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Term Structure of Defaultable Bonds with Recovery of Market Value

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Abstract

This paper reproduces the main result of Duffie and Singleton [1] and extends it to defaultable bonds with both continuous and periodic coupon payments. Specifically, if the recovery of a defaultable bond after default follows the recovery of market value (RMV) assumption, its implied term structure of interest rates takes the form $\overline{r}(t) = r(t) + (1-R)\lambda(t)$, where $r(t)$ is the risk-free rate, $\lambda(t)$ is the entity's default intensity, and R is the recovery rate of market value. These results are derived within the risk-neutral pricing framework using straightforward and elementary method.

Keywords

Credit Risk, Defaultable Bond, Recovery of Market Value, Risk-Neutral Pricing, Term Structure of Interest Rate

1. Introduction

The term structure of interest rates for defaultable bonds provides insights into the credit spreads of bonds. Credit spreads, to some extent, reflect the credit risk of the debtor. Therefore, analyzing the observed term structure of interest rates for defaultable bonds in the market helps in understanding the trend of the debtor's default risk, providing valuable information for market participants in risk management and investment analysis.

Duffie and Singleton [1] studied the implied term structure of defaultable claim. Under the recovery of market value assumption (RMV assumption, which is one of several standard recovery-rate modeling assumptions, see Section 2.1 or [1] for further detail), they proposed replacing the risk-free

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short-rate process r with the default-adjusted short-rate process $R = r + hL$, where h is the default intensity of defaultable bonds, and L is the expected percentage loss of market value upon default. This substitution allows the standard risk-free interest rate term structure model to be directly applied to defaultable bonds. For instance, Duffie and Singleton [2] applied the result of [1] and developed a multi-factor econometric model of the term structure of interest-rate swap yields. By parameterizing a model of swap rates directly, they computed model-based estimates of the defaultable zero-coupon bond rates implicit in the swap market without specifying a priori the dependence of these rates on default hazard or recovery rates.

Duffie and Singleton [1] employed the martingale technique to derive their result. They constructed a discounted gain process, which must be a martingale under the risk-neutral measure, applied Itô's formula to this process, and derived an equation for the pre-default market value of the defaultable claim. Their result follows from this equation. By taking the derivative of the risk-neutral pricing formula for defaultable bonds, this paper derives ordinary differential equations (ODEs) governing bond prices. The results are obtained by solving these ODEs. The methodology employed here is both straightforward and elementary, as the risk-neutral pricing formula is obtained by directly applying the risk-neutral pricing principle to defaultable bonds, and the results follow from solving the corresponding ODEs. For an introduction to the risk-neutral pricing principle, refer to Chapter 5 of [3].

Subsequent works, such as Belanger *et al.* [4], defined the term structure of instantaneous interest rate as $\bar{r}(t,T) = r_t + (1-\rho(t,T))\lambda_t$, where ρ is the recovery rate. They provided a pricing formula for defaultable bonds, which is consistent in form with the pricing expression for risk-free zero-coupon bonds. Bazgour and Platania [5] expanded the term structure of interest rate proposed by [1], using affine functions to model the spread process and introducing economic cycle factors into the modeling of credit risk premiums, which allows for a more accurate simulation of credit risk changes in the real market. Brignone *et al.* [6] proposed the form of the short rate under the arbitrage-free tenor-dependent Nelson-Siegel model and derived the default-free zero-coupon bond price, facilitating the pricing of interest rate derivatives. Hölzermann [7] extended the traditional HJM model, addressing the modeling of interest rate term structure under volatility uncertainty, where the uncertainty was driven by G-Brownian motion.

The study of the term structure of interest rates can be traced back to the structural models [8] [9]. The main idea is to link the pricing of corporate bonds with the asset value of the corporate and evaluate credit risk through option pricing theory. Moreover, the market value of the corporate serves as the primary source driving the uncertainty of credit risk. Specifically, when the value of the company exceeds the default boundary, the bond is paid at face value. Otherwise, a proportion of the bond is paid. However, Duffie and Lando [10] pointed out that one of the issues with the structural approach is that when a defaultable bond approaches its maturity, its credit spreads approach zero.

Reduced form models are another method for analyzing credit risk, including [1] [11]-[13]. Jarrow and Turnbull [12] treated credit risk as a spot exchange rate risk, thereby allowing traditional derivative pricing techniques to be applied to credit-risky products, providing novel insights for subsequent reduced form credit models. Jarrow *et al.* [13] extended the model proposed by [12], characterizing the bankruptcy process as a finite-state Markov process. Empirical analysis showed that the Markov-based model could better fit the credit spread term structure in real markets.

The third approach to analyzing credit risk is the incomplete information approach, which combines the structural and reduced form approaches [14] [15]. Hyong-Chol *et al.* [14] combined the features of structural and reduced form models, considering the dynamic changes in the company's asset value and default intensity, and solved the pricing problem for defaultable bonds with discrete coupon payments. Zarban *et al.* [15] used a regime switching model to describe different states of the economy and incorporated imperfect information to price defaultable zero-coupon bonds.

This paper derives the pricing formulas for defaultable bonds and the corresponding term structure of interest rates in the risk-neutral pricing framework. The rest of paper is organized as follows: Section 2 outlines the basic assumptions and provides the pricing formulas for risk-free bonds under three circumstances: zero-coupon, continuous coupon payments, and periodic coupon payments. Sections 3 and 4 provide the bond pricing formulas under deterministic and stochastic assumptions, respectively. Section 5 presents the main conclusions of this paper.

2. Background Knowledge

2.1. Basic Assumptions

Consider a defaultable bond with maturity T. A defaultable bond means that the debtor may fail to pay the promised payments due to some credit events occurred before maturity. We consider the implied term structure of interest rates for this bond under two different assumptions.

Deterministic Assumption: Assume that under the risk-neutral measure, the default time τ of the bond is the first jump time of a non-homogeneous Poisson point process $N(t)$, where the intensity $\lambda(t)$ of $N(t)$ is a deterministic function of time t, and $\lambda(t)$ has at most a finite number of jump discontinuities in the interval $[0,T]$. The function $\lambda(t)$ is also referred to as the default intensity. In other words, there exists an exponential random variable $Y \sim \text{Exp}(1)$ such that

$$\tau = \min\left\{t > 0; \int_0^t \lambda(u)\mathrm{d}u \geq Y\right\}.$$

In particular, when $\lambda(t)$ is constant and equal to λ, $\tau \sim \text{Exp}(\lambda)$.

Let $P(t,T)$ denote the value of the bond at time t given that no default has occurred ($\tau > t$). Assume that the bond value after default follows the recovery of market value, *i.e.*, there exists a constant $R \in [0,1)$ such that the value of the bond after default is $RP(\tau-,T)$, where $P(\tau-,T)$ is the left-sided limit of $P(t,T)$ as t approaches τ, that is $P(\tau-,T) = \lim_{t \uparrow \tau} P(t,T)$. R is called the recovery-rate. In the Deterministic Assumption, risk-free interest rate $r(t)$ is assumed to be a deterministic function of time t.

Stochastic Assumption: The risk-free interest rate $r(t)$ and default intensity $\lambda(t)$ follow some non-negative stochastic processes, and the recovery rate $R \in [0,1)$ is a random variable that follows some certain distribution. Further, $r(t)$, $\lambda(t)$ and R are independent of the exponential random variable Y introduced above.

Stochastic Assumption is more realistic. We consider the Deterministic Assumption only for simplicity. To some extent, the Deterministic Assumption serves as preparation for the Deterministic Assumption. In fact, given the filtration generated by stochastic $r(t)$, $\lambda(t)$ and R, the conditional expectation of the Stochastic Assumption can be transformed into a deterministic one. Please refer to Section 4 for details.

2.2. Calculations of Default Probability

Consider the calculations of several probabilities under the Deterministic Assumption. For $0 \le t \le T$, the probability that the bond has not defaulted by time t is given by

$$\Pr(\tau > t) = \mathrm{e}^{-\int_0^t \lambda(u)\mathrm{d}u}.$$

Given $\tau > t$, the conditional probability that the bond has not defaulted by time T is

$$\Pr(\tau > T \mid \tau > t) = \mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}.$$

For $t \le s < s + \Delta s \le T$, assume that $\lambda(t)$ is continuous on the time interval $(s, s+\Delta s]$. The conditional probability that the bond defaults in the interval $(s, s+\Delta s]$, given the condition $\tau > t$ and Δs is sufficiently small, is given by

$$\begin{aligned}\Pr(\tau \in (s, s+\Delta s] \mid \tau > t) &= \Pr(\tau > s \mid \tau > t) - \Pr(\tau > s+\Delta s \mid \tau > t) \\ &= \mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u} - \mathrm{e}^{-\int_t^{s+\Delta s} \lambda(u)\mathrm{d}u} \\ &\approx \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\Delta s.\end{aligned}$$

2.3. Pricing of Default-Free Bonds

Consider the price of a default-free bond under Deterministic Assumption. The price is derived in three different circumstances: zero-coupon bond, continuous coupon-paying bond and periodic coupon-paying bond.

2.3.1. Zero-Coupon Bonds

Consider a default-free zero-coupon bond which pays a face value $1 only at maturity T. The price of the bond is given by

$$P(t,T) = \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}. \tag{1}$$

2.3.2. Continuous Coupon-Paying Bonds

Consider a default-free bond with continuous coupon payments and a face value $1. Assume the bond matures in T years and pays a coupon $c(t)\mathrm{d}t$ during the time interval $[t, t+\mathrm{d}t]$, where $c(t)$ is referred to the coupon rate. The price of the bond is given by

$$P(t,T) = \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} + \int_t^T c(s)\mathrm{e}^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}s. \tag{2}$$

2.3.3. Periodic Coupon-Paying Bonds

Consider a default-free bond with annual coupon payments and a face value $1. Assume the bond matures in T years, where T is an integer, and the coupon payment in the k-th year is $c(k)$. The price of the bond is given by

$$P(t,T) = \mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} + \int_{(t,T]} \mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v), \tag{3}$$

where $\mathcal{C}(t)$ represents the cumulative coupon payments up to time t, *i.e.*,

$$\mathcal{C}(t) = \sum_{k \le t} c(k).$$

Clearly, $\mathcal{C}(t)$ is right-continuous with respect to t.

When t is not an integer, $P(t,T)$ is continuous and differentiable with respect to t, and satisfies

$$P'(t,T) = r(t)P(t,T). \tag{4}$$

When t is an integer, $P(t,T)$ is right-continuous with respect to t, and satisfies

$$\Delta P(t,T) = P(t,T) - P(t-,T) = -c(t). \tag{5}$$

In other words, the expression for $P(t,T)$ given in Equation (3) is the solution that satisfies Equations (4)-(5) and terminal condition $P(T,T) = 1$.

3. Term Structure under the Deterministic Assumption

Consider the implied term structure of interest rate of a defaultable bond under Deterministic Assumption. The term structure is discussed in three different circumstances: zero-coupon bond, continuous coupon-paying bond and periodic coupon-paying bond.

3.1. Zero-Coupon Bonds

Consider a defaultable zero-coupon bond. The bond pays a face value $1 at maturity T if no default occurs by T, and there is no payments made prior to that.

Theorem 1. The price of a defaultable zero-coupon bond at time t is given by

$$P(t,T)=\mathrm{e}^{-\int_t^T \overline{r}(u)\mathrm{d}u}, \tag{6}$$

where $\overline{r}(t)=r(t)+(1-R)\lambda(t)$.

Proof. By the risk-neutral pricing principle (see [3] for reference), the price of the bond under the risk-neutral probability, when $\tau>t$, is

$$\begin{aligned} P(t,T) &= \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\mathbf{I}_{\{\tau>T\}}+\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}RP(\tau-,T)\mathbf{I}_{\{\tau\le T\}} \mid \tau>t\right] \\ &= A(t)+C(t), \end{aligned} \tag{7}$$

where

$$A(t)=\mathrm{e}^{-\int_t^T r(u)+\lambda(u)\mathrm{d}u},$$

$$C(t)=R\int_t^T P(s-,T)\lambda(s)\mathrm{e}^{-\int_t^s r(u)+\lambda(u)\mathrm{d}u}\mathrm{d}s.$$

Assume that $P(t,T)$ is continuously differentiable with respect to t. Note that

$$A'(t)=\left[r(t)+\lambda(t)\right]A(t),$$

$$C'(t)=\left[r(t)+\lambda(t)\right]C(t)-R\lambda(t)P(t,T),$$

differentiating both sides of Equation (7) with respect to t, we obtain

$$\begin{aligned} P'(t,T) &= A'(t)+C'(t) \\ &= \left[r(t)+\lambda(t)\right]A(t)+\left[r(t)+\lambda(t)\right]C(t)-R\lambda(t)P(t,T) \\ &= \left[r(t)+\lambda(t)\right]P(t,T)-R\lambda(t)P(t,T) \\ &= \overline{r}(t)P(t,T). \end{aligned} \tag{8}$$

Equation (8) is a homogeneous first-order linear differential equation, the solution that satisfies Equation (8) and terminal condition $P(T,T)=1$ is given by

$$P(t,T)=\mathrm{e}^{-\int_t^T \overline{r}(u)\mathrm{d}u}. \tag{6}$$

□

Remark 1. When $r(t)$, $\lambda(t)$ are constants, and equal to r, λ respectively, we have

$$P(t,T)=\mathrm{e}^{-\overline{r}(T-t)},$$

where $\overline{r}=r+(1-R)\lambda$.

3.2. Continuous Coupon-Paying Bonds

Consider a defaultable bond with continuous coupon payments, where the coupon rate $c(t)$ is a deterministic function of time t. The bond pays a face value $1 at maturity T if no default occurs by T. In addition, the bond pays a coupon $c(t)\mathrm{d}t$ during the time interval $[t,t+\mathrm{d}t]$ if no default occurs by that time.

Theorem 2. The price at time t of a defaultable bond with continuous coupon payments is given by

$$P(t,T)=\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_t^T c(s)\mathrm{e}^{-\int_t^s \bar{r}(u)\mathrm{d}u}\mathrm{d}s, \tag{9}$$

where $\bar{r}(t)=r(t)+(1-R)\lambda(t)$.

Proof. Under the risk-neutral probability, when $\tau>t$, we have

$$\begin{aligned} P(t,T)=\mathbb{E}\Big[&\Big(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}+\int_t^T c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\Big)\mathbf{I}_{\{\tau>T\}} \\ &+\Big(RP(\tau-,T)\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}+\int_t^\tau c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\Big)\mathbf{I}_{\{\tau\le T\}}\mid\tau>t\Big] \\ =&\,A(t)+B(t)+C(t)+D(t), \end{aligned} \tag{10}$$

where

$$\begin{aligned} A(t)&=\mathrm{e}^{-\int_t^T r(u)+\lambda(u)\mathrm{d}u}, \\ B(t)&=\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}\int_t^T c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v, \\ C(t)&=R\int_t^T P(s-,T)\lambda(s)\mathrm{e}^{-\int_t^s r(u)+\lambda(u)\mathrm{d}u}\mathrm{d}s, \\ D(t)&=\int_t^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\Big(\int_t^s c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\Big)\mathrm{d}s \\ &=\int_t^T c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\Big(\int_v^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s\Big)\mathrm{d}v. \end{aligned}$$

Assume that $P(t,T)$ is continuously differentiable with respect to t. Note that

$$\begin{aligned} A'(t)&=\big[r(t)+\lambda(t)\big]A(t), \\ B'(t)&=\big[r(t)+\lambda(t)\big]B(t)-c(t)\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}, \\ C'(t)&=\big[r(t)+\lambda(t)\big]C(t)-R\lambda(t)P(t,T), \\ D'(t)&=\big[r(t)+\lambda(t)\big]D(t)-c(t)\int_t^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s \\ &=\big[r(t)+\lambda(t)\big]D(t)-c(t)\Big[1-\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}\Big], \end{aligned}$$

differentiating both sides of Equation (10) with respect to t, we obtain

$$\begin{aligned} P'(t,T)&=A'(t)+B'(t)+C'(t)+D'(t) \\ &=\big[r(t)+\lambda(t)\big]A(t) \\ &\quad+\big[r(t)+\lambda(t)\big]B(t)-c(t)\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u} \\ &\quad+\big[r(t)+\lambda(t)\big]C(t)-R\lambda(t)P(t,T) \\ &\quad+\big[r(t)+\lambda(t)\big]D(t)-c(t)\Big[1-\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}\Big] \\ &=\big[r(t)+\lambda(t)\big]P(t,T)-R\lambda(t)P(t,T)-c(t) \\ &=\bar{r}(t)P(t,T)-c(t). \end{aligned} \tag{11}$$

Equation (11) is a first-order linear differential equation, and the solution satisfying Equation (11) and the terminal condition $P(T,T)=1$ is given by

$$P(t,T)=\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_t^T c(s)\mathrm{e}^{-\int_t^s \bar{r}(u)\mathrm{d}u}\mathrm{d}s. \tag{9}$$

□

Remark 2. When $r(t)$, $\lambda(t)$ and $c(t)$ are constants, and equal to r, λ and c respectively, the price of the bond is given by

$$P(t,T)=\left(1-\frac{c}{\bar{r}}\right)\mathrm{e}^{-\bar{r}(T-t)}+\frac{c}{\bar{r}},$$

where $\bar{r}=r+(1-R)\lambda$.

3.3. Periodic Coupon-Paying Bonds

Consider a defaultable bond with annual coupon payments and a face value \$1. Assume the bond matures in T years, where T is an integer. The coupon payment in the k-th year is constant $c(k)$.

Let $P(t,T)$ be the ex-coupon bond price, *i.e.*, the bond price at time t, excluding the coupon payment made on that day, when t is a coupon payment date. The bond value, given that default occurs on a coupon payment date t, is assumed to be $RP(t-,T)+c(t)$, meaning that the default does not affect the coupon payment itself.

Denote the cumulative coupons up to time t by $\mathcal{C}(t)$, *i.e.*,

$$\mathcal{C}(t)=\sum_{k\le t}c(k).$$

The function $\mathcal{C}(t)$ is right-continuous with respect to t, and for any function $F(t)$ and $s>t$,

$$\sum_{k=\lfloor t\rfloor+1}^{\lfloor s\rfloor}F(k)c(k)=\int_{(t,s]}F(v)\mathrm{d}\mathcal{C}(v),$$

where $\lfloor t\rfloor$ denotes the integer part of t. The integral on the right-hand side is a Stieltjes integral, which is right-continuous with respect to the upper limit of integration s.

Theorem 3. The price at time t of a defaultable bond with periodic coupon payments is given by

$$P(t,T)=\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v \bar{r}(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v), \tag{12}$$

where $\bar{r}(t)=r(t)+(1-R)\lambda(t)$.

Proof. Under the risk-neutral probability, when $\tau>t$, the ex-coupon bond price is given by

$$\begin{aligned}P(t,T)=\mathbb{E}\Bigg[&\left(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}+\int_{(t,\tau]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathbf{I}_{\{\tau>T\}}\\&+\left(RP(\tau-,T)\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}+\int_{(t,\tau]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathbf{I}_{\{\tau\le T\}}\mid\tau>t\Bigg]\\=&\,A(t)+B(t)+C(t)+D(t),\end{aligned} \tag{13}$$

where

$$A(t)=\mathrm{e}^{-\int_t^T r(u)+\lambda(u)\mathrm{d}u},$$

$$B(t)=\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}\int_{(t,T]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v),$$

$$C(t)=R\int_t^T P(s-,T)\lambda(s)\mathrm{e}^{-\int_t^s r(u)+\lambda(u)\mathrm{d}u}\mathrm{d}s,$$

$$\begin{aligned}D(t)&=\int_t^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\left(\int_{(t,s]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathrm{d}s\\&=\int_{(t,T]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\left(\int_v^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s\right)\mathrm{d}\mathcal{C}(v).\end{aligned}$$

The calculation of $D(t)$ in the above equation involves the exchange of the order of double integration. Please refer to Appendix A for further details.

Assume that $P(t,T)$ is continuous and differentiable with respect to t when t is not an integer, and right-continuous with respect to t when t is an integer.

When t is not an integer, since

$$A'(t)=\left[r(t)+\lambda(t)\right]A(t),$$

$$B'(t)=\left[r(t)+\lambda(t)\right]B(t),$$

$$C'(t)=\left[r(t)+\lambda(t)\right]C(t)-R\lambda(t)P(t,T),$$

$$D'(t)=\left[r(t)+\lambda(t)\right]D(t),$$

we have

$$\begin{aligned}P'(t,T)&=A'(t)+B'(t)+C'(t)+D'(t)\\&=\left[r(t)+\lambda(t)\right]\left[A(t)+B(t)+C(t)+D(t)\right]-R\lambda(t)P(t,T)\\&=\left[r(t)+\lambda(t)\right]P(t,T)-R\lambda(t)P(t,T)\\&=\overline{r}(t)P(t,T).\end{aligned}\tag{14}$$

When t is an integer, since

$$\Delta A(t)=A(t)-A(t-)=0,$$

$$\Delta B(t)=B(t)-B(t-)=-c(t)\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u},$$

$$\Delta C(t)=C(t)-C(t-)=0,$$

$$\begin{aligned}\Delta D(t)&=D(t)-D(t-)=-c(t)\int_k^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s\\&=-c(t)\left[1-\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}\right],\end{aligned}$$

where the calculation of $\Delta B(t)$ and $\Delta D(t)$ can be further referred to Appendix B, it follows that

$$\begin{aligned}\Delta P(t,T)&=P(t,T)-P(t-,T)=\Delta A(t)+\Delta B(t)+\Delta C(t)+\Delta D(t)\\&=-c(t).\end{aligned}\tag{15}$$

The solution that satisfies Equations (14)-(15), and the terminal condition $P(T,T)=1$ is given by

$$P(t,T)=\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v \bar{r}(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v). \tag{12}$$

□

4. Term Structure under the Stochastic Assumption

Consider the implied term structure of interest rate of a defaultable bond under Stochastic Assumption. Assume that the risk-free interest rate $r(t)$ and the default intensity $\lambda(t)$ follow certain stochastic processes, and the recovery rate R is a random variable distributed over the interval $[0,1)$. Let $\mathcal{F}_t=\sigma(r(s),\lambda(s):0\le s\le t)$, $\mathcal{H}_t=\sigma\left(\mathbf{I}_{\{\tau>s\}}:0\le s\le t\right)$, $\mathcal{R}=\sigma(R)$, and $\mathcal{G}_t=\mathcal{H}_t\vee\mathcal{F}_t$. In the following theorems, we derive the defaultable bond prices by applying conditional expectation technique.

Theorem 4. The price at time t of a defaultable zero-coupon bond is given by

$$P(t,T)=\mathbb{E}\left[\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}\mid\mathcal{G}_t\right], \tag{16}$$

where $\bar{r}(t)=r(t)+(1-R)\lambda(t)$.

Proof. For random variables X and σ-algebra $\mathcal{G}\subset\mathcal{F}$, we have the following relationship: $E\left[E(X\mid\mathcal{F})\mid\mathcal{G}\right]=E(X\mid\mathcal{G})$. From Equation (6) and risk-neutral pricing principle, we obtain

$$\begin{aligned}P(t,T)&=\mathbb{E}\left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\mathbf{I}_{\{\tau>T\}}+\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}RP(\tau-,T)\mathbf{I}_{\{\tau\le T\}}\mid\mathcal{G}_t\right]\\&=\mathbb{E}\left[E\left(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}\mathbf{I}_{\{\tau>T\}}+\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}RP(\tau-,T)\mathbf{I}_{\{\tau\le T\}}\mid\mathcal{G}_t\vee\mathcal{F}_T\vee\mathcal{R}\right)\mid\mathcal{G}_t\right]\\&=\mathbb{E}\left[\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}\mid\mathcal{G}_t\right].\end{aligned}$$

□

Remark 3. For the price at time t of a defaultable contingent claim X maturing at time T, the same technique can be applied by replacing $\mathcal{G}_t\vee\mathcal{F}_T\vee\mathcal{R}$ with $\mathcal{G}_t\vee\mathcal{F}_T\vee\mathcal{R}\vee\mathcal{X}$, where $\mathcal{X}=\sigma(X)$ and X is independent of the exponential random variable Y introduced in Section 2.1.

Theorem 5. The price at time t of a defaultable bond with continuous coupon payments is given by

$$P(t,T)=\mathbb{E}\left[\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_t^T c(s)\mathrm{e}^{-\int_t^s \bar{r}(u)\mathrm{d}u}\mathrm{d}s\mid\mathcal{G}_t\right], \tag{17}$$

where $\bar{r}(t)=r(t)+(1-R)\lambda(t)$.

Proof. From Equation (9), we have

$$
\begin{aligned}
P(t,T) &= \mathbb{E}\Bigg[\left(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}+\int_t^T c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\right)\mathbf{I}_{\{\tau>T\}} \\
&\quad +\left(RP(\tau-,T)\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}+\int_t^\tau c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\right)\mathbf{I}_{\{\tau\le T\}}\mid\mathcal{G}_t\Bigg] \\
&= \mathbb{E}\Bigg[E\Bigg(\left(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}+\int_t^T c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\right)\mathbf{I}_{\{\tau>T\}} \\
&\quad +\left(RP(\tau-,T)\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}+\int_t^\tau c(v)\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}v\right)\mathbf{I}_{\{\tau\le T\}}\mid\mathcal{G}_t\vee\mathcal{F}_T\vee\mathcal{R}\Bigg)\mid\mathcal{G}_t\Bigg] \\
&= \mathbb{E}\left[\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_t^T c(s)\mathrm{e}^{-\int_t^s \bar{r}(u)\mathrm{d}u}\mathrm{d}s\mid\mathcal{G}_t\right].
\end{aligned}
$$

□

Theorem 6. The price at time t of a defaultable bond with periodic coupon payments is given by

$$P(t,T)=\mathbb{E}\left[\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v \bar{r}(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\mid\mathcal{G}_t\right], \tag{18}$$

where $\bar{r}(t)=r(t)+(1-R)\lambda(t)$.

Proof. From Equation (12), the price of the bond is

$$
\begin{aligned}
P(t,T) &= \mathbb{E}\Bigg[\left(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathbf{I}_{\{\tau>T\}} \\
&\quad +\left(RP(\tau-,T)\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathbf{I}_{\{\tau\le T\}}\mid\mathcal{G}_t\Bigg] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Bigg(\left(\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathbf{I}_{\{\tau>T\}} \\
&\quad +\left(RP(\tau-,T)\mathrm{e}^{-\int_t^\tau r(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathbf{I}_{\{\tau\le T\}}\mid\mathcal{G}_t\vee\mathcal{F}_T\vee\mathcal{R}\Bigg)\mid\mathcal{G}_t\Bigg] \\
&= \mathbb{E}\left[\mathrm{e}^{-\int_t^T \bar{r}(u)\mathrm{d}u}+\int_{(t,T]}\mathrm{e}^{-\int_t^v \bar{r}(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\mid\mathcal{G}_t\right].
\end{aligned}
$$

□

Remark 4. For simplicity, the continuous coupon payments $c(t)$ in Theorem 5 and the cumulative coupons $\mathcal{C}(t)$ in Theorem 6 are assumed to be deterministic. However, the conditional expectation technique can be extended to stochastic $c(t)$ and $\mathcal{C}(t)$, provided they are independent of the exponential random variable Y introduced in Section 2.1.

5. Conclusion

This paper provides pricing formulas for defaultable bonds under both deterministic and stochastic assumptions for the risk-free rate $r(t)$, default intensity $\lambda(t)$, and recovery rate R. The results indicate that whether for zero-coupon bonds, continuously coupon-paying bonds, or periodic coupon-paying bonds, the implied term structure of interest rates for defaultable bonds follows the form $\bar{r}(t)=r(t)+(1-R)\lambda(t)$. This form is consistent with the term structure of interest rates for default-free bonds, suggesting that methods from risk-free term structure analysis can be extended to study defaultable bonds. For instance, calibrating the term structure of defaultable bonds issued by a single entity is as straightforward as calibrating the term structure of Treasury bonds. Moreover, complex derivatives like interest rate swaps, defaultable bond options and credit spread options can be priced with significantly greater ease.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A: Exchange the Order of Double Integration

In Section 3.3, when calculating $D(t)$, the exchange of integration order is involved:

$$D(t)=\int_t^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\left(\int_{(t,s]} \mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathrm{d}s$$
$$=\int_{(t,T]} \mathrm{e}^{-\int_t^v r(u)\mathrm{d}u}\left(\int_v^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s\right)\mathrm{d}\mathcal{C}(v).$$

In order to calculate the above expression, we only need to prove

$$\int_t^T \lambda(s)\mathrm{e}^{-\int_0^s \lambda(u)\mathrm{d}u}\left(\int_{(t,s]} \mathrm{e}^{-\int_0^v r(u)\mathrm{d}u}\mathrm{d}\mathcal{C}(v)\right)\mathrm{d}s$$
$$=\int_{(t,T]} \mathrm{e}^{-\int_0^v r(u)\mathrm{d}u}\left(\int_v^T \lambda(s)\mathrm{e}^{-\int_0^s \lambda(u)\mathrm{d}u}\mathrm{d}s\right)\mathrm{d}\mathcal{C}(v).$$

Let $\Lambda(t)=\lambda(t)\mathrm{e}^{-\int_0^t \lambda(u)\mathrm{d}u}$ and $F(t)=\mathrm{e}^{-\int_0^t r(u)\mathrm{d}u}$. The above expression can be rewritten as

$$\int_t^T \Lambda(s)\left(\int_{[t,s)} F(v)\mathrm{d}\mathcal{C}(v)\right)\mathrm{d}s=\int_{(t,T]} F(v)\left(\int_v^T \Lambda(s)\mathrm{d}s\right)\mathrm{d}\mathcal{C}(v). \tag{19}$$

We make a slight modification to the form of Equation (19), rewriting it as the following lamma.

Lemma. The following equality holds

$$\int_{[t,T)} \Lambda(s)\left(\int_{[t,s)} F(v)\mathrm{d}\mathcal{C}(v)\right)\mathrm{d}s=\int_{[t,T)} F(v)\left(\int_{[v,T)} \Lambda(s)\mathrm{d}s\right)\mathrm{d}\mathcal{C}(v). \tag{20}$$

Proof. Note that:

$$\int_{[t,s)} F(v)\mathrm{d}\mathcal{C}(v)=\sum_{k=\lfloor t\rfloor+1}^{\lfloor s\rfloor} F(k)c(k).$$

When $s<\lfloor t\rfloor+1$, we have $\int_{[t,s)} F(v)\mathrm{d}\mathcal{C}(v)=0$, thus

$$\int_{[t,\lfloor t\rfloor+1)} \Lambda(s)\left(\int_{[t,s)} F(v)\mathrm{d}\mathcal{C}(v)\right)\mathrm{d}s=0. \tag{21}$$

The left-hand side of Equation (20) is a Lebesgue integral over the interval $[t,T)$ with respect to the variable s, this can be rewritten as a sum of integrals over subintervals $[t,\lfloor t\rfloor+1)$, $[\lfloor t\rfloor+1,\lfloor t\rfloor+2)$, $\cdots$, $[T-1,T)$. Define $H(i)=\int_{[i,i+1)} \Lambda(s)\mathrm{d}s$, then

$$\text{the left-hand side of equation (20)}$$
$$=\int_{[t,T)} \Lambda(s)\sum_{k=\lfloor t\rfloor+1}^{s} F(k)c(k)\mathrm{d}s$$
$$=\sum_{k=\lfloor t\rfloor+1}^{T-1}\int_{[i,i+1)} \Lambda(s)\sum_{k=\lfloor t\rfloor+1}^{i} F(k)c(k)\mathrm{d}s$$

$$
\begin{aligned}
&= \sum_{k=\lfloor t \rfloor+1}^{T-1} \sum_{k=\lfloor t \rfloor+1}^{i} F(k)c(k) \int_{[i,i+1)} \Lambda(s)\mathrm{d}s \\
&= \sum_{k=\lfloor t \rfloor+1}^{T-1} \sum_{k=\lfloor t \rfloor+1}^{i} F(k)c(k)H(i) \\
&= \sum_{k=\lfloor t \rfloor+1}^{T-1} \sum_{i=k}^{T-1} F(k)c(k)H(i) \\
&= \sum_{k=\lfloor t \rfloor+1}^{T-1} F(k)c(k) \sum_{i=k}^{T-1} H(i) \\
&= \sum_{k=\lfloor t \rfloor+1}^{T-1} F(k)c(k) \sum_{i=k}^{T-1} \int_{[i,i+1)} \Lambda(s)\mathrm{d}s \\
&= \sum_{k=\lfloor t \rfloor+1}^{T-1} F(k)c(k) \int_{[k,T)} \Lambda(s)\mathrm{d}s \\
&= \sum_{k=\lfloor t \rfloor+1}^{T} F(k)c(k) \int_{[k,T)} \Lambda(s)\mathrm{d}s \\
&= \text{the right-hand side of equation } (20),
\end{aligned}
$$

where the second-to-last equality utilizes the fact that $\int_{[T,T)} \Lambda(s)\mathrm{d}s = 0$. □

Appendix B: Calculations of $\Delta B(t)$ and $\Delta D(t)$

We explain the calculations of $\Delta B(t)$ and $\Delta D(t)$ in Section 3.3 for more details. When t is an integer,

$$
\begin{aligned}
B(t) &= \mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u} \sum_{k=\lfloor t \rfloor+1}^{T} c(k)\mathrm{e}^{-\int_t^k r(u)\mathrm{d}u}, \\
B(t-) &= \mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u} \sum_{k=\lfloor t \rfloor}^{T} c(k)\mathrm{e}^{-\int_t^k r(u)\mathrm{d}u}, \\
D(t) &= \sum_{k=\lfloor t \rfloor+1}^{T} c(k)\mathrm{e}^{-\int_t^k r(u)\mathrm{d}u} \left(\int_k^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s \right) \\
D(t-) &= \sum_{k=\lfloor t \rfloor}^{T} c(k)\mathrm{e}^{-\int_t^k r(u)\mathrm{d}u} \left(\int_k^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s \right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\Delta B(t) &= B(t) - B(t-) = -c(t)\mathrm{e}^{-\int_t^T \lambda(u)\mathrm{d}u}, \\
\Delta D(t) &= D(t) - D(t-) = -c(t)\int_t^T \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}\mathrm{d}s.
\end{aligned}
$$