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Optimization of Financial Asset Portfolio Using GARCH-EVT-Copula-CVaR Model

Immaculate Ngina Kyalo*, **Cyprian O. Omari**, **Anthony Ngunyi**

Department of Statistics and Actuarial Science, Dedan Kimathi University of Technology, Nyeri, Kenya
Email: *nginakyalo2015@gmail.com, cyomari@dkut.ac.ke, antony.ngunyi@dkut.ac.ke

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Abstract

Since the pioneering work of Markowitz on portfolio theory in 1950s, numerous developments have advanced to improve the original technique of portfolio optimisation. Current research on the topic focuses on integrating models to capture real-world financial characteristics, including volatility clustering, heavy tails, and non-linear dependencies. Portfolio optimisation is a critical aspect of financial risk management, requiring sophisticated models to accurately assess risk and optimise asset allocation. This study implemented an integrated approach utilising Generalised Autoregressive Conditional Heteroskedasticity (GARCH) for volatility estimation, Extreme Value Theory (EVT) for modelling extreme market movements, Copula functions for capturing dependencies between financial assets, and Conditional Value at Risk (CVaR) for robust risk assessment in the portfolio of financial assets. By applying this methodology to a portfolio of financial assets, the empirical results demonstrate that the GARCH-EVT-Copula-CVaR model significantly improves risk estimation, portfolio selection, and optimisation. The empirical results also confirm its superiority over conventional models, highlighting its potential for enhanced risk management and portfolio asset allocation. The integrated model can be recommended for utilisation by stakeholders in the financial markets, investors, and regulators for policy formulation and informed decision-making.

Keywords

Copula, Regular Vines, C-Vine, D-Vine, Stock Indices, Currency Exchange Rates, Commodities, Tail Dependence, Pair-Copula Constructions, Portfolio Optimization

1. Introduction

Since the pioneering work of [1] on portfolio theory, numerous developments have emerged to enhance the original technique of portfolio optimisation. Portfolio optimisation is an integral part of portfolio selection and management, especially in financial and investment markets, where the goal is to choose a mix of financial asset classes that can yield the maximum possible return while maintaining low risk exposure. Conventional methods, notably the Mean-variance optimisation model, assume that financial asset returns are normally distributed and correlations between portfolio assets remain constant over time. Nonetheless, empirical evidence in the literature has demonstrated that financial return series often exhibit stylised features such as volatility clustering, heavy-tailed distributions, and non-linear dependence structures, aspects that conventional financial time series models frequently fail to capture when modelling financial time series data.

In recent years, more researchers have shown interest in exploring the possibility of using more sophisticated portfolio optimization frameworks that integrate various models to accurately reflect the statistical properties of financial returns. Among these, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is employed to model time-varying volatility dynamics present in financial time series data. While GARCH-based models enhance volatility estimation and the leverage effect through existing GARCH model extensions such as the EGARCH model and GJR-GARCH models, they do not fully account for extreme losses, which are rare but can have catastrophic impacts on financial markets, thus compromising the portfolio optimization strategies used by investment managers and investors in the financial markets.

Extreme events such as the Global Financial Crisis (GFC) of 2007-2008, the COVID-19 pandemic and the recent Ukraine-Russia war have caused major disruptions in the operations of global financial markets. Such financial disruptions underscore the importance of incorporating Extreme Value Theory (EVT) into financial modelling and portfolio selection, optimisation, and hedging. EVT provides a robust statistical framework for modelling the tail behaviour of financial asset returns beyond a certain threshold, making it highly effective in identifying extreme risks and improving portfolio allocation decisions. However, EVT alone does not capture the interdependence between assets, which is essential for diversification and optimisation of portfolio assets.

Copula functions have been proposed to provide a more flexible method for modelling the joint distribution of asset returns and accurately representing non-linear dependence structure and asymmetric relationships among financial assets, particularly in extreme market conditions, making them highly relevant for portfolio optimization strategies.

In portfolio optimization, risk measures are essential in determining asset allocation. While Value at Risk (VaR) is commonly used, it has significant limitations, particularly in its inability to capture the magnitude of extreme losses. Conditional Value at Risk (CVaR), on the other hand, addresses this limitation by considering the expected loss beyond the VaR threshold, providing a more comprehensive approach to risk assessment in portfolio optimization. Several empirical studies have attempted to enhance portfolio optimization. [2] demonstrated that GARCH models improve volatility estimation, while [3] applied EVT to financial returns and found that traditional VaR methods underestimate extreme risks. Recent work by [4] showed that integrating GARCH and EVT leads to more accurate tail risk estimates, particularly in highly volatile markets.

While GARCH and EVT improve risk estimation, they do not fully capture the dependence structures among financial assets. The concept of copulas was first introduced by [5], but its application in finance gained traction through the work of [6], who emphasized its advantages over conventional correlation measures. [7] further advanced this field by developing time-varying copulas, allowing researchers to model changes in dependence structures over time. [8] demonstrated that Vine Copulas offer significant advantages in capturing high-dimensional dependencies, making them particularly useful for portfolio optimization. Additionally, [9] applied GARCH-Copula models to cryptocurrency markets and found that copula-based dependency modeling enhances portfolio diversification and risk-adjusted returns, further reinforcing the importance of this approach.

In terms of risk measures, [10] and [11] showed that CVaR provides a more comprehensive risk assessment than VaR, particularly during financial downturns. These findings suggest that combining GARCH, EVT, Copulas and CVaR into a unified portfolio optimization model could provide significant improvements in risk-adjusted returns. [12] applied copula theory to model the dependence structure of currency exchange rates. They utilized the GARCH-EVT-Copula framework to estimate the Value-at-Risk (VaR) of currency exchange rate portfolios.

[13] combined GARCH for volatility modelling and EVT to account for extreme risks and model tail distributions in financial portfolios. Using Conditional Value-at-Risk (CVaR) as a key risk measure, they demonstrated that this combination enhances portfolio risk management by improving risk estimation accuracy, particularly for extreme market events. [14] employs vine copulas to model both symmetric and asymmetric dependencies in financial returns from 2001 to 2022, encompassing multiple financial crises. The authors analyze various portfolio strategies, including maximum Sharpe ratio, minimum variance and minimum conditional value at risk, demonstrating that vine copulas can effectively reduce portfolio risk compared to traditional methods.

In addressing the limitations and challenges of conventional portfolio optimisation models, this study proposes the implementation of an integrated GARCH-EVT-Copula-CVaR framework for financial asset portfolio optimisation. The GARCH-EVT-Copula-CVaR model represents a significant advancement in portfolio optimisation by combining volatility forecasting, tail risk modelling, dependency structure, and risk measurement into one cohesive framework. It offers a more comprehensive approach to managing risk, particularly in extreme market conditions. By merging these methodologies, this study aims to: model time-varying volatility using the GARCH model, improve tail risk estimation through

EVT, and model non-linear dependencies between assets using Copula functions. The proposed GARCH-EVT-Copula model will be employed to optimise portfolio allocation by minimising CVaR as a risk measure. This study utilises the GARCH-EVT-Copula-CVaR model for portfolio optimisation, demonstrating how this integrated model can be applied to optimise both risk and returns across different market regimes and asset types, ultimately providing a more resilient framework for portfolio management in uncertain financial environments.

The remainder of this paper is organised as follows: Section 2 presents the methodology of the study, including the GARCH model specifications, extreme value theory, copula functions, and the GARCH-EVT-Copula CVaR model. Section 3 discusses the empirical results and the application of the model specifications to a portfolio of financial time series data. Section 4 implements portfolio selection and optimisation using the GARCH-EVT-Copula CVaR model, comparing it to conventional techniques. Finally, Section 5 summarises the conclusions and offers suggestions for future research.

2. Methodology

This study employs an integrated modeling framework that brings together four complementary techniques to enhance financial asset portfolio optimization and risk management. The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is used to estimate the time-varying volatility of financial assets, effectively capturing the persistence and clustering of volatility often observed in financial time series. To address the occurrence of rare but significant market events, Extreme Value Theory (EVT) is applied to model the tail behavior of asset returns. EVT focuses on the extreme ends of the return distribution, allowing the model to better quantify risks associated with severe market downturns.

In addition to capturing individual asset behavior, it is important to account for the dependency structure between assets, especially during periods of financial stress. This is achieved using copula functions, which offer a flexible way to model complex, non-linear and asymmetric relationships between financial variables. Specifically, regular vine (R-vine) copulas are employed to construct a high-dimensional dependence model based on bivariate copula pairs arranged hierarchically. This structure allows for more accurate modeling of joint movements in asset returns compared to traditional correlation-based approaches.

To measure and minimize portfolio risk, the Conditional Value at Risk (CVaR) metric is adopted. CVaR improves upon the more traditional Value at Risk (VaR) by estimating the expected loss in the tail of the distribution, thereby providing a more comprehensive view of downside risk. By integrating GARCH, EVT, copula theory and CVaR into a unified framework, this methodology provides a robust approach for modeling and optimizing financial portfolios under realistic market conditions, including extreme events and complex interdependencies.

2.1. GARCH Model Specification

Financial markets often exhibit periods of high and low volatility that cluster over time. To model this changing volatility, we use the Generalised Autoregressive Conditional Heteroscedastic (GARCH) model introduced by [15] which is widely utilized in the financial literature to capture time-varying volatility and to estimate risk measures such as Value at Risk (VaR). Let $r_t \in R$ be the returns of the financial asset that is of interest at time t.

$$r_t = \mu_t + \sigma_t z_t, \tag{1}$$

where μ_t denotes the mean component, σ_t denotes a conditional volatility process and z_t is the error term.

The standard GARCH(1,1) model is the most commonly used in modelling volatility dynamics in financial time series is given by:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \sigma_{t-1}^2 \tag{2}$$

where $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $(\alpha_1 + \beta_1) < 1$. This model is effective at capturing volatility clustering often observed in financial time series. The persistence of volatility is quantified by $\alpha_1 + \beta_1$, and for the model to be weakly stationary, the specified parameter conditions must be satisfied. Under stationarity, the unconditional variance is given by $\omega / (1 - (\alpha + \beta))$, which allows for the existence of higher-order moments.

Given that financial returns exhibit heavy-tailed distribution, excess kurtosis and leverage effects where negative returns tend to increase volatility more than positive ones, asymmetric GARCH models were introduced to address these weaknesses of the GARCH-type model.

The exponential GARCH (EGARCH) model proposed by [16] captures the leverage effects using threshold component. The variance component of the EGARCH(1,1) model is given by

$$\log_e \sigma_t^2 = \omega + \alpha_1 z_{t-1} + \gamma_1 \left(|z_{t-1}| - E|z_{t-1}| \right) + \beta_1 \log_e \left(\sigma_{t-1}^2 \right) \tag{3}$$

where parameter α_1 coefficient captures the sign effect of past shocks and $\gamma_1 > 0$ reflects the magnitude of the leverage effect. In this case, the persistence of volatility is determined by the coefficient β_1.

The Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model by [17] captures both positive and negative shocks on the conditional variance asymmetrically using an indicator function I. The variance process of the GJR-GARCH (1,1) model is given as:

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \gamma_1 I_{t-1} \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{4}$$

where γ_1 now represents the leverage term. The indicator function I takes on value of 1 for $\varepsilon_{t-1} \leq 0$ and 0 otherwise. The persistence depends on the parameter γ_1, through $\alpha_1 + \beta_1 + \gamma_1 \kappa$, where κ denotes the expected standardized residual value.

The asymmetric power ARCH (APARCH) model by [18] allows for both leverage and the Taylor effect, named after Taylor (1986) who observed that the sample autocorrelation of absolute returns was usually larger than that of squared returns. The APARCH (1,1) model can be expressed as:

$$\sigma_t^{\delta} = \omega + \alpha_1 \left(|z_{t-1}| - \gamma_1 z_{t-1} \right)^{\delta} + \beta_1 \sigma_{t-1}^{\delta}, \tag{5}$$

where $\delta \in R^{+}$, is a Box-Cox transformation of σ_t, and $-1 < \gamma_1 < 1$ is the coefficient in the leverage term. The persistence parameter is equal to $\beta_1 + \alpha_1 \kappa_1$, where κ_1 is the expected value of the standardized residuals under the Box-Cox transformation of the term, which includes the leverage parameter γ_1.

The component standard GARCH (CS-GARCH) model by [19] decomposes the component of the conditional variance so as to investigate the long and short-run movements of volatility. Let q_t represent the permanent component of the conditional variance, the component model can be written as

$$\sigma_t^2 = q_t + \alpha_1 \left(z_{t-1}^2 - q_{t-1} \right) + \beta_1 \left(\sigma_{t-1}^2 - q_{t-1} \right) \tag{6}$$
$$q_t = \alpha_0 + \rho q_{t-1} + \phi \left(z_{t-1} - \sigma_{t-1}^2 \right)$$

where effectively the intercept of the GARCH model is now time-varying following first-order autoregressive type dynamics.

In fitting the GARCH model, the innovations distribution is part of the volatility component that requires to be selected. In this study, the following innovations distributions were considered in fitting the GARCH-type models; standardised Student's *t*-distribution, skewed Student's *t*-distribution and generalized error distributions (GED) in places of the Gaussian distribution assumed in the original GARCH-type model. The distribution function of the standardized Student's *t*-distribution is given by:

$$f(\epsilon_t, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}} \left[1 + \frac{\epsilon_t^2}{(\nu-2)} \right]^{-\left(\frac{\nu+1}{2}\right)} \tag{7}$$

with degrees of freedom parameter $\nu > 0$, $\Gamma(z) = \int_0^{\infty} x^{z-1} \mathrm{e}^{-x} \mathrm{d}x$, is a standard gamma function controlling the thickness of the tail of the distribution.

The skewed Student's *t*-distribution is an alternative distribution for modelling skew and heavy-tailed data. [20] proposed a skew extension to the Student's *t*-distribution for modelling financial returns. The skewed Student's *t* (SST)-distribution by [21] is given by

$$f(x; \mu, \alpha, v, \mu, \sigma) = \begin{cases} \frac{1}{\sigma} K(v) \left[1 + \frac{1}{v} \left(\frac{x-\mu}{2\alpha\sigma} \right)^2 \right]^{-(v+1)/2}, & x \le \mu \\ \frac{1}{\sigma} K(v) \left[1 + \frac{1}{v} \left(\frac{x-\mu}{2(1-\alpha)\sigma} \right)^2 \right]^{-(v+1)/2}, & x > \mu \end{cases} \tag{8}$$

This parametrization of the SST is equivalent to those of [20] and [22].

The probability distribution of the generalised error distribution (GED) is given by

$$f(x,\mu,\alpha,\beta)=\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}\mathrm{e}^{-\left(\frac{|x-\mu|}{\alpha}\right)^{\beta}} \tag{9}$$

The GARCH-type model parameters $(\mu,\omega,\alpha,\gamma,\beta)$ are estimated simultaneously using Quasi-Maximum Likelihood (QML) method by maximizing the log likelihood. The log-likelihood function under the assumption that the random error term follows the standardized Student's *t*-distribution is given by

$$\begin{aligned} l(\Theta) &= \log\Gamma\left(\frac{\nu+1}{2}\right)-\frac{1}{2}\log\left(\pi(\nu-2)\right)\log\Gamma\left(\frac{\nu}{2}\right) \\ &\quad -\frac{\nu+1}{2}\log\left(1+\frac{\epsilon_t^2}{\nu-2}\right)-\frac{1}{2}\log\sigma_t^2 \end{aligned} \tag{10}$$

where $\Theta=(\mu,\omega,\alpha,\gamma,\beta)$ denote the unknown parameters in GARCH-type models to be estimated. Solving the first-order conditions of the log-likelihood function with respect to the parameters $\omega,\alpha_1,\gamma_1,\beta_1$, we obtain a specified optimal GARCH-type model. While GARCH model specification effectively models volatility dynamics in financial time series data, it does not explicitly account for extreme movements observed in financial markets. Therefore, extreme value theory (EVT) is applied to the standardised residuals from a fitted GARCH model.

2.2. Extreme Value Theory

In financial markets, most of the time returns behave normally, but it's the rare, extreme events (like crashes) that cause the most damage. Traditional models often underestimate these rare events. That's where Extreme Value Theory (EVT) comes in. Extreme Value Theory (EVT) is a statistical framework developed to model the tail behaviour of extreme observations in a financial time series. EVT focuses exclusively on estimating the distribution of tail observations instead of modelling the entire distribution of observations in a dataset. There are two main methods of implementing EVT: the block maxima method (BMM) and the peak over threshold (POT) approach. The methods primarily differ in how they define extreme observations and how the extreme values are determined. The Block maxima method first splits the data set into pre-determined blocks before extracting the maximum value within a specified period or block, whereas the POT approach considers all observations that exceed a predefined threshold, beyond which any observation that exceeds is considered an extreme observation. Therefore, POT is a more effective strategy given that it considers not only the most extreme observation but also all values that surpass a significant threshold. In the present study, the Peaks Over Threshold method is adopted for modelling the tail distributions of extreme observations.

The Peaks Over Threshold (POT) method classifies observations exceeding a specified threshold as extreme values, concentrating on these exceedances to estimate parameters of the tail distribution rather than analysing the entire dataset. Let $X_1,X_2,\cdots,X_n$ represent a sequence of independent and identically distributed (i.i.d.) random variables following an unknown distribution function F. The POT approach aims to estimate the distribution function F_u, which describes values of x that surpass a given high threshold u. The distribution of exceedance values over a given threshold u is defined as:

$$F_u(y) = \Pr(X - u \le | X > u) = \frac{F(u+y) - F(u)}{1 - F(u)},\ 0 < y < x_0 - u, \tag{11}$$

where $x_0 \le \infty$ is the right endpoint of F.

According to the theorem of [23] and [24], for a large class of underlying distributions functions F the conditional excess distribution function $F_u(y)$, for a large threshold, u, is well approximated by $F_u(y) \approx G(y)$ with $u \to \infty$, *i.e.*,

$$\lim_{u \to x_F} \sup_{0 \le y \le x_F - u} \left| F_u(y) - G_{\xi,\sigma(u)}(y) \right| = 0 \tag{12}$$

where $G_{\xi,\sigma(u)}(y)$, the generalized Pareto distribution (GPD), given by

$$G_{\xi,\sigma(u)}(y) = \begin{cases} 1 - \left(1 + \xi \dfrac{y}{\sigma(u)}\right)^{-\frac{1}{\xi}} & \text{if } \xi \ne 0, \\ 1 - \exp\left(-\dfrac{y}{\sigma(u)}\right) & \text{if } \xi = 0, \end{cases} \tag{13}$$

In the context of the Generalised Pareto Distribution (GPD), the variable y is defined with different conditions based on the value of the shape parameter ξ. Specifically, if $\xi \ge 0$, then $y \ge 0$. Conversely, if $\xi < 0$, the value of y is constrained such that $0 \le y \le -\frac{\sigma(u)}{\xi}$. The GPD is characterized by two parameters: σ is the scale parameter, while ξ is the shape parameter. This distribution includes several well-known distributions based on the values of these parameters. When $\xi > 0$, the GPD behaves as a heavy-tailed variant of the ordinary Pareto distribution. For $\xi = 0$, it simplifies to the light-tailed exponential distribution, and when $\xi < 0$, it corresponds to a short-tailed Pareto Type II distribution.

In the field of financial risk management, the case where $\xi > 0$ is often the most relevant, as it effectively models heavy-tailed data. The parameters ξ and $\sigma(u)$ can be estimated using the maximum likelihood method, as discussed by Embrechts *et al.* (1999). Additionally, research by Hosking *et al.* (1987) indicates that for $\xi = -0.5$, the regularity conditions for maximum likelihood estimation are satisfied. This ensures that the estimates are asymptotically normally distributed.

By setting $x = u + y$, an approximation of $F(x)$, for $x > u$, can be obtained from Equation (11):

$$F(x) = (1 - F(u)) G_{\xi,\sigma(u)}(y) + F(u), \tag{14}$$

The function $F(u)$ can be estimated non-parametrically using the empirical c.d.f:

$$\hat{F}(u) = \frac{n - n_u}{n} \tag{15}$$

where n_u represents the number of exceedances over the threshold u and n is the sample. By substituting Equations (13) and (15) into Equation (14), we get the estimate for $F(x)$ as follows:

$$\hat{F}(x) = 1 - \frac{n_u}{n}\left(1 + \hat{\xi}\left(\frac{x - \hat{u}}{\hat{\sigma}}\right)\right)^{-\frac{1}{\hat{\xi}}}, \tag{16}$$

where $\hat{\xi}$ and $\hat{\sigma}$ are estimates of ξ and σ, respectively, which can be estimated by the method of maximum likelihood.

For $p > F(u)$, x_p can be obtained from Equation (16) by solving for x;

$$x_p = \hat{u} + \frac{\hat{\sigma}}{\hat{\xi}}\left[\left(\frac{n}{n_u}(1-p)\right)^{-\hat{\xi}} - 1\right] \tag{17}$$

where u represents the threshold, while $\hat{\sigma}$ and $\hat{\xi}$ correspond to the estimated scale and shape parameters, respectively.

The challenge in applying the Peak Over Threshold (POT) method lies in selecting the most appropriate threshold value for the given data set. This decision is crucial, as an unsuitable threshold can significantly impact model accuracy. If the threshold is set too low, the model may diverge from the expected behaviour of the Generalized Pareto Distribution (GPD), leading to biased estimates. Conversely, setting the threshold too high results in fewer exceedances, which decreases the sample size and increases variability in the resulting parameter estimates. Therefore, it is essential to find a balance between minimizing bias and controlling variance. In this study, we utilize a quantile-based approach by setting the threshold at the 90th percentile (the upper 10%), a method commonly used in practical applications.

2.3. Copula Functions

While GARCH and EVT model each asset's behavior individually (volatility and tail risk), copulas help us understand how assets behave together, especially under stress. This is critical for portfolio optimization, since diversification depends not just on individual risk, but on how asset returns depend on each other. A copula is a function that links univariate marginal distributions to form a multivariate distribution function. Mathematically, an n-dimensional copula is a distribution function on $[0,1]^n$ with uniformly distributed margins on $[0,1]$. The primary role of a copula is to describe the dependence structure among random variables, as first formalized by [5]. Let F be an *n-dimensional joint distribution function* with marginal distributions $F_1, F_2, \cdots, F_n$. Then, there exists a *unique copula function* C such that:

$$F(x_1, x_2, \cdots, x_n) = C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)) \tag{18}$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$.

Equation (18) establishes that the joint distribution of multiple variables can be decomposed into their individual marginal distributions and a copula function.

The copula thus fully characterizes the dependence structure, making it highly effective for portfolio optimization and risk management. Copulas can be categorized into various families depending on their structural properties and ability to capture dependence in financial data. The two major families of copulas used in financial applications are Elliptical Copulas and Archimedean Copulas.

2.3.1. Elliptical Copulas

These copulas originate from elliptical distributions, such as the Gaussian and Student's *t*-distributions. While they effectively model symmetric dependence structures, they have limitations in modeling tail dependence, which is crucial for portfolio optimization in extreme market conditions.

Gaussian Copula: This copula assumes normally distributed marginals and is defined as:

$$C_N\left(u_1,u_2,\rho\right)=\Phi_\rho\left(\Phi^{-1}\left(u_1\right),\Phi^{-1}\left(u_2\right)\right) \tag{19}$$

where Φ is the standard normal CDF and ρ is the correlation parameter. The Gaussian copula is tail independent, meaning it does not capture extreme co-movements in asset prices.

Student's t Copula: It extends the Gaussian copula by introducing heavy tails, making it more suitable for financial applications where extreme events are more frequent. It is defined as:

$$C_t\left(u_1,u_2,\rho,v\right)=\int_{-\infty}^{t^{-1}(u_1)}\int_{-\infty}^{t^{-1}(u_2)} f_{t,\rho,v}\left(x,y\right)\mathrm{d}x\mathrm{d}y \tag{20}$$

where v represents degrees of freedom, and $f_{t,\rho,v}$ is the bivariate Student's t density function.

2.3.2. Archimedean Copulas

Elliptical copulas provide a good first approximation of asset dependencies, but they struggle with asymmetric relationships, particularly in financial downturns when assets exhibit stronger lower tail dependence than upper tail dependence. Archimedean copulas address this limitation by using generator functions that allow for asymmetric tail behavior, making them well-suited for financial risk modeling.

Clayton Copula: Suitable for modeling strong lower tail dependence, meaning assets tend to crash together during financial downturns. It is given by:

$$C_C\left(u_1,u_2,\theta\right)=\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-1/\theta} \tag{21}$$

where $\theta>0$ controls the strength of dependence.

Gumbel Copula: Models upper tail dependence, meaning it captures situations where asset prices rise together. It is defined as:

$$C_G\left(u_1,u_2,\theta\right)=\exp\left(-\left[\left(\ln u_1\right)^\theta+\left(\ln u_2\right)^\theta\right]^{1/\theta}\right) \tag{22}$$

where $\theta\geq1$ controls the dependence level.

Frank Copula: Unlike Clayton and Gumbel, the Frank copula does not exhibit tail dependence but is useful for modelling moderate dependencies across the entire distribution. It is given by:

$$C_F\left(u_1,u_2,\theta\right)=-\frac{1}{\theta}\ln\left(1+\frac{\left(e^{-\theta u_1}-1\right)\left(e^{-\theta u_2}-1\right)}{e^{-\theta}-1}\right) \tag{23}$$

where θ controls dependence strength.

While elliptical and Archimedean copulas are useful for modelling bivariate (two-variable) dependencies, real-world financial portfolios consist of multiple assets with complex interdependencies. Standard copulas struggle to adequately capture high-dimensional dependencies, leading to oversimplified risk assessments.

To address this limitation, we transition to Vine Copulas, which provide a hierarchical and flexible approach to modelling high-dimensional dependency structures. Vine copulas allow for pairwise modelling of dependencies, making them highly effective in multi-asset portfolio optimization.

2.4. Regular Vine Copulas

Real-world portfolios include many assets, each interacting in complex ways. Regular vine copulas provide a structured approach to model high-dimensional dependencies by breaking them into simpler, pairwise relationships. This hierarchical structure allows for greater flexibility and accuracy when modeling multi-asset portfolios. A Regular Vine (R-Vine) is a special case of a vine that follows the *proximity condition*, meaning that in any tree T_i, an edge can only connect two nodes if they share a common node in the previous tree. This ensures a hierarchical dependency structure that is consistent and mathematically sound.

A Regular Vine (R-Vine) copula specification fully describes a multivariate dependency structure through a hierarchical representation of pairwise copulas. This specification provides a systematic way to construct high-dimensional copulas while preserving flexibility in choosing different bivariate copulas for each pair of variables.

A Regular Vine (R-Vine) is a structured framework used to model high-dimensional dependencies by decomposing a multivariate distribution into a sequence of bivariate copulas. A vine is a structured representation of dependencies in terms of trees, where each level of the vine defines a new layer of conditional relationships between variables. A vine on d elements is denoted as:

$$V=\left(T_1,T_2,\cdots,T_{d-1}\right) \tag{24}$$

where T_1 represents the first tree, which contains all variables as nodes and edges representing bivariate dependencies and $T_2,T_3,\cdots,T_{d-1}$ are successive trees capturing conditional dependencies, with each node representing a pairwise relationship conditioned on previous variables.

According to [25], an R-Vine copula specification is formally defined as follows:

$$\left(F,V,B\right) \tag{25}$$

where $F = (F_1, \cdots, F_n)$ is a vector of continuous invertible distribution functions, V is a regular vine structure on n elements and $B = B_e \mid e \in E_i, i = 1, \cdots, n$, where B_e is a bivariate copula assigned to each edge e in tree T_i. Thus, a copula is assigned to every edge in the trees, ensuring that dependencies are modelled hierarchically.

Regular Vines can be further classified into Canonical Vines (C-Vines) and Drawable Vines (D-Vines). *C-Vines* have a central node that connects to all other nodes in each tree, making them suitable for cases where one variable (e.g., a market index) strongly influences all others. *D-Vines* arrange dependencies in a sequential chain, which is useful when relationships are primarily between neighbouring variables rather than a single dominant variable.

For a d-dimensional C-Vine, there are $\frac{d!}{2}$ possible tree structures. A C-Vine structure consists of one central root node connected to all other nodes in the first tree. The dependency structure is modelled hierarchically, where the root node influences subsequent dependencies. This structure is useful when one variable strongly influences all others, such as a major market index affecting multiple assets. Unlike C-Vine, D-Vine structures arrange dependencies sequentially. Instead of a single dominant variable, the structure models strong dependencies between neighbouring variables. This is particularly useful when modelling time series or spatial dependencies.

2.4.1. Tree Structure Construction

The construction of regular vines involves selecting pairs of variables that will be linked using copulas. The selection of an appropriate vine structure is crucial, as it influences the dependencies modelled in each tree level. A stepwise sequential construction method, as described by [26], is used to determine the optimal tree structure. The key objective is to capture the strongest dependencies in the first tree (T_1). The selection process follows these steps:

1) Compute the empirical Kendall's tau for each pair of variables.

2) Select the spanning tree that maximises the sum of the absolute values of empirical Kendall's tau coefficients.

3) Proceed to the next tree level, conditioning on previously selected dependencies.

In a C-Vine structure, a root node must be chosen at each tree level. This is achieved by summing the values in the empirical Kendall's tau matrix and selecting the node with the highest sum as the root node. On the other hand, a D-Vine structure does not require a root node; it follows a sequential path that prioritises the dependencies between adjacent variables. The choice of tree structure significantly influences the selection of pair-copula families in the next stage, ensuring an accurate and well-fitted dependence model.

2.4.2. Copula Selection

To select a suitable copula for each pair-copula, a range of bivariate copula families is considered, which include commonly used options such as Gaussian, Student's t, Clayton, Gumbel, Frank, and Joe, as well as more flexible families like BB1 (Clayton-Gumbel), BB6 (Joe-Gumbel), BB7 (Joe-Clayton), and BB8 (Frank-Joe). Their survival counterparts, such as Survival Clayton, Survival Gumbel, are also taken into consideration for modelling dependencies between the dataset pairs. The selection of the most appropriate copula for each pair is carried out independently using the Akaike Information Criterion (AIC), which balances model fit against model complexity by penalising the inclusion of additional parameters. Alternatively, the Bayesian Information Criterion (BIC) may be employed, which does not depend on sample size. Once the optimal copula families are selected for both conditional and unconditional variable pairs based on the structure of the regular vine, the estimation of the corresponding pair-copula parameters follows.

2.4.3. Parameter Estimation

The estimation of both copula parameters and marginal specifications is carried out using a two-step procedure known as the Inference Functions for Margins (IFM) method, originally introduced by Joe [27]. This approach is based on maximum likelihood estimation (MLE). In the first stage, the parameters of the marginal distributions are estimated independently. Once the marginal models are specified, the second stage focuses on estimating the parameters of the copula functions. This method is widely applied in the context of regular vine copulas, where the selection of appropriate pair-copula families and the estimation of their parameters are conducted simultaneously, as described by [28].

2.5. Portfolio Optimization

Once risk and dependencies are modeled, the next step is to decide how to allocate assets efficiently. Portfolio optimization is the process of selecting the best combination of assets from various potential portfolios to achieve a specific investment goal, such as maximizing expected returns or minimizing overall risk. Investors aim to allocate their assets efficiently by considering the relationships between different assets and applying risk management techniques. The Modern Portfolio Theory (MPT), introduced by [1], states that investors should diversify assets based on expected return and variance. The expected return of a portfolio is given by:

$$E\left(R_p\right)=\sum_{i=1}^{n} w_i E\left(R_i\right) \tag{26}$$

where w_i represents the weight of asset i, and $E\left(R_i\right)$ is its expected return. The portfolio variance, representing risk, is given by:

$$\operatorname{Var}\left(R_p\right)=\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \operatorname{Cov}\left(R_i, R_j\right) \tag{27}$$

This framework allows the construction of efficient portfolios that optimize risk-return trade-offs. In this study, the following methods were also considered for portfolio optimization.

2.5.1. Conditional Value-at-Risk Model

The Conditional Value-at-Risk (CVaR) model is a risk measure that quantifies extreme losses beyond the Value-at-Risk (VaR) threshold. It provides a more comprehensive measure of downside risk. Mathematically, CVaR is given by:

$$\text{CVaR}_{\beta}(W) = \min_{\alpha \in \mathbb{R}} F_{\beta}(W, \alpha) \tag{28}$$

where

$$F_{\beta}(W, \alpha) = \alpha + \frac{1}{1-\beta} \int_{r \in R^m} \max\left[f(W, r) - \alpha, 0 \right] p(r) \mathrm{d}r \tag{29}$$

This model ensures that portfolio optimization is conducted with a focus on minimizing potential extreme losses.

2.5.2. Mean-Variance Analysis

Under mean-variance analysis, investors seek to minimize risk given a target level of return. The optimization problem is formulated as:

$$\min_{w} w^{\mathrm{T}} \Sigma w \tag{30}$$

subject to:

$$w^{\mathrm{T}} \mu \geq \mu_0, \ \ w^{\mathrm{T}} w = 1, \ \ w_j \geq 0, \ \ \forall j \in 1, 2, \cdots, d \tag{31}$$

where μ_0 is the target return, Σ is the covariance matrix and w represents portfolio weights. The efficient frontier is derived from this formulation.

2.5.3. Global Minimum Variance Portfolio

The Global Minimum Variance (GMV) portfolio seeks to minimize risk without imposing return constraints. The optimization problem is formulated as:

$$\min_{W} W^{\mathrm{T}} \Sigma W \tag{32}$$

subject to:

$$W^{\mathrm{T}} 1 = 1, \ \ w_j \geq 0, \ \ \forall j \in 1, 2, \cdots, d \tag{33}$$

This approach is preferred over the mean-variance strategy due to its robustness in minimizing portfolio risk.

2.5.4. Sharpe Ratio

The Sharpe ratio, introduced by Sharpe (1994), measures the risk-adjusted return of a portfolio. It is defined as:

$$\max_{W} \frac{W^{\mathrm{T}} r}{\sqrt{W^{\mathrm{T}} \Sigma W}} \tag{34}$$

subject to:

$$W^{\mathrm{T}} 1 = 1, \ \ w_j \geq 0, \ \ \forall j \in 1, 2, \cdots, d \tag{35}$$

A higher Sharpe ratio indicates a more efficient portfolio with better risk-adjusted performance.

The goal of portfolio optimization is to balance return and risk by choosing how much to invest in each asset. Traditional models like Mean-Variance Optimization (Markowitz, 1952) focus on minimizing risk (variance) for a given expected return.

However, variance penalizes upside and downside equally. In practice, investors care more about downside risk, which is where more advanced methods like CVaR come in.

Financial risk is multifaceted—no single model can capture all its aspects effectively. This next section outlines an integrated approach that combines volatility modeling (GARCH), extreme risk estimation (EVT), dependency modeling (Regular Vine Copulas) and risk optimization (CVaR). By linking these components, the framework aims to produce a more comprehensive and realistic method for portfolio optimization, especially under uncertain and turbulent market conditions.

2.6. The GARCH-EVT-Copula-CVaR Model

This section illustrates the steps to follow in solving the portfolio optimization problem under the constraint within the GARCH-EVT-Copula-CVaR framework. The process begins with estimating the parameters of the GARCH-EVT-Copula model. Once the model is fitted, it is employed to generate simulated daily return scenarios for the assets in the portfolio. These simulated returns are then used as inputs in the optimization process, where portfolio weights are determined by minimizing Conditional Value at Risk (CVaR) while targeting a specified level of expected return.

Volatility Modelling Using GARCH-Type Specification

Consider the following model for daily log-returns $r_{i,t}$ for asset $i = 1, \cdots, d$ at day $t = 1, \cdots, n$:

$$r_{i,t} = \mu_i + \epsilon_{i,t}. \tag{36}$$

Here, μ_i is the mean component, and $\epsilon_{i,t}$ represents the error term following a GARCH(1,1) process. The GARCH(1,1) model is specified as follows:

$$\epsilon_{i,t} = \sigma_{i,t} z_{i,t}, \quad z_{i,t} \sim \text{i.i.d.} \tag{37}$$

$$\sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1}\epsilon_{i,t-1}^2 + \beta_{i,1}\sigma_{i,t-1}^2. \tag{38}$$

where $\{z_{i,t}\}$ is a sequence of i.i.d. white noise with mean 0 and variance 1. In the context of this thesis, $z_{i,t}$ is distributed according to Generalized Error Distribution (GED), Student's *t*-distribution, or skew Student's *t*-distribution, depending on the dataset.

Extreme Value Theory (EVT) for Tail Risk Estimation

Since financial return distributions often exhibit heavy tails, EVT is applied to model extreme losses. The Peak Over Threshold (POT) approach is used to estimate the Generalized Pareto Distribution (GPD) of excess losses:

$$G(y) = 1 - \left(1 + \xi \frac{y - u}{\sigma}\right)^{-1/\xi}, \tag{39}$$

where u is the selected threshold, σ is the scale parameter, and ξ is the shape parameter. The EVT approach improves risk management by focusing on extreme events that standard models may overlook.

Dependence modelling using Regular Vine Copulas

To capture the complex dependency structure among asset returns, Regular Vine (R-Vine) copulas are employed. R-Vine copulas allow for more flexible pairwise dependence structures compared to traditional Gaussian or Student-t copulas. The joint distribution function is decomposed into a set of bivariate copulas structured in a tree representation.

Minimizing CVaR

Conditional Value-at-Risk (CVaR) is used as the risk measure to minimize potential losses beyond Value-at-Risk (VaR):

$$\text{CVaR}_\alpha = E\left[R \mid R \le \text{VaR}_\alpha\right]. \tag{40}$$

The portfolio optimization problem is formulated as:

$$\min_w \text{CVaR}_\alpha(w) \quad \text{subject to } \sum w_i = 1,\ w_i \ge 0. \tag{41}$$

where w represents asset weights.

3. Empirical Results

3.1. Data Description

The dataset comprises a portfolio of nine financial assets, which includes four stock market indices: S&P 500, FTSE100, N225, and CAC 40. It also features three currency exchange rates (GBP-USD, EUR-USD, JPY-USD) and two commodities (Gold and WTI Oil). This data spans from January 1, 2006, to May 1, 2023, totalling 4254 daily observations, while excluding weekends and public holidays. The dataset for this study was constructed to reflect a global, cross-asset investment environment, comprising stock indices, currency exchange rates and commodities. These asset classes were selected for their exposure to broad macroeconomic, geopolitical and financial risks, which aligns with our objective of developing a portfolio optimization model capable of performing under diverse and volatile market conditions. The period from 2006 to 2023 includes key global events such as the Global Financial Crisis, the European debt crisis, the COVID-19 pandemic and geopolitical shocks like Brexit and the Russia-Ukraine conflict. These market conditions provide a rich context to assess the model's robustness in capturing volatility clustering, tail risk and dynamic dependencies across asset types. The data was sourced from https://www.investing.com/news/.

The selected period includes significant financial events, such as the Global Financial Crisis from 2007 to 2009, the Asian Crisis, and the COVID-19 pandemic in 2020-2021. These events introduced high volatility and triggered structural shifts in global financial markets, making the dataset ideal for analyzing risk dynamics and dependence structures.

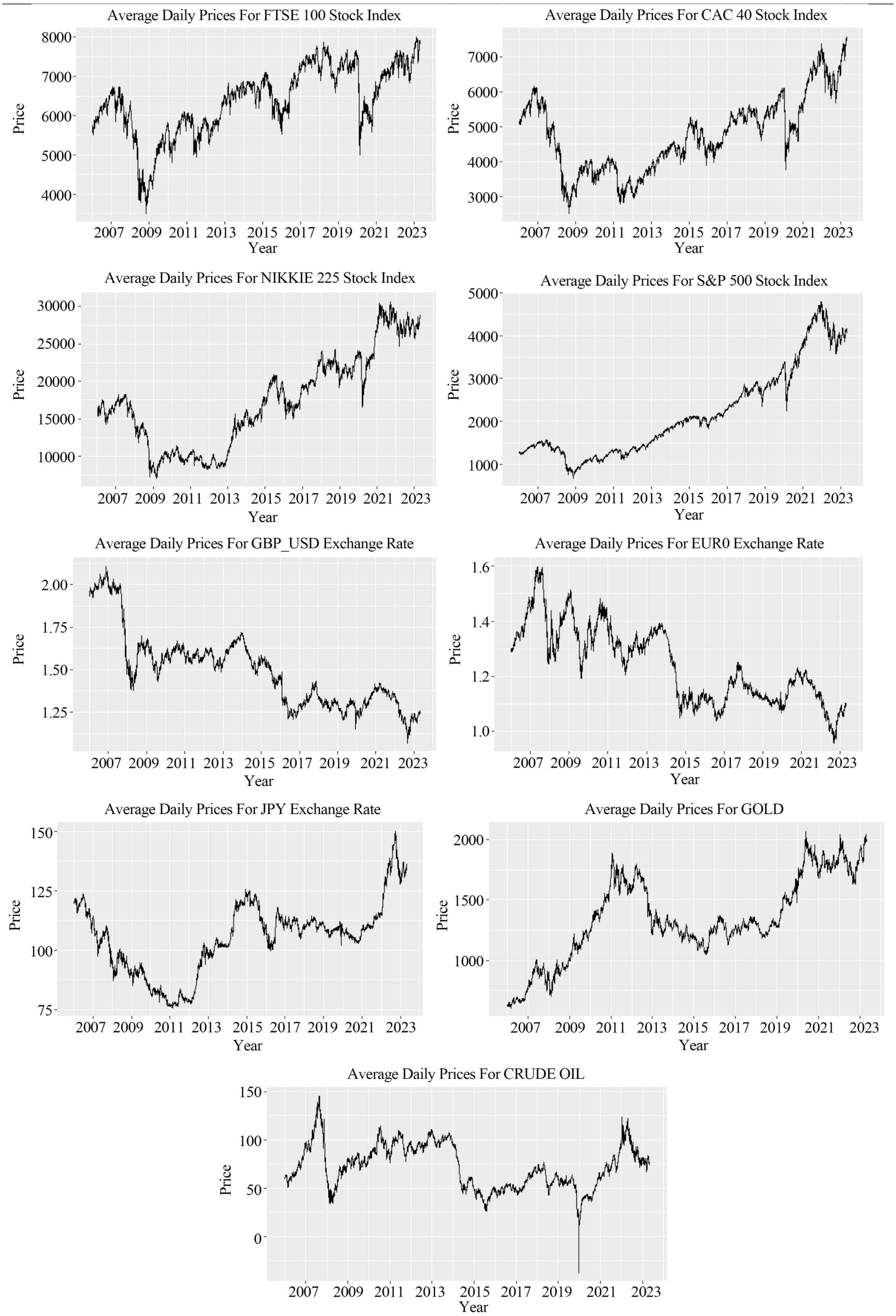

Figure 1. Time series plot for the daily market indices of the four securities markets, three currency exchange rates and two commodities for the period starting from 1st January 2006 to 1st May 2023.

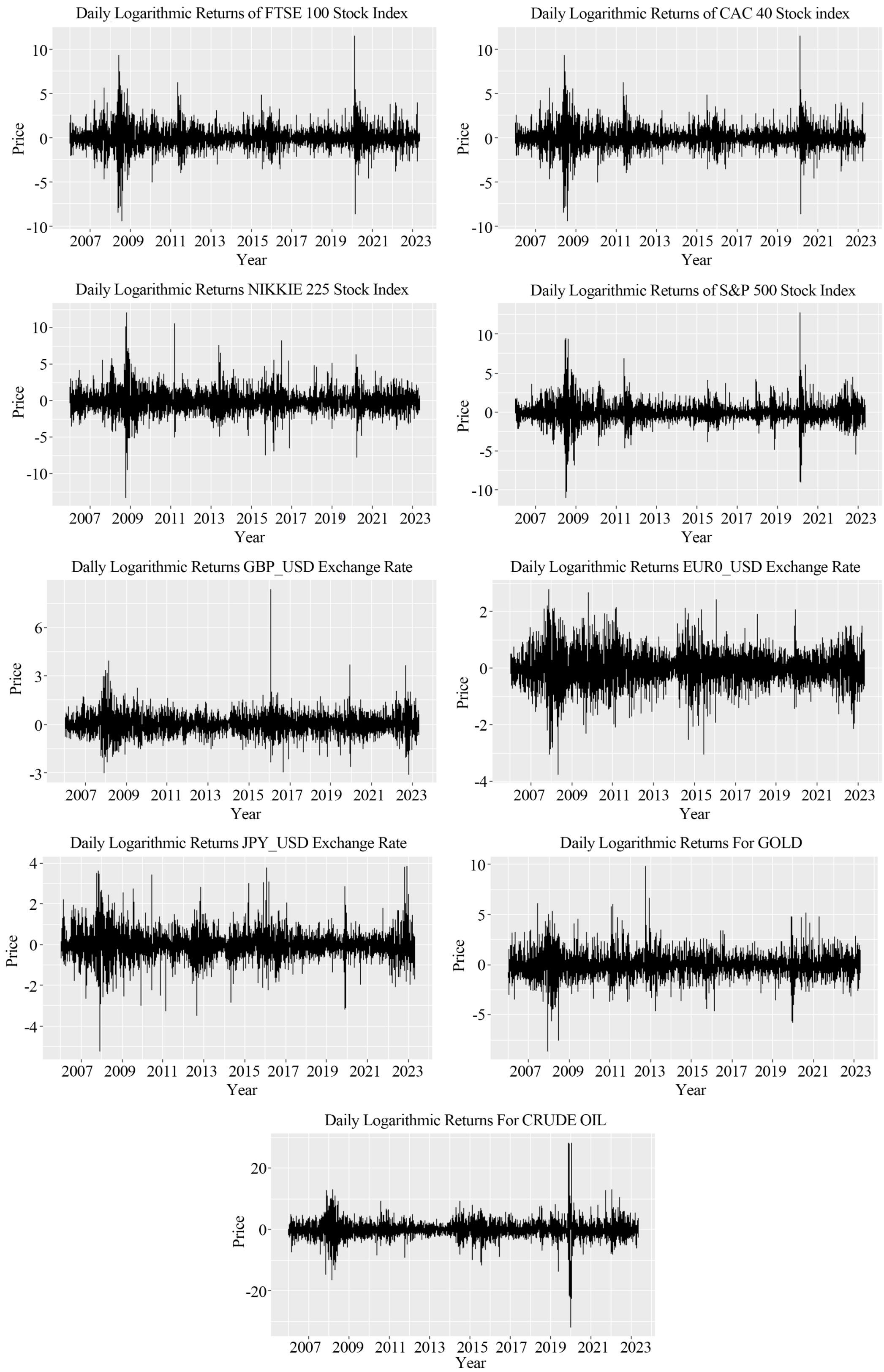

Figure 2. Continuously compounded returns series plot for the daily market indices of the four securities markets, three currency exchange rates and two commodities for the period starting from 1st January 2006 to 1st May 2023.

Figure 1 displays time series plots of the selected financial assets, offering a visual representation of their movements during the study period. The stock indices show an overall increasing trend, with notable declines observed during the 2008 Global Financial Crisis and the COVID-19 pandemic. The S&P 500, FTSE 100, NIKKEI 225, and CAC 40 all follow similar patterns, demonstrating recovery periods after each downturn. In contrast, the currency exchange rates for GBP/USD, EUR/USD, and JPY/USD show fluctuations influenced by macroeconomic events, monetary policies, and geopolitical factors. Particularly, the GBP/USD exchange rate experienced significant depreciation following the Brexit referendum in 2016. The commodity prices for gold and crude oil showcase distinct behaviours. Gold serves as a safe-haven asset during periods of market uncertainty, while crude oil prices exhibit sharp declines, especially in 2008 and 2020, reflecting the impacts of economic downturns and imbalances in supply and demand.

The daily average closing prices of the financial assets were transformed into continuously compounded returns using the formula; $r_{i,t} = \log\left(p_{i,t}/p_{i,t-1}\right)$, where $p_{i,t}$ represents the price of asset i at time t. **Table 1** provides the summary descriptive statistics for the portfolio log return series, along with the results of selected statistical tests. The mean returns for most assets are close to zero or slightly negative, indicating that the data is primarily influenced by risk rather than consistent upward trends. The standard deviations confirm significant volatility across all assets, highlighting the presence of time-varying risk dynamics. The skewness values indicate that most assets exhibit positive skewness, with the exception of the EUR/USD exchange rate, which shows negative skewness. Additionally, the high kurtosis values indicate fat tails, meaning extreme returns occur more frequently than in a normal distribution-reinforcing the need for EVT in tail modeling. **Figure 2** illustrates the continuously compounded returns of the selected assets, highlighting the volatility clustering observed throughout the dataset. Periods of market distress, such as the 2008 financial crisis and the COVID-19 pandemic, are characterized by increased fluctuations in asset returns. The return distributions are non-Gaussian, reinforcing the necessity for advanced models that effectively capture tail risk and extreme events.

Table 1. Descriptive summary statistics and statistical tests for the daily market indices of the four securities markets, three currency exchange rates and two commodities for the period starting from 1st January 2006 to 1st May 2023.

Statistics	S&P500	FTSE100	CAC40	N225	GBP-USD	EURO-USD	JPY-USD	GOLD	WTI
Complete dataset									
Obs.	4253	4253	4253	4253	4253	4253	4253	4253	4253
Max.	12.7652	11.5124	13.0983	12.1110	8.3965	3.1252	3.8578	9.8105	56.6017
Min.	−10.9572	−9.3842	−10.5945	−13.2346	−3.2790	−3.7740	−5.2157	−8.5890	−31.9634
Mean	−0.0275	−0.0073	−0.0089	−0.0133	0.0101	0.0039	−0.0031	−0.0276	−0.0061
Std.Dev	1.2814	1.1792	1.4045	1.4680	0.6709	0.6604	0.6390	1.1105	2.8480
Skew	0.51790	0.4112	0.2529	0.4264	0.7088	−0.0592	0.1430	0.2614	1.7248
Kurt	12.0164	9.6534	7.9375	7.5384	7.8362	1.4786	5.1056	5.8027	51.9396

Continued

Statistical tests									
JB	25808.0832	16654.3693	11224.733	10212.5377	22721.9593	1114.3575	4640.7676	6023.8685	6052.6071
p-value	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)
ADF p-value	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100
LBQ (*10*)	109.36	46.922	36.45	15.865	23.753	9.679	20.071	19.607	31.244
p-value	(*0.000*)	(*0.000*)	(*0.000*)	(*0.1036*)	(*0.0083*)	(*0.4691*)	(*0.0286*)	(*0.0332*)	(*0.0005*)
LBQ (*20*)	171.28	68.752	52.191	23.493	35.63	16.159	44.597	41.49	58.778
p-value	(*0.0000*)	(*0.0000*)	(*0.0001*)	(*0.02653*)	(*0.0017*)	(*0.7057*)	(*0.0000*)	(*0.0032*)	(*0.0000*)
ARCH-LM	4419.985	2425.873	1789.372	3308.566	377.0401	974.4848	722.0357	467.0275	570.5848
p-value	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)	(*0.0000*)

Table 2 presents the correlation coefficients between asset returns, offering insights into their interdependencies. Stock indices exhibit strong positive correlations, particularly among the S&P 500, FTSE 100, NIKKEI 225, and CAC 40, indicating that global equity markets tend to move in tandem. The exchange rates demonstrate weaker correlations with stock indices; however, the EUR/USD and GBP/USD pairs show some degree of co-movement. Crude oil prices exhibit moderate correlations with stock indices, reflecting their sensitivity to global economic conditions. In contrast, gold demonstrates weak or negative correlations with most assets, reinforcing its role as a hedge during market downturns.

Table 2. The correlation coefficients for each pair of stock market indices, currency exchange rates and commodities from 2006 to 2023.

	S&P500	FTSE100	CAC40	N225	GBP-USD	EUR-USD	JPY-USD	GOLD	WTI
Complete dataset									
SP500		0.7317	0.8186	**0.9293**	−0.6637	−0.7004	0.6372	0.6327	−0.2053
FTSE100			0.7601	0.7336	−0.4634	−0.6102	0.6570	0.2980	−0.1551
CAC40				0.8835	−0.2760	−0.4603	0.7617	0.2214	−0.1318
N225					−0.5252	−0.6427	0.7572	0.3733	−0.3014
GBP-USD						0.8327	−0.2863	−0.6203	0.4394
EUR-USD							−0.5977	−0.4535	0.5616
JPY-USD								**−0.0102**	−0.3008
GOLD									0.0968

Results from the Jarque-Bera test show that all return series significantly differ from a normal distribution, highlighting the need for using models that account for non-Gaussian behaviour. The Augmented Dickey-Fuller (ADF) test suggests that each return series is stationary, making them appropriate for time-series analysis. Furthermore, the Ljung-Box Q test reveals evidence of autocorrelation in the squared returns, indicating volatility clustering. This justifies the use of GARCH models, which are well-suited for modelling time-varying volatility in financial data.

Overall, the data analysis highlights the complex nature of financial markets, characterized by extreme events, volatility clustering and intricate dependence structures. These findings underscore the importance of employing advanced risk modelling techniques, such as the GARCH-EVT-Copula-CVaR framework, to accurately assess and optimize portfolio risk.

3.2. Parameter Estimates for ARMA-GARCH Models

The initial step in modelling financial returns involves fitting an appropriate autoregressive moving average (ARMA) model to capture the conditional mean dynamics. To determine the best-fitting ARMA(p, q) model for each asset, both the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are used. The model with the lowest values of AIC and BIC is selected as the most appropriate specification. Table 3 presents the AIC and BIC values for different ARMA model specifications applied to the log returns of stock indices, exchange rates and commodities. The results indicate variations in the best-fitting ARMA models across different assets. The CAC40 and Gold return series exhibit characteristics best modelled by an ARMA(0,0) process, meaning that the returns are essentially white noise. The S&P 500, EUR-USD and JPY-USD are optimally modelled using an ARMA(0,1) specification, which implies that their returns follow a moving average process with one lag. The FTSE 100 and NIKKEI 225 indices follow an ARMA(1,1) structure, indicating that both past returns and past shocks influence current returns. The GBP-USD exchange rate is best captured by an ARMA(2,1) process, suggesting a more complex dependency structure. Finally, the WTI crude oil returns are best described by an ARMA(2,0) model. For consistency in the subsequent volatility modelling process, an ARMA(1,1) model is assumed across all assets when fitting the GARCH models. This choice ensures comparability across asset classes while maintaining an appropriate balance between model complexity and predictive performance.

Table 3. The ARMA(p,q) model specification with the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values.

Model	Value	S&P500	FTSE100	CAC40	N225	GBP-USD	EURO-USD	JPY-USD	GOLD	WTI
ARMA(0,0)	*AIC*	14181.88	13474.66	**14952.04**	15337.74	8677.63	8543.28	8263.54	**12964.13**	20550.65
	BIC	14194.60	13487.37	**14974.75**	15350.45	8690.34	8555.99	8276.25	**12976.84**	20563.37
ARMA(1,0)	*AIC*	14113.99	13472.19	14960.15	15334.17	8670.41	8496.33	8256.85	12966.11	20552.43
	BIC	14133.05	13491.26	14979.21	15353.24	8689.48	8515.40	8275.91	12985.18	20571.49
ARMA(0,1)	*AIC*	**14113**.45	13471.8	14960.01	15334.36	8670.2	**8494.85**	**8256.44**	12966.11	20552.41
	BIC	**14132.52**	13490.87	14979.08	15353.43	8689.27	**8513.92**	**8275.50**	12985.18	20571.48
ARMA(1,1)	*AIC*	14115.34	**13465.56**	14958.02	**15332.32**	8668.29	8496.23	8256.24	12968.11	20552.10
	BIC	14140.76	**13490.98**	14983.44	**15357.74**	8693.72	8521.65	8281.67	12993.53	20577.55

Continued

ARMA(1,2)	*AIC*	14117.37	13466.62	14959.83	15334.31	8669.04	8498.01	8258.44	12968.81	20550.40
	BIC	14149.15	13498.39	14991.61	15366.09	8700.81	8529.78	8290.21	13000.59	20582.17
ARMA(2,0)	*AIC*	14115.23	13467.57	14961.06	15334.7	8671.85	8496.89	8256.45	12966.84	**20549.44**
	BIC	14140.66	13492.99	14986.49	15360.12	8697.27	8522.31	8281.87	12992.26	**20574.87**
ARMA(0,2)	*AIC*	14115.33	13467.20	14960.86	15335.02	8671.69	8496.39	8256.42	12966.81	20549.62
	BIC	14140.75	13492.62	14986.28	15360.44	8697.11	8521.81	8281.84	12992.23	20575.04
ARMA(2,1)	*AIC*	14116.26	13466.57	14960.37	15334.31	**8669.00**	8497.88	8258.47	12968.84	20550.00
	BIC	14143.04	13498.35	14992.15	15366.09	**8700.77**	8529.66	8290.24	13000.62	20581.78
ARMA(2,2)	*AIC*	14117.86	13468.53	14955.07	15334.09	8669.98	8499.2	8259.04	12961.94	20549.92
	BIC	14155.99	13506.66	14993.20	15372.22	8708.11	8537.33	8297.17	13000.08	20588.05

The smallest AIC and BIC values are presented in bold.

Once the conditional mean model is selected, a EGARCH(1,1) model is applied to the residuals of the ARMA process to capture time-varying volatility. Table 4 presents the estimated parameters of the ARMA-EGARCH models for each asset. The key parameters of interest include α_1, which measures the impact of past squared residuals (ARCH effect), and β_1, which captures the persistence of volatility shocks (GARCH effect). The sum $\alpha_1 + \beta_1$ indicates the degree of volatility persistence.

Table 4. Parameter estimate values and residual diagnostic tests results of the fitted optimal ARMA(1,1)-EGARCH(1,1) with Student's *t* innovations distribution.

Par.est	S&P500	FTSE100	CAC40	N225	GBP-USD	EURO-USD	JPY-USD	GOLD	WTI
Mean component parameter estimates									
μ	−0.1202	−0.0632	−0.1085	−0.1029	−0.0068	0.0000	−0.0141	−0.0458	−0.1151
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.9996	(0.0442)	(0.0002)	(0.0000)
ϕ_1		0.9241		−0.7126	0.8004				−0.0156
		(0.0000)		(0.0000)	(0.0000)				(0.0490)
ϕ_2					0.0912				−0.0049
					(0.0000)				(0.0482)
Volatility component parameter estimates									
ω	0.0061	0.0032	0.0053	0.0106	−0.0032	0.0100	−0.0167	0.00164	0.0164
	(0.0275)	(0.1788)	(0.0217)	(0.0001)	(0.0012)	(0.0000)	(0.0008)	(0.0001)	(0.0000)
α_1	−0.1851	−0.1457	−0.1608	−0.1204	−0.0377	−0.1008	−0.0263	−0.0126	−0.0824
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0023)	(0.0249)	(0.1584)	(0.0000)
β_1	0.9930	0.9917	0.9960	0.9864	0.9915	1.000	0.9849	0.9928	0.9903
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
γ	0.2024	0.1463	0.1390	0.1766	0.0671	0.3651	0.1798	0.0925	0.1467
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Continued

Shape	7.4814	7.7440	7.0291	8.4157	8.5756	2.1000	5.4591	4.2291	7.9018
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Residual diagnostic tests results									
LBQ(10)	15.746	3.5233	8.8038	3.1776	4.8661	43.219	7.4419	8.6771	6.5659
p-value	(0.0461)	(0.8974)	(0.3591)	(0.9227)	(0.7718)	(0.00000	(0.4898)	(0.3703)	(0.5841)
LBQ(20)	28.361	10.7690	13.892	14.2370	14.0340	49.0870	22.6260	19.4700	13.2310
p-value	(0.0567)	(0.9039)	(0.7361)	(0.7135)	(0.7269)	(0.0001)	(0.2054)	(0.3634)	(0.7777)
$LBQ^2(10)$	8.1536	4.7867	6.7361	5.2569	49.262	39.4820	4.7537	24.6750	27.1700
p-value	(0.4186)	(0.7801)	(0.5654)	(0.7298)	(0.0000)	(0.0000)	(0.7835)	(0.0017)	(0.0007)
$LBQ^2(20)$	13.284	14.107	11.1770	24.8440	53.856	133.8600	8.9557	26.8220	30.1100
p-value	(0.7744)	(0.7221)	(0.8867)	(0.1293)	(0.0000)	(0.0000)	(0.9608)	(0.0824)	(0.0363)
ARCH-LM	1.3141	0.8705	0.2530	0.3280	0.73602	37.2838	0.2749	11.9005	5.4219
p-value	(0.9994)	(0.9999)	(0.9999)	(0.9999)	(0.9999)	(0.0000)	(0.9999)	(0.2917)	(0.8612)

The results in Table 4 confirm that the volatility persistence is high across all assets, with $\alpha_1 + \beta_1$ values approaching one. This implies that volatility shocks diminish gradually over time, a typical feature of financial return series. The estimated parameters of the GARCH model are statistically significant, supporting its suitability for modelling asset return volatility.

The choice of error distribution is critical when modelling financial returns due to the presence of fat tails. The models are estimated under different distributions, including the Student's *t*-distribution, the skewed Student's *t*-distribution, and the Generalized Error Distribution (GED). The results show that non-Gaussian distributions provide a better fit, as indicated by lower AIC and BIC values. The estimated shape parameters of these distributions confirm that the return series exhibit heavy tails, highlighting the necessity of using a model that accounts for extreme events.

Figure 3 illustrates the standardized residuals of the fitted ARMA(1,1)-EGARCH(1,1) model with Student's t-distributed innovations. The plot highlights periods of heightened volatility, particularly during financial crises, such as the 2008 Global Financial Crisis and the 2020 COVID-19 pandemic. The residuals are clustered, indicating volatility clustering, a feature effectively captured by the GARCH framework.

Overall, the ARMA-GARCH modelling framework effectively captures both the conditional mean and volatility structure of financial asset returns. These results provide a strong foundation for the subsequent modelling steps, including Extreme Value Theory for tail risk estimation and copula models for dependence structure analysis.

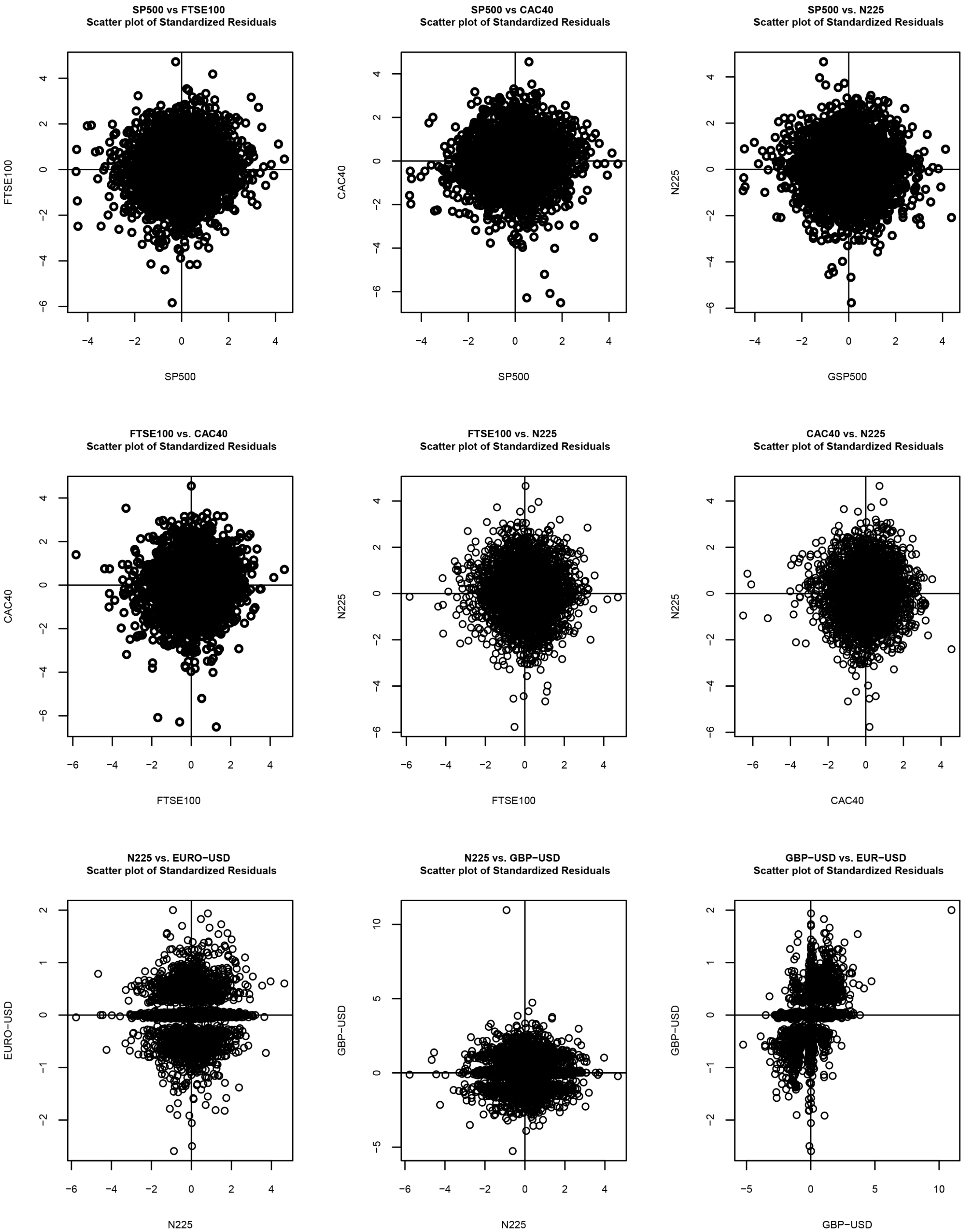

Figure 3. The standardized residuals of the fitted ARMA(1,1)-EGARCH(1,1) with Student's *t* innovations distribution for the portfolio assets for the entire period.

3.3. Parameter Estimates for ARMA-GARCH-EVT Models

To improve the accuracy of copula modelling and estimation, it is essential to correctly fit the standardised residuals. A parametric ARMA-GARCH-EVT model is constructed to model the standardised residuals obtained from the previous stage. This model leverages the advantages of both estimation methods: the GARCH model effectively captures heteroscedasticity, while the generalised Pareto distribution (GPD) provides an accurate representation of the tail distributions. The estimation process follows a two-step procedure. First, a GARCH model is fitted to the return series to remove volatility clustering and obtain standardised residuals. Then, EVT is applied to these residuals to model the extreme tails. Specifically, the Peaks Over Threshold (POT) approach is used to fit the Generalised Pareto Distribution (GPD) to the upper and lower tails of the standardised residuals. The selection of the threshold is based on the 90th and 10th quantiles, following the approach suggested by McNeil *et al.* (2005), to ensure a sufficient number of extreme observations while maintaining statistical reliability.

Table 5 presents the parameter estimates of the ARMA(1,1)-EGARCH(1,1)-EVT model. The table includes estimates for the tail index ξ and scale parameter β for both the upper and lower tails. The threshold value u used for each tail is also reported. The results indicate that the estimated tail indices are positive for all assets, confirming the presence of heavy-tailed distributions. A higher tail index reflects a greater degree of tail risk, which is particularly evident in the currency exchange rates and crude oil prices, where extreme events are more frequent. The scale parameter is also positive across all return series, reinforcing the heavy-tailed nature of financial asset returns and supports using EVT for tail risk modeling.

Table 5. Parameter estimates of the fitted ARMA-EGARCH(1,1)-EVT model with Student's *t* innovations distribution for stock market indices, currency exchange rates and commodities for the entire period.

Parameters	S&P500	FTSE100	CAC40	N225	GBP-USD	EUR-USD	JPY-USD	GOLD	WTI
Complete dataset									
Upper Tail									
No. of Obs.	426	426	425	426	426	425	426	426	426
ϵ	0.0007	−0.0577	0.0529	−0.0444	0.0336	0.1204	0.0575	0.0267	0.0515
β	0.4133	0.5054	0.4433	0.4851	0.4051	0.1920	0.5854	0.5739	0.4557
u	1.126	1.1462	1.1263	1.1890	1.2598	0.5678	1.1214	1.0797	1.1558
Lower Tail									
No. of Obs.	3828	3829	3829	3829	3829	3828	3828	3829	3829
ϵ	0.0872	0.0326	0.0694	0.0342	0.2498	0.1925	0.0860	0.0983	0.1034
β	0.6715	0.6455	0.6591	0.6079	0.3967	0.1960	0.5865	0.6343	0.5727
u	−1.2826	−1.2768	−1.2468	−1.2991	−1.2802	−0.5704	−1.1828	−1.1625	−1.2818
Statistical tests									
KS	0.0149	0.0112	0.0119	0.0107	0.1180	0.1648	0.0124	0.0143	0.0102
CvM	0.0663	0.0371	0.0204	0.0434	10.7702	30.5742	0.0610	0.1093	0.04020

Figure 4 provides a semi-parametric cumulative distribution function (CDF) for the standardized residuals. The figure compares the empirical kernel density estimates for the central distribution with the GPD estimates for the lower and upper tails. The visual inspection confirms that EVT adequately models the extreme returns, as the GPD fits well to the empirical tail behavior.

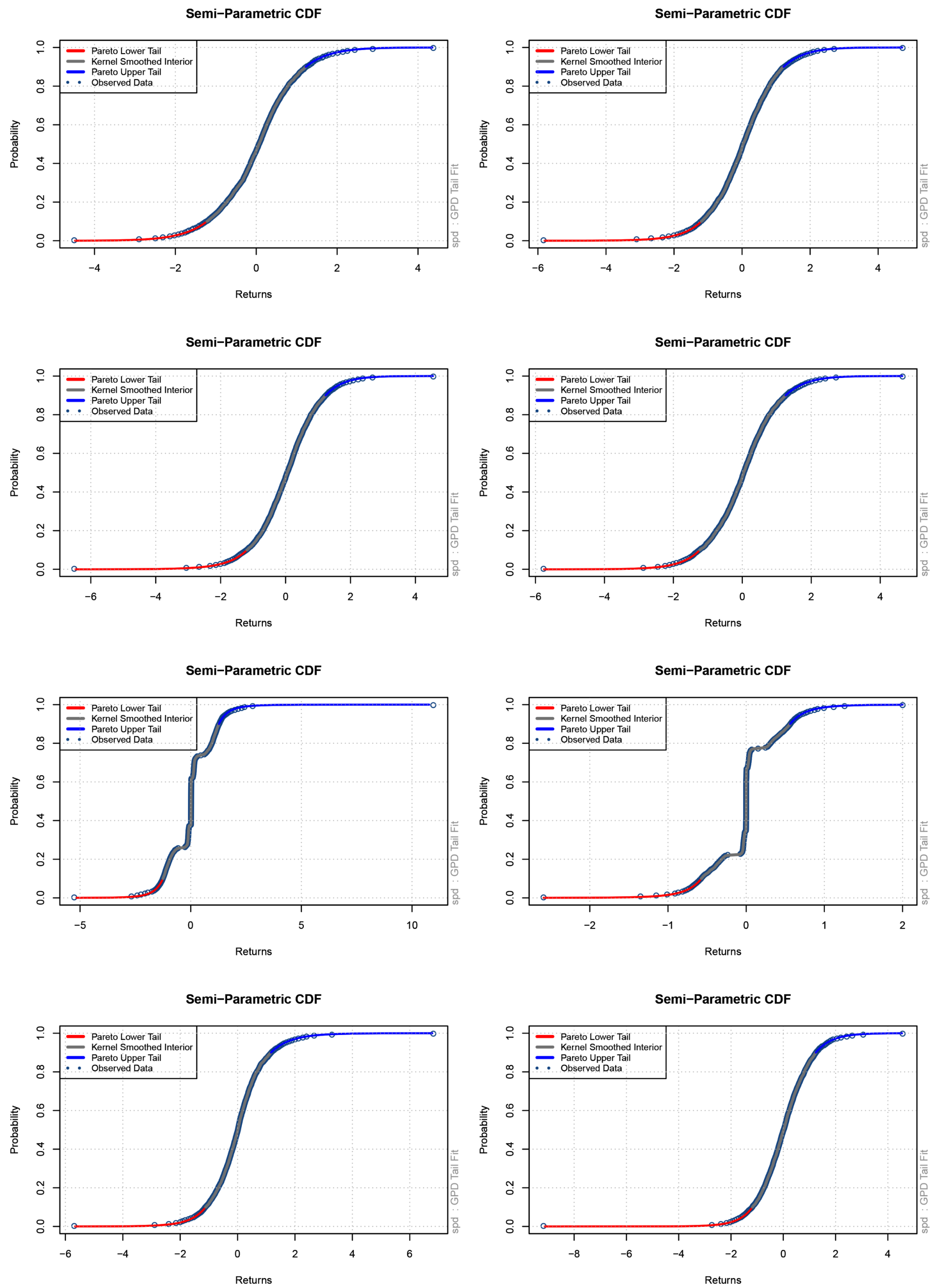

Figure 4. The empirical semi-parametric cumulative distribution function for the portfolio assets.

In order to model the dependence structure among stock indices, exchange rates, and commodity returns, the standardized residuals obtained from the fitted ARMA-GARCH-EVT model are transformed into the unit interval $[0,1]$ using the Probability Integral Transformation (PIT). This transformation is a necessary step before copula estimation, ensuring that the marginal distributions conform to a uniform scale. The transformed residuals serve as inputs for the copula modelling framework, which will be used in the subsequent section to estimate dependence structures using regular vine copulas.

The ARMA-GARCH-EVT model effectively captures the heteroscedasticity and tail risk present in financial markets. The parameter estimates confirm the presence of heavy-tailed return distributions, justifying the need for extreme value modelling. The results from this section provide a robust foundation for copula estimation, which will enable a detailed analysis of dependency structures across asset classes.

3.4. Dependence Results using Regular Vine Copulas

This section presents the empirical results for the fitted dependence models, focusing on the Regular Vine (R-vine) copula specifications. The copula models employed include R-vine, C-vine, and D-vine copulas. These models offer a flexible approach to capturing complex dependencies between financial assets, going beyond simple linear correlations. The sequential estimation procedure proposed by Dissmann *et al.* (2013) is used, allowing for an efficient decomposition of the regular vine copula structure into multiple trees. This stepwise process ensures computational efficiency and improved accuracy in modelling interdependencies.

Table 6. The empirical Kendall's τ correlation coefficient values and the total over each row of the stock market indices, currency exchange rates and commodities copula data for the entire period.

Assets	S&P500	FTSE100	CAC 40	N225	GBP-USD	EUR-USD	JPY-USD	GOLD	WTI	Sum
Complete dataset										
S&P500		0.0523	0.0309	−0.0032	−0.0006	−0.0040	0.0124	0.0114	−0.0014	0.0978
FTSE100			0.0343	−0.0102	0.0030	0.0037	−0.0013	−0.0052	0.0210	0.0453
CAC40				0.0142	0.0071	0.0188	0.0127	−0.0006	−0.0046	0.0476
N225					0.0123	−0.0027	−0.0218	0.0043	0.0057	−0.0022
GBP-USD						**0.3010**	−0.1104	0.0176	0.01895	**0.2272**
EUR-USD							−0.1568	0.0161	0.0131	−0.1276
JPY-USD								−0.0267	−0.0027	−0.0294
GOLD									0.0064	0.0064

Table 6 summarizes the empirical Kendall's τ values for all asset pairs. The strongest dependence is observed between the GBP-USD and EUR-USD currency pairs, with a Kendall's τ value of 0.3010. This suggests a relatively strong positive relationship, likely driven by the economic ties between the UK and the Eu-

rozone. Stock indices like NIKKEI 225 and CAC40, however, show modest correlations, reflecting the influence of regional economic factors.

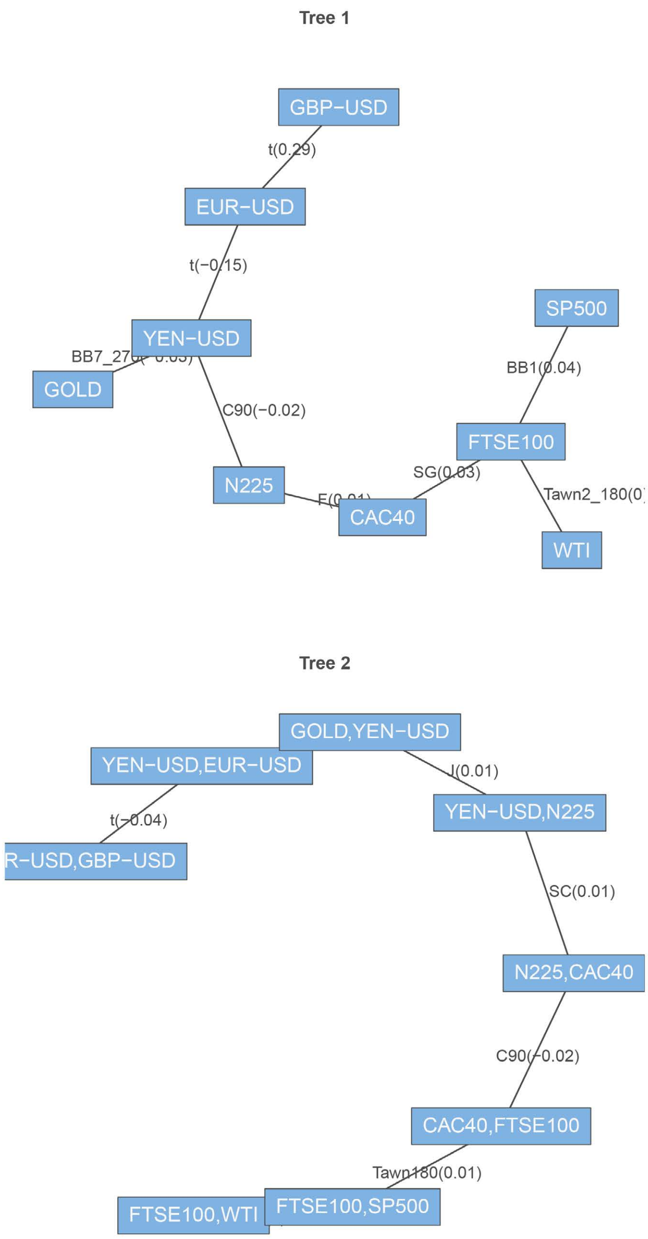

Figure 5. First and second tree of the R-vine structure V_1.

To further explore the dependency structure, the R-vine, C-vine, and D-vine copula models are fitted using a stepwise tree-wise selection procedure. Each tree is constructed to maximize the sum of absolute Kendall's τ values for variable pairs, allowing for an adaptive representation of both symmetric and asymmetric dependencies.

Figure 5 illustrates the R-vine copula, which provides the most flexible framework for modelling dependencies. The first tree captures the strongest pairwise dependencies between assets, while subsequent trees model conditional dependencies. Unlike C-vines and D-vines, R-vines do not require a fixed root node or sequential path, allowing the model to freely adapt to complex dependency structures. The first tree highlights the significant dependency between GBP-USD and EUR-USD, while the second tree introduces conditional relationships, such as the dependence of S&P 500 and FTSE 100 conditioned on CAC40.

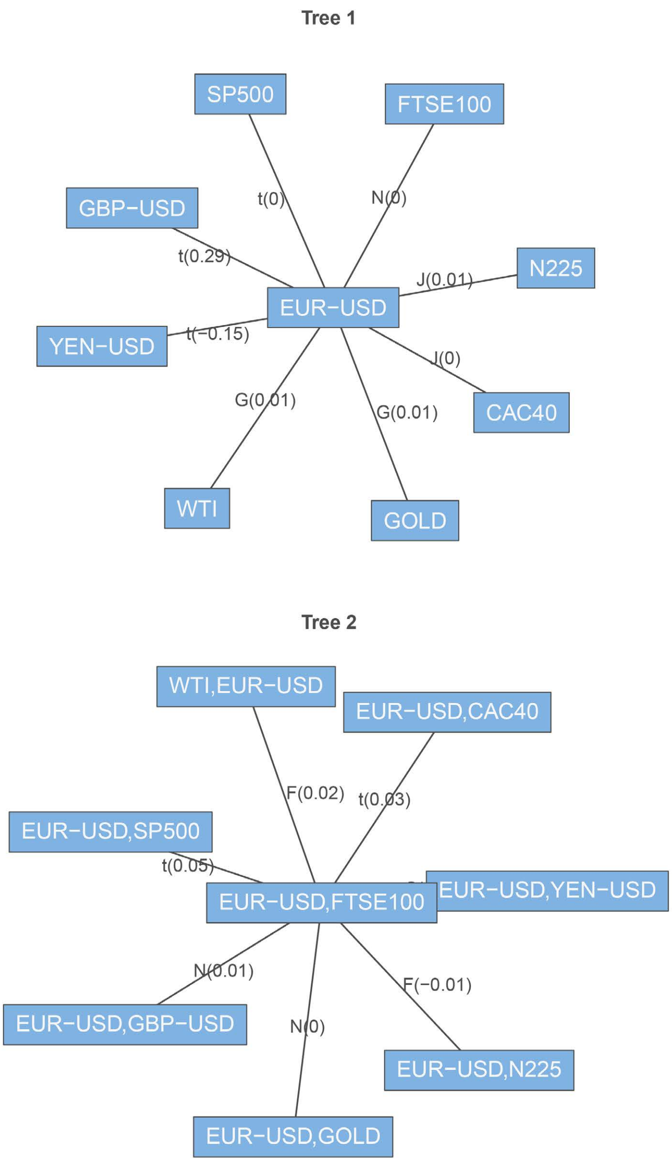

Figure 6. First and second tree of the C-vine structure V_1.

Figure 6 presents the C-vine copula structure, which organizes dependencies hierarchically with one central node at each tree level. The first tree identifies GBP-USD as the root node, as it has the strongest average dependence with other assets. This structure assumes that the relationship between any pair of assets can be explained through their connection to a central variable. The second tree continues this pattern by conditioning relationships on the central node, reflecting a star-like dependency pattern.

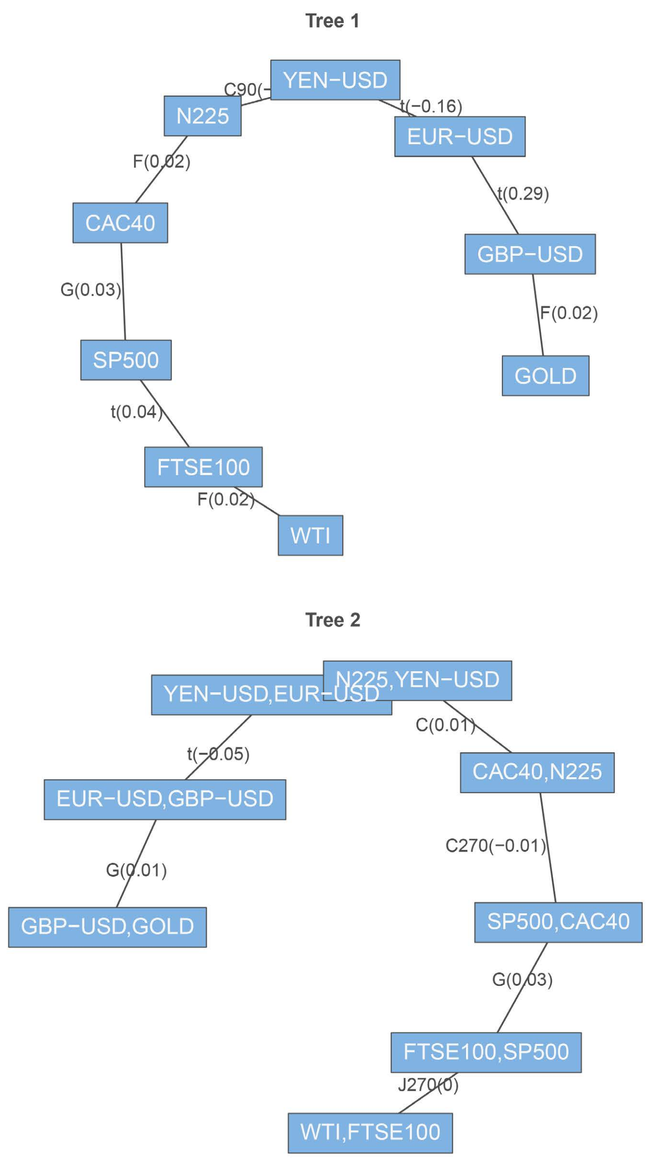

Figure 7. First and second tree of the D-vine structure V_1.

Figure 7 displays the D-vine copula structure, where assets are arranged sequentially, forming a path where dependencies flow through intermediary nodes. In the first tree, S&P 500 and FTSE 100 are directly linked, with subsequent pairs joined sequentially. This structure is particularly useful for modelling dependencies with a natural ordering or temporal relationship. The second tree adds conditional dependencies, providing further flexibility in capturing tail dependencies and indirect relationships.

The selection of the optimal vine structure and copula families is guided by model selection criteria, including the Akaike Information Criterion (AIC) and

Bayesian Information Criterion (BIC). The R-vine copula, due to its flexibility, often yields the best fit, though the C-vine and D-vine structures provide useful alternatives for capturing specific dependency patterns.

The fitted copula models specify bivariate copula families for each asset pair. These include the Gaussian copula for symmetric dependencies, the Clayton copula for capturing lower tail dependence, and the Gumbel copula for upper tail dependence. The choice of copula family depends on the nature of the observed dependencies, ensuring that the model can accurately reflect both the strength and direction of asset relationships.

In summary, the R-vine, C-vine, and D-vine copula models reveal a rich and complex dependence structure across asset classes. The R-vine structure provides the most flexible representation, while the C-vine and D-vine models offer alternative ways to capture hierarchical and sequential dependencies. These insights are crucial for understanding joint risk dynamics and will be integrated into the GARCH-EVT-Copula-CVaR framework in the next section to optimize portfolio risk management.

4. Portfolio Optimization Results

This section presents the results of portfolio optimization using the GARCH-EVT-Copula-CVaR framework. The copula models—Gaussian, Student-t, C-vine, D-vine and R-vine are employed to capture the dependencies between asset returns. The performance of portfolios under both mean-variance optimization and Conditional Value at Risk (CVaR) optimization strategies is analyzed, comparing equal weighting and optimal weighting approaches.

The optimization process involved 10,000 simulations of portfolio returns. This number was chosen based on empirical tests showing that increasing the number of simulations beyond this point produced minimal changes in the optimized weights, confirming the robustness of the results.

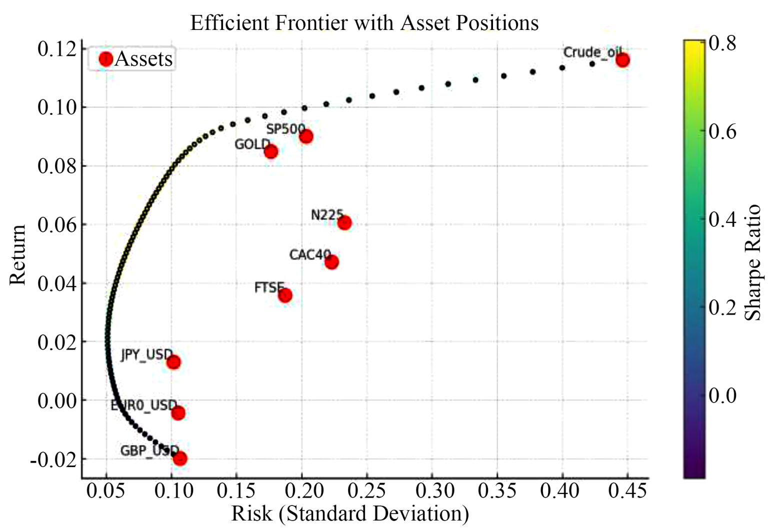

Figure 8. Efficient frontier.

Figure 8 illustrates the efficient frontier, which represents the set of optimal portfolios offering the highest expected return for a given level of risk. The assets, such as GBP-USD, JPY-USD and EUR-USD, are positioned towards the lower-left corner of the plot, reflecting their lower risk and minimal or negative returns. Stock indices, including FTSE100, CAC40 and N225, display higher risk levels and moderate returns, while WTI crude oil is placed at the extreme right, highlighting its higher risk and return profile.

The efficient frontier, shown as a blue curve, moves from low-risk, low-return portfolios to high-risk, high-return ones. Portfolios along this curve are considered optimal, achieving the best risk-return trade-off, while those below the frontier are inefficient. The shape of the frontier illustrates how diversification can push the curve outward, improving portfolio performance without a proportional increase in risk.

Table 7. Equal weighting mean variance optimization.

	GARCH-EVT-Gaussian copula	GARCH-EVT-Student-t copula	GARCH-EVT-C vine copula	GARCH-EVT-D vine copula	GARCH-EVT-R vine copula
P-ret	3.1424	3.4408	2.3599	3.8056	3.9731
P-volatility	5.4233	4.7990	4.8347	4.6626	4.8429
Sharpe Ratio	3.7943	4.1698	3.8812	5.1620	5.2041

The results of the mean-variance optimization are presented in Table 7 and Table 8. The equal weighting strategy assigns the same weight to each asset, while the optimal weighting approach seeks to maximize the portfolio's Sharpe ratio by adjusting asset allocations according to their expected returns and risk levels.

In the equal weighting approach, portfolio returns range from 2.3599% under the C-vine copula to 3.9731% with the R-vine copula. The R-vine copula yields the highest return, suggesting that it captures dependencies more effectively, particularly in periods of extreme market movements. Portfolio volatility ranges from 4.6626% with the D-vine copula to 5.4233% with the Gaussian copula. The Sharpe ratio, a measure of risk-adjusted return, spans from 4.8812 for the C-vine copula to 8.2041 for the R-vine copula. These results highlight that the R-vine copula not only provides higher expected returns but also delivers better risk-adjusted performance.

Table 8. Empirical results for the GARCH-EVT-Copula CVaR model with mean variance optimization.

Data	GARCH-EVT-Gaussian copula	GARCH-EVT-Student-t copula	GARCH-EVT-C vine copula	GARCH-EVT-D vine copula	GARCH-EVT-R vine copula
S&P500	6.0925%	7.84026%	7.8082%	7.2980%	7.5724%
FTSE100	8.8832%	7.43170%	8.2509%	8.9105%	7.2143%
CAC40	5.6411%	4.84528%	4.7749%	3.7334%	4.9407%
N225	6.2348%	5.18619%	4.6323%	4.8529%	4.5355%

Continued

GBP-USD	20.0208%	20.61955%	20.3963%	18.0511%	23.6189%
EUR-USD	26.2150%	26.15772%	25.1214%	29.3439%	22.8427%
JPY-USD	18.8281%	19.23123%	19.3567%	18.6632%	19.7699%
GOLD	7.1908%	7.73054%	8.8428%	8.3277%	8.4884%
WTI	0.8939%	0.95754%	0.8165%	0.8193%	1.0172%
P-ret	1.4872	1.8373	1.9535	1.1827	1.5599
P-volatility	3.3332	3.2755	3.1076	3.1508	3.2315
Sharpe ratio	4.46230	5.60877	6.2861	3.7534	4.8270

For the optimal weighting strategy, portfolio returns vary between 1.1827% (D-vine copula) and 1.9535% (C-vine copula). The C-vine copula achieves the highest return, while the R-vine copula produces a Sharpe ratio of 6.2861, the strongest risk-adjusted return among the copula models. The volatility estimates show that the C-vine copula maintains the lowest level of risk at 3.1076%, reinforcing its ability to construct stable portfolios.

Table 9. Empirical results for the GARCH-EVT-Copula CVaR model with equal weighting CvaR.

	GARCH-EVT-Gaussian copula	GARCH-EVT-Student-t copula	GARCH-EVT-C vine copula	GARCH-EVT-D vine copula	GARCH-EVT-R vine copula
P-ret	4.0687	2.9892	3.6751	2.2308	3.2964
P-volatility	5.2202	4.8375	4.5654	4.6748	4.7647
VaR	1.2454	1.1575	1.0710	1.0969	1.1223
CVaR	1.7464	1.4871	1.2638	1.3702	1.3316

The empirical results of the CVaR optimization are presented in **Table 9** and **Table 10**. Portfolio returns range from 2.2308% (D-vine copula) to 4.0687% (Gaussian copula) under equal weighting. The Gaussian copula delivers the highest expected return, though the C-vine copula minimizes risk with the lowest volatility at 4.5654%. For the optimal weighting CVaR strategy, portfolios are adjusted to minimize tail risk by focusing on Conditional Value at Risk rather than variance alone. Portfolio returns range from 1.6527% (R-vine copula) to 1.9627% (C-vine copula). The Gaussian copula shows the lowest volatility at 3.2336%, while the C-vine copula produces the best risk-adjusted returns with a Sharpe ratio of 6.2861. The Value at Risk (VaR) measuring potential losses at a 95% confidence level ranges from 0.7336 (Gaussian copula) to 0.7889 (R-vine copula). The Conditional Value at Risk (CVaR) values, which capture the expected losses beyond the VaR threshold, range from 0.8471 (Gaussian copula) to 0.9654 (D-vine copula). The Gaussian copula consistently shows the lowest CVaR, indicating better protection against extreme losses.

Table 10. Empirical results for the GARCH-EVT-Copula CVaR model portfolio optimization.

Data	GARCH-EVT-Gaussian copula	GARCH-EVT-Student-t copula	GARCH-EVT-C vine copula	GARCH-EVT-D vine copula	GARCH-EVT-R vine copula
S&P500	6.0626%	6.2417%	4.5620%	8.0232%	4.1854%
FTSE100	7.0225%	8.7194%	9.1554%	7.0622%	9.2856%
CAC40	4.5171%	6.0421%	5.5445%	5.9431%	5.5658%
N225	4.2782%	5.4557%	5.7041%	5.3246%	4.7594%
GBP-USD	22.5865%	20.0882%	16.3483%	23.3258%	20.1796%
EUR-USD	28.0033%	23.5695%	25.8230%	31.2537%	27.7357%
JPY-USD	17.5081%	20.0629%	22.7841%	7.6867%	17.6820%
GOLD	9.0752%	8.8018%	9.0037%	10.2155%	9.4299%
WTI	0.9465%	1.0187%	1.0749%	1.1650%	1.1766%
P-ret	1.8877	1.7956	1.9627.	1.6799	1.6527
P-volatility	3.2336	3.2391	3.2971	3.3961	3.3972
VaR	0.7336	0.7664	0.7622	0.7811	0.7889
CVaR	0.8471	0.8971	0.9027	0.9654	0.9311

The empirical results indicate notable differences in portfolio performance among various GARCH-EVT-Copula based VaR models. The R-vine copula generally yields the highest returns and Sharpe ratios when using mean-variance optimization, demonstrating its flexibility in modelling complex dependencies. In contrast, the C-vine copula offers the lowest risk and the strongest risk-adjusted returns when employing CVaR optimization, emphasizing its ability to mitigate tail risk. When comparing the two optimization methods, CVaR optimization emerges as a superior strategy for managing extreme market risks. Therefore, the proposed model yields a lower CVaR and higher Sharpe ratio than traditional models, suggesting it provides better downside protection while maintaining return efficiency. While mean-variance optimization emphasizes variance as the primary measure of risk, CVaR optimization provides a more comprehensive approach by considering potential tail losses. This feature is especially valuable for portfolios that are exposed to the non-linear dependencies and extreme events of financial markets. Ultimately, the GARCH-EVT-Copula-CVaR framework effectively balances risk and return, with both the R-vine and C-vine copulas standing out for their capability to model complex asset dependencies and generate robust, optimized portfolios.

In addition to the empirical evaluation, it is useful to conceptually compare our model with other modern portfolio optimization frameworks. While recent models such as DCC-GARCH, GARCH-EVT and copula-CVaR address specific aspects of financial risk, they each have limitations that reduce their applicability to full portfolio optimization. For instance, DCC-GARCH models capture time-varying correlations but rely on elliptical assumptions and do not explicitly model tail risk or asymmetric dependence. Copula-based CVaR models capture non-linear dependence but often assume constant volatility and may not adequately address extreme loss events. GARCH-EVT models handle tail behavior and volatility clustering but typically overlook inter-asset dependencies. These limitations pose challenges when the objective is robust and comprehensive portfolio optimization. Our GARCH-EVT-Copula-CVaR model was designed specifically to overcome these gaps by integrating all key dimensions volatility dynamics, extreme risk, dependency structure and downside risk minimization, within a single optimization framework.

In contrast, traditional models such as Mean-Variance, GMV and Sharpe Ratio-based optimization approaches are widely used as baseline methods and remain common in practice. We included them in our empirical comparison to demonstrate the practical benefits of our model relative to established benchmarks. Together, the conceptual comparison with modern approaches and the empirical evaluation against traditional ones position our model as a robust and realistic alternative for portfolio optimization.

5. Conclusion

This study investigated portfolio optimization using the GARCH-EVT-Copula-CVaR framework, which integrates GARCH models for volatility estimation, the peak-over-threshold method of Extreme Value Theory (EVT) for capturing tail risks, and regular vine copulas to model complex dependencies among asset returns in a portfolio. The empirical results showed that Conditional Value at Risk (CVaR) optimization outperformed traditional mean-variance optimization by effectively capturing extreme market movements and the volatility dynamics observed in financial markets. The R-vine copula produced the highest returns with mean-variance optimization, while the C-vine copula offered the best risk-adjusted returns with lower tail risk under CVaR optimization. Although Gaussian copulas are limited in capturing tail dependencies, they resulted in lower Value at Risk (VaR) and CVaR values, making them suitable for conservative portfolios. The optimized portfolios consistently achieved higher Sharpe Ratios and lower volatility compared to equal-weighted portfolios. This highlights the importance of dynamic asset selection strategies and effective portfolio optimization. By integrating GARCH, EVT, copulas and CVaR, the proposed model provides a more realistic and robust tool for portfolio managers. It captures tail risks and inter-asset dependencies more effectively than traditional models, helping investors manage extreme losses and allocate capital more prudently. In conclusion, the GARCH-EVT-Copula-CVaR framework provides a comprehensive approach to balancing the risk-return trade-off, offering a reliable model for portfolio management in volatile financial markets. The findings are especially relevant in today's volatile global markets, where extreme events and contagion effects are common. This model equips decision-makers with a framework that reflects such realities more accurately.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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