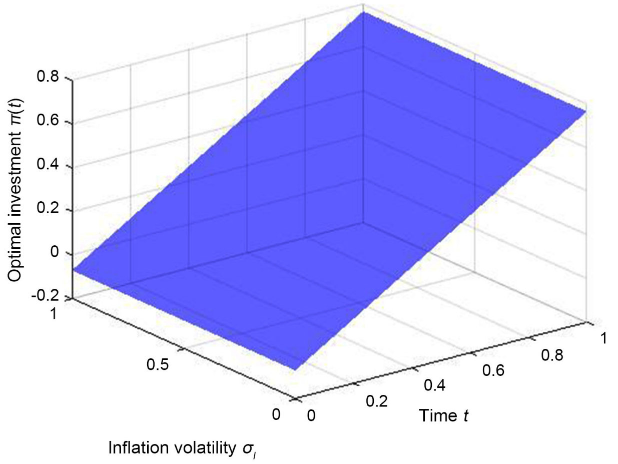

Figure 5. The effects of inflation volatility σ_I on optimal investment $\pi^*(t)$ over time *t*.

9. Conclusion

In this study, we investigated stochastic optimal control problems with stochastic wage income and inflation for CRRA investors. The agent invested in the financial market. The goal was to allocate initial wealth between investment securities in order to maximize the expected discounted utilities derived from intermediate consumption and terminal wealth. This stochastic optimal control problem was linked to the non-linear second-order PDE called the HJB PDE via Bellman's optimality principle. Upon solving HJB PDE, the value function and optimal controls were obtained. Numerical results showed that the correlation coefficient ρ has a positive effect on optimal investment $\pi^*(t)$. Furthermore, an increase in the risk aversion factor δ, the growth rate of risky asset μ_s and an increase in inflation volatility σ_I affected the optimal investment $\pi^*(t)$ in a positive way. When the inflation volatility is increasing, an agent becomes aggressive and invests more in the risky asset for more wealth. There are many related topics that may be worthy to study in the future, for example, employing other techniques such as viscosity solutions and Backward Stochastic Differential Equations (BSDEs) would yield interesting results.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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