Journal of Mathematical Finance, 2022, 12, 566-581
https://www.scirp.org/journal/jmf
ISSN Online: 2162-2442
ISSN Print: 2162-2434

The Sharpe Ratio's Upper Bound of the Portfolios in the Presence of a Benchmark: Application to the US Financial Market

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How to cite this paper: Ye, J., Wang, Y.W. and Raza, M.W. (2022) The Sharpe Ratio's Upper Bound of the Portfolios in the Presence of a Benchmark: Application to the US Financial Market. *Journal of Mathematical Finance*, **12**, 566-581.
https://doi.org/10.4236/jmf.2022.123030

Received: April 1, 2021
Accepted: August 22, 2022
Published: August 25, 2022

Abstract

[1] analyzed the performance of Madoff's investment strategy using the Sharpe ratio. Going a further step, [2] calculated the upper bound of the Sharpe ratio given different conditions. The upper bound is the maximum of the Sharpe ratio that a portfolio can realize. The US financial market is one of the well developed and diversified market across the globe. Significant numbers of funds are based on the broader market index and its derivatives. In this article, the upper bound of the Sharpe ratio for the portfolio depending on the broader index is calculated. The upper bound estimated in this study will help investors and regulators in US and across the globe in general to evaluate the Sharpe ratio with caution and identify investment vehicles that are promising fictitious returns.

Keywords

Sharpe Ratio, Mean-Variance Efficient Portfolio, Correlation Constraint, Copula Constraint

1. Introduction

An efficient investment strategy maximizes the returns at a given level of risk or minimizes the variability at a given level of returns. [3] was the first to propose a quantitative approach to determine the optimal trade-off between risk and returns of an investment strategy. He won the Nobel Prize for original contribution to the portfolio management. Following the seminal work, [4] proposed the Sharpe ratio for performance valuation of investment portfolios. Sharpe ratio shows the ability of a fund/portfolio to generate a specific amount of returns at a given level of risk. A higher Sharpe ratio shows superior performance of a fund relative to its peers. The lesser amount of input, simplicity and ease in application makes the Sharpe ratio very influential in practice.

In last few decades considerable work has been done to search for investment strategies that can optimize the Sharpe ratio. For example, [5] introduced a rank measure to pick up the stocks. Applying his strategy in practice, it results a significantly higher Sharpe ratio in the Swedish stock market. [6] proposed a learning-based trading strategy to maximize the Sharpe ratio. [7] measured the Sharpe ratio in Multi-Period settings. [8] introduced an approach which generates a nonlinear strategy that explicitly maximizes the Sharpe ratio. [9] calculated the Sharpe ratio in a log-normal framework and the maximum of the Sharpe ratio is given in an explicit formulation. [10] bore in mind the investor's expectations and developed a simple and intuitive formula for the squared Sharpe ratio. [11] evaluated the probability at which an estimated Sharpe ratio can exceed a given threshold in presence of non-normal returns. [12] applied an improved bootstrap-based estimator to improve the out-of-sample Sharpe ratio. [13] studied the relationship between alpha decay and the Sharpe ratio. Their model suggests measuring the performance of fund managers by both alpha decay and the Sharpe ratio in practice. [14] found that regardless of the setting, there is no statistically significant difference between MinVar and MaxSR portfolio performance. [15] made use of the Cornish Fisher expansion to analyze the non-normality in portfolio performance. [16] proposed a fast procedure to reduce the complexity of covariance measuring. They build a new portfolio performing better than several existing portfolio allocation methods. All these papers are focusing on improving the Sharpe ratio by developing new statistical methods.

Solutions that are focusing on optimizing the Sharpe ratio can help the investors find better investment portfolios. However, the process is not a plain-vanilla. It requires understanding of different investment priorities, settings and cumbersome mathematical estimations. In addition to this, investors should also consider the practice oriented outcomes of such optimization techniques as some of the proposed techniques can be merely Ponzi schemes. For example, one such investment solution was presented by Bernie Madoff. He was successful in attracting a wide following because the proposed investment strategy was able to deliver considerably higher returns with very low volatility for an extended period of time. The consistent low volatility of Madoff's returns leads to an unusually high Sharpe ratio. Thus the strategy was able to report a higher Sharpe ratio relative to its peers. He attributes the success of this strategy to a split-strike conversion strategy. However, later it was exposed to be nothing more than a Ponzi scheme [1].

The failure of finance industry to detect fraud in Madoff's strategy is a matter of concern for investors as well as for regulatory bodies. Therefore, research studies have searched for solutions to caution investors about the success and transparency of dubious strategies ([17] and [18]). Going a further step, [2] found optimal portfolio with the maximum possible Sharpe ratio in the presence of different conditions. They argued that upper bounds on the Sharpe ratio can be a useful technique for the regulators to detect frauds in investment strategies.

As we know, the US financial market is one of the well structured and developed financial markets across the globe. According to an estimate, there are around 3527 hedge funds and 2096 Exchange Traded Funds (ETFs) in the US market with almost 3 trillion and 4.4 trillion USD Asset under management (AUM) respectively[1]. The considerable number of hedge funds and ETFs makes it one of the most attractive markets for the global investors. These funds are implemented following a diverse set of strategies. It is very crucial for investors to find, not only an optimum but also a reliable hedge fund. A possible source to gather relevant information for investigating the reliability of funds is financial statements. No doubt, these statements provide handful of useful information. However, the leniency in accounting principles provides a slight edge to the accountants to window dress these statements for creditors and tax purposes. Relying solely on financial statements for performance evaluation can be misleading in times, thus the reliability of the performance of such hedge funds is an open question.

This research work is an effort to fill this gap by providing robust estimates for upper bound on Sharpe ratio in the US market. Our first contribution is that, we apply the model of [2] and estimate the maximum possible Sharpe ratio of portfolio that depends/tracks the performance of broader market index or its ETFs. It is important to mention that the optimum portfolio may not provide protection to the investors, specifically in bear markets. In practice, the investors reward strategies that offer protection or exhibit some dependence with a benchmark. Therefore, following [2] we extend the model by imposing a correlation constraint between the proposed strategy and the overall market. The correlation constraints only highlight the linear relationship and fail to characterize the dependence fully. Therefore, we further extend the model and derived optimal mean-variance strategy with copula constraint.

The upper bound estimation of the Sharpe ratio depends on the estimation of the expected return and volatility. Previous studies have estimated these parameters for whole sample period. In practice, these estimates are time variant. Therefore, this study also estimates the time varying expected returns and volatility for the estimation of upper bounds on Sharpe ratio. Such time varying estimation enables the model to grasp the effect of regimes on the upper bound. Finally, we estimate the upper bound for different indexes altogether. Such analysis helps to identify an index with higher bound on Sharpe ratio in the presence of correlation and copula constraint.

The results show that the portfolio depending on the Nasdaq Index can reach the highest Sharpe ratio in the last decade as compared to Dow Jones Index, S&P

[1]https://www.Statistica.com.

500 Index and Vanguard total stock market ETF. The upper bound on Sharpe ratio of US market is not constant and it varies across market regimes. This study has implications for investors as well as for regulatory bodies. The upper bound on Sharpe ratio can help investors to find reliable vehicles of investments. The investors also need to pay close attention to the time interval when they estimate the upper bound of the Sharpe ratio. Finally, the regulatory bodies can use these estimate to report portfolios or their derivatives with dubious or exaggerated reported Sharpe ratio.

The rest of the paper is organized as follows. The optimal portfolio problem and the assumptions on the financial market are presented in Section 2. Section 3 provides explicit expressions for the maximum Sharpe ratio in different conditions. The conclusion is presented in Section 4.

2. Market Setting

Our theoretical set-up assumes an arbitrage-free, perfectly liquid and frictionless financial market with a fixed investment horizon $T>0$. Let $(\Omega,\mathcal{F}_t,\mathbb{P})$ be the corresponding probability space. Let ξ_T be the pricing kernel that is agreed by all agents. It has a positive density on $\mathbb{R}^+\setminus\{0\}$. The initial cost of a strategy with terminal payoff X_T is given by

$$c(X_T)=\mathbb{E}_{\mathbb{P}}[\xi_T X_T], \tag{1}$$

where $\mathbb{P}$ is the real-world probability measure. We only consider terminal payoffs X_T with finite initial cost $c(X_T)$.

In the remainder of the paper, we consider an investor with a fixed horizon $T>0$ without intermediate consumption. We denote the initial wealth of investor by $W_0>0$ initial wealth. For convenience, all the results are illustrated in a Black-Scholes market. In the Black-Scholes setting, there is a bank account earning a constant risk-free rate $r>0$. The dynamics of the Index under the physical measure $\mathbb{P}$ is given by

$$\frac{\mathrm{d}S_t}{S_t}=\mu\mathrm{d}t+\sigma\mathrm{d}W_t. \tag{2}$$

Here W_t is a standard Brownian motion, $\mu>r$ is the instantaneous expected return and σ is the volatility. The unique stochastic discount factor process (ξ_t) in the Black-Scholes setting is given by

$$\xi_t=\mathrm{e}^{-rt}\mathrm{e}^{-\theta W_t-\frac{1}{2}\theta^2 t},\quad \theta=\frac{\mu-r}{\sigma}. \tag{3}$$

Furthermore, the Market Index S_t^* is characterized in the following dynamics:

$$\frac{\mathrm{d}S_t^*}{S_t^*}=\left(\frac{\theta}{\sigma}\mu+\left(1-\frac{\theta}{\sigma}\right)r\right)\mathrm{d}t+\theta\mathrm{d}W_t. \tag{4}$$

S_t^* amounts to a constant mix strategy, where at time t a fraction $\frac{\theta}{\sigma}$ is invested in the Index and the remaining fraction $1-\frac{\theta}{\sigma}$ in the bank account.

3. The Sharpe Ratio in US Financial Market

In order to estimate the upper bound of Sharpe ratio, we collect price data of three different broader market indexes and an ETF. These are S&P 500, Nasdaq Index, Dow Jones Index and Vanguard total stock market ETF. The back-tests are carried out for the time period 2006-2020. The Sharpe ratio is a well-known measure balancing risk and return of a portfolio X_T (see [4]). It is defined as

$$SR(X_T)=\frac{\mathbb{E}[X_T]-W_0\mathrm{e}^{rT}}{\mathrm{std}(X_T)}, \tag{5}$$

where r is the risk-free rate and $\mathrm{std}(X_T)$ is the standard deviation of X_T.

3.1. The Sharpe Ratio's Upper Bound of the Portfolio without Constraint

In this section we estimate the upper bound on Sharpe ratio in the absence of any constraints. For estimating the maximum of Sharpe ratio we need a mean-variance efficient portfolio. [2] defines a mean-variance efficient portfolio as given in the following definition.

Definition 1 (Mean-variance Efficient Portfolio). *Assume that investor aims for a strategy that maximizes the expected return for a given variance* s^2 *for* $s\geq 0$ *and a given initial cost* W_0. *The solution of the following mean-variance optimization problem*

$$\min_{\substack{\mathrm{var}(X_T)=s^2\\ c(X_T)=W_0}} \mathbb{E}[X_T], \tag{6}$$

is defined as mean-variance efficient portfolio.

Using mean-variance efficient portfolio, the maximal Sharpe ratio is given by the following equation (See Proposition 3.3 in [2]):

$$SR^*=\sqrt{\frac{\mathrm{var}(\xi_T)}{\mathbb{E}^2[\xi_T]}}=\mathrm{e}^{rT}\mathrm{std}(\xi_T). \tag{7}$$

In the Black-Scholes market, we can infer from (3) that $\mathbb{E}[\xi_T]=\mathrm{e}^{-rT}$ and $\mathbb{E}[\xi_T^2]=\mathrm{e}^{-2rT+\theta^2T}$. Hence, in this setting the expression (7) for the maximum of the Sharpe ratio is

$$SR^*=\sqrt{\mathrm{e}^{\theta^2T}-1}, \tag{8}$$

where $\theta=\frac{\mu-r}{\sigma}$. (8) can also be found in [19].

To apply (8) in the portfolio depending on the broader Index, the expected return μ and the volatility σ of S&P 500 Index, Nasdaq Index, Dow Jones Index and Vanguard total stock market ETF needs to be estimated. Choosing the daily Index data from 2006-2020, the expected return μ and the volatility σ are estimated using the following formulas:

$$\begin{cases} r_i = \ln\left(\dfrac{I_i}{I_{i-1}}\right) \\ \mu = 250 * \dfrac{\sum_{i=2}^{N} r_i}{N-1} \quad i = 2, \cdots, N \\ \sigma = \mathrm{std}\left(\{r_i\}\right) * \sqrt{250} \end{cases}$$

where $I_i\left(i = 2, \cdots, N\right)$ is the daily Index data[2]. $\mathrm{std}\left(\{r_i\}\right)$ is the standard deviation of the time series $\{r_i\}$. According to the daily index data, the daily log return r_i is calculated. The annualized returns are estimated by multiplying daily returns with 250. Similarly, the annualized volatility is estimated with square root of time rule. The estimated μ and σ of the three indexes and the ETF are listed in the second and third columns of Table 1. The maximum of the Sharpe ratio is also estimated in the 4th column of Table 1 using Equation (8).

The results reported in Table 1 are based on whole sample period. In practice the μ and σ are time variant. To show this we also report the same statistics for an alternative choice of sample period, 2013-1-4 to 2020-9-30.

Results in Table 1 and Table 2 show that the estimated μ is different from each other. Similarly, we can notice improvement in stability (lower standard deviation in returns) in Table 2. This is due to the exclusion of crisis period from the estimation window. The improvement in μ and σ leads to almost 50% increase in the value of maximum Sharpe ratio. These results are important because it shows that the estimate of maximum Sharpe ratio is time dependent. An interesting finding is that in comparison to other indexes, Nasdaq index results in higher value for maximum Sharpe ratio. A possible reason for such superior performance is relatively higher weights allocation to technology stocks.

Table 1. μ, σ are estimated using the daily index data from 2006-1-4 to 2020-9-30. 1-year US Treasury yield is the risk-free rate $r = 0.137\%$ and the investment horizon is $T = 1$.

Index	μ	σ	Maximum of the Sharpe ratio
S&P500	6.55%	20.30%	0.324
Nasdaq	10.76%	21.86%	0.516
Dow Jones Index	6.32%	19.44%	0.326
Vanguard total stock market ETF	6.70%	20.28%	0.322

Table 2. μ, σ are estimated using the daily index data from 2013-1-4 to 2020-9-30. 1-year US Treasury yield is the risk-free rate $r = 0.137\%$ and the investment horizon is $T = 1$.

Index	μ	σ	Maximum of the Sharpe ratio
S&P500	10.65%	17.13%	0.676
Nasdaq	16.44%	19.07%	1.038
Dow Jones Index	9.3%	17.68%	0.555
Vanguard total stock market ETF	10.46%	17.17%	0.660

[2]The price data is obtained from finance.yahoo.com on daily basis.

The estimation of maximum Sharpe ratio in the 4th column is very important because it gives a clear picture of the potential financial performance of a fund. Assume a hedge fund that track the performance of S&P 500 index. In order to gauge a true picture of its performance we must compare its actual Sharpe ratio with its estimated upper bounds. A significant deviation in the actual Sharpe ratio of the hedge fund is a signal for investors and regulators to launch more investigation about the hedge fund financial performance.

It is important to understand that the violation of the upper bound given in (8) for the Black-Scholes model, has to be seen as an indication (a signal) that there could be a fraud, but not as a formal proof of it. The violation of this upper bound does not always imply a fraud subject to model error.

3.2. The Sharpe Ratio's Upper Bound of the Portfolio with a Correlation Constraint

By taking into account additional information, the fraud detection mechanism can further be enhanced. The maximum of the Sharpe ratio in Section 3.1 do not consider the dependence features between the investment strategy and the Market Index[3]. Regulators estimate the Sharpe ratio of a hedge fund but can also investigate correlations of the fund with the Market Index, and this additional source of information may be useful for refining the process of fraud detection. A popular strategy amongst hedge funds is the so-called "market-neutral" strategy. One of its key properties is that it typically ensures very low correlation with the Market Index. We show that, in the presence of correlation constraint the maximum possible Sharpe ratio is reduced compared to the unconstrained case. A mean-variance efficient portfolio in the presence of a correlation constraint is defined in the following way.

Definition 2 (Mean-variance efficient portfolio with a correlation constraint). *Take* S_T^* *as the market index. Let* $|\rho|<1$ *and* $s>0$. *The solution to the following mean-variance optimization problem*

$$\min_{\substack{\operatorname{var}(X_T)=s^2 \\ c(X_T)=W_0 \\ \operatorname{corr}(X_T,S_T^*)=\rho}} \mathbb{E}[X_T] \tag{9}$$

is defined as mean-variance efficient portfolio X_T^* *with a correlation constraint.*

The maximum possible Sharpe ratio of the mean-variance efficient portfolio with a correlation constraint ρ is given by the following equation (See Proposition 4.2 in [2]):

$$SR_\rho^* = \mathrm{e}^{rT}\frac{\operatorname{cov}\left(\xi_T,\xi_T-cS_T^*\right)}{\operatorname{std}\left(\xi_T-cS_T^*\right)} \le SR^* = \mathrm{e}^{rT}\operatorname{std}\left(\xi_T\right). \tag{10}$$

where SR^* is the unconstrained Sharpe ratio found in (7) and c is determined uniquely by the equation $\operatorname{corr}\left(\xi_T-cS_T^*,S_T^*\right)=\rho$.

[3]The market index is defined by Equation (4).

In Black-Scholes market model, c could be calculated.

$$c = \frac{-\left(1-\rho^2\right)+\rho\sqrt{\left(1-\rho^2\right)\left(e^{2\theta^2 T}-1\right)}}{e^{\left(2r+2\theta^2\right)T}\left(1-\rho^2\right)}, |\rho|<1. \tag{11}$$

and the maximum of the Sharpe ratio is given by:

$$SR^* = \frac{\sqrt{e^{\theta^2 T}-1}\left(e^{-rT}+ce^{rT}\right)}{\sqrt{c^2 e^{\left(2r+2\theta^2\right)T}+2c+e^{-2rT}}}. \tag{12}$$

Combined with (11) and (12), the relationship between SR^* and ρ can be plotted in **Figure 1**. The plot can help us understand the influence of the correlation coefficient ρ on the Sharpe ratio in the constrained case.

Four figures are created for the indexes respectively. In each chart, the red dash line delegates the relationship between SR^* and ρ using Equation (12). The black solid line in each graph is the maximum of the Sharpe ratio without constraint for each index. The black dash line points out the Sharpe ratio when the correlation between the portfolio and the market index is low ($|\rho|<0.1$). SR^* reaches the maximum when $c=0$ and c is the increasing function of ρ, so SR^* reaches the maximum $\sqrt{e^{\theta^2 T}-1}$ only when $\rho = e^{-\theta^2 T}$. The proposition shows that for fraud detection it is useful to incorporate correlation features of displayed returns. **Figure 1** displays the maximum Sharpe ratio for the unconstrained case (See **Table 1**) and constrained Sharpe ratio for different levels of correlation constraints ρ when the market index S_T^* is the benchmark.

Observe that for low correlation levels $\left(\rho\in[-0.1,0.1]\right)$, the maximum of the Sharpe ratio for S&P 500 Index, Dow Jones Index and Vanguard total stock market ETF, is reduced to half in the absence of any constraints. For the Nasdaq Index, the maximum of the Sharpe ratio is reduced to 60 percent in the absence of any constraints. One of the main reasons for such behavior is the higher expected return of Nasdaq Index. In terms of volatility all the four indexes exhibit similar results. For those investors who prefer the "market-neutral" strategy, Nasdaq Index and its derivatives is a better choice as compared to the other indexes considered in this study, for the time period 2006-2020.

Adding correlation constraint has the advantage that it provides more information to the model, thus enabling it to detect fraud with relatively higher accuracy. Observing that for negative correlation levels, the maximum Sharpe ratio can be negative[4], thus if hedge fund returns display a negative correlation with the financial market and a positive Sharpe ratio, then there could be some suspicion about these returns. This is strongly different from what we observed in the unconstrained case as the maximum Sharpe ratio is always positive.

Figure 1 reports the upper bound on the Sharpe ratio for full sample period.

[4]At first, it might look counter intuitive that an optimal strategy has a lower return than what we can achieve by investing in risk-free securities. However, enforcing a negative dependence with the market comes at some cost. A similar observation can be drawn for the put option: it has a low expected return but provides protection when markets fall.

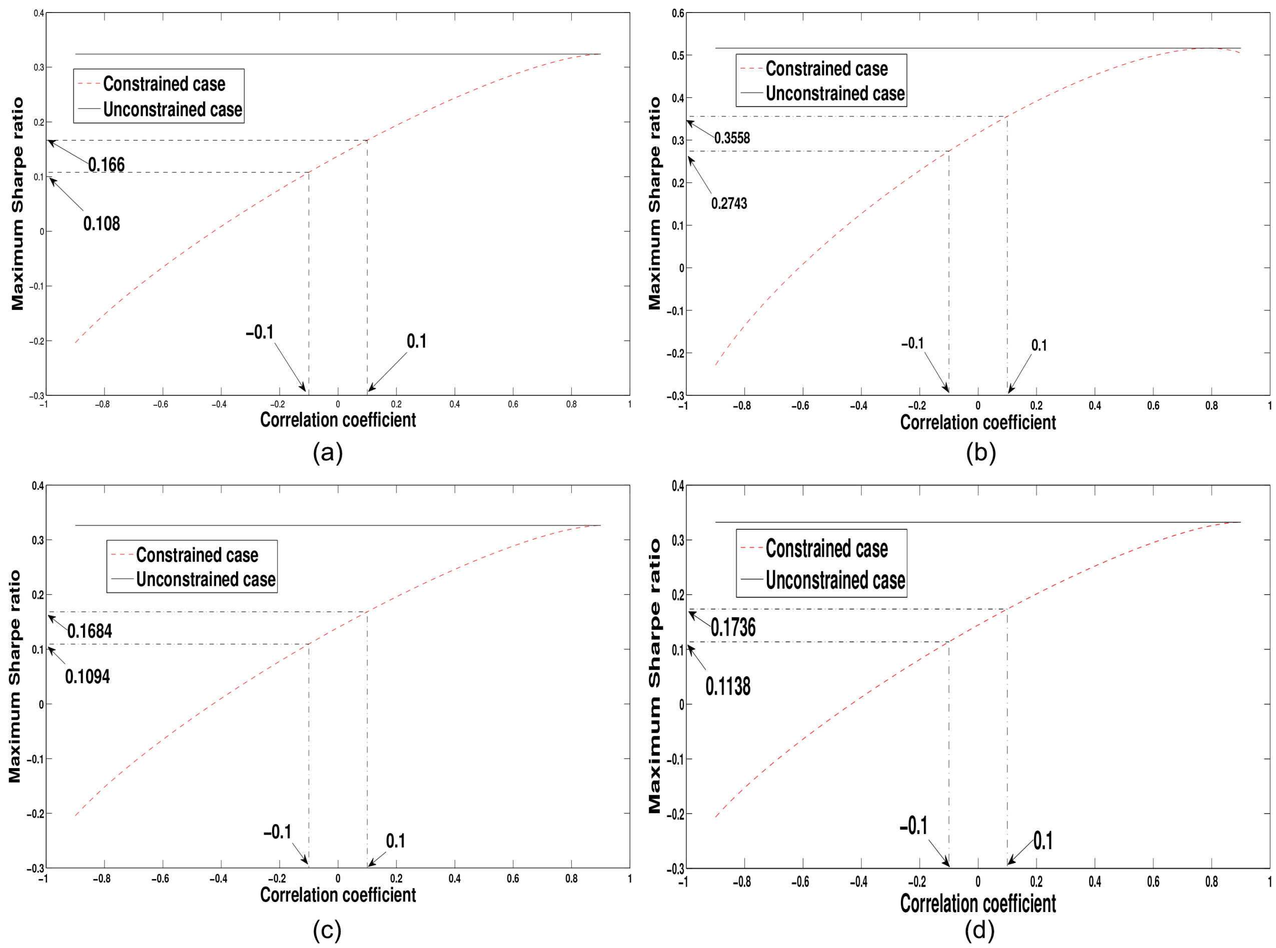


Figure 1. Maximum Sharpe ratio SR^* for different values of the correlation ρ when the benchmark is S_T^*. The parameters are taken from **Table 1**. The maximum of the Sharpe ratio for the unconstrained case is calculated in **Table 1** and it is the flat black solid line in this Figure. (a) S&P 500 Index; (b) Nasdaq Index; (c) Dow Jones Index; (d) Vanguard total stock market ETF.

To shows that the estimation of upper bound is time varying, we plot the results for an alternative choice of time period in **Figure 2**.

Based on the parameters in **Table 2**, the maximum Sharpe ratio SR^* is calculated in **Figure 2**. Note that for low correlation levels $\left(\rho \in [-0.1, 0.1]\right)$, the maximum of the Sharpe ratio for S&P 500 Index, Dow Jones Index and Vanguard total stock market ETF, is reduced to around 70 percent in the absence of any constraints. The case of Nasdaq Index is a bit different as the upper bound on its Sharpe ratio is decreased to 90% as compared to the case when there was no constraint. This is interesting because we can infer that the decrease in Sharpe ratio for market neutral strategy is time dependent. These findings also suggest that proper timing is very important for the success of any investment decision.

In **Figure 1** and **Figure 2**, the relationship between the maximum of the Sharpe ratio and the correlation coefficient ρ are drawn for different indexes separately. In order to compare the upper bound of indices for different value of the correlation, we plot **Figure 3**. This Figure also highlights the importance of a suitable timing strategy for investors.

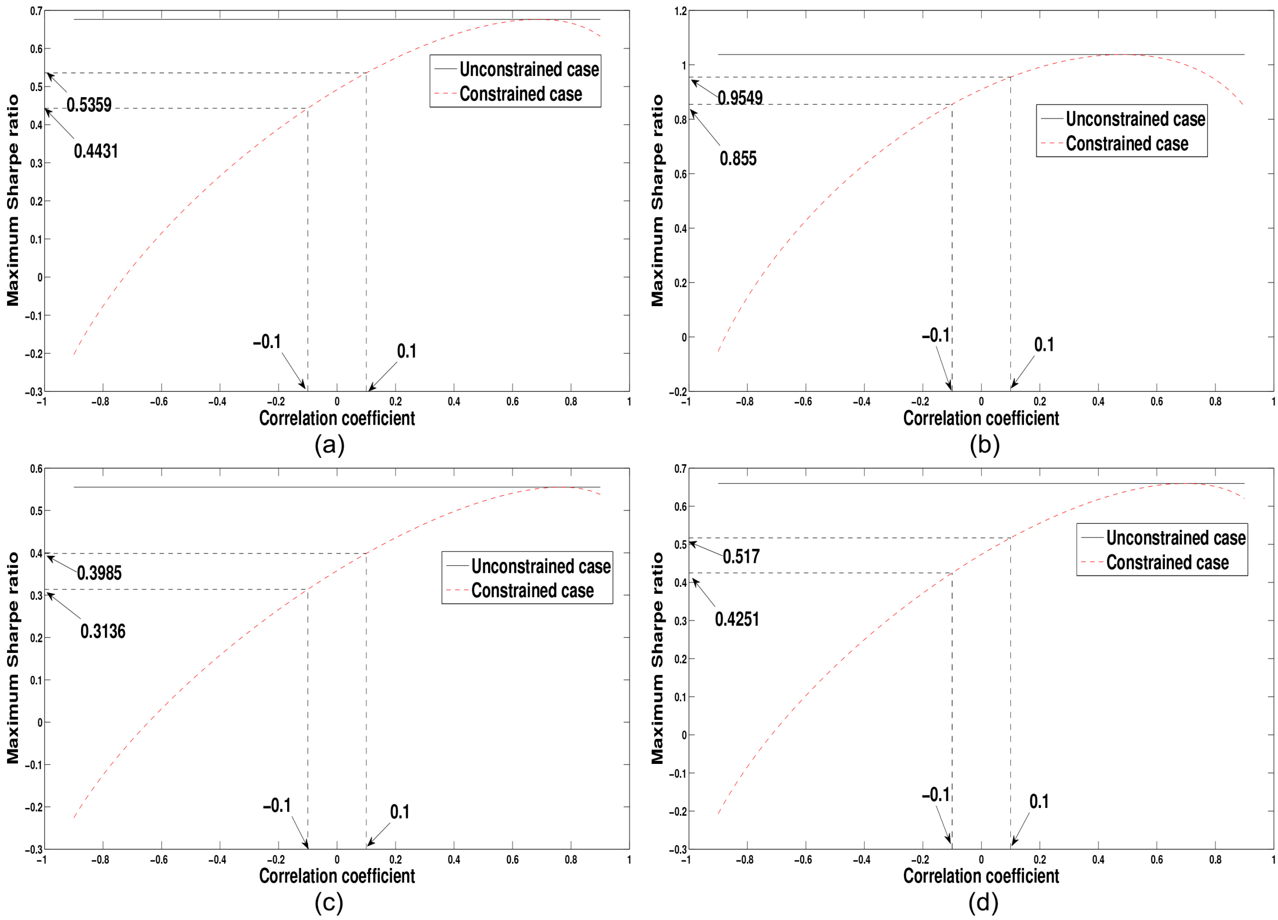


Figure 2. Maximum Sharpe ratio SR^* for different values of the correlation ρ when the benchmark is S_T^*. The parameters are taken from **Table 2**. The black solid line represents the maximum of the Sharpe ratio for the unconstrained case, reported in **Table 2**. (a) S&P 500 Index; (b) Nasdaq Index; (c) Dow Jones Index; (d) Vanguard total stock market ETF.

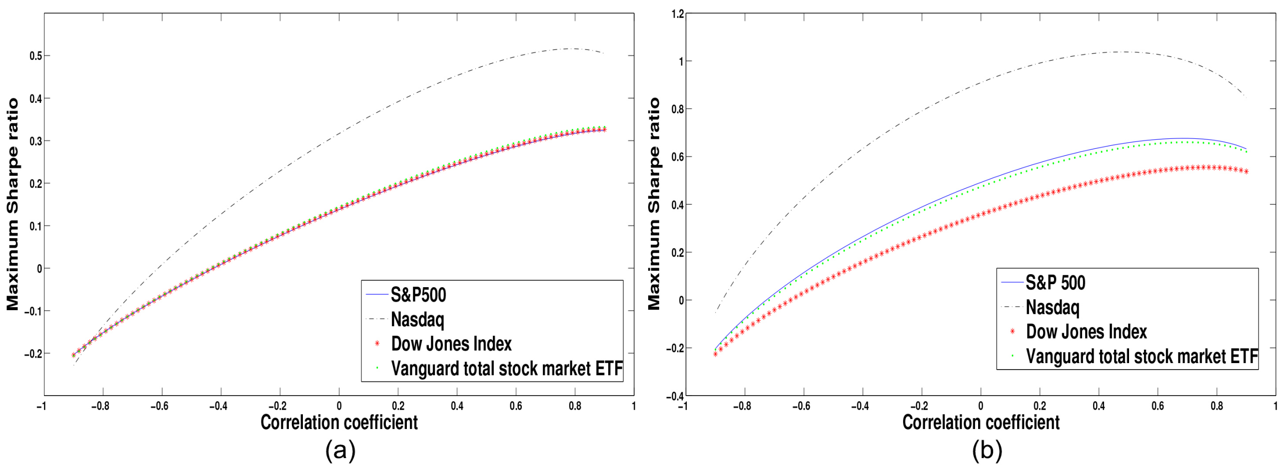


Figure 3. Maximum Sharpe ratio SR^* for different values of the correlation ρ when the benchmark is S_T^*. The parameters are taken from **Table 1** and **Table 2** in Panel A and Panel B respectively. (a) Panel A; (b) Panel B.

For the two different time intervals, we can put the Sharpe ratio's upper bound of the portfolio with correlation constraint for the three indexes and one

ETF together. The upper bound of the Sharpe ratio for S&P 500 Index, Dow Jones Index and Vanguard total stock market ETF are more or less the same, however the upper bound of the Sharpe ratio for Nasdaq Index is higher than the other three indexes when $\rho > -0.8$ in Panel A. Nasdaq index is still the best choice in Panel B. The Sharpe ratio's upper bound for the Nasdaq Index is much higher than the other three indexes. Due to the lower expected return of Dow Jones Index comparing to S&P 500 Index and Vanguard total stock market ETF, the Sharpe ratio's upper bound for Dow Jones Index is lower than S&P 500 Index and Vanguard total stock market ETF in Panel B. The Sharpe ratio's upper bound for S&P 500 Index, Dow Jones Index and Vanguard total stock market ETF are similar from 2006-1-4 to 2020-9-30, however the Sharpe ratio's upper bound for the three indexes are different from 2013-1-4 to 2020-9-30. It proves that timing is very important.

From **Figure 3**, we can find the optimal choice for the portfolio with correlation constraint is still the portfolio depending on Nasdaq index. The main reason is that the expected return of Nasdaq index is much higher than the other three indexes, however their volatility is slightly different.

3.3. The Sharpe Ratio's Upper Bound of the Portfolio with a Copula Constraint

Correlation is only one property related to dependence. It measures the linear relationship between strategies but falls short in depicting dependence fully. A useful device for reflecting the interaction between the strategy's payoff X_T and S_T^* is the copula. Sklar's theorem shows that the joint distribution of $\left(S_T^*, X_T\right)$ can be decomposed as

$$\mathbb{P}\left(S_T^* \le y, X_T \le x\right) = C\left(F_{S_T^*}\left(y\right), F_{X_T}\left(x\right)\right),$$

where C is the joint distribution (also called the copula) for a pair of uniform random variables U and V over $(0,1)$ and where $F_{S_T^*}$ and F_{X_T} denote respectively the (marginal) cdf of S_T^* and X_T. A copula is a function containing full information about the interaction between the two variables.

Definition 3 (Mean-Variance Efficient Portfolio with a Copula Constraint). *Take* $B_T = S_t^*$. *For* $s > 0$, *a solution of the following constrained mean-variance optimization problem*

$$\min_{\substack{\operatorname{var}(X_T)=s^2 \\ c(X_T)=W_0 \\ C\text{:copula between } X_T \text{ and } S_t^*}} \mathbb{E}\left[X_T\right] \tag{13}$$

is defined as a mean-variance efficient portfolio X_T^* *with a copula constraint.*

Assume the copula between mean-variance efficient portfolios X_T^* and the market index S_t^* is the Gaussian copula with correlation ρ_0, the Black-Scholes setting allows to derive an explicit expression for the Maximal Sharpe ratio (See Proposition 5.5 in [2]):

$$SR^*_{\rho_0,G} = \frac{e^{rT}\mathbb{E}\left[\xi_T G_T^c\right] - \mathbb{E}\left[G_T^c\right]}{\operatorname{std}\left(G_T^c\right)} \le SR^*. \tag{14}$$

Here $G_T = \left(S_t^*\right)^\alpha S_T^*$, $\alpha = \rho_0\sqrt{\frac{T-t}{t}\frac{1}{1-\rho_0^2}} - 1$, $c = -\frac{\alpha t + T}{(\alpha+1)^2 t + (T-t)}$, $\mathbb{E}\left[G_T^c\right] = e^{M+\frac{V}{2}}$ and $\operatorname{var}\left(G_T^c\right) = \left(e^V - 1\right)e^{2M+V}$, with $M := \mathbb{E}\left[\ln\left(G_T^c\right)\right] = c\left(r + \frac{\theta^2}{2}\right)(\alpha t + T)$ and $V := \operatorname{var}\left(\ln\left(G_T^c\right)\right) = c^2\theta^2\left(\alpha^2 t + T + 2\alpha t\right)$. Moreover $\mathbb{E}\left[\ln\left(\xi_T\right) + \ln\left(G_T^c\right)\right] = M - rT - \frac{\theta^2}{2}T$ and $\operatorname{var}\left(\ln\left(\xi_T\right) + \ln\left(G_T^c\right)\right) = \theta^2\left(c^2\alpha^2 t + (c-1)^2 T + 2c(c-1)\alpha t\right)$ so that $\mathbb{E}\left[\xi_T G_T^c\right]$ reflects the expectation of a lognormal which can be computed from these two first log-moments similarly as we did for G_T^c. Using the parameters of the Index in Table 1, the maximum of the Sharpe ratio could be drawn related to $\rho \in \left[-\sqrt{1-\frac{t}{T}}, 1\right)$.

It is not easy to see the relationship between ρ and the maximum of the Sharpe ratio, so Figure 4 is plotted to describe the relationship between ρ and maximum of the Sharpe ratio.

$$\begin{cases} \alpha = \rho\sqrt{\frac{T-t}{t}\frac{1}{1-\rho^2}} - 1 \\ c = -\frac{\alpha t + T}{(\alpha+1)^2 t + (T-t)} \\ M = \mathbb{E}\left[\ln\left(G_T^c\right)\right] = c\left(r + \frac{\theta^2}{2}\right)(\alpha t + T) \\ V = \operatorname{var}\left(\ln\left(G_T^c\right)\right) = c^2\theta^2\left(\alpha^2 t + T + 2\alpha t\right) \\ SR^*_{\rho_0,G} = \frac{e^{rT + M - rT - \frac{\theta^2}{2}T + \frac{\theta^2\left(c^2\alpha^2 t + (c-1)^2 T + 2c(c-1)\alpha t\right)}{2}} - e^{M+\frac{V}{2}}}{\sqrt{\left(e^V - 1\right)e^{2M+V}}} \end{cases} \tag{15}$$

For the same reason as Figure 1 and Figure 2, different time periods are chosen to estimate the maximum of the Sharpe ratio.

Figure 4 and Figure 5 are plotted based on two different time intervals. In each chart, the red dash line delegates the relationship between $SR^*_{\rho_0,G}$ and ρ using Equation (15). The flat black solid line in each graph is the maximum of the Sharpe ratio without constraint for each index. Observe that the constrained case reduces to the unconstrained maximum Sharpe ratio when the correlation in the Gaussian copula is $\rho_0 = \sqrt{t/T}$. The reason is that the copula between the unconstrained optimum and S_t^* is the Gaussian copula with correlation $\rho_0 = \sqrt{t/T}$. The constraint is thus redundant in that case. Figure 6 reports the upper bound on Sharpe ratio SR^* for different values of the correlation ρ when the benchmark is S_T^*.

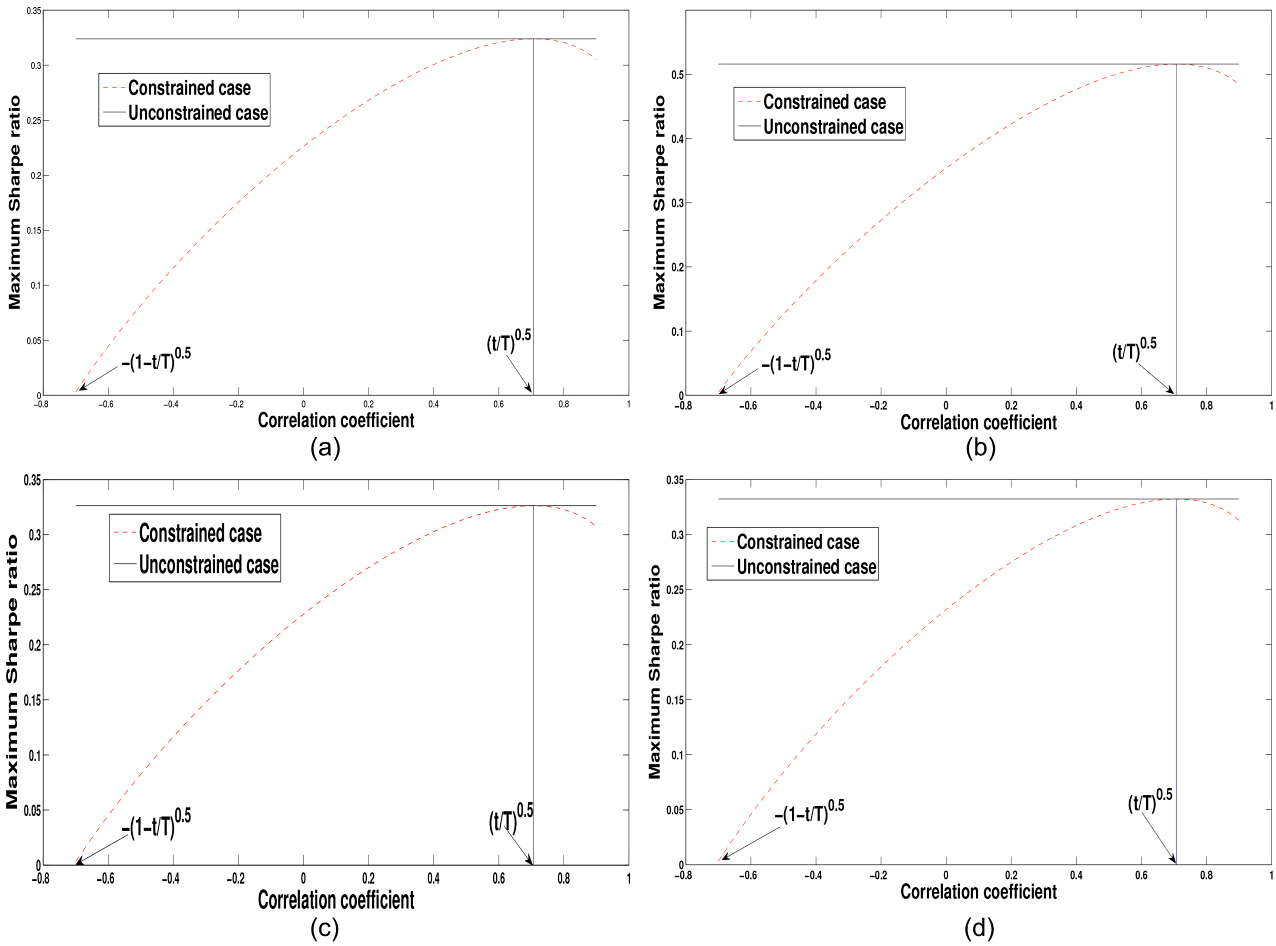
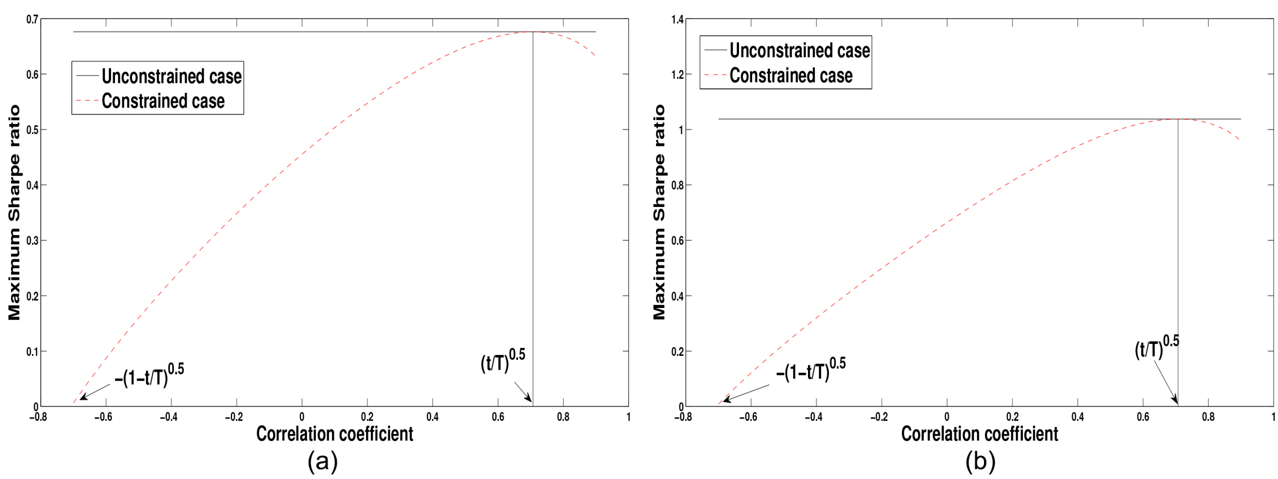


Figure 4. Maximum Sharpe ratio $SR^*_{\rho,G}$ given by (14) for different values of the correlation ρ when the benchmark is $B_T = S_t^*$. Besides the parameters from **Table 1**, we also use the following parameters: $t = 1/2$, $\sqrt{t/T} = 0.707$, $-\sqrt{1-t/T} = -0.707$, $T = 1$. The maximum of the Sharpe ratio for the unconstrained case is in **Table 1** and it is the flat black line in the Figure. (a) S&P 500; (b) Nasdaq; (c) Dow Jones Index; (d) Vanguard total stock market ETF.



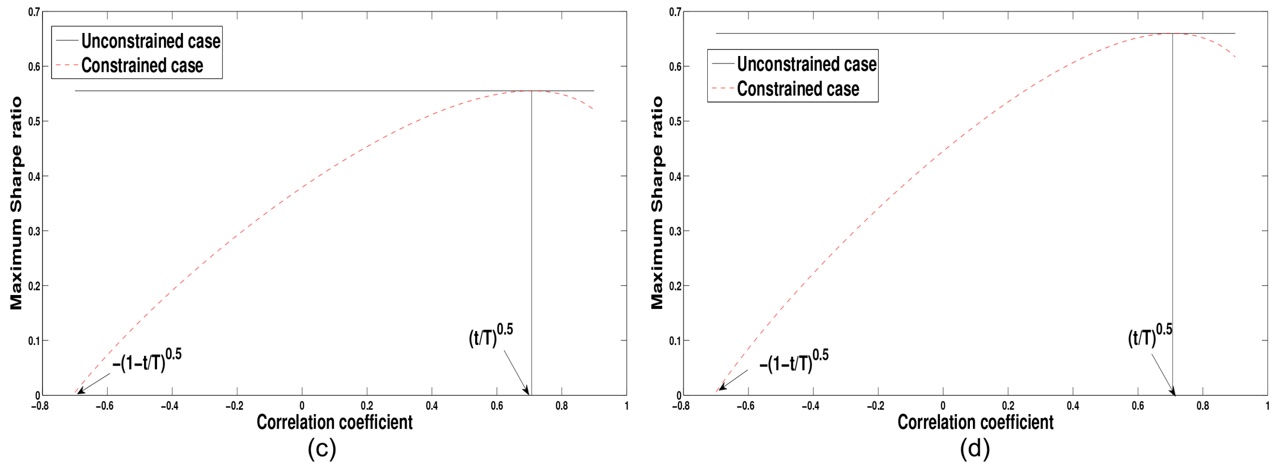


Figure 5. Maximum Sharpe ratio $SR^*_{\rho,G}$ given by (14) for different values of the correlation ρ when the benchmark is $B_T = S_t^*$. Besides the parameters from **Table 2**, we also use the following parameters: $t = 1/2$, $\sqrt{t/T} = 0.707$, $-\sqrt{1-t/T} = -0.707$, $T = 1$. The maximum of the Sharpe ratio for the unconstrained case is in **Table 2** and it is the flat black line in the Figure. (a) S&P 500; (b) Nasdaq; (c) Dow Jones Index; (d) Vanguard total stock market ETF.

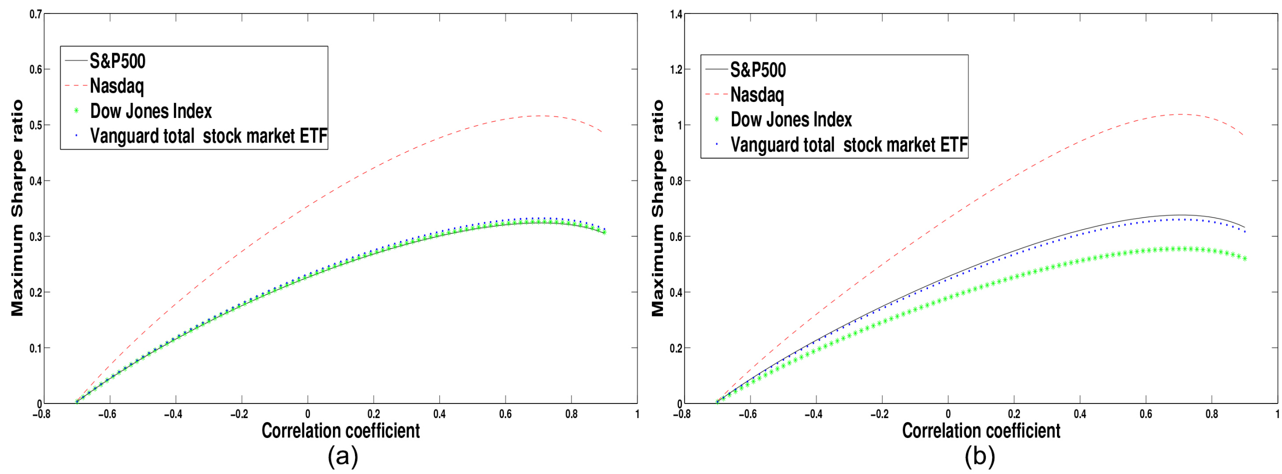


Figure 6. Maximum Sharpe ratio SR^* for different values of the correlation ρ when the benchmark is S_T^*. The parameters are taken from **Table 1** and **Table 2** in Panel A and Panel B respectively. (a) Panel A; (b) Panel B.

From **Figure 6**, we find the Sharpe ratio's upper bound for Nasdaq Index is much higher than the other three indexes. The Sharpe ratio's upper bound for Dow Jones Index is lower than S&P 500 Index and Vanguard total stock market ETF in Panel B. This is similar to what we find in **Figure 3**.

4. Conclusions

The Sharpe ratio is a widely used index in the financial industry. To maximize the Sharpe ratio is an interesting topic. Applying the model in [2], the maximum of the Sharpe ratio to three indexes and one ETF (S&P 500 Index, Nasdaq Index, Dow Jones Index and Vanguard total stock market ETF) are calculated in this study. The Sharpe ratio of a hedge fund is estimated in three different conditions: no constraint, correlation constraint and copula constraint. This study estimates the time varying expected returns and volatility for the estimation of upper bounds on Sharpe ratio. Such time varying estimation enables the model to grasp the effect of regimes on the upper bound. We also estimate the upper bound for different indexes altogether. Such analysis helps to identify an index with higher bound on Sharpe ratio in the presence of correlation and copula constraint. In our example, the Sharpe ratio's upper bound for Nasdaq index is higher than the other three indexes. It is a good choice to invest in Nasdaq index in the last decade.

The maximum of the Sharpe ratio could be seen as a signal for the regulator to monitor the performance of the hedge funds. However, it is important to mention that such an estimation does not provide a solid proof for detecting frauds. When the correlation constraint is added, we can find the maximum of the Sharpe ratio decreases when the hedge fund is market-neutral or negative correlation with the market index. One point we need pay attention to is that the drop percentage of the Sharpe ratio for the market-neutral strategy depends on the time interval used for expected return and volatility estimation.

Future studies should extend the findings of this study to other markets (e.g. China, Russia, Brazil, India and South Africa). The upper bound of the Sharpe ratio for these countries needs to be estimated and compared to ETFs and broader market indexes.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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