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Valuation Model of the Expected SBDA as a Forward-Looking Performance Measure for PE Funds

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Abstract

In this study, we adopt Expected SBDA as a forward-looking measure of excess return of PE funds, and attempt to examine the mechanism how expected SBDA is influenced by factors such as drift and volatility of the investee company return, expected time to find investee companies, and target exit multiple. Implications from the numerical examples are that it is better to find investee companies with large drift to increase expected SBDA, even if you pay some sourcing time and volatility risk of the investee company return and that the target exit multiple maximizing expected SBDA is generally between 3 and 6, although it depends on the magnitude of the other parameters.

Keywords

Expected SBDA, Forward-Looking Performance Measure, PE Funds, First-Hitting-Time, Target Exit Multiple

1. Introduction

In the global low-interest rate environment, PE funds are of great interest to investors as a part of alternative investments seeking higher absolute returns. For this reason, the Internal Rate of Return (IRR) and Total Value per $ invested (TVPI) with respect to absolute return have been widely used as performance indicators for PE funds (e.g., Gilligan and Wright [1]). In recent years, PE funds have been the useful tool to enhance absolute return in the public pension investment. For example, the GPIF in Japan has set a target absolute return (annualized over the five-year accounting period from 2025 to 2029) of 1.9% over the wage growth rate. In addition to the target absolute return, the GPIF also tries to seek an excess return over the policy benchmark (e.g., Government Pension Investment

Fund [2]).

Because the cash flows from the investment in PE funds are complex, it is more difficult to evaluate the relative performance of PE funds than active funds, which invest in listed equities. Prior studies on relative performance measures are PME (Public Market Equivalent) such as PME (e.g., Long and Nickeles [3]), PME+ (e.g., Rouvinez [4]), mPME (e.g., Cambridge Associates [5]), Direct Alpha (e.g., Gredi *et al.* [6]), and Spread Based Direct Alpha (SBDA) (e.g., Miyazaki and Shimada [7] [8]). Among these PMEs, SBDA has following three attractive features 1) The excess return (SBDA) is obtained by discounting the cash flows with the benchmark return from the time they occur to the time of commitment, like the credit spread of a corporate bond; 2) The distributed cash flows are not reinvested in the PE fund, but return to the benchmark, so that the cash flow after the time of distribution does not affect the excess return, and 3) Not only percentage display but also dollar value of the excess return can be easily obtained. For the derivation of the dollar value of the excess return, refer to Miyazaki and Shimada [7].

While SBDA plays a role in measuring the ex-post excess return of PE funds, it is difficult to capture the mechanism by which factors have influence on SBDA, because cash flows of PE funds are complex and subject to many factors. Therefore, in this study, we newly introduce expected SBDA as a forward-looking measure of excess return of PE funds, and formulate factors such as drift and volatility of the investee companies, time for PE fund to find them, and target exit multiple to evaluate the expected SBDA, and discuss the mechanism to generate expected SBDA by examining the sensitivity of the key parameters of the model to the expected SBDA.

This paper is organized as follows. Section 2 reviews SBDA, the ex-post relative performance measure of PE funds. Section 3, as Valuation Model I, proposes the valuation model of the expected SBDA focusing on the case where the PE fund invests in only one investee company. Section 4 extends Valuation Model I to the case of multiple investee companies as Valuation Model II. Section 5 gives a numerical example focusing on Valuation Model I and derives the implications from the sensitivity of the key parameters of the model to the expected SBDA. In the final section, summary and future issues are added.

2. Spread Based Direct Alpha (SBDA)

Since PE funds invest mainly in unlisted stocks, the benchmark of PE funds is an equity index. For example, for a PE fund that invests in unlisted stocks worldwide, the benchmark would be the MSCI All Country World Index (ACWI). When an investor commits to invest in a PE fund, he or she first sets the commitment amount, sourcing period (e.g., five years after commitment), and investment period, which is the total commitment period minus the sourcing period (e.g., up to 10 years after commitment). The PE fund finds investee companies during the sourcing period and then makes a capital call to raise funds from investors to invest in the investee company, and when the company's business takes off, it exits (through IPO, M&A, etc.) and distributes the results back to the investors within the investment period. Thus, unlike investments in ordinary active equity funds, investments in PE funds have complex cash flows, and it is difficult to evaluate how well their performance is relative to the benchmark (relative performance).

When an investor who is evaluated against ACWI as a benchmark tries to invest in PE funds, the idea of SBDA is to "capture the excess return obtained on average over the commitment period". Therefore, the commitment period of SBDA for a PE fund is defined as the period from the time point when the investor commits to invest in the PE fund (when the investor contracts to transfer funds from ACWI to the PE fund in response to a capital call) up to the time point when all investment results have been distributed by the last exit. Based on the above, the definition of SBDA is given by Equation (1).

(Definition 1) SBDA

SBDA is s that satisfies the following equation.

$$\sum_{j=1}^{m}\frac{Call(j)}{(1+r_j)^j}=\sum_{i=1}^{n}\frac{Dist(i)}{(1+r_i+s)^i}+\frac{NAV}{(1+r_i+s)^n} \tag{1}$$

where $Call(j)$ and r_j are the amount of funds invested in response to the capital call at j and the annualized benchmark return such that the cumulative return from time 0 (the time of commitment to the PE fund) to time j is $(1+r_j)^j$, respectively; $Dist(i)$ and r_i are the amount of the distribution and the annualized benchmark rate such that the cumulative return on the benchmark from time 0 to time i is $(1+r_i)^i$ respectively, and NAV represents the valuation of the last distribution. The above formula shows that there are m capital calls and n distributions. In addition, the PE fund must meet the following conditions as stipulated in the original agreement: $\sum_{j=1}^{m}Call(j)$ is within the commitment amount, m indicating the time of the last capital call is within the investment period, and n expressing the time of the last distribution within the investment period. For more information on the mechanism of SBDA and other related issues, see Miyazaki and Shimada [7].

3. Valuation Model I (Case of Investing in Only One Company)

The point in time when the PE fund invests the fund raised from investor through capital call in the investee companies is represented by the stochastic variable $\tilde{t}_C$, and the sourcing period $[0,T_1]$ ($0\le\tilde{t}_C\le T_1$) is the period for the PE fund to find the investee companies. The stochastic variable $\tilde{t}_D$ represents the investment period from the time the PE fund invests at $\tilde{t}_C$ to the time it exits, and the commitment to the PE fund is completed at T_2. The total commitment period to the PE fund is $[0,T_2]$ ($T_1<T_2$), so $0\le\tilde{t}_C+\tilde{t}_D\le T_2$ must be satisfied.

(Modeling the stochastic variable $\tilde{t}_C$ which represents the beginning of the investment)

Following a standard approach for modeling random, independent events (see for example, Shreve [9], Chan *et al.* [10]), a lump-sum group of companies consisting of venture companies and buyout targets sourced by the PE fund will be assumed to emerge according to Poisson's arrival with some arrival rate depending on the macroeconomic environment and the PE fund's ability. $X(t)$ is the number of investee companies for the PE fund in the period [0, t] and this is modeled by the Poisson arrival in Equation (2) (Poisson distribution with parameter λt).

$$P\left(X(t)=k\right)=\frac{\mathrm{e}^{-\lambda t}(\lambda t)^k}{k!} \quad k=0,1,2,\cdots \tag{2}$$

Let T denote the stochastic variable that represents the arrival interval of a group of firms that come one after another according to Poisson arrival, and consider its probability density function $f(t)$. Since the event $\{T>t\}$ represents that the arrival interval T is greater than t, it is the same as the event $\{X(t)=0\}$ because it represents that the next group of firms has not appeared up to the time t, and we obtain Equation (3) by putting $k=0$ in Equation (2).

$$\int_t^\infty f(x)\mathrm{d}x=P(T>t)=P\left(X(t)=0\right)=\mathrm{e}^{-\lambda t} \tag{3}$$

By differentiating $F(t)=\int_0^t f(x)\mathrm{d}x=1-\mathrm{e}^{-\lambda t}$ by t, the probability density function or probability of the stochastic variable $\tilde{t}_C$ representing the investment point in time is modeled by Equations (4) and (5).

Cases in which investment is made:

$$f_{t_C}(t)=\lambda\mathrm{e}^{-\lambda t} \quad \left(0\le t<T_1\right) \tag{4}$$

Probability that will not be invested:

$$P\left(t_C>T_1\right)=\int_{T_1}^\infty \lambda\mathrm{e}^{-\lambda t}\mathrm{d}t=\mathrm{e}^{-\lambda T_1} \tag{5}$$

(Modeling benchmark ($i=0$) and investee company ($i=1$) stock prices using stochastic process $S_i(t)$).

The drift and the volatility of the investee company return usually larger than those of the benchmark return, *i.e.* $\mu_1>\mu_0$ and $\sigma_1>\sigma_0$.

$$\frac{\mathrm{d}S_i(t)}{S_i(t)}=\mu_i\mathrm{d}t+\sigma_i\mathrm{d}B_i(t) \tag{6}$$

where μ_i, $\sigma_i>0$, and $B_i(t)$ are the drift, volatility, and standard Brownian motion, respectively.

(Modeling the stochastic variable $\tilde{t}_D$ representing the investment period)

PE funds typically attempt to exit when the stock price of the investee company reaches several times the stock price $S_1(\tilde{t}_C)$ at the starting time of investment $\tilde{t}_C$. Here, we assume that the PE funds exit at time $\tilde{t}_C+\tilde{t}_D$ when the stock price reaches $\gamma S_1(\tilde{t}_C)$, which is γ times $S_1(\tilde{t}_C)$, for the first time (first arrival time). If the stock price never reaches $\gamma S_1(\tilde{t}_C)$ during the commitment period $[0,T_2]$, we assume exit with the stock price $S_1(T_2-\tilde{t}_C)$ at T_2. If the stock price of the investee company follows the stochastic process in Equation (6), the stock price

$S_1(t)$ ($0 < t < T_2 - \tilde{t}_C$) at time t from the starting time of investment $\tilde{t}_C$ is given by Equation (7).

$$S_1(t) = S_1(t_C)\mathrm{e}^{\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 B_1(t)} \tag{7}$$

The probability density function or probability of the stochastic variable $\tilde{t}_D$ representing the investment period in the stock $S_1(t)$ is given by Lemma 1.

Lemma 1

The probability density function or not hitting probability of the stochastic variable $\tilde{t}_D$ representing the investment period from the starting point of investment $\tilde{t}_C$ is given by

Case 1 (the stock price reaches $\gamma S_1(t_C)$ within the investment period $[t_C, T_2]$)

The probability density function for the random variable $\tilde{t}_D$ is given by Equation (8).

$$f_{t_D}(t_D) = \frac{\log\gamma}{\sqrt{2\pi t_D^3}\sigma_1}\exp\left[-\frac{\left\{\log\gamma - \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)t_D\right\}^2}{2\sigma_1^2 t_D}\right] \quad \left(0 < t_D \le T_2 - \tilde{t}_C\right) \tag{8}$$

Case 2 (the stock price does not reach $\gamma S_1(t_C)$ within the investment period $[t_C, T_2]$)

The probability $P(\tilde{t}_D = T_2 - \tilde{t}_C)$ is given by Equation (9).

$$P(\tilde{t}_D = T_2 - \tilde{t}_C) \equiv P(\tilde{t}_D > T_2 - \tilde{t}_C) = 1 - \int_0^{T_2 - \tilde{t}_C} f_{t_D}(t_D)\mathrm{d}t_D \tag{9}$$

(Proof)

From Equation (7), the stock price $S_1(t)$ reaching $\gamma S_1(\tilde{t}_C)$ is the same as the Brownian motion with drift $\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 B_1(t)$ reaching $\log\gamma$.

Regarding the standard Brownian motion $\{B(t), t \ge 0\}$, let

$M^*(t) \overset{\text{def}}{=} \max_{0 \le u \le t} B(u)$ be the maximum value up to the point in time t. Then, from the mirror image principle, we obtain Equation (10).

$$P\left[M^*(t) \ge x, B(t) \le x \mid B(0) = 0\right] = P\left[B(t) \ge x \mid B(0) = 0\right] \quad (x > 0) \tag{10}$$

Using Equation (10),

$$\begin{aligned}
&P\left[M^*(t) \ge x \mid B(0) = 0\right] \\
&= P\left[M^*(t) \ge x, B(t) \le x \mid B(0) = 0\right] + P\left[M^*(t) \ge x, B(t) \ge x \mid B(0) = 0\right] \\
&= P\left[M^*(t) \ge x, B(t) \le x \mid B(0) = 0\right] + P\left[B(t) \ge x \mid B(0) = 0\right] \\
&= 2P\left[B(t) \ge x \mid B(0) = 0\right] \\
&= 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)
\end{aligned} \tag{11}$$

is obtained, where $\Phi(\cdot)$ represents the distribution function of the standard normal distribution. From this,

$$P\left[M^*(t) \le x \mid B(0)=0\right] = 1-2\left(1-\Phi\left(\frac{x}{\sqrt{t}}\right)\right) = 2\Phi\left(\frac{x}{\sqrt{t}}\right)-1 \tag{12}$$

Denoting the first arrival time of $\{B(t), t \ge 0\}$ to the state x as T_x, we see $\{T_x \le t\} = \{M^*(t) \ge x\}$, so from Equation (11)

$$P\left[T_x \le t \mid B(0)=0\right] = 2\left(1-\Phi\left(\frac{x}{\sqrt{t}}\right)\right) \tag{13}$$

is obtained. Furthermore, by differentiating Equation (13) with time t, the probability density function $f_x(t)$ of T_x is attained as,

$$f_x(t) = \frac{x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2}{2t}\right\}, \quad (t>0) \tag{14}$$

From this result, the transition probability density of the (μ, σ_1)-Brownian motion with a drift of μ and a volatility of σ_1 is given by Equation (15).

$$f_x(t) = \frac{x}{\sqrt{2\pi t^3}\sigma_1} \exp\left\{-\frac{(x-\mu t)^2}{2\sigma_1^2 t}\right\}, \quad (t>0) \tag{15}$$

Substituting $\mu = \mu_1 - \frac{1}{2}\sigma_1^2$ into Equation (15), we obtain the probability density function Equation (8) for the stochastic variable $\tilde{t}_D$.

The probability that the stock price does not reach $\gamma S_1(t_C)$ at $[0, T_2]$ is obtained by integrating Equation (8) over the interval $\tilde{t}_D > T_2 - \tilde{t}_C$ as Equation (9).

(QED)

In Case 2, where the stock price $S_1(t)$ at time t from the time $\tilde{t}_C$ does not reach $\gamma S_1(\tilde{t}_C)$ within the investment period $[t_C, T_2]$, the expected exit price is assumed to be the expected price of $S_1(T_2 - \tilde{t}_C)$ at the end of the investment period T_2 and given by Lemma 2.

Lemma 2

When the stock price $S_1(u) = S_1(\tilde{t}_c)\exp\left\{\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)u + \sigma_1 B_1(u)\right\}$ does not reach $\gamma S_1(\tilde{t}_C)$ in the investment period $[t_C, T_2]$, with the definition $R_1(u) \overset{\text{def}}{=} \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)u + \sigma_1 B_1(u) = \mu u + \sigma_1 B_1(u)$, the expected exit price of $S_1(T_2 - \tilde{t}_C)$ at time T_2 is given as Equation (16) by calculating

$$E\left[S_1(\tilde{t}_c)e^{R_1(T_2-\tilde{t}_C)} 1_{\{R_1(T_2-\tilde{t}_C)\le \log\gamma\}} \cdot 1_{\left\{\max\limits_{0\le u\le T_2-\tilde{t}_C} R_1(u)\le\log\gamma\right\}} \mid \mathcal{F}_{\tilde{t}_c}\right].$$

$$S_1(\tilde{t}_c)e^{\mu_1(T_2-\tilde{t}_C)}\left\{\Phi(K_1') - \gamma^{\left(1+\frac{2\mu_1}{\sigma^2}\right)}\left(1-\Phi(K_2')\right)\right\} \tag{16}$$

where $K_1' = \frac{\log\gamma - \left(\mu_1 + \frac{1}{2}\sigma_1^2\right)\left(T_2 - \tilde{t}_C\right)}{\sigma_1\sqrt{T_2 - \tilde{t}_C}}$ and $K_2' = \frac{\log\gamma + \left(\mu_1 + \frac{1}{2}\sigma_1^2\right)\left(T_2 - \tilde{t}_C\right)}{\sigma_1\sqrt{T_2 - \tilde{t}_C}}$

and $\Phi(\cdot)$ represents the distribution function of the standard normal distribution.

(Proof)

To calculate the expected value

$\left[S_1\left(\tilde{t}_c\right)\mathrm{e}^{R_1(T_2-\tilde{t}_C)}1_{\{R_1(T_2-\tilde{t}_C)\le\log\gamma\}}\cdot 1_{\left\{\max_{0\le u\le T_2-\tilde{t}_C} R_1(u)\le\log\gamma\right\}} \mid \mathcal{F}_{\tilde{t}_c}\right]$, we need the transition probability density of the (μ,σ_1)-Brownian motion with an absorbing boundary at $\log\gamma$. Referring to Kijima [11], the transition probability density $a(x,y,t)$ of the (μ,σ_1)-Brownian motion with an absorbing boundary at $\log\gamma$ from state x at time 0 to state y at time t is given by Equation (17). $x, y > 0$; at $t > 0$ time,

$$\begin{aligned} a(x,y,t) &= \frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y-x-\mu t)^2}{2\sigma_1^2 t}\right\} \\ &\quad - \mathrm{e}^{-\frac{2\mu(x-\log\gamma)}{\sigma_1^2}}\frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y+x-2\log\gamma-\mu t)^2}{2\sigma_1^2 t}\right\} \end{aligned} \tag{17}$$

The derivation of Equation (17) is provided in **Appendix**.

Since we are interested in the transition probability density $\tilde{a}(0,y,t)$ for the (μ,σ_1)-Brownian motion with an absorbing boundary at state $\log\gamma$ starting from the state $R_1(0) = \mu\cdot 0 + \sigma_1 B_1(0) = 0$ at time 0, to the state y at time t, it is given by Equation (18).

$$\begin{aligned} \tilde{a}(0,y,t) &= \frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y-\mu t)^2}{2\sigma_1^2 t}\right\} \\ &\quad - \mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}}\frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y-2\log\gamma-\mu t)^2}{2\sigma_1^2 t}\right\} \end{aligned} \tag{18}$$

To obtain the expected value, we utilize Equation (18) to calculate Equation (19).

$$\begin{aligned} &\int_{-\infty}^{\log\gamma}\mathrm{e}^y\tilde{a}(0,y,t)\,\mathrm{d}y \\ &= \int_{-\infty}^{\log\gamma}\mathrm{e}^y\frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y-\mu t)^2}{2\sigma_1^2 t}\right\}\mathrm{d}y \\ &\quad - \mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}}\int_{-\infty}^{\log\gamma}\mathrm{e}^y\frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y-2\log\gamma-\mu t)^2}{2\sigma_1^2 t}\right\}\mathrm{d}y \end{aligned} \tag{19}$$

Compute the first term on the right-hand side of Equation (19),

$$\int_{-\infty}^{\log\gamma}\mathrm{e}^y\frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{(y-\mu t)^2}{2\sigma_1^2 t}\right\}\mathrm{d}y.$$

Convert the variable to $\frac{y-\mu t}{\sigma_1\sqrt{t}}=z_1$ and proceed with the calculation, paying attention to $\mathrm{d}y=\sigma_1\sqrt{t}\mathrm{d}z_1$ and $z_1:-\infty\to\frac{\log\gamma-\mu t}{\sigma_1\sqrt{t}}$, to obtain

$\int_{-\infty}^{\frac{\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\mathrm{e}^{\mu t+\sigma_1\sqrt{t}z_1}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^2}{2}\right\}\mathrm{d}z_1$.

$$\begin{aligned}&\int_{-\infty}^{\frac{\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\mathrm{e}^{\mu t+\sigma_1\sqrt{t}z_1}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^2}{2}\right\}\mathrm{d}z_1\\&=\mathrm{e}^{\left(\mu+\frac{1}{2}\sigma_1^2\right)t}\int_{-\infty}^{\frac{\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(z_1-\sigma_1\sqrt{t}\right)^2}{2}\right\}\mathrm{d}z_1\end{aligned}$$

Now, convert the variable to $z_1-\sigma_1\sqrt{t}=v_1$ and proceed with the calculation, paying attention to $\mathrm{d}z_1=\mathrm{d}v_1$, and $v_1:-\infty\to\frac{\log\gamma-\left(\mu+\sigma_1^2\right)t}{\sigma_1\sqrt{t}}$.

$$\begin{aligned}&\mathrm{e}^{\left(\mu+\frac{1}{2}\sigma_1^2\right)t}\int_{-\infty}^{\frac{\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(z_1-\sigma_1\sqrt{t}\right)^2}{2}\right\}\mathrm{d}z_1\\&=\mathrm{e}^{\left(\mu+\frac{1}{2}\sigma_1^2\right)t}\int_{-\infty}^{\frac{\log\gamma-\left(\mu+\sigma_1^2\right)t}{\sigma_1\sqrt{t}}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{v_1^2}{2}}\mathrm{d}v_1\end{aligned}$$

Finally, substituting $\mu=\mu_1-\frac{1}{2}\sigma_1^2$, the first term on the right side of Equation (19) becomes Equation (20).

$$\mathrm{e}^{\mu t+\frac{\sigma_1^2 t}{2}}\Phi\left(\frac{\log\gamma-\left(\mu+\sigma_1^2\right)t}{\sigma_1\sqrt{t}}\right)=\mathrm{e}^{\mu_1 t}\Phi\left(\frac{\log\gamma-\left(\mu_1+\frac{\sigma_1^2}{2}\right)t}{\sigma_1\sqrt{t}}\right)\tag{20}$$

In a similar manner, the second term on the right-hand side of Equation (19) is calculated.

$$\mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}}\int_{-\infty}^{\log\gamma}\mathrm{e}^{y}\frac{1}{\sqrt{2\pi t}\cdot\sigma_1}\exp\left\{-\frac{\left(y-2\log\gamma-\mu t\right)^2}{2\sigma_1^2 t}\right\}\mathrm{d}y$$

is calculated as follows.

Convert the variable as $\frac{y-2\log\gamma-\mu t}{\sigma_1\sqrt{t}}=v_2$ and proceed with the calculation, paying attention to $\mathrm{d}y=\sigma_1\sqrt{t}\mathrm{d}v_2$ and $v_2:-\infty\to\frac{-\log\gamma-\mu t}{\sigma_1\sqrt{t}}$.

$$\begin{aligned}&\mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}}\int_{-\infty}^{\frac{-\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\mathrm{e}^{2\log\gamma+\mu t+\sigma_1\sqrt{t}v_2}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{v_2^2}{2}}\mathrm{d}v_2\\&=\mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}+2\log\gamma+\mu t+\frac{1}{2}\sigma_1^2 t}\int_{-\infty}^{\frac{-\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\left(v_2-\sigma_1\sqrt{t}\right)^2}{2}}\mathrm{d}v_2\end{aligned}$$

Finally, after variable conversion and setting $v_2 - \sigma_1\sqrt{t} = w_2$, noting $\mathrm{d}v_2 = \mathrm{d}w_2$ and $w_2 : -\infty \to \frac{-\log\gamma - \left(\mu + \sigma_1^2\right)t}{\sigma_1\sqrt{t}}$, and further substituting $\mu = \mu_1 - \frac{1}{2}\sigma_1^2$, the first term on the right side of Equation (19) becomes Equation (21).

$$\begin{aligned}
&\mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}+2\log\gamma+\mu t+\frac{1}{2}\sigma_1^2 t}\int_{-\infty}^{\frac{-\log\gamma-\mu t}{\sigma_1\sqrt{t}}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\left(v_2-\sigma_1\sqrt{t}\right)^2}{2}}\mathrm{d}v_2 \\
&=\mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}+2\log\gamma+\mu t+\frac{\sigma_1^2 t}{2}}\int_{-\infty}^{\frac{-\log\gamma-\left(\mu+\sigma_1^2\right)t}{\sigma_1\sqrt{t}}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{w_2^2}{2}}\mathrm{d}w_2 \\
&=\mathrm{e}^{\frac{2\mu\log\gamma}{\sigma_1^2}+2\log\gamma+\left(\mu+\frac{\sigma_1^2}{2}\right)t}\left\{1-\Phi\left(\frac{\log\gamma+\left(\mu+\sigma_1^2\right)t}{\sigma_1\sqrt{t}}\right)\right\} \\
&=\mathrm{e}^{\frac{2\left(\mu_1-\frac{1}{2}\sigma_1^2\right)\log\gamma}{\sigma_1^2}+2\log\gamma}\mathrm{e}^{\left(\mu_1-\frac{1}{2}\sigma_1^2+\frac{\sigma_1^2}{2}\right)t}\left\{1-\Phi\left(\frac{\log\gamma+\left(\mu_1-\frac{1}{2}\sigma_1^2+\sigma_1^2\right)t}{\sigma_1\sqrt{t}}\right)\right\} \\
&=\gamma^{1+\frac{2\mu}{\sigma_1^2}}\cdot\mathrm{e}^{\mu_1 t}\left\{1-\Phi\left(\frac{\log\gamma+\left(\mu_1+\frac{1}{2}\sigma_1^2\right)t}{\sigma_1\sqrt{t}}\right)\right\}
\end{aligned} \tag{21}$$

In Equations (20) and (21), substituting $t = T_2 - \tilde{t}_C$ and subtracting Equation (21) from Equation (20), we obtain Equation (16). (QED)

The expected return from committing to a PE fund is shown in Proposition 1, and the expected time from the beginning of commitment to exit is shown in Proposition 2.

Proposition 1

The expected absolute return $E\left(\tilde{R}_{PE}\right)$ obtained from committing to the PE fund at time 0 until the PE fund exits at time $\tilde{t}_C + \tilde{t}_D$ is given by Equation (22).

$$\begin{aligned}
E\left(\tilde{R}_{PE}\right) = \mu_0 T_1 \mathrm{e}^{-\lambda T_1} + \int_0^{T_1}\Big\{&\mu_0 t_C \cdot \left(1+\int_0^{T_2-t_C} f_{t_D}\left(t_D\right)\mathrm{d}t_D\right) \\
&+\left(\log\gamma\right)\int_0^{T_2-t_C} f_{t_D}\left(t_D\right)\mathrm{d}t_D + \log\Psi\left(t_C\right)\Big\}\lambda\mathrm{e}^{-\lambda t_C}\mathrm{d}t_C
\end{aligned} \tag{22}$$

where $\Psi\left(t_C\right) = \mathrm{e}^{\mu_1\left(T_2-t_C\right)}\left\{\Phi\left(K_1'\right) - \gamma^{\left(1+\frac{2\mu_1}{\sigma^2}\right)}\left(1-\Phi\left(K_2'\right)\right)\right\}$.

(Proof)

Derive the expected returns from committing to a PE fund in three cases and sum the three of them with appropriate probability weight.

1) PE fund cannot find an investee company within the sourcing period $\left[0, T_1\right]$ ($T_1 \le \tilde{t}_C$)

In this case, the commitment to the PE fund ends at the time T_1, so the return for this period is the return on the benchmark only, and given by Equation (23).

$$\left\{\log\frac{E\left[S_0(0)\mathrm{e}^{\left(\mu_0-\frac{1}{2}\sigma_0^2\right)T_1+\sigma_0 B(T_1)}\right]}{S_0(0)}\right\}\cdot\int_{T_1}^{\infty}\lambda\mathrm{e}^{-\lambda t_c}\mathrm{d}t_c=\mu_0 T_1\mathrm{e}^{-\lambda T_1} \tag{23}$$

2) PE fund invests an amount $S_1(\tilde{t}_C)=S_0(0)\mathrm{e}^{\left(\mu_0-\frac{1}{2}\sigma_0^2\right)\tilde{t}_C+\sigma_0 B(\tilde{t}_C)}$ in an investee company at $\tilde{t}_C$ in the sourcing period $[0,T_1]$ and is able to exit at $\tilde{t}_C+\tilde{t}_D$ during the investment period $[\tilde{t}_C,T_2]$ at $\gamma S_1(\tilde{t}_C)$.

In this case, the return at a given investment point in time $\tilde{t}_C=t_c$ is $\left\{\log\frac{E\left[\gamma\cdot S_0(0)\mathrm{e}^{\left(\mu_0-\frac{1}{2}\sigma_0^2\right)\tilde{t}_C+\sigma_0 B(\tilde{t}_C)}\right]}{S_0(0)}\right\}\cdot\int_0^{T_2-\tilde{t}_C}f_{t_D}(t_D)\mathrm{d}t_D$ using the exit probability $\int_0^{T_2-\tilde{t}_C}f_{t_D}(t_D)\mathrm{d}t_D$ from Lemma 1, so the expected return is given by Equation (24).

$$\begin{aligned}&\int_0^{T_1}\left\{\log\frac{E\left[\gamma\cdot S_0(0)\mathrm{e}^{\left(\mu_0-\frac{1}{2}\sigma_0^2\right)\tilde{t}_C+\sigma_0 B(\tilde{t}_C)}\right]}{S_0(0)}\right\}\cdot\int_0^{T_2-\tilde{t}_C}f_{t_D}(t_D)\mathrm{d}t_D\lambda\mathrm{e}^{-\lambda t_c}\mathrm{d}t_c\\&=\int_0^{T_1}(\mu_0 t_c+\log\gamma)\cdot\int_0^{T_2-\tilde{t}_C}f_{t_D}(t_D)\mathrm{d}t_D\lambda\mathrm{e}^{-\lambda t_c}\mathrm{d}t_c\end{aligned} \tag{24}$$

3) PE fund invests an amount $S_1(\tilde{t}_C)=S_0(0)\mathrm{e}^{\left(\mu_0-\frac{1}{2}\sigma_0^2\right)T_1+\sigma_0 B(T_1)}$ in an investee company at $\tilde{t}_C$ in the sourcing period $[0,T_1]$ but fails to exit in the investment period $[\tilde{t}_C,T_2]$ and sells it with the price $S_0(0)\mathrm{e}^{\left(\mu_0-\frac{1}{2}\sigma_0^2\right)\tilde{t}_C+\sigma_0 B(\tilde{t}_C)+\left(\mu_1-\frac{1}{2}\sigma_1^2\right)(T_2-\tilde{t}_C)+\sigma_1 B(T_2-\tilde{t}_C)}=S_1(\tilde{t}_C)\mathrm{e}^{R_1(T_2-\tilde{t}_C)}$ at the point of time T_2.

The expected return is given by Equation (25), using Equation (16) in Lemma 2.

$$\int_0^{T_1}\log\frac{E\left[S_1(\tilde{t}_c)\mathrm{e}^{R_1(T_2-\tilde{t}_C)}1_{\{R_1(T_2-\tilde{t}_C)\le\log\gamma\}}\cdot 1_{\left\{\max\limits_{0\le u\le T_2-\tilde{t}_C}R_1(u)\le\log\gamma\right\}}\mid\mathcal{F}_{\tilde{t}_c}\right]}{S_0(0)}\mathrm{d}\tilde{t}_c \tag{25}$$

Summing Equations (23)-(25), we obtain Equation (22). (QED)

Proposition 2

The expected time $E(\tilde{T}_{PE})$ from the beginning of the commitment to the PE fund at time 0 to the final exiting at time $\tilde{t}_C+\tilde{t}_D$ is given by Equation (26).

$$\begin{aligned}E(\tilde{T}_{PE})&=E(\tilde{t}_C+\tilde{t}_D)=T_1\mathrm{e}^{-\lambda T_1}+T_2\left(1-\mathrm{e}^{-\lambda T_1}\right)\\&+\int_0^{T_1}\left(\left(\left(t_C+\int_0^{T_2-t_C}t_D f_{t_D}(t_D)\mathrm{d}t_D\right)-T_2\right)\left(\int_0^{T_2-t_C}f_{t_D}(t_D)\mathrm{d}t_D\right)\right)\lambda\mathrm{e}^{-\lambda t_c}\mathrm{d}t_c\end{aligned} \tag{26}$$

(Proof)

Derive the expected times committing to a PE fund in three cases and sum the three of them with appropriate probability weights.

1) PE fund cannot find an investee company in the sourcing period $\left[0, T_1\right]$ ($T_1 \le \tilde{t}_C$).

In this case, the expected time is given by Equation (27) because the commitment to the PE fund ends at the time T_1.

$$T_1 \cdot \int_{T_1}^{\infty} \lambda \mathrm{e}^{-\lambda t_c} \mathrm{d}t_c = T_1 \mathrm{e}^{-\lambda T_1} \tag{27}$$

2) PE fund invests in an investee company at $\tilde{t}_C$ in the sourcing period $\left[0, T_1\right]$ and exits at $\tilde{t}_C + \tilde{t}_D$ in the investment period $\left[\tilde{t}_C, T_2\right]$.

In this case, the expected time is given by Equation (28), since the probability density function of the time to exit given the point in time of investment $\tilde{t}_C = t_c$ is Equation (8) in Lemma 1.

$$\int_0^{T_1} \left(t_C + \int_0^{T_2 - t_C} t_D f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) \left(\int_0^{T_2 - t_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) \lambda \mathrm{e}^{-\lambda t_c} \mathrm{d}t_c \tag{28}$$

3) PE fund invests in an investee company at $\tilde{t}_C$ in the sourcing period $\left[0, T_1\right]$ but fails to exit in the investment period $\left[\tilde{t}_C, T_2\right]$ and sells it at time T_2.

The expected time is given by Equation (29), using Equation (9) in Lemma 1.

$$\begin{aligned}
&\int_0^{T_1} \left(t_C + \left(T_2 - t_C\right) \right) \left(1 - \int_0^{T_2 - t_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) \lambda \mathrm{e}^{-\lambda t_c} \mathrm{d}t_c \\
&= T_2 \int_0^{T_1} \left(1 - \int_0^{T_2 - t_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) \lambda \mathrm{e}^{-\lambda t_c} \mathrm{d}t_c \\
&= T_2 \left(1 - \mathrm{e}^{-\lambda T_1}\right) - \int_0^{T_1} T_2 \left(\int_0^{T_2 - t_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) \lambda \mathrm{e}^{-\lambda t_c} \mathrm{d}t_c
\end{aligned} \tag{29}$$

Summing Equations (27)-(29), we obtain Equation (26). (QED)

Here, corresponding to SBDA in (Definition 1) that evaluates the ex-post return of the PE fund, we propose the expected SBDA given by Equation (30) in (Definition 2) that allows us to compare the expected excess return of the PE funds with that of the regular active equity funds.

(Definition 2) Expected SBDA

The expected SBDA is α that satisfies the following equation

$$\left(\mu_0 + \alpha\right) E\left(\tilde{T}_{PE}\right) = E\left(\tilde{R}_{PE}\right) \tag{30}$$

where μ_0, $E\left(\tilde{R}_{PE}\right)$, and $E\left(\tilde{T}_{PE}\right)$ are the benchmark drift, Equation (22) in Proposition 1, and Equation (26) in Proposition 2, respectively.

The expected SBDA is given by Equation (31).

$$-\mu_0 + \frac{\mu_0 T_1 \mathrm{e}^{-\lambda T_1} + \int_0^{T_1} \left\{ \mu_0 t_C \cdot \left(1 + \int_0^{T_2 - \tilde{t}_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) + \left(\log \gamma\right) \int_0^{T_2 - \tilde{t}_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D + \log \Psi\left(t_C\right) \right\} \lambda \mathrm{e}^{-\lambda t_C} \mathrm{d}t_C}{T_1 \mathrm{e}^{-\lambda T_1} + T_2 \left(1 - \mathrm{e}^{-\lambda T_1}\right) + \int_0^{T_1} \left(\left(\left(t_C + \int_0^{T_2 - t_C} t_D f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) - T_2 \right) \left(\int_0^{T_2 - t_C} f_{t_D}\left(t_D\right) \mathrm{d}t_D \right) \right) \lambda \mathrm{e}^{-\lambda t_c} \mathrm{d}t_c} \tag{31}$$

(Proof)

We have only to substitute Equation (22) and Equation (26) into

$$\alpha = -\mu_0 + \frac{E\left(\tilde{R}_{PE}\right)}{E\left(\tilde{T}_{PE}\right)}.$$

(QED)

4. Valuation Model II (Case of Investing in Multiple Investee Companies at Multiple Points in Time)

First, using Valuation Model I, we consider the case in which a PE fund invests in two investee companies whose returns are assumed to be independent at two points in time. The assumptions introduced in Valuation Model I are retained. However, for the purpose of distinction between the first and second investee companies, they are represented by adding suffixes 1 and 2, respectively. More specifically, the stochastic variable $\tilde{t}_C$ representing the point in time of investment becomes $\tilde{t}_{C1}$ for the first time and $\tilde{t}_{C2} = \tilde{t}_{C1} + \tilde{t}_{C1\to2}$ for the second time, where $\tilde{t}_{C1\to2}$ is a stochastic variable representing the time interval from the first investment point in time to the second one and follows the same, independent exponential distribution as $\tilde{t}_{C1}$. $\tilde{t}_{D1}$ and $\tilde{t}_{D2}$ are stochastic variables representing the investment period from the time the PE fund invests at $\tilde{t}_{C1}$ and $\tilde{t}_{C2}$ until it exits, respectively. $\gamma_1 S_1(\tilde{t}_{C1})$ and $\gamma_2 S_1(\tilde{t}_{C2})$ are stochastic variables representing the exit price of the first investee company and the second one, respectively. The weights for the first and second investee companies are w_1 and w_2 ($w_1 \ge 0$, $w_2 \ge 0$, and $w_1 + w_2 = 1$), respectively.

The probability that no single investee company arrives in the sourcing period, which is calculated $P(\tilde{t}_{C1} > T_1)$ as in Equation (5), resulting in Equation (32).

$$P(\tilde{t}_{C1} > T_1) = \int_{T_1}^{\infty} \lambda e^{-\lambda t} dt = e^{-\lambda T_1} \tag{32}$$

The probability that both the first and the second investee companies are invested is given by the probability $P(\tilde{t}_{C2} \le T_1)$ representing that the second investment time point $\tilde{t}_{C2}$ falls on the sourcing period $[0, T_1]$. The probability density function of $\tilde{t}_{C2} = \tilde{t}_{C1} + \tilde{t}_{C1\to2}$ composed by $\tilde{t}_{C1\to2}$ and $\tilde{t}_{C1}$, which follow the same independent exponential distribution with parameter λ, is the gamma distribution with parameters 2, λ and is given by Equation (33).

$$f_{tC2}(t) = \begin{cases} \dfrac{\lambda^2}{\Gamma(2)} t^{2-1} e^{-\lambda t} = \lambda^2 t e^{-\lambda t} & t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

Integrating the probability density function in Equation (33) gives $P(\tilde{t}_{C2} \le T_1)$ as Equation (34).

$$P(\tilde{t}_{C2} \le T_1) = \int_0^{T_1} \lambda^2 t e^{-\lambda t} dt = 1 - e^{-\lambda T_1} - \lambda T_1 e^{-\lambda T_1} \tag{34}$$

Find the probability that the first investment is made in the investment period $[0, T_1]$ but not the second one. This probability is obtained by subtracting the probability of not making a single investment $e^{-\lambda T_1}$ and the probability of making both the first and second investments $1 - e^{-\lambda T_1} - \lambda T_1 e^{-\lambda T_1}$ from the total probability 1, resulting in Equation (35).

$$1 - e^{-\lambda T_1} - \left(1 - e^{-\lambda T_1} - \lambda T_1 e^{-\lambda T_1}\right) = \lambda T_1 e^{-\lambda T_1} \tag{35}$$

This probability is the same as in Equation (36), which is obtained by integrating the probability density function $g_{\tilde{t}_{C1\times}}(t_{C1\times}) = \lambda e^{-\lambda t_{C1\times}} - \lambda^2 t_{C1\times} e^{-\lambda t_{C1\times}}$ of the random variable $\tilde{t}_{C1\times}$ representing the point in time of the first investment when the second investment is not made, over the investment period $[0, T_1]$.

$$P(\tilde{t}_{C1\times} \le T_1) = \int_0^{T_1} \lambda e^{-\lambda t_{C1\times}} - \lambda^2 \tilde{t}_{C1\times} e^{-\lambda \tilde{t}_{C1\times}} d\tilde{t}_{C1\times} = \lambda T_1 e^{-\lambda T_1} \tag{36}$$

Based on the above, Proposition 3 shows the expected absolute return from committing to the PE fund in the case where the PE fund invests in two investee companies at two points in time, and Proposition 4 shows the expected commitment time to the PE fund.

Proposition 3

The expected absolute return $E(\tilde{R}_{PE2})$ obtained from committing to the PE fund at time 0 until the PE fund exits the first and the second investee companies at time $\tilde{t}_{C1} + \tilde{t}_{D1}$ and $\tilde{t}_{C2} + \tilde{t}_{D2}$, respectively is given by Equation (37).

$$\begin{aligned} E(\tilde{R}_{PE}) = {} & \mu_0 T_1 e^{-\lambda T_1} + w_1 \int_0^{T_1} \Big\{ \mu_0 t_{C1} \cdot \Big(1 + \int_0^{T_2 - t_{C1}} f_{t_{D1}}(t_{D1}) dt_{D1}\Big) \\ & + (\log \gamma_1) \int_0^{T_2 - t_{C1}} f_{t_{D1}}(t_{D1}) dt_{D1} + \log \Psi(t_{C1}) \Big\} \lambda e^{-\lambda t_{C1}} dt_{C1} \\ & + w_2 \int_0^{T_1} \Big\{ \mu_0 t_{C2} \cdot \Big(1 + \int_0^{T_2 - t_{C2}} f_{t_{D2}}(t_{D2}) dt_{D2}\Big) \\ & + (\log \gamma_2) \int_0^{T_2 - t_{C2}} f_{t_{D2}}(t_{D2}) dt_{D2} + \log \Psi(t_{C2}) \Big\} \lambda^2 t_{C2} e^{-\lambda t_{C2}} dt_{C2} \end{aligned} \tag{37}$$

where, $\Psi(t_{Ci}) = e^{\mu_1 (T_2 - t_{Ci})} \left\{ \Phi(K_1') - \gamma^{\left(1 + \frac{2\mu_1}{\sigma^2}\right)} \left(1 - \Phi(K_2')\right) \right\}$, ($i = 1, 2$).

(Proof)

We have only to apply Proposition 1 separately to the investment points in time $\tilde{t}_{C1}$ and $\tilde{t}_{C2}$ sum the results of them. In doing so, note that we should use the probability density function $g_{\tilde{t}_{C1\times}}(t_{C1\times}) = \lambda e^{-\lambda t_{C1\times}} - \lambda^2 t_{C1\times} e^{-\lambda t_{C1\times}}$ of the stochastic variable $\tilde{t}_{C1\times}$ representing the first investment time, when the second investment is not made. (QED)

Proposition 4

The expected time $E(\tilde{T}_{PE2})$ for the PE fund to exit the first and the second investee companies at time $\tilde{t}_{C1} + \tilde{t}_{D1}$ and $\tilde{t}_{C2} + \tilde{t}_{D2}$, respectively after committing to the PE fund at time 0 is given by Equation (38).

$$\begin{aligned} E(\tilde{T}_{PE2}) = {} & E\big(w_1(\tilde{t}_{C1} + \tilde{t}_{D1}) + w_2(\tilde{t}_{C2} + \tilde{t}_{D2})\big) = T_1 e^{-\lambda T_1} + T_2\left(1 - e^{-\lambda T_1}\right) \\ & + w_1 \int_0^{T_1} \left(\left(\left(t_{C1} + \int_0^{T_2 - t_{C1}} t_{D1} f_{t_{D1}}(t_{D1}) dt_{D1} \right) - T_2 \right) \left(\int_0^{T_2 - t_{C1}} f_{t_{D1}}(t_{D1}) dt_{D1} \right) \right) \lambda e^{-\lambda t_{c1}} dt_{c1} \\ & + w_2 \int_0^{T_1} \left(\left(\left(t_{C2} + \int_0^{T_2 - t_{C2}} t_{D2} f_{t_{D2}}(t_{D2}) dt_{D2} \right) - T_2 \right) \left(\int_0^{T_2 - t_{C2}} f_{t_{D2}}(t_{D2}) dt_{D2} \right) \right) \lambda e^{-\lambda t_{c2}} dt_{c2} \end{aligned} \tag{38}$$

(Proof)

With similar fashion that Proposition 1 is used in the proof of Proposition 3, we also have only to use Proposition 2 here. (QED)

We extend the expected SBDA in (Definition 2) to the case where a PE fund invests in two investee companies at two different points in time.

(Definition 3) Expected SBDA (the case of investing in two investee companies at two different points in time).

The expected SBDA is α that satisfies the following equation

$$\left(\mu_0+\alpha\right)E\left(\tilde{T}_{PE2}\right)=E\left(\tilde{R}_{PE2}\right) \tag{39}$$

where μ_0, $E\left(\tilde{R}_{PE2}\right)$, and $E\left(\tilde{T}_{PE2}\right)$ are the benchmark drift, Equation (37) in Proposition 3, and Equation (38) in Proposition 4, respectively.

Solving Equation (39) with respect to α yields the expected SBDA corresponding to Equation (31) of the theorem in this case, but we do not describe it because of complicated notation.

Based on the previous discussion, the expected SBDA can be easily obtained in the same way as above for the case where a PE fund invests in n investee companies at n different points in time. Although not shown here due to the complicated notation, it should be noted that $g_k\left(t\right)=\frac{\lambda^k}{\Gamma\left(k\right)}t^{k-1}\mathrm{e}^{-\lambda t}-\frac{\lambda^{k+1}}{\Gamma\left(k+1\right)}t^{(k+1)-1}\mathrm{e}^{-\lambda t}$ should be used as the probability density function of the stochastic variable representing the k-th investment time ($k=1,\cdots,n-1$), and $\frac{\lambda^n}{\Gamma\left(n\right)}t^{n-1}\mathrm{e}^{-\lambda t}$ should be used as the probability density function of the stochastic variable representing the n-th investment time.

5. Numerical Example

5.1. Setup

In this numerical example, the basic setup for the PE fund is a 5-year sourcing period and a 10-year total commitment period. The drift and volatility of the benchmark return, which is used to evaluate relative performance of the PE fund, are set to 8% and 20%, respectively. The expected time for the PE fund to find an investee company, which refers to $E\left(\tilde{t}_C\right)$ and is hereafter called the expected sourcing time is set to 2.5 years ($\lambda=0.4$) as the standard sourcing time, in addition, 1 year ($\lambda=1$) or 4 years ($\lambda=0.25$) as possible early or late sourcing times. The drift of an investee company return is set to 15% as the standard growth rate, in addition, 20% or 10% as possible high or low growth rates, respectively. In all cases, the drift of the benchmark return will be set lower than that of the investee company. The volatility of an investee company return is set to 25% as the standard risk, in addition, 30% or 20% as possible high or low risks, respectively. In all cases, the volatility of the investee company return is set higher than that of the benchmark return. In a word, the parameters of the investee company return are set to high risk/high return relative to that of the benchmark return in all the cases.

We analyze the sensitivity of each of the three parameters, such as drift and volatility of the investee company return and the expected sourcing time of the PE fund, to the expected absolute return of the PE fund ($E\left(\tilde{R}_{PE}\right)$ in Proposition 1), the expected commitment period ($E\left(\tilde{t}_C+\tilde{t}_D\right)$ in Proposition 2) and expected SBDA (Equation (31) in the theorem) with respect to a broad range of γ, which is the target exit multiple. Since three levels are set for each of drift, volatility, and expected time to find investee companies, and in the sensitivity analysis standard values for the two parameters other than the one of interest are adopted, the sensitivity analysis attempts a total of 9 cases, consisting of 3 levels for 3 parameters to be analyzed. The purpose of the sensitivity analysis is to compare numerically how the level of γ, the target exit multiple, should be set in each case in order to optimize the expected SBDA, which represents the relative performance of the PE fund. The correspondence between the cases of the above setups and the figures representing the results of the sensitivity analysis are listed in Table 1. As shown in Table 1, the results of the sensitivity analysis on the expected absolute return of the PE fund with respect to the drift and volatility of the investee company return and the expected sourcing time of the PE fund are shown in Figure 1 through Figure 3, respectively. The results of the sensitivity analysis on the expected commitment period of the PE fund with respect to the drift and volatility of the investee company return and the expected sourcing time of the PE fund are shown in Figure 4 through Figure 6, respectively. Finally, Figure 7 through Figure 9 show the results of the sensitivity analysis on the expected SBDA of the PE fund with respect to the drift and volatility of the investee company return and the expected sourcing time of the PE fund, respectively.

Table 1. The correspondence between the cases of the setups and the figures.

Parameters	Drift	Volatility	Arrival Rate
Benchmark	8%	20%	-
Target Company	15%; 10%, 20%	25%; 20%, 30%	2.5Y; 1Y, 4Y
Expected Return	Figure 1	Figure 2	Figure 3
Expected Commitment time	Figure 4	Figure 5	Figure 6
Expected SBDA	Figure 7	Figure 8	Figure 9

5.2. Results and Their Implications

5.2.1. Sensitivity of Each Parameter to the Expected Absolute Return of the PE Fund

From the scaling of the vertical axis in each figure, among the three parameters, the drift of the investee company return has the largest sensitivity to the expected absolute return of the PE fund, while the sensitivity to volatility or expected sourcing time is relatively small. Therefore, it is better to find investee companies with as large a drift as possible, even if the PE fund takes a little longer time to find them or they are a little bit riskier.

The sensitivity of parameters related to drift or expected sourcing time to expected absolute return increases as the target exit multiple increases from 5 to 6 and remains almost flat above 6, while the sensitivity of volatility to expected absolute return appears as the target exit multiple is around 3 to 6. The sensitivity of volatility to expected absolute return decreases as the target exit multiple increases, and is almost nonexistent above 6.

Figure 1 shows the sensitivity of drift to expected absolute return in more detail. When the drift is 10%, increasing the target exit multiple to about 4 improves the expected absolute return, and increasing the target exit multiple beyond that does not improve the expected absolute return as much. Similarly, when the drift is 15% or 20%, the target exit multiple is better to be increased to about 6 or 8, respectively. One of the reasons behind the results is that if the target exit multiple is raised too high, there is almost no possibility for the investee company with its drift to reach the target exit price, and the expected absolute return is not affected by the target exit multiple.

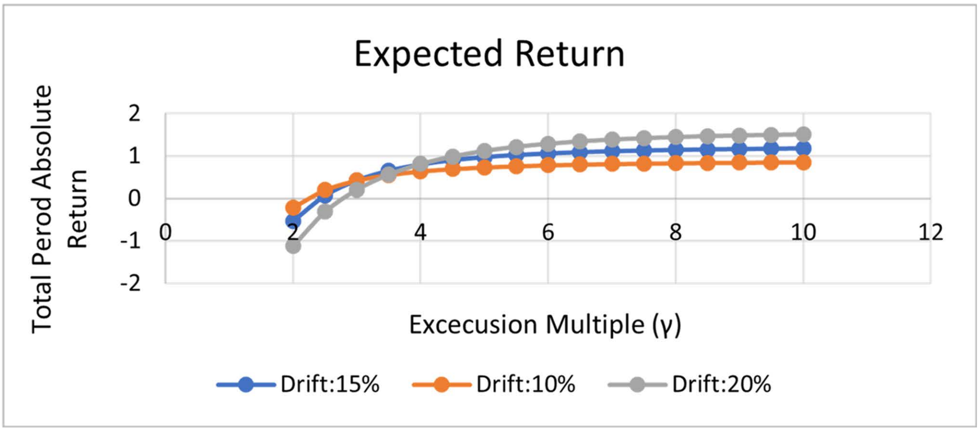

Figure 1. Sensitivity of drift to the expected absolute return of the PE fund.

Figure 2 shows the sensitivity of volatility to expected absolute return in more detail. When the target exit multiple is around 3 to 6 times, the expected absolute return improves as volatility decreases from 30% to 20%. However, when the target exit multiple is other than that, the expected absolute return remains the same regardless of the volatility level. This is because, when the target exit multiple is around 3 to 6 times, the volatility level has a significant impact on the probability of reaching the target exit multiple and exiting, as well as the probability of a downward return when the target exit multiple is not reached.

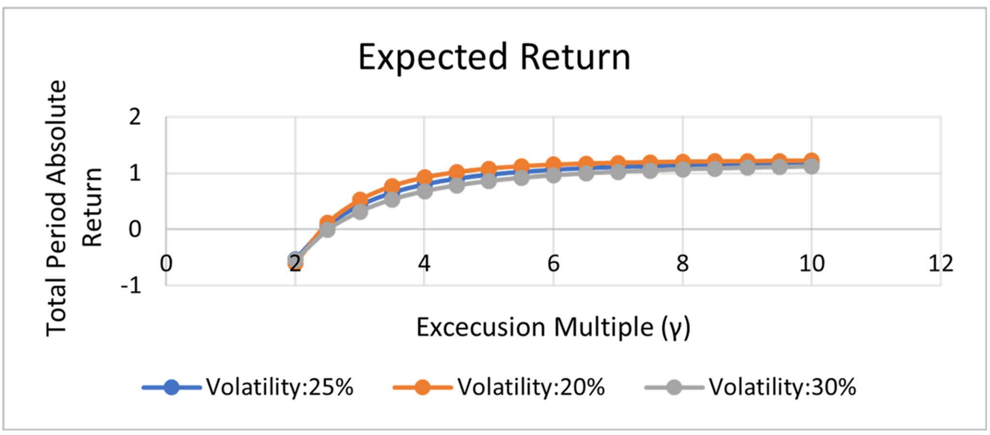

Figure 2. Sensitivity of volatility to the expected absolute return of the PE fund.

Figure 3 shows the sensitivity of expected sourcing time to expected absolute return. The expected absolute return is higher when the target exit multiple is higher than 4. One of the reasons behind the results is the fact that short expected sourcing time allows for a long investment period in the investee company with a larger drift than that of the benchmark.

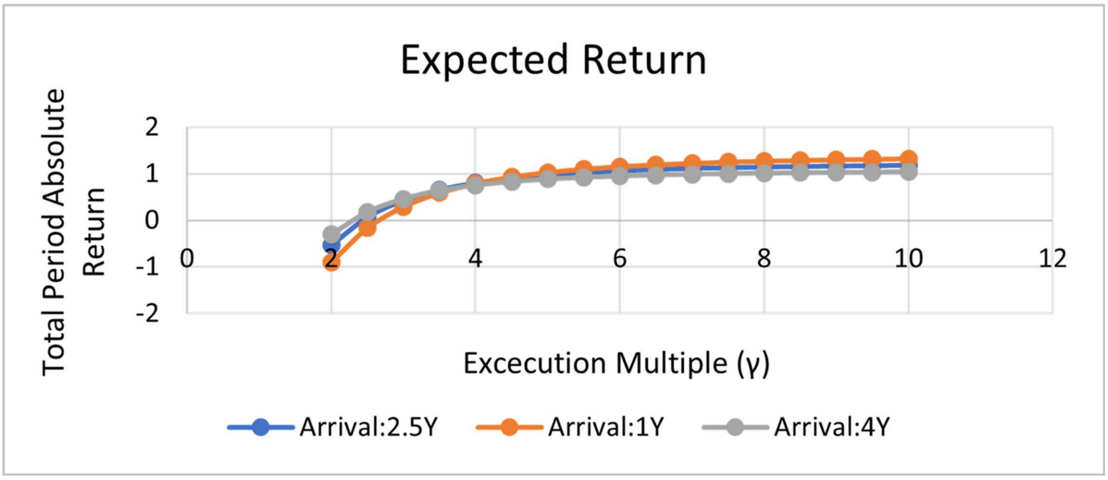

Figure 3. Sensitivity of expected arrival time to the expected absolute return of the PE fund.

5.2.2. Sensitivity of Each Parameter to the Expected Commitment Period of the PE Fund

The same scaling of the vertical axis in each figure indicates that, among the three parameters, the drift of the investee company return and the expected sourcing time of the PE fund show some sensitivity to the expected commitment period, but the sensitivity of the volatility to the expected commitment period is almost negligible at any target exit multiple.

Figure 4 shows that the sensitivity of drift to the expected commitment period decreases with increasing drift, regardless of the target exit multiple. In more detail, the sensitivity of the drift to the expected commitment period is greater at levels of nearly 4 to 6 than at low levels of nearly 2 or high levels of nearly 10. Some of the reasons behind the results are the following: for the same target exit multiple, the higher the drift, the faster the target exit price can be reached; at low target exit multiples of nearly 2, it does not take much time to reach the target exit price, so the difference between high and low drift is not significant; and at high target exit multiple of nearly 10, the probability of not reaching the target exit price during the commitment period seems to be high, no matter what the drift is.

Figure 6 shows the sensitivity of the expected sourcing time to the expected commitment period in more detail. When the target exit multiple is less than 4, the shorter the expected sourcing time, the slightly lower the expected commitment period, while when the target exit multiple is higher than 4, the result is opposite. One of the reasons behind the results is the fact that when the target exit multiple is larger than 4, starting the investment as early as possible may have a strong influence on the possibility to reach the exit stock price level.

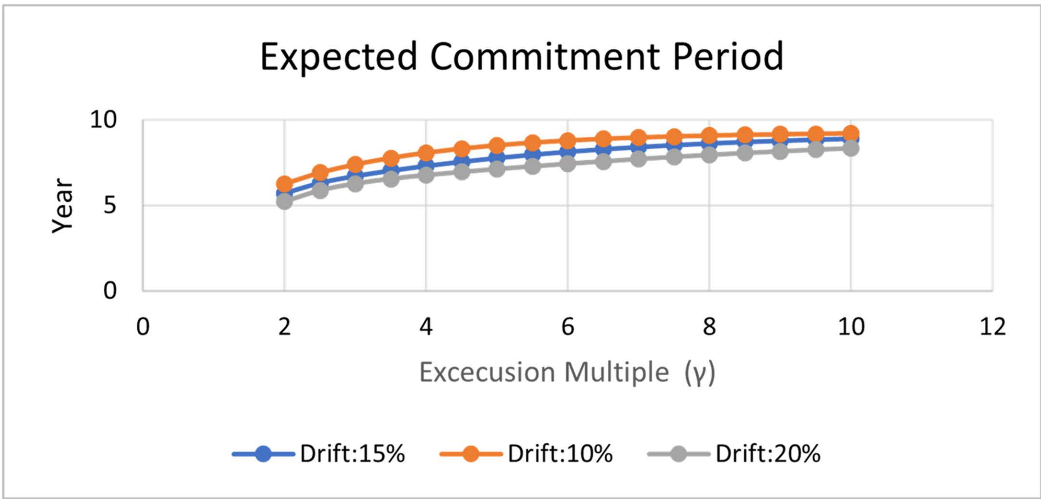

Figure 4. Sensitivity of drift to the expected commitment period of the PE fund.

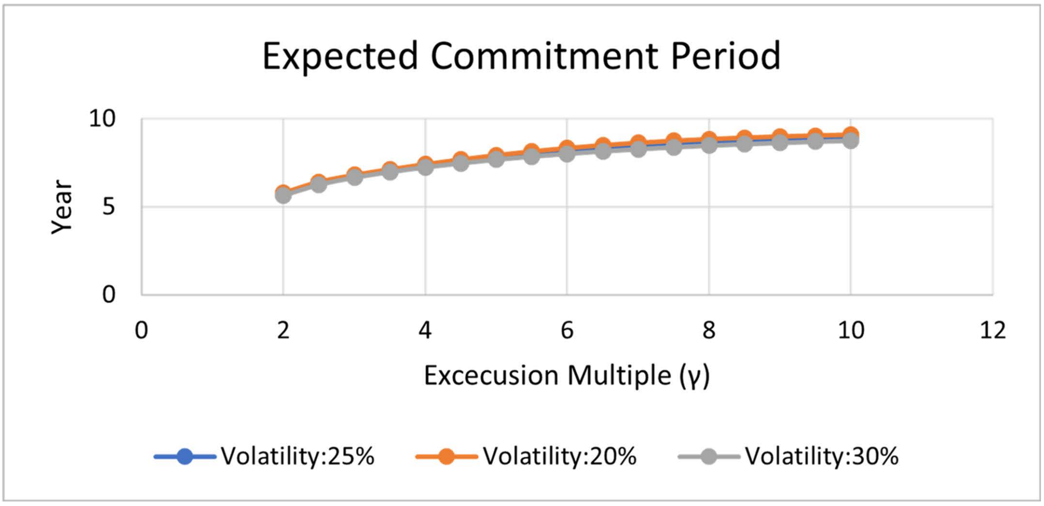

Figure 5. Sensitivity of volatility to the expected commitment period of the PE fund.

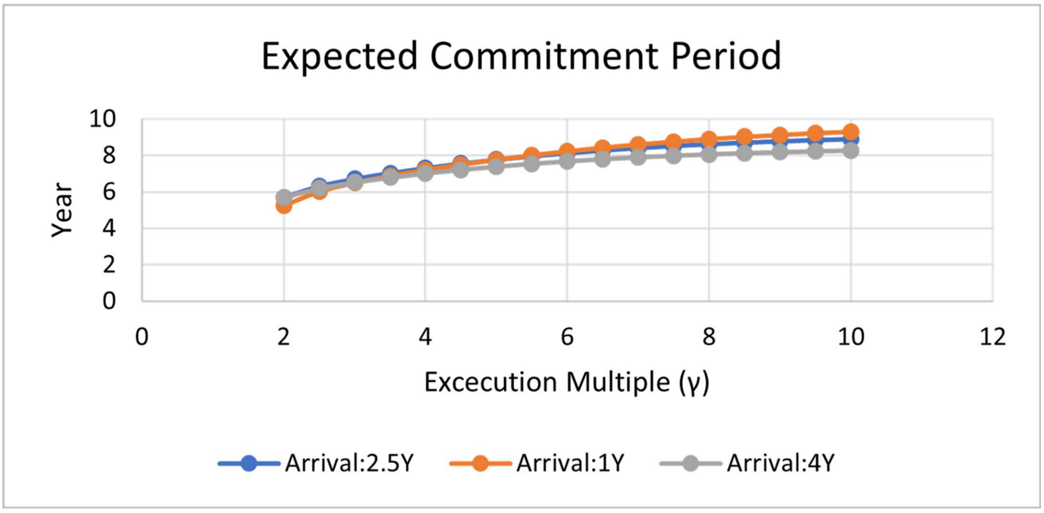

Figure 6. Sensitivity of expected arrival time to the expected commitment period of the PE fund.

5.2.3. Sensitivity of Each Parameter to the Expected SBDA of the PE Fund

From the scaling of the vertical axis in each figure, the drift of the investee company return has the largest sensitivity to the expected SBDA of the PE fund, while the impact of volatility and expected sourcing time is relatively small. This is the same picture for the impact of each parameter on the absolute return of the PE fund as seen in Section 5.2.1. Therefore, in order to maximize the expected excess returns, it is better to find investee companies with as large a drift as possible, even if the PE fund takes a little longer sourcing time or the investee companies are a little riskier.

Figure 7 provides the sensitivity of the drift to the expected SBDA in more detail. When the target exit multiple is less than about 3, the expected SBDA will be negative at any drift level, and the absolute return of the PE fund will be lower than that of the benchmark. As the target exit multiple is increased, the expected SBDA rapidly increases when the drift is 20%, reaching a maximum at a target exit multiple of about 6, and the expected SBDA stays the same as the target exit multiple is increased beyond that. Also, when the drift is 15%, the expected SBDA increases as the target exit multiple is increased, reaching the maximum when the target exit multiple is about 5, and the expected SBDA stays the same as the target exit multiple is increased beyond that. In the case of a 10% drift, increasing the target exit multiple to 4 will result in a small positive value for expected SBDA, but increasing the target multiple larger than 4 does not improve expected SBDA, which remains at a small positive value. These results suggest that when the drift is large it is better to set the target exit multiple relatively large to enjoy the spread between the drift of the investee company return and that of the benchmark return as long as possible, but if the target exit multiple is set too high, due to the increasing possibility not to exit at the target exit price and the investment period may become long even though the return becomes not that large, leading to a decline in expected SBDA, which is the excess return per unit of expected commitment time.

Figure 8 shows the sensitivity of volatility to expected SBDA in more detail. The graph in Figure 8 is very similar in shape to the graph in Figure 2. When the target multiple for exit is around 3 to 6 times, the expected SBDA improves as volatility decreases from 30% to 20%. However, when the target multiple is other than that, the expected SBDA remains the same regardless of the volatility level. This is because, when the target multiple is around 3 to 6 times, the volatility level has a significant impact on the expected absolute return and in addition the sensitivity of the volatility to the expected commitment period is almost negligible at any target exit multiple.

Figure 9 provides the sensitivity of the expected sourcing time to the expected SBDA in more detail. The graph in Figure 9 is similar in shape to the graph in Figure 3, but the sensitivity of the expected sourcing period to the expected SBDA becomes almost constant when the target multiplier exceeds 5 times. This is thought to be because when the target multiplier exceeds 5 times, the improvement in expected absolute return and the extension of the expected commitment period balance each other out.

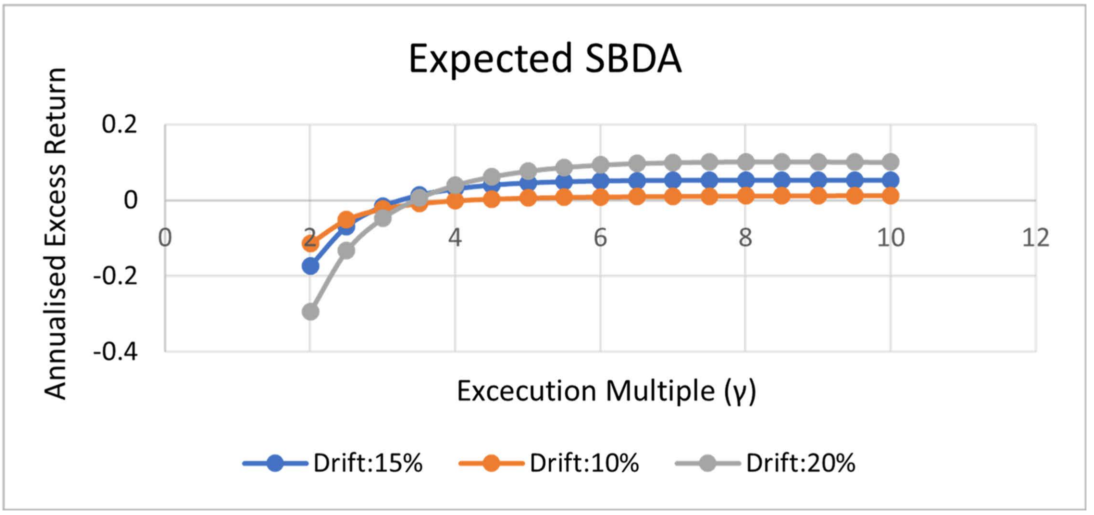

Figure 7. Sensitivity of drift to the expected SBDA of the PE fund.

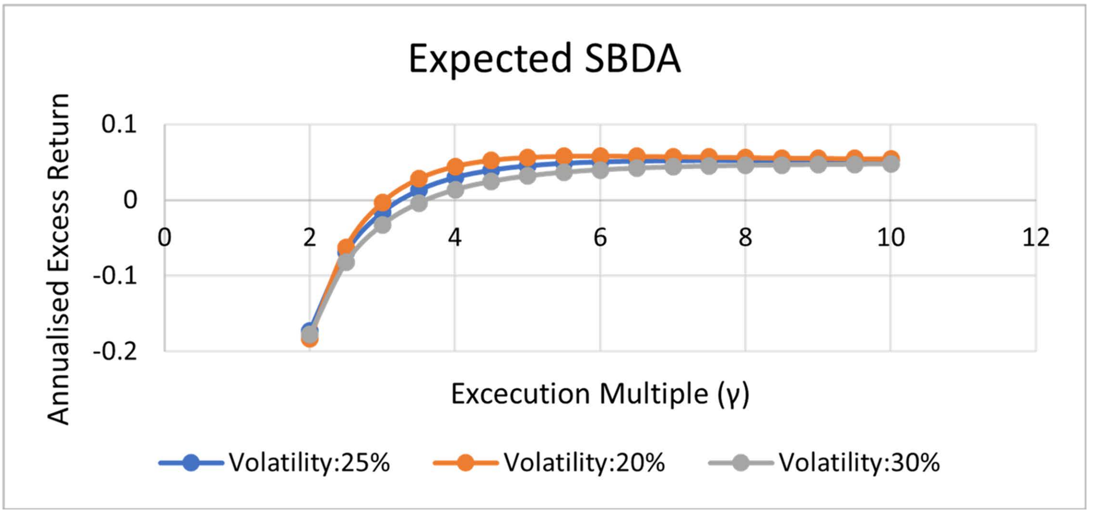

Figure 8. Sensitivity of volatility to the expected SBDA of the PE fund.

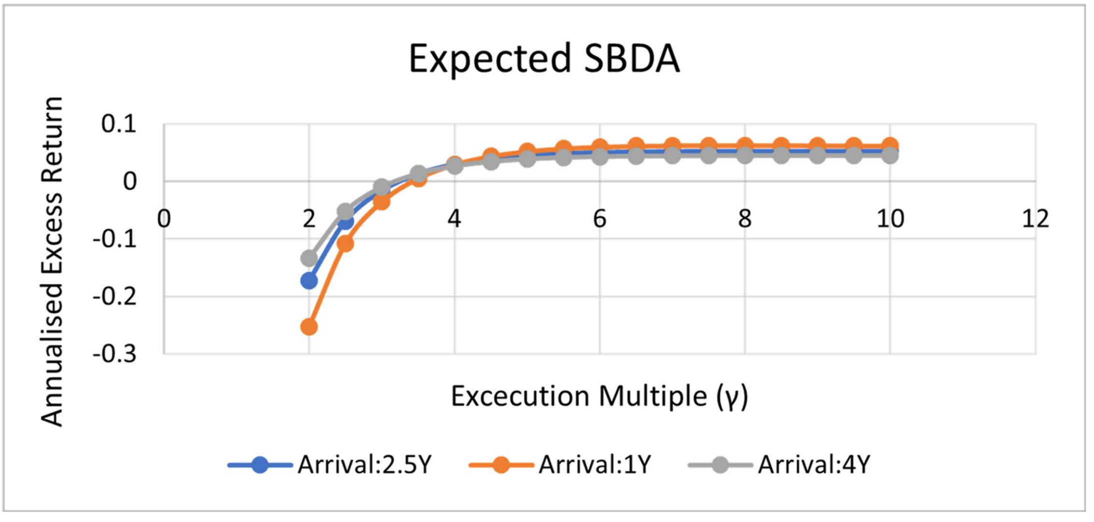

Figure 9. Sensitivity of expected arrival time to the expected SBDA of the PE fund.

6. Summary and Future Issues

In this study, we adopted SBDA as a PME or measure of excess return of PE funds, and attempted to examine the mechanism how expected SBDA is influenced by factors such as drift and volatility of the investee company return, expected time to find investee companies, and target exit multiple.

We adopted a model-based approach rather than accumulating empirical analyses on ex-post SBDA of PE funds. We constructed a valuation model for expected SBDA by introducing geometric Brownian motion as the stochastic process for both benchmark return and investee company return and exponential process as the one for the time to find investee company, and then discussed the sensitivity of parameters related to drift and volatility of investee company return, expected time to find an investee company, and target exit multiple to the expected SBDA.

From the numerical examples, it was found that, in general, the drift of investee company return has the greatest sensitivity to the expected absolute return of a PE fund, while the sensitivities of volatility and expected sourcing time is relatively small, so it is better to find investee companies with large drift to increase expected SBDA, even if you pay some sourcing time and volatility risk. It was also confirmed that the target exit multiple that maximizes expected SBDA is generally between 3 and 5, although it depends on the magnitude of the other parameters.

For future issue, it is an interesting research question to make the model more sophisticated one that overcomes the several simplifying assumptions, such as geometric Brownian motion for returns and a single lump-sum investment and to estimate the parameters of the valuation model based on cashflow information of PE funds and to accurately identify their skills by comparing the optimal expected SBDA derived by the model with their actual SBDA.

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Conflicts of Interest

The author declares no conflicts of interest.

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Appendix

Derivation of Equation (17)

Step 1) Find the transition probability of a random walk with an absorbing wall.

Lemma 1A

When there is an absorbing wall at state b, the transition probability of a symmetric random walk $a_{ij}(n)$ (the probability of transitioning from state i to state j after n periods starting from state at time 0) is given by

$$a_{ij}(n) = v_{j-i}(n) - v_{j+i-2b}(n) \qquad i, j < b$$

where, $v_j(n)$ is the transition probability of a symmetric random walk defined by

$$v_j(n) = Pr[W_n = j \mid W_0 = 0]; \quad p = q = \frac{1}{2}$$

and W_n is the random walk defined by

$$W_n = W_0 + \sum_{i=1}^{n} X_i, \quad n = 1, 2, \cdots$$

where X_i is a Bernoulli trial with $Pr[X_i = 1] = p, (0 < p < 1)$ and $Pr[X_i = -1] = q, i = 1, 2, \cdots$.

(Proof)

If there is an absorbing wall at state b, then to be in state $j < b$ at time n, it is necessary that the walk has not visited the wall b by that time. In other words, if we set $M_n^* = \max_{0 \le k \le n} W_k$ and $n = 1, 2, \cdots$, then $M_n^* < b$ must hold. In this case,

$$a_{ij}(n) = Pr\left[W_n = j, M_n^* < b \mid W_0 = i\right] \quad i, j < b, \quad n = 1, 2, \cdots$$

The event $\{W_n = j\}$ is divided into mutually exclusive events $\{W_n = j\} = \{W_n = j, M_n^* < b\} + \{W_n = j, M_n^* \ge b\}$, so

$$a_{ij}(n) = Pr[W_n = j \mid W_0 = i] - Pr\left[W_n = j, M_n^* \ge b \mid W_0 = i\right]$$

holds. Here, $Pr\left[W_n = j, M_n^* \ge b \mid W_0 = i\right]$ is the probability that a random walk $\{W_n, n = 1, 2, \cdots\}$ starting from state i reaches state b and then moves to state j. Using the mirror principle, this is equal to the probability of starting from state i and crossing $y = b$ to reach state $2b - j$,

$$Pr\left[W_n = j, M_n^* \ge b \mid W_0 = i\right] = Pr[W_n = 2b - j \mid W_0 = i] = v_{2b-j-i}(n)$$

Therefore,

$$a_{ij}(n) = v_{j-i}(n) - v_{2b-j-i}(n),$$

and since W_n is a symmetric random walk, $v_{-j}(n) = v_j(n)$, $j = 0, \pm 1, \cdots, \pm n$, $n = 0, 1, 2, \cdots$,

we obtain

$$a_{ij}(n) = v_{j-i}(n) - v_{j+i-2b}(n).$$

(QED)

(Step 2) Use a measure change to find the transition probability of the asymmetric random walk ($p \neq \frac{1}{2}$).

Lemma 2A

Let the transition probability of the p-random walk with no state space restrictions be denoted by $u_j(n)$. If the state b has an absorbing wall, then the transition probability of the p-random walk is given by

$$a_{ij}(n) = u_{j-i}(n) - \left(\frac{q}{p}\right)^{i-b} u_{j+i-2b}(n)$$

(Proof)

Let $m(\theta)$ be the moment generating function of X_k.

$$m(\theta) = E\left[e^{\theta X_k}\right] = pe^{\theta} + qe^{-\theta}, \quad k = 1, 2, \cdots$$

exists for all θ. Now, let $W_0 = i$, and

$$Y_n = m(\theta)^{-n} e^{\theta(W_n - i)}, \quad n = 1, 2, \cdots.$$

Since $W_n - i = \sum_{i=1}^{n} X_i$,

$$Y_n = \prod_{i=1}^{n} m(\theta)^{-1} e^{\theta X_i}, \quad n = 1, 2, \cdots,$$

and since $X_1, \cdots, X_n$ are independent, $e^{\theta X_1}, \cdots, e^{\theta X_n}$ is also independent, and therefore,

$$E[Y_n] = \prod_{i=1}^{n} m(\theta)^{-1} E\left[e^{\theta X_i}\right] = 1, \quad n = 1, 2, \cdots$$

holds.

Let T be a sufficiently large point in time, and for a given event A, $\tilde{P}(A) = E[I_A Y_T]$ converts the probability measure from P to $\tilde{P}$. Kijima [11] P. 71 Theorem 2.5 Corollary indicates that putting $\theta = \log\sqrt{\frac{p}{q}}$, the measure change transforms symmetric random walk $\{W_n, n = 1, 2, \cdots\}$ to p-random walk $\{W_n, n = 1, 2, \cdots\}$.

When there is an absorbing wall at state b, the transition probability of the p-random walk is

$$a_{ij}(n) = \tilde{P}\left[W_n = j, M_n^* < b \mid W_0 = i\right] = E\left[I_{\{W_n = j, M_n^* < b\}} Y_T \mid W_0 = i\right]$$

$$i, j < b, \quad n = 1, 2, \cdots.$$

Since $W_T = W_n + \sum_{i=n+1}^{T} X_i$ and W_n and $X_{n+1}, \cdots, X_T$ are independent under P,

$$\begin{aligned}
a_{ij}(n) &= E\left[I_{\{W_n = j, M_n^* < b\}} Y_n \mid W_0 = i\right] E\left[\prod_{i=n+1}^{T} m(\theta)^{-1} e^{\theta X_i}\right] \\
&= m(\theta)^{-n} E\left[I_{\{W_n = j, M_n^* < b\}} e^{\theta(W_n - i)} \mid W_0 = i\right] \\
&= m(\theta)^{-n} e^{\theta(j-i)} Pr\left[W_n = j, M_n^* < b \mid W_0 = i\right] \\
&= m(\theta)^{-n} e^{\theta(j-i)} \left(v_{j-i}(n) - v_{j+i-2b}(n)\right)
\end{aligned}$$

$\theta = \log \sqrt{\frac{p}{q}}$ and therefore, $e^{\theta j} = \left(\frac{p}{q}\right)^{\frac{j}{2}}$ and $m(\theta) = \frac{e^{\theta} + e^{-\theta}}{2} = (4pq)^{-\frac{1}{2}}$,

$a_{ij}(n) = 2^n p^{\frac{n}{2}} q^{\frac{n}{2}} p^{\frac{j-i}{2}} q^{\frac{i-j}{2}} \left\{ v_{j-i}(n) - v_{j+i-2b}(n) \right\}$.

By à Kijima [11] Theorem 2.1 on page 60,

$$u_j(n) = {}_nC_{\frac{n+j}{2}} p^{\frac{n+j}{2}} q^{\frac{n-j}{2}} \qquad j = 0, \pm 1, \cdots, \pm n,\ n = 0, 1, 2, \cdots$$

$$v_j(n) = {}_nC_{\frac{n+j}{2}} 2^{-n} \qquad j = 0, \pm 1, \cdots, \pm n,\ n = 0, 1, 2, \cdots$$

We have $u_j(n) = 2^n v_j(n) p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}$, so here, we can replace j with $j-i$ or $j+i-2b$,

$$u_{j-i}(n) = 2^n v_{j-i}(n) p^{\frac{n+j-i}{2}} q^{\frac{n-j+i}{2}} = 2^n p^{\frac{n}{2}} q^{\frac{n}{2}} p^{\frac{j-i}{2}} q^{\frac{i-j}{2}} v_{j-i}(n)$$

$$\begin{aligned} u_{j+i-2b}(n) &= 2^n v_{j+i-2b}(n) p^{\frac{n+j+i-2b}{2}} q^{\frac{n-j-i+2b}{2}} \\ &= 2^n p^{\frac{n}{2}} q^{\frac{n}{2}} p^{\frac{j-i}{2}} p^{\frac{2i-2b}{2}} q^{\frac{i-j}{2}} q^{\frac{2b-2i}{2}} v_{j+i-2b}(n) \\ &= p^{i-b} q^{b-i} 2^n p^{\frac{n}{2}} q^{\frac{n}{2}} p^{\frac{j-i}{2}} q^{\frac{i-j}{2}} v_{j+i-2b}(n) \end{aligned}$$

Therefore, since $\left(\frac{q}{p}\right)^{i-b} u_{j+i-2b}(n) = 2^n p^{\frac{n}{2}} q^{\frac{n}{2}} p^{\frac{j-i}{2}} q^{\frac{i-j}{2}} v_{j+i-2b}(n)$, we obtain

$$a_{ij}(n) = u_{j-i}(n) - \left(\frac{q}{p}\right)^{i-b} u_{j+i-2b}(n). \qquad \text{(QED)}$$

(Step 3) From the transition probability of the p-random walk, we obtain the transition probability density of (μ, σ^2)-Brownian motion $\{X(t), t \geq 0\}$.

Putting $i = \frac{x}{\Delta x}, j = \frac{y}{\Delta x}, n = \frac{t}{\Delta t}$ $p(x, y, t) = \frac{1}{\Delta x} u_{\frac{x}{\Delta x}, \frac{y}{\Delta x}}\left(\frac{t}{\Delta t}\right)$, the transition probability $u_{ij}(n)$ of the random walk $\{W_n, n = 1, 2, \cdots\}$ converges to the transition probability density $p(x, y, t)$ of (μ, σ^2)-Brownian motion $\{X(t), t \geq 0\}$.

Note that $\left(\frac{q}{p}\right)^{i-b} = \left(\frac{\sigma^2 - \mu \Delta x}{\sigma^2 + \mu \Delta x}\right)^{\frac{x - \log \gamma}{\Delta x}} \to e^{-\frac{2\mu(x - \log \gamma)}{\sigma^2}}$,

$$\begin{aligned} a(x, y, t) &= p(y - x, t) - e^{-\frac{2\mu(x - \log \gamma)}{\sigma^2}} p(y + x - 2\log \gamma, t) \\ &= \frac{1}{\sqrt{2\pi t}\sigma} \exp\left\{-\frac{(y - x - \mu t)^2}{2\sigma^2 t}\right\} \\ &\quad - e^{-\frac{2\mu(x - \log \gamma)}{\sigma^2}} \frac{1}{\sqrt{2\pi t}\sigma} \exp\left\{-\frac{(y + x - 2\log \gamma - \mu t)^2}{2\sigma^2 t}\right\} \end{aligned}$$

is obtained. (QED)