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Random Premiums Risk Process with Dividends and Investment

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Abstract

We extend the discrete time risk model studied in [1] by introducing an investment component and show that the combined dividend-investment model provides higher expected returns for both the insurer and the shareholders. Furthermore, we show that given two strategies that have the same probability of ultimate ruin on the infinite time horizon, the strategy with larger initial capital and smaller loading factor is less risky than the strategy with smaller initial capital and larger loading factor in that it has a smaller lower bound for ruin probability on the finite time horizon.

Keywords

Discrete Time Surplus Process, Random Premiums, Dividend Barrier, Total Expected Discounted Dividends Prior to Ruin, Investment Strategy

1. Introduction

Actuarial risk was first modeled by the Crámer-Lundberg surplus process, defined by

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k \tag{1.1}$$

with $U(t)$ representing the capital at time $t > 0$ given the initial capital $U(t) = u \geq 0$, and incoming random claim payments X_k in the interval (0, *t*] according to a Poisson process $N(t)$. The premium income stream ct is deterministic with premium rate *c* per unit of time. $U(t)$ represents the risk reserve of a company at time *t*. The main objective is to calculate the odds of company's reserve becoming negative, referred to as the probability of ultimate ruin.

Over the years, the Crámer-Lundberg model has evolved by considering either discrete time or random premiums. To that end, the reader may consult [2]-[9]

for historical accounts and generalizations. More recently, [10] considered a model with both random premiums and random claims in discrete time. We remark, that while it may be reasonable for an insurer to collect premiums according to deterministic formula *ct*, given customers' contractual obligation to pay premiums to receive coverage for their claims, it however may not be a reasonable assumption for the insurer's revenue, due to the fact the future number of customers and their respective premium payments cannot be guaranteed. Furthermore, while it may be true "on average" that the insurer receives premiums as a continuous stream, it is still possible that the total premiums collected by time *t* may be substantially smaller than *ct*, at some future times *t*.

As a remedy for model (1.1) limitations, [10] and [11] considered instead a discrete-time surplus process, where in addition, the premium income ct was replaced by a suitable stochastic component. Namely,

$$U_n = u + \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} Y_k, \quad n = 0,1,2,\cdots \quad X_k \in [0,\infty),\ Y_k \in [0,\infty) \tag{1.2}$$

with *i.i.d.* random premiums X_k and *i.i.d.* claims Y_k respectively, for which the probability of ultimate ruin on infinite and finite time horizons were obtained.

Subsequently, [1] extended (1.2) to an Integer-valued surplus process given by

$$U(t) = u + \sum_{k=1}^{t} X_k - \sum_{k=1}^{t} Y_k - \sum_{k=0}^{t} d_k, \quad t = 0,1,2,\cdots \tag{1.3}$$

where $U(0) = u \in \mathbb{N} = \{0,1,2,\cdots\}$ is the initial capital, X_k are *i.i.d.* premiums, Y_k are *i.i.d.* claims, $d_k = \max\{U(k) - b, 0\}$ are dividends for a given constant barrier $b \in \mathbb{N}$.

The paper is organized as follows. Section 2 begins with the necessary notation, details and assumptions associated with the surplus process (1.3). Section 3 establishes approximation of the total expected discounted dividend prior to ruin by its counterpart on finite time horizon. In Section 4, we introduce a hybrid dividend-investment model which delivers higher expected return to the insurer and the shareholders when compared to the dividends only model. Section 5 addresses the probability of ruin for (1.3) whereas Section 6 is a summary conclusion.

2. Model Description

Before analyzing a central object of this paper, *i.e.*, total expected discounted dividends prior to ruin, we recall following [1]:

Definition 2.1. A *Random Discount Factor*, assumed to be independent of the surplus process $U(t)$, is defined by

$$D(t) = \frac{1}{M(t)} = \mathrm{e}^{-\left(r - \frac{1}{2}\sigma^2\right)t - \sigma W(t)} \tag{2.1}$$

The corresponding expected discount factor is given by

$$E\left[D(t)\right] = E\mathrm{e}^{-\left(r - \frac{1}{2}\sigma^2\right)t - \sigma W(t)} = \mathrm{e}^{-\left(r - \sigma^2\right)t}, \quad 0 < \sigma^2 < r \tag{2.2}$$

Here $M(t)=\mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W(t)}$ with $E\left[M(t)\right]=\mathrm{e}^{rt}$ and standard Brownian Motion $W(t)$ solves Stochastic Differential Equation: $\mathrm{d}M(t)=rM(t)\mathrm{d}t+\sigma M(t)\mathrm{d}W(t), M(0)=1$.

Remark 2.2. In the absence of noise, *i.e.*, $\sigma=0, D(t)=\mathrm{e}^{-rt}$.

In what follows, we utilize the *Random Discount Factor* and its expected value for discrete times $t=k=0,1,2,\cdots$

$$E\left[D(k)\right]=\mathrm{e}^{-\left(r-\sigma^2\right)k}=\nu^k,\ \nu\equiv\nu(r,\sigma)=\mathrm{e}^{-\left(r-\sigma^2\right)}<1 \tag{2.3}$$

In addition, the stream of premiums in the surplus process (1.3) is assumed independent of the stream of claims. The distributions of $X_k\sim X, Y_k\sim Y$, are denoted by

$$\begin{aligned} &f(n)=P(X=n),\ n=0,1,\cdots,N<\infty \\ &g(n)=P(Y=n),\ n=0,1,\cdots \end{aligned} \tag{2.4}$$

Furthermore, we assume that at any time *t* the surplus process updates are done in the following order: premiums first, dividends next, claims last. The ruin time of $U(t)$ is defined by

$$T_{u,b}=\inf\left\{k\geq 1 \mid U(k)<0\right\} \tag{2.5}$$

By (2.1) the total discounted dividends prior to ruin reads

$$\sum_{k=0}^{T_{u,b}}\mathrm{e}^{-\left(r-\frac{1}{2}\sigma^2\right)k-\sigma W(k)}d_k=\sum_{k=0}^{T_{u,b}}D(k)d_k \tag{2.6}$$

Finally, the total expected discounted dividends prior to ruin, given the initial surplus *u*, is defined as follows:

$$\begin{aligned} V(u,b) &= E\left[\sum_{k=0}^{T_{u,b}}D(k)d_k \mid U(0)=u\right] \\ &= E\left[\sum_{k=0}^{T_{u,b}}E\left[D(k)\right]d_k \mid U(0)=u\right] \\ &= E\left[\sum_{k=0}^{T_{u,b}}\nu^k d_k \mid U(0)=u\right] \end{aligned} \tag{2.7}$$

where the second equality is the result of $\left\{D(k)\right\}$ being independent of $\left\{U(k)\right\}$ (thus in particular independent of ruin time $T_{u,b}$ and dividends $\left\{d_k\right\}$), whereas the third equality comes by substitution from (2.3).

Remark 2.3. A somewhat similar to our model was a model introduced by Nie [9], which considered premiums and claims modulated by two-state Markov chain under constant discount factor. By contrast, our accounting of dividends is modulated by a random discount factor associated with the varying interest rates in the underlying economy. Moreover, while the binary premiums in [9] were limited to {0, 1}, the range of our premiums is extended to {0, 1, …, *N*}.

3. Expected Discounted Dividends on Finite Time Horizon

In this section we consider the total expected discounted dividend payments on the interval [0, *T*] defined by

$$V_T(u,b) = E\left[\sum_{k=0}^{\min(T,T_{u,b})} v^k d_k \mid U(0) = u\right] \tag{3.1}$$

for some positive integer T and its approximation to

$$V(u,b) = E\left[\sum_{k=0}^{T_{u,b}} v^k d_k \mid U(0) = u\right]$$

Lemma 3.1. Let premiums $X_k \le c$ for some positive integer *c*. Then for any $\varepsilon > 0$,

$$0 \le V(u,b) - V_T(u,b) < \varepsilon \quad \text{whenever} \quad T \ge \left[\frac{\ln\left(\frac{\varepsilon(1-v)}{c}\right)}{\ln v}\right] \tag{3.2}$$

where [ ] stands for the integer part.

Proof.

$$\begin{aligned}
0 &\le V(u,b) - E\left[\sum_{k=0}^{\min(T,T_{u,b})} v^k d_k \mid U(0) = u\right] \\
&= E\left[\sum_{k=0}^{T_{u,b}} v^k d_k \mid U(0) = u\right] - E\left[\sum_{k=0}^{\min(T,T_{u,b})} v^k d_k \mid U(0) = u\right] \\
&= E\left[\sum_{k=0}^{T_{u,b}} v^k d_k - \sum_{k=0}^{\min(T,T_{u,b})} v^k d_k \mid U(0) = u\right]
\end{aligned}$$

If $T \ge T_{u,b}$

$$= E\left[\sum_{k=0}^{T_{u,b}} v^k d_k - \sum_{k=0}^{T_{u,b}} v^k d_k \mid U(0) = u\right] = 0$$

If $T < T_{u,b}$

$$\begin{aligned}
&= E\left[\sum_{k=0}^{T_{u,b}} v^k d_k - \sum_{k=0}^{T} v^k d_k \mid U(0) = u\right] \le E\left[\sum_{k=0}^{\infty} v^k d_k - \sum_{k=0}^{T} v^k d_k \mid U(0) = u\right] \\
&= E\left[\sum_{k=T+1}^{\infty} v^k d_k \mid U(0) = u\right] = v^{T+1} \sum_{k=0}^{\infty} v^k E\left[d_{T+1+k} \mid U(0) = u\right] \le \frac{cv^{T+1}}{1-v} < \varepsilon
\end{aligned}$$

Solving the last inequality for T gives

$$T > \frac{\ln\left(\frac{\varepsilon(1-v)}{c}\right)}{\ln v} - 1$$

Therefore

$$T \ge \left[\frac{\ln\left(\frac{\varepsilon(1-v)}{c}\right)}{\ln v}\right]$$

and the proof is complete.

Remark 3.2. We point out that the lower bound for *T* in (3.2) is valid even if the premium distribution $f(n) = P(X = n)$ and claim distribution $g(n) = P(X = n)$ are unknown.

Fact 3.3. $F(c) = \dfrac{\ln\left(\dfrac{\varepsilon(1-v)}{c}\right)}{\ln v}$ is increasing in *c* for $c > 0$, therefore smaller *c* gives a smaller lower bound for *T*.

Proof. One checks $F'(c) > 0$.

Based on the above fact we can obtain a better estimate for *N* (*i.e.*, smaller *N*) if we establish the following.

Lemma 3.4. If premiums $X_k \le c$ for some positive integer *c*, then the conditional expectations of dividends satisfy

$$E\left[d_k \mid U(0) = u\right] < E\left[d_k \mid U(k-1) = b, U(0) = u\right] < c, k = N+1, N+2, \cdots \quad (3.3)$$

Proof. Since we only consider premiums with values in {0, 1, 2, 3}, the analysis below is based on *c* = 3. Given $U(k-1) = b$ we have the following conditional distributions of d_k. Namely, by independence of premiums X_k and claims Y_k we obtain

$$d_k = 3$$

$$X_k = 3 \quad Y_k = 0$$

$$P(d_k = 3) = f(3)g(0)$$

$$d_k = 2$$

$$X_k = 2 \quad Y_k = 0$$

$$X_k = 3 \quad Y_k = 1$$

$$P(d_k = 2) = f(2)g(0) + f(3)g(1)$$

$$d_k = 1$$

$$X_k = 3 \quad Y_k = 2$$

$$X_k = 2 \quad Y_k = 1$$

$$X_k = 1 \quad Y_k = 0$$

$$P(d_k = 1) = f(3)g(2) + f(2)g(1) + f(1)g(0)$$

summarized as follows

$$d_k = \begin{cases} 3 & f(3)g(0) \\ 2 & f(2)g(0) + f(3)g(1) \\ 1 & f(3)g(2) + f(2)g(1) + f(1)g(0) \\ 0 & P(\text{elsewhere}) \end{cases}$$

with $P(\text{elsewhere}) = P(\{1 \le X \le 3, X \le Y\}) = \sum_{\{1 \le x \le 3, x \le y\}} f(x)g(y)$.

Therefore

$$
\begin{aligned}
&E\left[d_k \mid U(k-1)=b, U(0)=u\right] \\
&=3f(3)g(0)+2\left(f(2)g(0)+f(3)g(1)\right) \\
&\quad +1\left(f(3)g(2)+f(2)g(1)+f(1)g(0)\right).
\end{aligned}
$$

Given $U(k-1)=b-1$ we have the following conditional distributions of d_k. Namely, by independence of premiums X_k and claims Y_k we obtain

$$d_k = 2$$

$$X_k = 3 \quad Y_k = 0$$

$$P(d_k = 2) = f(3)g(0)$$

$$d_k = 1$$

$$X_k = 3 \quad Y_k = 1$$

$$X_k = 2 \quad Y_k = 0$$

$$P(d_k = 1) = f(3)g(1) + f(2)g(0)$$

summarized as follows

$$
d_k = \begin{cases} 2 & f(3)g(0) \\ 1 & f(3)g(1)+f(2)g(0) \\ 0 & P(\text{elsewhere}) \end{cases}
$$

with $P(\text{elsewhere}) = P(\{2 \le X \le 3, X-1 \le Y\}) = \sum_{\{2\le x\le 3, x-1\le y\}} f(x)g(y)$ and

$$E\left[d_k \mid U(k-1)=b-1, U(0)=u\right] = 2f(3)g(0)+f(3)g(1)+f(2)g(0).$$

Given $U(k-1)=b-2$ we have the following conditional distributions of d_k. Namely, by independence of premiums X_k and claims Y_k we obtain

$$d_k = 1$$

$$X_k = 3 \quad Y_k = 0$$

$$P(d_k = 1) = f(3)g(0)$$

$$
d_k = \begin{cases} 1 & f(3)g(0) \\ 0 & P(\text{elsewhere}) \end{cases}
$$

with $P(\text{elsewhere}) = P(\{X=3, 1 \le Y\}) = \sum_{\{x=3, 1\le y\}} f(x)g(y)$ and

$$E\left[d_k \mid U(k-1)=b-2, U(0)=u\right] = f(3)g(0).$$

Now

$$
\begin{aligned}
&E\left[d_k \mid U(0)=u\right] = E\left\{E\left[d_k \mid U(k-1)=u_{k-1}, U(0)=u\right]\right\} \\
&= E\left[d_k \mid U(k-1)=b, U(0)=u\right] P\left(U(k-1)=b\right) \\
&\quad + E\left[d_k \mid U(k-1)=b-1, U(0)=u\right] P\left(U(k-1)=b-1\right) \\
&\quad + E\left[d_k \mid U(k-1)=b-2, U(0)=u\right] P\left(U(k-1)=b-2\right).
\end{aligned}
$$

Since

$$\begin{aligned} E\left[d_k \mid U(k-1)=b, U(0)=u\right] &> E\left[d_k \mid U(k-1)=b-1, U(0)=u\right] \\ &> E\left[d_k \mid U(k-1)=b-2, U(0)=u\right] \end{aligned}$$

$$\begin{aligned} &E\left[d_k \mid U(0)=u\right] \\ &< E\left[d_k \mid U(k-1)=b, U(0)=u\right]\Big(P\big(U(k-1)=b\big) \\ &\quad + P\big(U(k-1)=b-1\big) + P\big(U(k-1)=b-2\big)\Big) \\ &\le E\left[d_k \mid U(k-1)=b, U(0)=u\right] \end{aligned}$$

which proves the first inequality in (3.3).

Assuming $f(i) > 0$ for $i = 0,1,2,3$ and $g(i) > 0$ for $i = 0,1,2$ we have

$$E\left[d_k \mid U(k-1)=b, U(0)=u\right] < 3$$

because

$$\begin{aligned} &E\left[d_k \mid U(k-1)=b, U(0)=u\right] \\ &= 3f(3)g(0) + 2\big(f(2)g(0) + f(3)g(1)\big) + 1\big(f(3)g(2) + f(2)g(1) + f(1)g(0)\big) \\ &< 3\big(f(3)g(0) + f(2)g(0) + f(3)g(1) + f(3)g(2) + f(2)g(1) + f(1)g(0)\big) \\ &= 3\big(f(3)\big(g(0)+g(1)+g(2)\big) + f(2)\big(g(0)+g(1)\big) + f(1)g(0)\big) \end{aligned}$$

proving the second inequality in (3.3).

Corollary 3.5. By Fact 3.3 and Lemma 3.4 we have

$$\left[\frac{\ln\left(\dfrac{\varepsilon(1-v)}{E\left[d_k \mid U(k-1)=b, U(0)=u\right]}\right)}{\ln v}\right] < \left[\frac{\ln\left(\dfrac{\varepsilon(1-v)}{3}\right)}{\ln v}\right] \tag{3.4}$$

Examples 3.6. Substituting $\varepsilon = 0.1$ and $v = 0.94$ into (3.2) with $c = 3$ with unknown distribution of premiums and claims gives

$$T \ge \left[\frac{\ln\left(\dfrac{0.1(1-0.94)}{3}\right)}{\ln v}\right] = [100.4375] = 100$$

On the other hand, for premiums $X \sim$ binomial $b(N,p)$, $f(n) = \binom{N}{n} p^n (1-p)^{N-n}$ for $N = 3$, $p = 2/5$, $n = 0,1,2,3$ and claims $Y \sim$ binomial $b(N,p)$, $g(n) = \binom{N}{n} p^n (1-p)^{N-n}$ for $N = 8$, $p = \frac{1}{8}$, $n = 0,1,2,3,4,5,6,7,8$ we have

$$\begin{aligned} &E\left[d_k \mid U(k-1)=b, U(0)=u\right] \\ &= 3f(3)g(0) + 2\big(f(2)g(0) + f(3)g(1)\big) \\ &\quad + 1\big(f(3)g(2) + f(2)g(1) + f(1)g(0)\big) \\ &= 0.588 \end{aligned}$$

Substituting $\varepsilon = 0.1$ and $v = 0.94$ with $c = E\left[d_k \mid U(k-1) = b, U(0) = u\right] = 0.588 < 3$ into (3.2) gives, by Corollary 3.5, a smaller lower bound

$$T \geq \left[\frac{\ln\left(\frac{0.1(1-0.94)}{0.588}\right)}{\ln(0.94)}\right] = [74.1000] = 74 \tag{3.5}$$

in agreement with Fact 3.3.

Substituting $\varepsilon = 0.5$ and $v = 0.94$ into (3.2) with $c = 3$ with unknown distribution of premiums and claims gives

$$T \geq \left[\frac{\ln\left(\frac{0.5(1-0.94)}{3}\right)}{\ln v}\right] = [74.4265] = 74$$

Substituting $\varepsilon = 0.5$ and $v = 0.94$ with $c = E\left[d_k \mid U(k-1) = b, U(0) = u\right] = 0.588 < 3$ into (3.2) gives, by Corollary 3.5, a smaller lower bound

$$T \geq \left[\frac{\ln\left(\frac{0.5(1-0.94)}{0.588}\right)}{\ln(0.94)}\right] = [48.0891] = 48$$

again in agreement with Fact 3.3.

Finally, substituting $\varepsilon = 1$ and $v = 0.94$ into (3.2) with $c = 3$ with unknown distribution of premiums and claims gives

$$T \geq \left[\frac{\ln\left(\frac{1-0.94}{3}\right)}{\ln v}\right] = [63.2242] = 63$$

whereas (3.2) with $\varepsilon = 1$ and $v = 0.94$ with $c = E\left[d_k \mid U(k-1) = b, U(0) = u\right] = 0.588 < 3$ gives, by Corollary 3.5, a smaller lower bound

$$T \geq \left[\frac{\ln\left(\frac{(1-0.94)}{0.588}\right)}{\ln(0.94)}\right] = [36.8867] = 36$$

as claimed by Fact 3.3.

4. Dividend-Investment Model

We define the total expected discounted dividends prior to ruin, given the initial surplus *u*, by the dividend gain function $G(\{d_k\})$ defined below

$$G(\{d_k\}) = E\left[\sum_{k=0}^{T_{u,b}} v^k d_k \mid U(0) = u\right] = V(u,b)$$

Then for any, $0 < \alpha < 1$, $\alpha\{d_k\} \equiv \{\alpha d_k\}$ expresses the reduction of dividend payment from $d_k \to \alpha d_k, k = 0, 1, \cdots$

$$\begin{aligned} G(\alpha\{d_k\}) = G(\{\alpha d_k\}) &= E\left[\sum_{k=0}^{T_{u,b}} v^k \alpha d_k \mid U(0) = u\right] \\ &= \alpha E\left[\sum_{k=0}^{T_{u,b}} v^k d_k \mid U(0) = u\right] = \alpha V(u,b) = \alpha G(\{d_k\}) \end{aligned} \tag{4.1}$$

which shows that $G(.)$ is homogenous in α and $G(\{\alpha d_k\}) = \alpha V(u,b)$.

Now the original dividend d_k will be split into dividend αd_k for shareholders and $(1-\alpha)d_k$ collected by insurer for investment.

Under reduction of dividend payments by the factor α the total expected discounted dividends by fixed time *T* is as follows

$$G_T(\{\alpha d_k\}) = E\left[\sum_{k=0}^{\min(T, T_{u,b})} v^k \alpha d_k \mid U(0) = u\right] = \alpha V_T(u,b)$$

and

$$G_T(\{(1-\alpha)d_k\}) = (1-\alpha)V_T(u,b)$$

In what follows we will assume that an insurer has an option to redirect portion α of dividend payouts into an investment portfolio $X(t)$ whose evolution satisfies the following Ito's Stochastic Differential Equation (SDE)

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),\ t \geq 0 \tag{4.2}$$

μ = portfolio average return rate;

σ = portfolio volatility;

$W(t)$ = standard Brownian motion on the real line.

Fact 4.1. ([12]). The solution to (4.2) is

$$X(t) = X(0)\mathrm{e}^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

with the expected return

$$EX(t) = X(0)\mathrm{e}^{\mu t} \tag{4.3}$$

Lemma 4.2. There is an investment strategy on [*T*, *T* + *t*] that realizes any predetermined expected profit $A > 0$ at time *T* + *t*.

Proof. By Fact 4.1 the expected return (4.3) at time *T* + *t* on $(1-\alpha)V_T(u,b)$ invested at time *T* is

$$(1-\alpha)V_T(u,b)\mathrm{e}^{\mu t} \tag{4.4}$$

Consequently, it suffices to device a portfolio with an average return rate μ such that

$$(1-\alpha)V_T(u,b)\mathrm{e}^{\mu t} = (1-\alpha)V_T(u,b) + A \tag{4.5}$$

which yields

$$\mu = \frac{\ln\left(1+\frac{A}{(1-\alpha)V_T(u,b)}\right)}{t} \tag{4.6}$$

The second part of the strategy is to share the investment profit with the shareholders. Namely, the profit A at time $T+t$ is split as bonus βA to shareholders and bonus $(1-\beta)A$ to the insurer, for some $0<\beta<1$ determined by the insurer. The investment strategy stipulates that by time $T+t$, the shareholders, on top of received dividends $\alpha V_{T+t}(u,b)$, will additionally receive $(1-\alpha)V_{T+t}(u,b)$ back from the insurer along with profit sharing bonus βA. This amounts to the following total shareholders revenue during $[0, T+t]$

$$S_R = \alpha V_{T+t}(u,b) + (1-\alpha)V_T(u,b) + \beta A > V(u,b) + \beta A - \varepsilon \tag{4.7}$$

where the lower bound follows from applying Lemma 3.1.

Table 1. Expected Revenue Comparison: *Dividends with Investment* vs *Sole Dividends.*

Dividends with Investment strategy on $[0, T]$	*Sole Dividends* on $[0, \infty)$
Shareholders Revenue: $S_R > V(u,b) + \beta A - \varepsilon$	$V(u,b)$
Insurer's Revenue: $I_R = (1-\beta)A$	0

Corollary 4.3. The Investment strategy benefits both the insurer and the shareholders in the sense that their expected revenue over the finite time horizon $[0, T+t]$ exceeds their respective expected revenue on the infinite time horizon $[0, \infty)$ where only the dividend are being considered. Clearly, under this strategy, expected revenue in finite time exceeds expected revenue over infinite time for both parties.

Remark 4.4. In order to select suitable A we express A in the right-hand side of (4.5) in terms of $V(u,b)$, whereby utilizing the approximation of $V(u,b)$ on $[0, \infty)$ by $V_{T+t}(u,b)$ on $[0, T+t]$. Namely, to account for $V(u,b) - V_{T+t}(u,b) \le V(u,b) - V_T(u,b) < \varepsilon$, (4.4) can be written as

$$\begin{aligned}(1-\alpha)V_T(u,b)\mathrm{e}^{\mu t} &= (1-\alpha)V(u,b) + A - (1-\alpha)\left(V(u,b) - V_T(u,b)\right) \\ &= (1-\alpha)V(u,b) + \hat{A}\end{aligned} \tag{4.8}$$

where $\hat{A} = A - (1-\alpha)\left(V(u,b) - V_T(u,b)\right) > A - \varepsilon$ or equivalently $\hat{A} = A - \delta$ for some $0<\delta<\varepsilon$. As a result, raising the desired profit from $\hat{A} = A - \delta$ by ε delivers profit of at least A. Based on the above, instead of (4.5)-(4.6) we consider

$$(1-\alpha)V_T(u,b)\mathrm{e}^{\mu t} = (1-\alpha)V(u,b) + \varepsilon\hat{A} + \varepsilon \tag{4.9}$$

$$\mu = \frac{\ln\left(1+\frac{\hat{A}+\varepsilon}{(1-\alpha)V(u,b)}\right)}{t} \tag{4.10}$$

for portfolio average return rate μ.

Remark 4.5. Large insurance companies can raise capital for investment by issuing zero-coupon bonds. In such cases an investment of the raised capital C begins at time 0 and the insurer may deliver a return relevant to implementation of the above strategy by the time T_C, which may be chosen significantly smaller than T found in Lemma 3.1. In other words, the insurer may achieve the return $(1-\alpha)V(u,b)+A$ by time $T_C + t$. That is, investing C at time 0 at rate of return μ gives the expected return at time T_C

$$C\mathrm{e}^{\mu T_C} = (1-\alpha)V(u,b)$$

and subsequently investing at rate $\tilde{\mu}$ to reach the expected return at time $T_c + t$

$$(1-\alpha)V_T(u,b)\mathrm{e}^{\tilde{\mu}t} = (1-\alpha)V(u,b)+A$$

The best case scenario corresponds to $C = (1-\alpha)V(u,b)$, in which case $T_C = T = 0$ and the Investment strategy concludes at time t without the delay time T.

Examples 4.6. Consider (3.5) in Examples 3.10 with premiums in {0, 1, 2, 3} and make following choices $A = 4$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ for the investment strategy. Then with quarters as units of time

$$T \ge \left[\frac{\ln\left(\frac{0.1(1-0.94)}{0.588}\right)}{\ln(0.94)}\right] = [74.1000] = 74$$

hence we choose $T = 74$ (18.5 years). We remark that in this case $V(2,4) = 2.7095$ as given in **Table 1** in {1}.

Furthermore, taking $t = 40$ (10 years) the average rate of return per quarter on the interval $[T, T+t] = [74, 114]$, by (4.10) we get the quarterly rate of return

$$\mu = \frac{\ln\left(1+\frac{4}{\left(\frac{1}{2}\right)2.7095}\right)}{40} = 0.0344$$

which is 4 (0.0344) = 0.1376 or 13.76%/year.

Consider Example 5.4 of [1] with $V(2,4) = 2.5767$ for premiums in {0, 1, 2} and make the following choices $A = 4$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ for the investment strategy. By Corollary 3.5 with quarters as units of time, $\varepsilon = 0.1$ and $v = 0.94$ we get

$$T \geq \left[\frac{\ln\left(\frac{\varepsilon(1-v)}{E\left[d_k \mid U(k-1)=b, U(0)=u\right]} \right)}{\ln v} \right] = \left[\frac{\ln\left(\frac{0.1(1-0.94)}{0.5507} \right)}{\ln(0.94)} \right] = [73.0408] = 73$$

$$E\left[d_k \mid U(k-1)=b, U(0)=u\right] = 2f(2)g(0) + f(1)g(0) + f(2)g(1) = 0.5507$$

and we choose $T = 73$ (18.25 years).

Furthermore, taking $t = 40$ (10 years) the average rate of return per quarter on the interval $[T, T+t] = [73, 113]$, by (4.10) we get the quarterly rate of return

$$\mu = \frac{\ln\left(1 + \frac{4}{\left(\frac{1}{2}\right)2.5767}\right)}{40} = 0.0353$$

which is 4(0.0353) = 0.1412 or 14.12%/year.

Consider Example 5.7. of [1] with $V(2,4) = 1.3811$ for premiums in {0, 1} and make the following choices $A = 4$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ for the investment strategy. By Corollary 3.5 with quarters as units of time, $\varepsilon = 0.1$ and $v = 0.94$ we get

$$T \geq \left[\frac{\ln\left(\frac{\varepsilon(1-v)}{E\left[d_k \mid U(k-1)=b, U(0)=u\right]} \right)}{\ln v} \right] = \left[\frac{\ln\left(\frac{0.1(1-0.94)}{0.3560} \right)}{\ln(0.94)} \right] = [65.9902] = 65$$

where

$$E\left[d_k \mid U(k-1)=b, U(0)=u\right] = f(1)g(0) = 0.3560$$

and we choose $T = 65$ (16.25 years).

Furthermore, taking $t = 40$ (10 years) the average rate of return per quarter on the interval $[T, T+t] = [65, 105]$, by (4.10) we get the quarterly rate of return

$$\mu = \frac{\ln\left(1 + \frac{4}{\left(\frac{1}{2}\right)1.3811}\right)}{40} = 0.0479$$

which is 4(0.0479) = 0.1916 or 19.16%/year.

5. Probability of Ruin

Based on domination of the surplus process with dividends (1.2) by the surplus process without dividends (1.3)

$$U(t) = u + \sum_{k=1}^{t} X_k - \sum_{k=1}^{t} Y_k - \sum_{k=0}^{t} d_k \leq U_n = u + \sum_{k=1}^{n} X_k - \sum_{k=1}^{n} Y_k \qquad (5.1)$$

their ruin times satisfy

$$\tau^{U(\cdot)} \le \tau^{U.}$$

where

$$\tau^{U(\cdot)} = \inf\left\{t > 0 \mid U(t) < 0\right\} \quad \text{and} \quad \tau^{U.} = \inf\left\{t > 0 \mid U(t) < 0\right\} \tag{5.2}$$

Fact 5.1 ([11]). For $\tau_{U.}$ defined by (5.2) we have

$$P\left(\tau^{U.} \le t\right) = \int_0^t f_{\tau^{U.}}(s)\,\mathrm{d}s = \int_0^t \frac{\frac{u}{\sigma}}{\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{\left(\frac{u}{\sigma}+\frac{\theta\mu}{\sigma}s\right)^2}{2s}}\,\mathrm{d}s \tag{5.3}$$

with defective density $f_{\tau^{U.}}(\cdot)$ such that

$$P(ruin) = P\left(\tau^{U.} < \infty\right) = \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u} < 1 \quad \text{and} \quad P(no\ ruin) = P\left(\tau^{U.} = \infty\right) = 1 - \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u} > 0 \tag{5.4}$$

and

$$\sigma^2 = Var\left(X_i - Y_i + \theta\mu\right) = Var\left(X_i\right) + Var\left(Y_i\right) = \sigma_X^2 + \sigma_Y^2 \quad \text{with } \sigma = \sqrt{\sigma_X^2 + \sigma_Y^2} \tag{5.5}$$

Lemma 5.2. Probability of survival (*no ruin*) for the surplus process with dividends has the following upper bound

$$P\left(\tau^{U(\cdot)} = \infty\right) \le 1 - \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u}$$

Proof. By domination (5.1)

$$\tau_n^{U(\cdot)} = \infty \iff U(t) \ge 0,\ t \ge 0 \implies U_t \ge 0,\ t \ge 0 \iff \tau^{U\cdot} = \infty$$

gives

$$\left\{\tau^{U(\cdot)} = \infty\right\} \subset \left\{\tau^{U.} = \infty\right\}$$

whence by applying (5.3)

$$P\left(\tau^{U(\cdot)} = \infty\right) \le P\left(\tau^{U.} = \infty\right) = 1 - \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u}$$

which concludes the proof.

Example 5.3. Consider binomial premiums $X \sim b(N, p)$, $f(n) = \binom{N}{n} p^n (1-p)^{N-n}$
with $N = 1$, $p = 3/5$, for $n = 0, 1$ and binomial claims Y, $g(n) = \binom{N}{n} p^n (1-p)^{N-n}$
with $N = 6$, $p = 1/12$, for $n = 0, 1, 2, 3, 4, 5, 6$ from Example 5.7 in [1]. The absolute loading factor $EX - EY = 3/5 - 1/2 = 0.10$ gives EY relative loading factor $\theta = \frac{EX - EY}{EY}$ or $EX = (1+\theta)EY$.

Then

$$\begin{aligned} &\sigma_X^2 = Npq = \frac{6}{25},\ \sigma_Y^2 = Npq = \frac{11}{24},\ \sigma^2 = \sigma_X^2 + \sigma_Y^2 = \frac{419}{600} \\ &EY = \mu = \frac{1}{2},\ \theta = 0.2,\ u = 2 \end{aligned} \tag{5.6}$$

and by (5.2) and Lemma 5.2

$$P(ruin) = P\left(\tau^{U(\cdot)} < \infty\right) \geq \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u} = \mathrm{e}^{-\frac{2\times\frac{1}{5}\times\frac{1}{2}}{\frac{419}{600}}\times 2} = 0.5639$$

with

$$P(no\ ruin) = P\left(\tau^{U(\cdot)} = \infty\right) \leq 1 - \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u} = 0.4361$$

Since dividend payments with investment strategy involve finite time horizon, we will analyze ruin probability on [0, *t*] and derive probability of ruin estimates based on comparison with the risk process without dividends. Key observation here is that based on initial capital u and loading factor θ, $P(U_{\cdot}\ ruin) = \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u}$ is invariant in the product $u\theta$.

The distribution of ruin time given by (5.3) has significant consequences regarding the choice of parameters in the investment strategy. It turns out that given (u_1, θ_1) and (u_2, θ_2) with $u_1\theta_1 = u_2\theta_2$ *i.e.*, having the same probability of ruin in $[0, \infty)$), a strategy with larger initial capital and smaller loading factor is better by having a smaller probability of ruin in [0, *t*].

Example 5.4. By Fact 5.3, the defective density of the risk process $U_{\cdot}$ without dividends in Example 5.3 given by (5.1) and (5.6) has the form (Figure 1)

$$f_{\tau^{U_{\cdot}}}(t) = \frac{\frac{u}{\sigma}}{\sqrt{2\pi^3 t}}\mathrm{e}^{-\frac{\left(\frac{u}{\sigma}+\frac{\theta\mu}{\sigma}t\right)^2}{2t}} = \frac{\frac{2}{\sqrt{\frac{419}{600}}}}{\sqrt{2\pi^3 t}}\mathrm{e}^{-\frac{\left(\frac{2}{\sqrt{\frac{419}{600}}}+\frac{\frac{1}{10}}{\sqrt{\frac{419}{600}}}t\right)^2}{2t}}$$

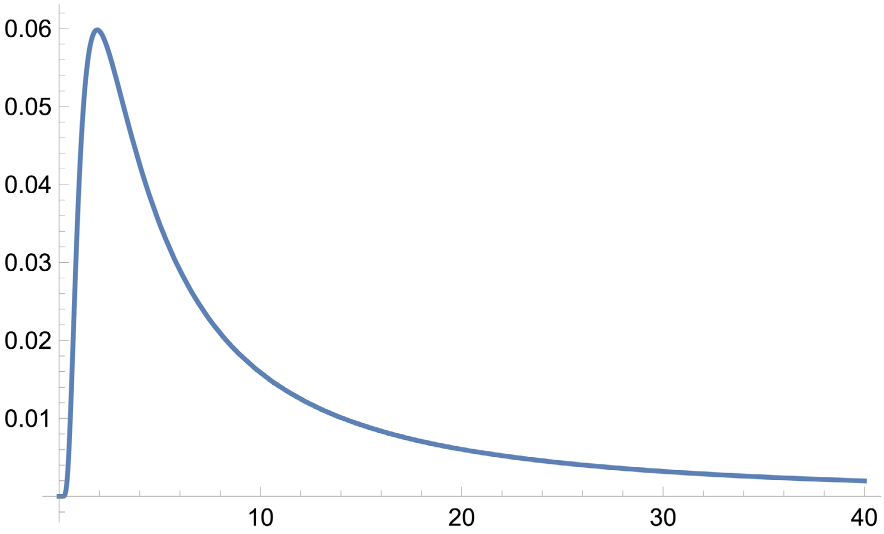

Figure 1. Defective density $f_{\tau^{U_{\cdot}}}(t)$ where $\theta = 0.2, u = 2$ and $\theta u = 0.4$.

We have

$$P\left(\tau^{U_{\cdot}} \leq 20\right) = \int_0^{20} f_{\tau^{U_{\cdot}}}(s)\,\mathrm{d}s = 0.4242$$

On the other hand for (**Figure 2**)

$$f_{\tau^{U.}}(t)=\frac{\frac{u}{\sigma}}{\sqrt{2\pi^3 t}}\mathrm{e}^{-\frac{\left(\frac{u}{\sigma}+\frac{\theta\mu}{\sigma}t\right)^2}{2t}}=\frac{\frac{4}{\sqrt{\frac{419}{600}}}}{\sqrt{2\pi^3 t}}\mathrm{e}^{-\frac{\left(\frac{4}{\sqrt{\frac{419}{600}}}+\frac{\frac{5}{100}}{\sqrt{\frac{419}{600}}}t\right)^2}{2t}} \quad \text{with } \theta=0.1,\ u=4,\ \theta u=0.4$$

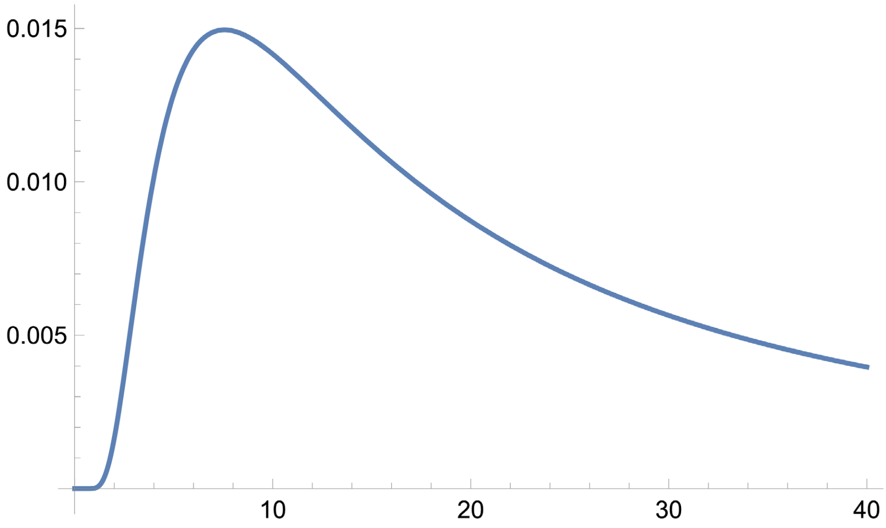

Figure 2. Defective density $f_{\tau^{U.}}(t)$ where for $\theta=0.1, u=4$ and $\theta u=0.4$.

Their combined plot (**Figure 3**) illustrates a pronounced difference between the two cases and shows that the second scenario has significantly smaller probability of ruin on [0, *t*] = [0, 20].

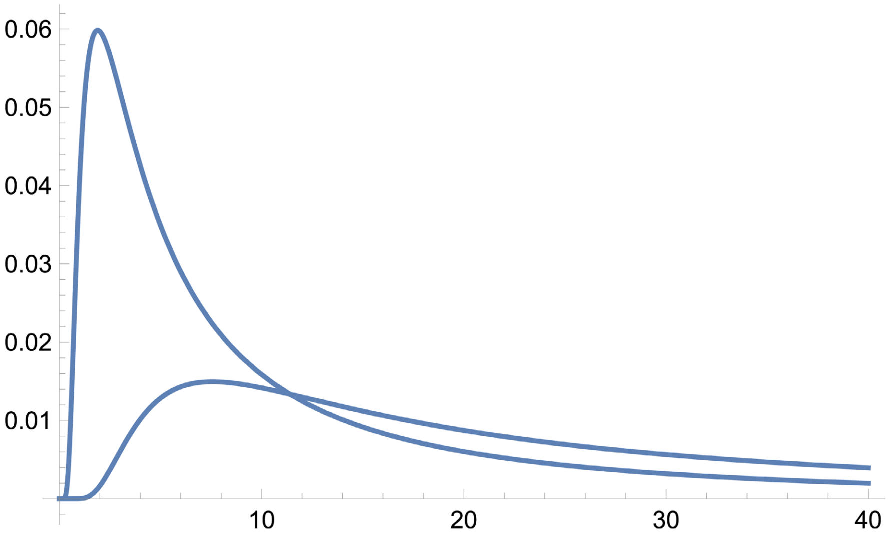

Figure 3. Defective density $f_{\tau^{U.}}(t)$ for $\theta=0.2, u=2$ and $\theta=0.1, u=4$.

Since the ruin time of the process with dividends $\tau^{U(\cdot)}<\tau^{U.}$—ruin time of the process without dividends, we have the following lower bounds

$$P\left(\tau^{U(\cdot)} \le 20\right) \ge P\left(\tau^{U.} \le 20\right) = \int_0^{20} f_{\tau^{U.}}(s)\,\mathrm{d}s = 0.4242 \quad \text{for} \quad \theta = 0.2, u = 2, \theta u = 0.4$$

and

$$P\left(\tau^{U(\cdot)} \le 20\right) \ge P\left(\tau^{U.} \le 20\right) = \int_0^{20} f_{\tau^{U.}}(s)\,\mathrm{d}s = 0.2095 \quad \text{for} \quad \theta = 0.1, u = 4, \theta u = 0.4$$

Again, similarly to the risk process without dividends, the second scenario is a better strategy because of smaller lower bound for the probability of ruin on [0, 20].

6. Summary Discussion

We introduced a new model by adding an investment strategy component to the sole dividend model and showed that the combined dividend-investment approach provides higher expected returns for both the insurer and the shareholders. Furthermore, we demonstrated that in the dividend with investment model, given two strategies that have the same probability of ultimate ruin on $[0,\infty)$, the strategy with the larger initial capital and the smaller loading factor is less risky than the strategy with the smaller initial capital and the larger loading factor in that it has a smaller lower bound for the probability of ruin on the finite time interval [0, *t*]. We intend to obtain real-world data to examine applicability of the proposed model, given a potential insurer would agree to implement our hybrid dividend-investment strategy and subsequently share the data. We note that the assumption of independence of premiums and claims could be relaxed by considering Markovian structure, however this will result in less tractable recursive expressions for the total expected discounted dividends. Future work is expected to shed some light on the current model generalization.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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