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Mean Reversion in Auction Markets

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Abstract

Mean reversion is a process that influences commodity prices in auction markets. This process captures price dynamics that the traditional static model of demand and supply cannot fully explain. By accounting for periodic shocks to demand and supply, mean reversion provides a more accurate pricing model under conditions of uncertainty. This article presents empirical evidence that various commodities traded in auction markets exhibit mean reversion in their pricing. This mean reversion is a significant component of total volatility. After accounting for commodity value, the derived volatility pattern aligns with the observed volatility pattern of a commodity.

Keywords

Demand and Supply, Commodity Price and Value, Volatility, Demeaned Price, Mean Reversion, First-Order Autoregression, Prediction

1. Introduction

In a competitive market in microeconomics, the demand and supply of a commodity determine its transaction price. The traditional model of demand and supply does not account for mean reversion in commodity prices within auction markets. In contrast to conventional fixed-price markets, auction markets are characterized by a competitive bidding process that inherently induces mean reversion in transaction prices. Speculators play a crucial role in determining prices in these markets. Mean reversion becomes a primary determinant of commodity prices under uncertainty, adding risks to the pricing process.

This article delves into mean reversion as the market mechanism that dictates the current prices of commodities traded in auction markets. These markets are characterized by their volatilities, underscoring the risks involved in investing in such commodities. Online data on commodity price movements reveal that prices in the agriculture, energy, metal, and foreign exchange sectors tend to revert to the mean, exhibiting consistent small price fluctuations.

Previous studies on commodity prices used stochastic models to examine central issues of mean reversion, volatilities, and convenience yields; see, for instance, Gibson and Schwartz [1], Schwartz [2], Pindyck [3], Schwartz and Smith [4], Cortazar and Schwartz [5], and Beck [6].

The continuous-time first-order autoregressive or AR (1) process is a valuation model that effectively captures observed volatility patterns. Commodity and currency price movements exhibit both significant fluctuations due to mean reversion and minor fluctuations throughout the day. While simpler than the stochastic models typically used to analyze commodity prices, the continuous-time AR (1) model remarkably projects a volatility pattern that closely resembles those observed in commodity and currency markets. Its unique treatment of commodity value distinguishes it from conventional approaches, adding to its analytical appeal.

The continuous-time AR (1) model assumes that the parameters, $\{\theta, \nu\}$, remain constant throughout the sample period. Nonetheless, substantial occurrences—such as prominent economic announcements spanning longer durations—may undermine this assumption and lead to sudden shifts, often described as "jumps" in either parameter.

In continuous-time models (like the Ornstein-Uhlenbeck process), the speed of mean reversion (θ) mathematically acts as a discount rate. When a mean-reversion process in continuous time is integrated, the pace of mean reversion becomes a discount rate. This concept is widely recognised within financial mathematics, with a corresponding proof provided in Appendix.

In continuous-time models, price changes are assumed to occur continuously rather than at discrete intervals. In auction markets—whether on a physical trading floor or an electronic trading platform—transactions happen at a high frequency. As a result, the observed transaction prices in these markets closely align with the price dynamics predicted by mean-reversion models in continuous time.

The actions of speculators play a pivotal role in auction markets. Through their bidding activity, speculators directly influence how quickly transaction prices revert to their average, a phenomenon known as mean reversion. When this bidding process is modeled using the continuous-time AR (1) framework, the rate at which current prices revert to the mean—known as the speed of mean reversion—is interpreted as a discount rate. This discount rate is then used to evaluate the underlying value of a commodity. In essence, speculators not only contribute to price volatility but also help the pace at which current prices return to fair or commodity value. The bidding process provides trading support and enables analysts to model both speculation and valuation.

In auction markets, the values and spot prices of commodities are shaped by the interplay of demand and supply under uncertainty, which inherently brings volatility. Both the supply schedules and commodity values account for storage costs. In the continuous-time AR (1) model, spot prices driven by demand and supply in an auction market establish the commodity value, and the residual error of this model is commonly referred to as volatility, making their influence central to the pricing dynamics in auction markets.

The continuous-time AR (1) model can be transformed into a prediction model by incorporating its error term into the prediction's error term. This model provides a clear understanding of speculative trading, where speculators make prediction errors that naturally revert to the mean. These errors, measured as volatilities, represent small portions of the daily prices. Meanwhile, the commodity values, which speculators aim to establish, constitute the larger portions. Prediction errors tend to revert to the mean when current prices also revert to the mean.

This article analyzes the daily commodity spot price. The value of a commodity is determined by discounted valuation or the interplay of demand and supply in an auction market, while volatility represents fluctuations or noise. Unlike volatility, which is not a component of value, the commodity's value is intrinsic to its current price.

Unlike the prediction model, the model of demand and supply in microeconomics does not determine a commodity price that reverts to its mean. In this competitive model, the commodity price remains stable and equals its commodity value. The prediction model complements the competitive model by introducing volatility, meaning the commodity price is the sum of commodity value and volatility. This volatility mirrors the usual pattern seen in commodity prices.

The prediction model dynamically represents demand and supply under uncertainty in auction markets. In this setting, the bidding process determines a bid price from the demand side and an ask price from the supply side, without requiring explicit demand and supply schedules. When these prices match, a trade is executed, and the corresponding quantity is both demanded and supplied.

Despite the uncertainties in auction markets, microeconomic theory remains dependable and robust, as commodity values consistently reflect significant portions of spot prices.

The following sections will explore the methodologies used to quantify mean reversion, offering a comprehensive overview of the mathematical and statistical tools employed. Additionally, we will discuss the implications of mean reversion for market participants, including speculators whose activities can intensify price volatility. By examining the relationship between speculative trading and mean reversion, we aim to provide insights into the pricing of commodities and the euro in auction markets.

Following the introduction, Section 2 specifies the mean-reversion process in continuous time and introduces the prediction model. Section 3 presents the statistical theory behind parameter estimation. Section 4 describes the functioning of auction markets and explains the role of electronic trading platforms. Section 5 details the data sources used in the empirical study. Section 6 reports the empirical findings and provides a descriptive summary of the results, including screenshots that illustrate daily spot price movements for soybeans, WTI oil, Henry Hub natural gas, copper, gold, and the EUR/USD exchange rate—showing substantial fluctuations due to mean reversion alongside frequent small price changes. Finally, Section 7 summarizes the findings and discusses practical applications of the results. Appendix illustrates the derivation of an AR (1) model in continuous time from the Ornstein-Uhlenbeck process.

2. Modeling Mean Reversion and Prediction

To start the modeling process, decompose a current price P_t into a mean price, μ, and a current demeaned price X_t : $P_t = \mu + X_t$.

X_t is mean reverting towards zero and follows the Ornstein-Uhleckbeck [7] process as shown in Equation (1).

$$\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \nu P_t \mathrm{d}z_t \tag{1}$$

The speed of mean reversion is θ, ν is volatility such that $\nu P_t = \sigma$, σ is overall volatility, P_t is the current price, and $\mathrm{d}z_t$ is a unit normal random variable.

Alternately, P_t is mean reverting towards μ as shown in Equation (2).

$$\mathrm{d}P_t = \theta(\mu - P_t)\mathrm{d}t + \nu P_t \mathrm{d}z_t \tag{2}$$

X_t and P_t are volatile due to volatility, ν. The volatility, ν, is measured by $\mathrm{d}X_t$ or $\mathrm{d}P_t$.

The integration of $\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \nu P_t \mathrm{d}z_t$ yields a continuous-time first-order autoregressive model or AR (1) model, as indicated in Equation (3). See Appendix.

$$X_t = \mathrm{e}^{-\theta} X_{t-1} + \varepsilon \tag{3}$$

and $\varepsilon = \nu P_t \int_{t-1}^{t} \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q)$ where $\mathrm{e}^{-\theta}$ is a discount factor in continuous time, θ is a discount rate, and ε is residual error. The integral qualifying volatility, ν, indicates that price fluctuations due to mean reversion are quantified as volatility, and this volatility is equal to the residual error, ε, which is commonly referred to as volatility. The residual error,
ε, is normally distributed, i.e., $\varepsilon \sim N\left\{0, \left(P_t \int_{t-1}^{t} \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q)\right)^2 \nu^2\right\}$.

The above model (3) can be interpreted as a prediction model. At time t–1, the current demeaned price, X_{t-1}, predicts the future demeaned price, X_t , at t, and is then discounted. At t, the current demeaned price, X_t, equals the discounted lagged demeaned price plus prediction error, $\nu P_t \int_{t-1}^{t} \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q)$. The prediction errors revert to the mean when current prices, whether demeaned, Equation (1) or not, Equation (2), do the same during the bidding process in speculative trading. The bidding process in auction markets introduces competition among participants, which can influence the dynamics of mean reversion and price volatility.

During mean reversion, investors and speculators set the pace, which acts as the discount rate. This discount rate, in turn, determines the commodity's value. When speculation is high, current prices can fluctuate sharply.

The integral is completed as

$$\lambda = \int_{t-1}^{t} e^{-\theta(t-q)} dz(q) = \sqrt{\left(1 - e^{-2\theta}\right)/2\theta} \tag{4}$$

The prediction model has a specific pattern of volatility. The total volatility, ν, is divided into two parts: the volatility, $\lambda\nu$, due to mean reversion and the net volatility, $(1-\lambda)\nu$. The price fluctuations caused by mean reversion are significant and are accompanied by constant, small price fluctuations. This creates a unique volatility pattern.

The model shown in Equation (3) is expanded to give

$$P_t - \mu = e^{-\theta}\left(P_{t-1} - \mu\right) + \lambda\nu P_t dz_t \tag{5}$$

The parameter μ on both sides of Equation (5) is the mean of prices from the same n observations. In estimation, with a sample size of n observations, read n+1 observations or the 1st observation again. Then compute μ from 1 to n observations or from 2 to $n+1$ observations.

In addition to prices, the prediction model can be applied to accounting and economic variables to establish and maintain equilibrium values. A time series of observations tends to be volatile, and mean reversion ensures the time series of observations is stable.

The commodity value is $\mu + e^{-\theta}\left(P_{t-1} - \mu\right)$ and aligns with the current price as the prediction error is small.

The current commodity price, P_t, is the sum of commodity value and volatility due to mean reversion.

The model presented in Equation (5) enhances the static competitive model of demand and supply in microeconomics by incorporating volatility. This means that the commodity price is influenced not only by its commodity value but also by price fluctuations initiated by mean reversion.

3. Estimation

The boundary conditions for the prediction model given in Equation (5) are that $\theta > 0$ and $\nu > 0$. Parameterize the parameters {θ, ν} to estimate them in continuous time. The original parameters {θ, ν} are transformed by letting that $\theta = e^{-x_1}$ and $\nu = e^{-x_2}$. The mean $\mu = x_3$.

Maximum likelihood estimates of {x_1, x_2, x_3} are obtained by maximizing a log-likelihood function using a log-density function, ld, derived from Equation (5).

$$\text{ld} = \ln\left\{1/\left(\lambda\nu P_t\sqrt{2\pi}\right)\right\} - \left\{P_t - \mu - e^{-\theta}\left(P_{t-1} - \mu\right)\right\}^2 / 2\left(\lambda\nu P_t\right)^2 \tag{6}$$

See GAUSS [8] and Maxlik [9].

The original parameters θ and ν are recovered from their exponential functions. For example, dropping the numbering of x, the point estimate, $\bar{\theta} \cong e^{-\bar{x}}$. The maximum likelihood estimate of x is $\bar{x}$. The asymptotic standard error of $e^{-\bar{x}}$ is $\left|\frac{d\left(e^{-\bar{x}}\right)}{d\bar{x}}\right| \times \text{S.E.}(\bar{x})$, where $\text{S.E.}(\bar{x})$ is the standard error of $\bar{x}$. The t-statistic for $\bar{\theta}$ is $1/\text{S.E.}(\bar{x})$. By eliminating the approximate mean $e^{-\bar{x}}$, the t-statistic for

$\overline{\theta}$ depends on the standard error of the exponent $\bar{x}$ of the mean transformed parameter $e^{-\bar{x}}$.

The exponent x of e^{-x} has a normal distribution with mean $\bar{x}$ and variance σ^2. In the derivation of a t-statistic for the point estimate, $\overline{\theta}$, the numerical approximation of e^{-x} is used because it has a normal distribution with mean $e^{-\bar{x}}$ and variance $e^{-2\bar{x}}\sigma^2$.

In estimating the parameters of the AR (1) model, prices follow this order (P_t, P_{t-1}, P_{t-2}, P_{t-3}, …).

4. Exchange and Electronic Trading Platform

An auction market is an exchange with a trading floor where participants, such as specialists or brokers, execute trades by matching bid and ask prices. This indicates that prices are competitive, and transactions are determined by demand and supply. The matching mechanism can be either manual or semi-automated.

Spot contracts for commodities such as soybeans, WTI oil, Henry Hub natural gas, copper, and gold are traded on electronic trading platforms. These platforms match bid and ask prices in real time through a fully automated, high-frequency system. Participants in these markets include producers, end users, merchants, traders, financial institutions, retail traders, exchanges, and clearing houses. Soybeans, WTI oil, and Henry Hub natural gas are traded on CME Globex, while copper and gold are traded over the counter (OTC) via online platforms.

The EUR/USD currency pair is traded over the counter (OTC) through an international network of banks, brokers, and electronic trading platforms. These platforms continuously match bid and ask prices. The EUR/USD currency pair is the most traded pair in the world and is quoted indirectly. For example, a quote like EUR/USD = 1.10 means 1 euro is worth 1.10 US dollars.

Since the essence of an auction is that a trade is executed when bid and ask prices are matched, I will use the term 'auction market' as a generic term to represent an exchange or an electronic trading platform.

5. Data

Daily reported spot prices of commodities and the euro traded on electronic trading platforms and OTC are obtained from the internet from various sources. The commodities are soybeans in the agricultural sector, West Texas Intermediate oil and Henry Hub natural gas in the energy sector, copper and gold in the metal sector, and the EUR/USD exchange rate.

Bloomberg soybeans spot prices (BCOMSY), Bloomberg copper spot prices (BCOMHG) and the EUR/USD exchange rates are from Investing.com; Cushing, OK WTI oil spot prices and Henry Hub natural gas spot prices are from U.S Energy Information Administration; and gold spot prices are from https://www.usagold.com daily gold-price history.

The samples cover a three-month period from April 1, 2025, to June 30, 2025, representing the final three months from the start of the empirical study in July

2025.

The parameters, $\{\theta, \nu\}$, capture the speed and volatility of mean reversion in commodity prices. In high-frequency auction markets, a sample of three months of daily data is typically sufficient to estimate these parameters and demonstrate that they are statistically significant.

Small samples of daily prices are utilized to demonstrate the efficacy of the prediction model in the short term. This model is specifically designed for speculative trading in auction markets.

6. Empirical Results

The empirical results indicate that the commodity spot price tends to revert to the mean. The speed of this mean reversion, θ, is significantly greater than zero, suggesting a consistent tendency for the commodity spot price to return to its average level. Additionally, the volatility, ν, is significantly greater than zero. The volatility, $\lambda\nu$, associated with mean reversion is positive, and the prediction error tends to revert to the mean. Throughout the day, prices experience constant, small fluctuations, with larger price fluctuations periodically reverse direction. These larger price fluctuations, combined with the small ones, contribute to the total volatility. The larger price fluctuations are due to mean reversion, and are quantified as volatility, $\lambda\nu P_t$. The volatility due to mean reversion is equal to the prediction-error volatility. The small price fluctuations are measured as net volatility, $(1-\lambda)\nu P_t$.

Commodity spot prices in the agricultural, energy, metal, and foreign exchange sectors tend to exhibit mean reversion. The prediction model is an effective time series model for speculative trading.

The average values for commodities such as soybeans, WTI oil, copper, gold, and the EUR/USD exchange rate, typically range between 97.30 % and 99.04% of their respective daily spot prices. A key finding is that the model's predicted values—approximately 97% or more of daily spot prices—suggest the prediction model effectively captures the underling demand and supply dynamics of auction markets. In such markets, commodity and currency values are primarily driven by real-time fluctuations in demand and supply, closely mirroring the behavior of spot prices.

The volatilities due to mean reversion are within a margin of 2¾% of spot prices, corresponding to prediction errors within a margin of 2¾% of spot prices.

The test results for natural gas are unusual. The commodity value for natural gas stands out as an outlier at 94.87 % of the daily spot price. The prediction error exhibits a volatility of 5.13% of the daily spot price, indicating greater uncertainty in the natural gas market.

The screenshots in **Figures 1-6** illustrate the mean reversion and constant, small price fluctuations. These screenshots depict the mean reversion of commodity prices for soybeans (agriculture), WTI oil (energy), natural gas (Henry Hub) (energy), copper (metal), and gold (metal), and the EUR/USD exchange rate. The images show larger price fluctuations that change direction, accompanied by constant, small price fluctuations. This distinct volatility pattern is common across soybeans, WTI oil, natural gas, copper, gold, and the EUR/USD exchange rate. Note that mean reversion is quantified as volatility, $\lambda v P_t$.

Soybeans

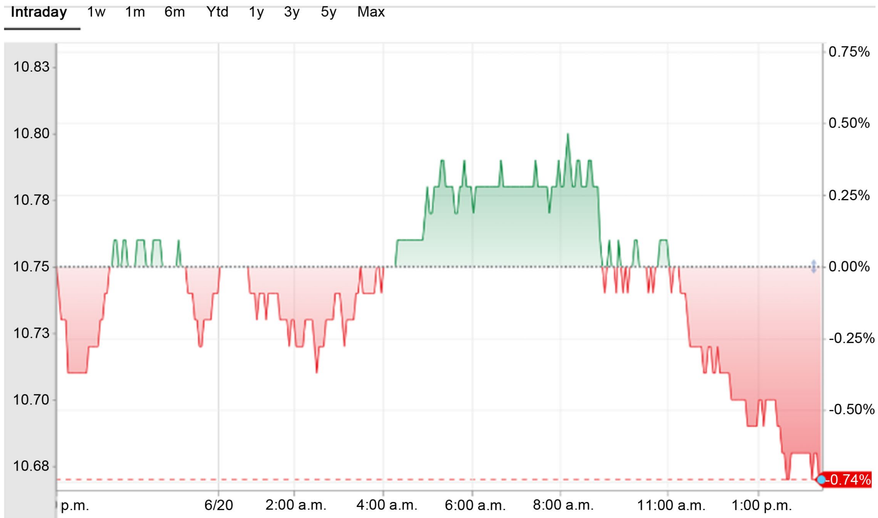

Figure 1. Changes in the spot price of soybeans on June 20, 2025.

WTI chart›

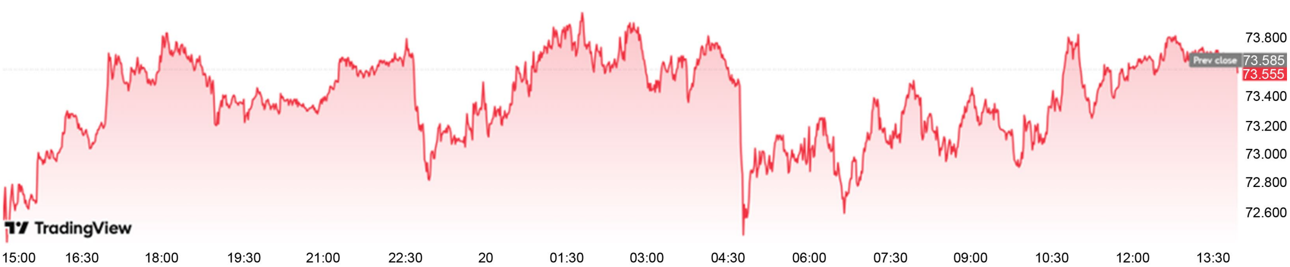

Figure 2. Changes in the spot price of WTI oil on June 20, 2025.

Natural Gas (Henry Hub) Snapshot

Figure 3. Changes in the spot price of natural gas (Henry Hub) on June 20, 2025.

Figure 4. Changes in the spot price of copper on June 24, 2025.

Figure 5. Changes in the spot price of gold on June 24, 2025.

Table 1 shows the estimates of the parameters $\{\theta, \nu, \mu\}$ of the prediction model, with t-statistics shown in parentheses below the point estimates. The t-statistics for θ and ν are greater than the critical value of 1.67 at the 5% level of significance.

Since the speed of mean reversion, θ, is positive, the commodity or currency spot price tends to revert to its mean (Equation (2)). Given that volatility, ν, is also positive, the volatility, $\lambda\nu$, due to mean reversion can be calculated. The test of mean reversion is complete when the spot price is shown to revert to the mean, and the volatility caused by mean reversion is independently verified. The net volatility, $(1-\lambda)\nu$, can then be computed.

Figure 6. Changes in the spot EUR/USD exchange rate on July 2, 2025.

Table 1. Estimates of daily parameters for the prediction model.

Commodity	n	θ	ν	μ	$\lambda\nu$	$(1-\lambda)\nu$
Soybeans	62	0.1725 (2.11)	0.0105 (10.23)	235 (129.43)	0.0096	0.0009
WTI oil	62	0.1296 (1.87)	0.0288 (10.44)	64 (31.83)	0.0270	0.0018
Natural gas	62	0.2162 (2.31)	0.0570 (10.03)	3.19 (23.99)	0.0513	0.0057
Copper	62	0.1608 (2.05)	0.0207 (10.29)	393 (56.76)	0.0191	0.0016
Gold	65	0.1679 (2.14)	0.0183 (10.50)	3288 (77.94)	0.0168	0.0015
EUR/USD	65	0.2911 (2.64)	0.0137 (9.91)	1.1357 (178.22)	0.0119	0.0018

Notes: The five commodities are soybeans, West Texas Intermediate oil (energy), natural gas (Henry Hub) (energy), copper (metal), and gold (metal), and the EUR/USD exchange rate. Daily reported spot prices cover April to June 2025. The number of observations is n. The daily parameters, $\{\theta, \nu, \mu\}$, are for the prediction model. The discount rate is θ and the volatility is ν. The parameter μ is the mean of prices. The t-statistics for the point estimates are shown in parentheses below the corresponding estimates. The critical value of the t-statistic for θ and ν is 1.67 at the 5% level of significance. The volatility due to mean reversion is $\lambda\nu$, and the net volatility is $(1-\lambda)\nu$. The value of λ is calculated from Equation (4).

As shown in **Table 1**, the volatilities, $\lambda\nu$, caused by mean reversion typically range between 0.96% and 2.70% of their respective daily commodity spot prices, except for natural gas. The net volatilities, $(1-\lambda)\nu$, range between 0.09% and 0.18% of their respective daily commodity spot prices, again except for natural gas. The higher volatility due to mean reversion is assumed to be the volatility made by speculators. The volatility pattern of the prediction model indicates significant price fluctuations due to mean reversion, accompanied by constant, small price fluctuations. This pattern aligns with the observed volatility trends.

The volatility due to mean reversion for natural gas stands at 5.13% of the daily spot price, indicating a higher level of uncertainty in the natural gas market. In contrast, commodity values for other resources typically range between 97.30% and 99.04% of their respective daily spot prices, suggesting that these values make up a larger portion of the daily prices compared to prediction errors. These commodity values align closely with the commodity prices. For natural gas, the commodity value is 94.87% of the daily spot price, which, while still high, is less common.

7. Conclusions

The prediction model projects a unique volatility pattern that aligns with the observed volatility patterns of commodity spot prices and the EUR/USD exchange rate. The spot prices and the EUR/USD exchange rate exhibit mean reversion dynamics, where significant price fluctuations due to mean reversion are accompanied by constant, small price changes. This observed volatility pattern highlights the consistent behavior of mean reversion in pricing.

The prediction model has two separable volatilities: volatility, $\lambda\nu$, due to mean reversion and net volatility, $(1-\lambda)\nu$. These two volatilities define the unique volatility pattern of the model.

The volatility, $\lambda\nu$, arising from fluctuations in mean reversion, is sufficiently high to be observed, enabling traders and risk managers to take advantage of market fluctuations. One effective approach is to wait for a market correction before buying and then sell once the market rises again. This waiting strategy can be profitable, even though the market itself remains unpredictable. Importantly, waiting is different from predicting—it is a disciplined response to observable conditions rather than a speculative forecast. The net volatility, $(1-\lambda)\nu$, is too low to be meaningfully detected or exploited in trading.

The parameters, $\{\theta,\nu,\mu\}$, of the prediction model were estimated from the daily spot prices of five commodities in the agricultural, energy, and metal sectors, as well as the EUR/USD exchange rate for three months in 2025. The volatilities due to mean reversion are substantially higher than those of net volatilities. This indicates that price fluctuations caused by mean reversion periodically change direction and are accompanied by constant, small price fluctuations.

Screenshots of the day's price movements of five commodities—soybeans, West Texas Intermediate oil, Henry Hub natural gas, copper, gold—as well as the

EUR/USD exchange rate, reveal observable volatility patterns. These patterns show significant price fluctuations that change direction, along with constant, small price fluctuations.

In auction markets, commodity and currency values—excluding outliers like natural gas with atypical test results—typically range from 97.30% to 99.04% of daily spot prices. This close alignment reflects the strong influence of supply and demand on commodity and currency values in auction markets. Because these markets are governed by competitive bidding, the prediction model can be employed to estimate the underlying values of traded assets.

Since common stocks are traded in auction markets, the prediction model can be employed to estimate the equity or fair value of a firm's common stock.

Speculators often make prediction errors that tend to revert to the mean. These prediction errors account for approximately 2¾% of the daily commodity spot prices.

Conflicts of Interest

The author declares no conflict of interest.

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Appendix

Proof

X_t is mean reverting towards zero and follows the Ornstein-Uhleckbeck [7] process as shown below.

$$\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \sigma \mathrm{d}z_t$$

A continuous-time first-order autoregressive process is derived from the above SDE.

Multiply both sides of the SDE by the integrating factor $\mathrm{e}^{\theta t}$.

$$\mathrm{e}^{\theta t}\mathrm{d}X_t = -\theta \mathrm{e}^{\theta t} X_t \mathrm{d}t + \sigma \mathrm{e}^{\theta t}\mathrm{d}z_t$$

Apply the product rule from calculus:

$$\mathrm{d}\left(\mathrm{e}^{\theta t} X_t\right) = \mathrm{e}^{\theta t}\mathrm{d}X_t + \theta \mathrm{e}^{\theta t} X_t \mathrm{d}t$$

Substitute the expression for $\mathrm{d}X_t$ as given in the SDE to remove the drift term:

$$\mathrm{d}\left(\mathrm{e}^{\theta t} X_t\right) = \mathrm{e}^{\theta t}\left(-\theta X_t \mathrm{d}t + \sigma \mathrm{d}z_t\right) + \theta \mathrm{e}^{\theta t} X_t \mathrm{d}t$$

Simplify:

$$\mathrm{d}\left(\mathrm{e}^{\theta t} X_t\right) = \sigma \mathrm{e}^{\theta t}\mathrm{d}z_t$$

Integrate both sides from s to t.

$$\mathrm{e}^{\theta t} X_t - \mathrm{e}^{\theta s} X_s = \sigma \int_s^t \mathrm{e}^{\theta q}\mathrm{d}z_q$$

Solve for X_t:

$$X_t = X_s \mathrm{e}^{-\theta(t-s)} + \sigma \int_s^t \mathrm{e}^{-\theta(t-q)}\mathrm{d}z_q$$

Set $s = t-1$:

$$X_t = \mathrm{e}^{-\theta} X_{t-1} + \sigma \int_{t-1}^t \mathrm{e}^{-\theta(t-q)}\mathrm{d}z_q$$