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An Analysis of Incomplete and Random Financial Networks

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Abstract

This paper contributes to the theoretical literature by analyzing the relationship between changes in sparsity and their impacts on financial networks with incomplete and random core-periphery structures, which are widely studied in finance. Sparsity, which measures edge density, reflects the level of connectivity: high sparsity indicates fewer connections between agents, while low sparsity signifies a denser web of interactions. Changes in sparsity result in variations in network impacts. Building on a linear network model inspired by spatial econometrics, we find that reducing sparsity amplifies network impacts in incomplete core-periphery structures through two strategies: 1) increasing the number of core agents and 2) merging two or more core-periphery components. For networks with specific incomplete core-periphery configurations, we derive theoretical results for the average total impact and validate other impact measures through simulations. Furthermore, our analysis extends to networks with randomly generated core-periphery structures, affirming the robustness of our findings.

Keywords

Incomplete Core-Periphery Structure, Random Core-Periphery Structure, Network Sparsity, Network Impact

1. Introduction

The collapse of Lehman Brothers underscored the role of financial network architecture in propagating systemic risk. Network sparsity, a feature of financial interconnectedness, creates pathways for risk transmission between institutions. During the crisis period, Brunetti, Harris, Mankad, and Michailidis [1] observed a significant decrease in interconnectedness within interbank lending networks.

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This shift highlights the need to explore interconnectedness changes and their impact. Yet, the relationship between the change in network sparsity and its impact is under-researched.

This paper investigates the theoretical relationship between sparsity and its impact on financial networks with core-periphery components, which are widely studied in finance literature (e.g., Li and Schürhoff [2], in't Veld and van Lelyveld [3]). Sparsity measures the edge density: high sparsity means fewer connections between agents, while low sparsity indicates a denser web of interactions. Network impact quantifies how changes in an independent variable for one agent influence the dependent variables of other agents through network interactions. Variations in sparsity lead to changes in network impacts. This paper focuses on the impact of the linear network model inspired by spatial econometrics (Bramoullé, Djebbari, and Fortin [4]). To our best knowledge, this study is among the first to explore theoretically the impact of financial networks with incomplete and random core-periphery structures.

Recently, Denbee, Julliard, Li, and Yuan [5], for example, studied the liquidity multiplier effect of a given interbank network. Our paper fills in the gap by investigating how the change in network sparsity influences the relationship between an explanatory variable for one agent and the dependent variable of all other agents in financial networks with core-periphery structures. This paper is an extension of Jie and Ma [6]. We relax the assumption of complete core-periphery structure of Jie and Ma [6] to examine the incomplete core-periphery structure.

We develop analytical solutions to quantify the relationship between sparsity and its effects on financial networks, specifically focusing on incomplete and random core-periphery structures. For scenarios where closed-form solutions are unattainable, we present simulation results to provide further insights.

Core-periphery networks, comprising two tiers of agents—an interconnected core and a sparse periphery—have been extensively studied in finance literature (Di Maggio, Kermani, and Song [7], Li and Schürhoff [2]). To maintain the core-periphery structure while reducing network sparsity, this paper examines two strategies: 1) promoting periphery agents to core agents and 2) integrating distinct core-periphery components. An example of the first strategy is illustrated by WeBank, a private Chinese bank founded by Tencent, which upgraded its periphery branch in Hong Kong to a core branch, serving as a regional headquarters in 2024. This strategic move strengthened its connections with other periphery branches in Hong Kong and the Greater Bay Area of mainland China.

The second strategy to reduce sparsity involves combining two or more sub-networks into a larger network by establishing links between core and periphery agents across sub-networks. For example, Levine, Lin, and Wang [8] demonstrate that merging two geographic banking networks into one can create value. In this study, we find that both sparsity reduction strategies significantly enhance the impacts of financial networks.

In practical applications, these two strategies can be combined in various ways to effectively reduce sparsity, enhance network influence, and balance the tradeoffs between them. For instance, upgrading a peripheral agent to a core agent can be a cost-effective approach to reducing network sparsity. However, the impact of this strategy is somewhat limited when it comes to addressing inter-network linkages. On the other hand, merging two networks into one can significantly reduce sparsity across networks and deliver more effective outcomes after the merger. However, this approach tends to be resource-intensive and costly, requiring substantial input and effort to implement successfully.

Finally, to relax the constraints of complete core-periphery structure in the analysis of Jie and Ma [6], and the specific incomplete core-periphery structure of our current theoretical analysis, we generate sparse networks with random core-periphery components. We decrease network sparsity by adding new links randomly among unconnected core and periphery agents. Simulation analysis reveals that reduced sparsity leads to increased network impacts.

The remainder of the paper is organized as follows. Section 2 introduces basic definitions and model setup. Section 3 examines sparse financial networks with incomplete core-periphery components. Section 4 explores networks with random core-periphery components. Finally, Section 5 concludes. Proofs are provided in the Appendix.

2. Definition and Model Setup

In this paper, we define the network adjacency matrix W as a symmetric matrix with binary elements, where $w_{ij} = w_{ji} = 1$ if there is a connection between agent i and j, and $w_{ij} = w_{ji} = 0$ otherwise (Jackson [9]). The diagonal entries of W are set to be zero, adhering to conventional definition (LeSage and Pace [10]). The network sparsity is defined as the proportion of zero entries in W, excluding the diagonal elements (Diestel [11], p. 164). We follow Elliott, Golub, and Jackson [12] and Craig and Ma [13] to define the core-periphery structure.

Definition 1. (Core-periphery Component) A core-periphery component has total number of agents p, where s and $p - s$ are the number of core and periphery agents respectively. It has the corresponding pp adjacency matrix W_i of the following structure:

$$W_i = \begin{pmatrix} CC & CP \\ CP' & PP \end{pmatrix} = \begin{pmatrix} \mathbf{1_0} & \boldsymbol{R} \\ \boldsymbol{R}' & \mathbf{0} \end{pmatrix} \tag{1}$$

The block CC defines the full connections among core agents, where $CC = \mathbf{1_0}$ with dimension of $s \times s$ comprises all ones except that the diagonal terms are zeros. The block PP is a zero matrix ($PP = \mathbf{0}$) with dimension of $(p - s) \times (p - s)$, indicating no linkage among periphery agents. The block $CP = \boldsymbol{R}$ with size $s \times (p - s)$. $\boldsymbol{R}$ is both row-regular and column-regular, indicating that each core agent is connected to at least one periphery agent, and vice versa.

Core-periphery components can be classified into *complete* and *incomplete* ones. Jie and Ma [6] examined the *complete* core-periphery component, which has a restrictive assumption that all entries of CP and R are one. They find that a sparsity reduction will lead to an increase in the network impact. In this paper, we further relax this restrictive assumption to investigate the *incomplete* core-periphery component and its network impact.

An *incomplete core-periphery component* exists when at least one zero appears in the CP block. This implies that not all core agents are connected to all periphery agents in an incomplete core-periphery component. Using the incomplete core-periphery financial network structure, existing literature has examined the risk transmission among financial institutions (Elliott, Golub, and Jackson [12]), fund flow of interbank market (Craig and Ma [13]), and interest rates of the federal funds market (Bech and Atalay [14]).

In this paper, we focus on a specific type of incomplete core-periphery component, which consists of two complete core-periphery sub-components where the two cores are interconnected but each core agent does not connect to periphery agents in the other sub-component. It is these missing links that constitute the *incomplete* core-periphery topology of the component. In other words, our incomplete core-periphery component can be regarded as interlinked core-periphery components. A recent application of this network structure can be found in Babus and Hu [15], who find that the interlinked star network is efficient and stable for over-the-counter transactions.

We consider a sparse network W comprising B independent homogeneous incomplete core-periphery components W_i of size p. The total number of agents in the network is therefore $N = Bp$. The adjacency matrix W takes a block-diagonal form:

$$W = \begin{pmatrix} W_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_B \end{pmatrix}_{N \times N} \quad \text{with} \quad W_1 = W_2 = \cdots = W_B. \tag{2}$$

The size of matrix W is $N \times N$ and the size of component matrix W_i is $p \times p$.

To introduce the concept of network impacts, we define our model of financial network as the following:

$$y = \rho W y + X\beta + u \tag{3}$$

where y and X are the dependent and the set of explanatory variables respectively, W is a network adjacency matrix, ρ and β are parameters, and u is the disturbance term.

We assume that $\rho \in \left(\frac{1}{\lambda_{\min}}, \frac{1}{\lambda_{\max}}\right)$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of W_i, respectively. This assumption ensures both the consistent estimation of the network model and invertibility of the matrix $I - \rho W$ (LeSage and Pace [10], p. 47).

We rewrite the network model above to have

$$y = (I - \rho W)^{-1} (X\beta + \varepsilon) = S(X\beta + \varepsilon) \tag{4}$$

where $S \overset{\text{def}}{=} (I - \rho W)^{-1}$.

In a network model, the influence of x_k on the dependent variable y differs across observations for all agents. To capture these varying effects, LeSage and Pace [10] introduced three measures: average direct impact, average indirect impact, and average total impact, to quantify the different aspects of network effects.

Definition 2. (Network Impacts)

For any independent variable $x_k \subset X$, define:

$$\text{Average Direct Impact} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial y_i}{\partial x_{ik}}\right) = \left(\frac{1}{N}\sum_{i=1}^{N} s_{ii}\right)\beta_k \tag{5}$$

$$\text{Average Indirect Impact} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j\neq i}\left(\frac{\partial y_i}{\partial x_{jk}}\right) = \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j\neq i} s_{ij}\right)\beta_k \tag{6}$$

$$\begin{aligned}\text{Average Total Impact} &= \text{Average Direct Impact} + \text{Average Indirect Impact}\\ &= \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} s_{ij}\right)\beta_k\end{aligned} \tag{7}$$

Financial network research has extensively utilized these network impact measures. For instance, di Giovanni and Hale [16] applied these measures to investigate the impacts of U.S. monetary policy shocks on global stock markets. Grieser, Hadlock, LeSage, and Zekhnini [17] employed these measures to analyze causal peer effects in capital structure decisions within a peer network. To simplify the analysis without loss of generality, we normalize the parameter $\beta_k = 1$, allowing us to concentrate primarily on the financial network impacts in the subsequent discussions.

3. Sparse Network with Incomplete Core-Periphery Components

In this section, we relax the assumption of complete core-periphery structure of Jie and Ma [6] to examine the incomplete core-periphery structure. The complete core-periphery structure imposes a restrictive assumption, as it requires all core agents to be linked with all periphery agents. In reality, core agents do not necessarily maintain uniform connections with the same set of periphery agents. For instance, each CEO is primarily connected to the shareholders of their own company rather than shareholders in other firms, and shareholder compositions vary across companies. Therefore, we extend our analysis to sparse networks with incomplete core-periphery components.

Specifically, we consider a network with N agents, where all agents are evenly grouped into B identical, independent, incomplete core-periphery components. Each component consists of two core-periphery structures with interconnected core agents. Specifically, each core-periphery structure has s core agents and $p - s$ periphery agents. As a result, each component consists of $2s$ core agents and $2(p - s)$ periphery agents, giving the total network size $N = 2pB$. Unlike the complete structure, where all core agents are linked to all periphery agents, we introduce a sparser connectivity pattern. The first set of s core agents are exclusively connected to the first set of $p - s$ periphery agents, while the second set of s core agents are linked only to the second set of $p - s$ periphery agents. It is regarded as an interlinked core-periphery component (Babus and Hu [15]). This modification better reflects real-world network structures, where core agents maintain selective, rather than uniform, connections to periphery agents.

As discussed in Section 1, we examine two strategies to reduce sparsity while preserving the incomplete core-periphery component structure: 1) increasing the number of core agents, and 2) merging small components into larger ones. Under both strategies, we derive closed-form solutions for the relationship between sparsity and network impacts. We formally prove that a decrease in the network sparsity leads to an increase in average total impact of the network in the Appendix. For average direct and indirect impacts, we rely on simulations to explore their network impacts.

3.1. Sparsity Reduction by Increasing the Number of Core Agents

In this sub-section, we analyze the sparsity reduction process by promoting some of the periphery agents to core agents in each component. For example, Figure 1 illustrates the process of increasing the number of core agents from $2s = 2$, $2s = 4$, and then $2s = 6$, while keeping component size $2p = 8$ fixed. As two previous periphery agents are promoted to core status in each component, sparsity of $\boldsymbol{W}_i$ decreases due to the formation of new links between the newly designated core agents and their corresponding remaining periphery sets.

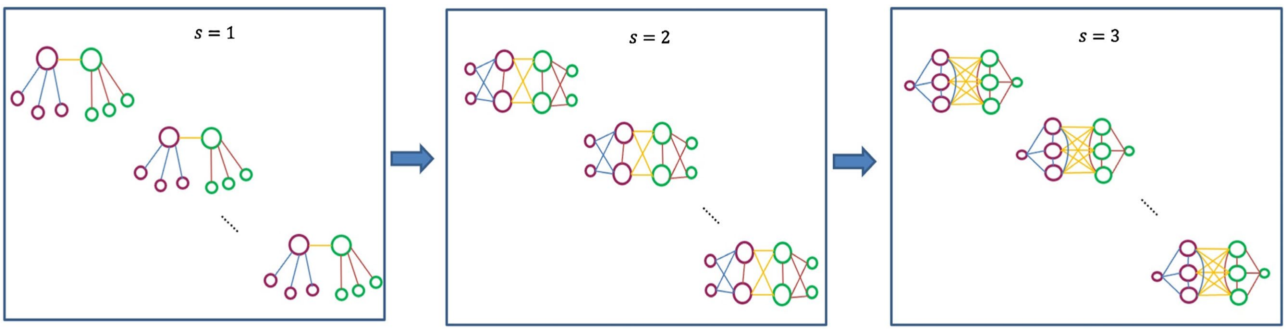

Figure 1. Sparsity reduction by increasing the number of core agents within incomplete core-periphery components.

In general, the formula for network sparsity is given by:

$$\text{Sparsity} = \frac{N(N-1) - \left[4s(p-s) + 2s(2s-1)\right]B}{N(N-1)} = 1 - \frac{(2p-1)s}{p(N-1)} \tag{8}$$

We have $\frac{\partial \text{Sparsity}}{\partial s} = -\frac{2p-1}{p(N-1)} < 0$, as $p > 1$. This confirms that sparsity is a decreasing function of the number of core agents s, implying that an increase in the number of core agents leads to a denser network. Consequently, s can serve as an inverse measure of sparsity.

Proposition 1. $\boldsymbol{W}$ is a block-diagonal matrix, where each block $\boldsymbol{W}_i$ represents an incomplete core-periphery component of size $2p$. Each $\boldsymbol{W}_i$ consists of two interconnected sets of core agents, each connecting to its corresponding set of periphery agents. The structure of $\boldsymbol{W}_i$ is as follows:

$$\boldsymbol{W}_i = \begin{pmatrix} \mathbf{1}_{\mathbf{0}_{2s\times 2s}} & \boldsymbol{R}_{\mathbf{0}_{2s\times 2(p-s)}} \\ \boldsymbol{R}'_{\mathbf{0}_{2(p-s)\times 2s}} & \mathbf{0}_{2(p-s)\times 2(p-s)} \end{pmatrix}_{2p\times 2p} \tag{9}$$

where

$$\boldsymbol{R}_{\mathbf{0}_{2s\times 2(p-s)}} = \begin{pmatrix} \mathbf{1}_{s\times(p-s)} & \mathbf{0}_{s\times(p-s)} \\ \mathbf{0}_{s\times(p-s)} & \mathbf{1}_{s\times(p-s)} \end{pmatrix}_{2s\times 2(p-s)}.$$

If $S \overset{\text{def}}{=} (I-\rho W)^{-1}$, then $\boldsymbol{S}$ is also a block-diagonal matrix with submatrix $\boldsymbol{S}_i$ of the following form:

$$\boldsymbol{S}_i = \begin{pmatrix} A_{2s\times 2s} & B_{2s\times 2(p-s)} \\ B'_{2(p-s)\times 2s} & D_{2(p-s)\times 2(p-s)} \end{pmatrix}_{p\times p}$$

$$= \begin{pmatrix} M_{s\times s} & F_{s\times s} & J_{s\times(p-s)} & H_{s\times(p-s)} \\ F_{s\times s} & M_{s\times s} & H_{s\times(p-s)} & J_{s\times(p-s)} \\ J'_{(p-s)\times s} & H'_{(p-s)\times s} & K_{(p-s)\times(p-s)} & G_{(p-s)\times(p-s)} \\ H'_{(p-s)\times s} & J'_{(p-s)\times s} & G_{(p-s)\times(p-s)} & K_{(p-s)\times(p-s)} \end{pmatrix} \tag{10}$$

where

$$A_{2s\times 2s} = \begin{pmatrix} M_{s\times s} & F_{s\times s} \\ F_{s\times s} & M_{s\times s} \end{pmatrix}$$

$$B_{2s\times 2(p-s)} = \begin{pmatrix} J_{s\times(p-s)} & H_{s\times(p-s)} \\ H_{s\times(p-s)} & J_{s\times(p-s)} \end{pmatrix}$$

$$D_{2(p-s)\times 2(p-s)} = \begin{pmatrix} K_{(p-s)\times(p-s)} & G_{(p-s)\times(p-s)} \\ G_{(p-s)\times(p-s)} & K_{(p-s)\times(p-s)} \end{pmatrix}$$

$$M_{s\times s} = a\boldsymbol{I}_{s\times s} + c\mathbf{1}_{\mathbf{0}_{s\times s}}$$

$$F_{s\times s} = f\mathbf{1}_{s\times s}$$

$$K_{(p-s)\times(p-s)} = b\boldsymbol{I}_{(p-s)\times(p-s)} + d\mathbf{1}_{\mathbf{0}_{(p-s)\times(p-s)}}$$

$$G_{(p-s)\times(p-s)} = g\mathbf{1}_{(p-s)\times(p-s)}$$

$$J_{s\times(p-s)} = e\mathbf{1}_{s\times(p-s)}$$

$$H_{s\times(p-s)} = h\mathbf{1}_{s\times(p-s)}$$

$\boldsymbol{I}$ is an identity matrix and $\mathbf{1}_{\mathbf{0}}$ consists of all ones except that the diagonal terms are zeros.

Detailed formulae of a to h are in the Appendix A.1.

From Proposition 1, we can calculate network impacts based on definition 2 in Section 2 and prove the following proposition in the Appendix A.1.

Proposition 2. Given the size p of incomplete core-periphery components, if the number of its core agents s increases that leads to a decrease in network sparsity, then the average total impact of the network increases.

Since average direct and indirect impacts, and their derivatives with respect to s are complicated, we rely on simulations to verify their signs. In the simulation analysis, we also verify the total impact to confirm Proposition 2.

We conducted two batches of simulations to examine the signs of these network impacts. In the first batch of simulation, we utilized the closed-form solutions for average direct, indirect, and total impacts from Proposition 2, as derived in Appendix A.1. By calculating their derivatives with respect to the number of core agents s—an inverse metric of network sparsity—we identified positive derivatives across all measures (see Panel A, B, and C of **Table 1**). This finding is consistent with our conclusion: as the number of core agents s increases, leading to reduced sparsity, all measures of network impact enlarge.

Table 1. Simulation results of adding core agents in the sparse network with incomplete core-periphery components. Panel A to Panel C reports simulation results of the derivatives of network impact with respect to s (the number of core agents), where s is an inverse measure of the network sparsity. The component sample size p is assigned to 10, 50,100, 500, 10,000, 50,000. 100,000, and 200,000, respectively. The number of cores agents in each component s is chosen as quintiles of $[1, p-1]$, *i.e.*, $p/5$, $2p/5$, $3p/5$, $4p/5$, and $p-1$. ρ is the median of $\left(0, \frac{1}{200000-1}\right)$, *i.e.*, $\rho = \frac{1}{200000-1} \times \frac{1}{2}$. This will make ρ within the consistent and nonsingular range. Note that our network matrix W is an unnormalized $N \times N$ matrix, hence the ρ is unnormalized and is relatively small. If we normalize W, say, by dividing all its elements by $N-1$, then the normalized $\hat{\rho} = \frac{1}{2}$ (see Jie and Ma [6], Corollary 1 in Appendix A.1). In Panel A, all numbers are multiplied by 10^11. In Panel B and Panel C, all numbers are multiplied by 10^6. Panel D reports simulation results of the relationship between sparsity and network impacts. $N = 200000$, $p = 1000$ and $B = 200$ are fixed. ρ is the same as in Panel A to Panel C. The formula for sparsity is $1 - \frac{(2p-1)s}{p(N-1)}$.

Panel A. Derivatives of direct impact with respect to s (the number of core agents).

s	$[p/5]$	$[2p/5]$	$[3p/5]$	$[4p/5]$	$p-1$
p					
10	1.1875	1.1875	1.1876	1.1876	1.1876
50	1.2376	1.2377	1.2379	1.2380	1.2382
100	1.2440	1.2442	1.2445	1.2448	1.2451
500	1.2498	1.2510	1.2524	1.2540	1.2558
1000	1.2514	1.2539	1.2567	1.2599	1.2635
5000	1.2603	1.2728	1.2873	1.3039	1.3226
10,000	1.2711	1.2966	1.3265	1.3611	1.4005
50,000	1.3675	1.5217	1.7207	1.9737	2.2916
100,000	1.5201	1.9292	2.5526	3.5092	4.9999

Panel B. Derivatives of indirect impact with respect to s (the number of core agents).

s	$[p/5]$	$[2p/5]$	$[3p/5]$	$[4p/5]$	$p-1$
p					
10	4.7501	4.7502	4.7503	4.7503	4.7504
50	4.9507	4.9511	4.9514	4.9518	4.9521
100	4.9764	4.9771	4.9779	4.9786	4.9793
500	5.0019	5.0056	5.0094	5.0132	5.0169
1000	5.0113	5.0188	5.0264	5.0340	5.0415
5000	5.0691	5.1078	5.1469	5.1864	5.2263
10,000	5.1406	5.2205	5.3022	5.3855	5.4706
50,000	5.7782	6.3024	6.8945	7.5667	8.3332
100,000	6.7857	8.3167	10.3890	13.2860	17.5000

Panel C. Derivatives of total impact with respect to s (the number of core agents).

s	$[p/5]$	$[2p/5]$	$[3p/5]$	$[4p/5]$	$p-1$
p					
10	4.7501	4.7502	4.7503	4.7504	4.7504
50	4.9507	4.9511	4.9514	4.9518	4.9521
100	4.9764	4.9771	4.9779	4.9786	4.9793
500	5.0019	5.0056	5.0094	5.0132	5.0169
1000	5.0113	5.0188	5.0264	5.0340	5.0415
5000	5.0691	5.1078	5.1469	5.1864	5.2263
10,000	5.1406	5.2205	5.3022	5.3855	5.4706
50,000	5.7783	6.3024	6.8946	7.5667	8.3332
100,000	6.7858	8.3168	10.3890	13.2860	17.5000

Panel D. Relationship between sparsity and network impacts.

s	Sparsity	Direct impact	Indirect impact	Total impact
1	0.999990005	1.000000000	0.000005004	1.000005004
100	0.999000495	1.000000001	0.000500563	1.000500564
200	0.998000990	1.000000003	0.001001503	1.001001505
300	0.997001485	1.000000004	0.001502819	1.001502822
400	0.996001980	1.000000005	0.002004512	1.002004517
500	0.995002475	1.000000006	0.002506583	1.002506589
600	0.994002970	1.000000008	0.003009032	1.003009039
700	0.993003465	1.000000009	0.003511859	1.003511867
800	0.992003960	1.000000010	0.004015064	1.004015074
900	0.991004455	1.000000011	0.004518649	1.004518660
1000	0.990004950	1.000000013	0.005022613	1.005022626

In the second batch of simulations, we calculated sparsity and network impacts, presenting the results in Panel D of Table 1 and Figure 2. It shows a negative relationship between sparsity and network impacts. Additionally, we observe from Panel D of Table 1 and Figure 2 that direct impacts contribute substantially to total impacts, while the reduction in sparsity was relatively small. This is because our simulated networks are highly sparse by our experiment design, indicating indirect impacts to be relatively low.

However, we find that the magnitude of the derivatives for indirect impacts significantly outweighs that of total impacts. In fact, the derivatives of total impacts in Panel C closely mirror those of indirect impacts presented in Panel B of Table 1. This indicates that indirect impacts are more sensitive to changes in sparsity than direct impacts. This finding is akin to those reported in Jie and Ma [6] who examined financial networks with complete core-periphery components. Taking together, it accentuates the importance of investigating both direct and indirect impacts to get a full picture of the network total impacts. We have arrived at the following conclusion.

Conclusion 1. Given the size p of incomplete core-periphery components, if the number of its core agents s increases that leads to a reduction of network sparsity, then both the average direct and indirect impacts of the network increase.

3.2. Sparsity Reduction by Merging Components

This sub-section explores the second strategy to reduce network sparsity: merging multiple small incomplete core-periphery components into a larger incomplete core-periphery component. For example, Figure 3 illustrates the merging process from two core agents ($2s = 2$) to four core agents ($2s = 4$). Before the merger, the network consists of B incomplete core-periphery components. Each component has a size of 6 with 2 core agents: the first core agent is connected to two periphery agents, while the second core agent is linked to the other two periphery agents. After the merger, every two components are merged into a single larger incomplete core-periphery component, increasing the number of core agents to four ($2s = 4$).

In this new structure, two core agents are connected to the four periphery agents, while the other two core agents are linked to the remaining four periphery agents. The four core agents are also interconnected. The sparsity of the new network decreases as new links are formed among core agents and between core and periphery agents. The number of components B is halved, while the component size p doubles.

Suppose before merging, the adjacency matrix contains $2s_0$ core agents per component, with each component size being $2p_0$. After merging μ components into a larger component, the number of core agents becomes $2s = 2\mu s_0$ and the component size becomes $2p = 2\mu p_0$. That is, $p = sp_0/s_0 = \lambda s$ where $\lambda = p_0/s_0 > 1$ is fixed. In this circumstance, p has a linear relationship with s.

In general, the network sparsity becomes

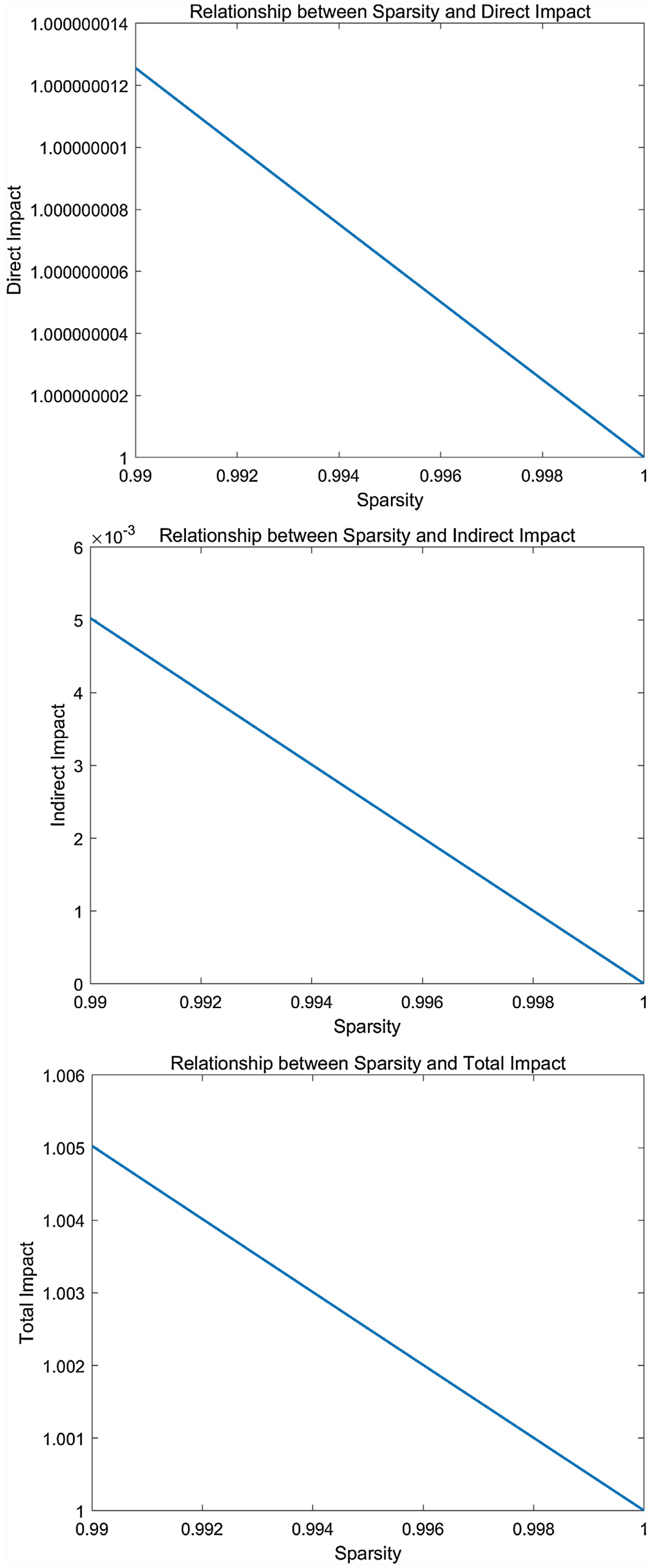

Figure 2. Sparsity and its impact on networks with incomplete core-periphery components.

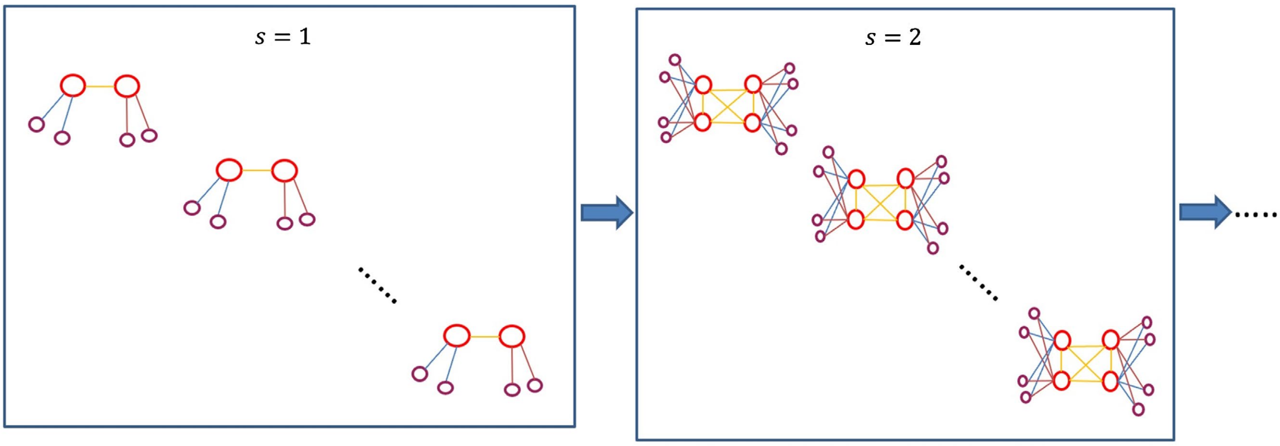

Figure 3. Sparsity reduction by merging incomplete core-periphery components.

$$\text{Sparsity} = \frac{N(N-1)-\left[4s(p-s)+2s(2s-1)\right]B}{N(N-1)} = 1-\frac{2\lambda s-1}{\lambda(N-1)} \tag{11}$$

The formula shows that sparsity decreases as s increases: $\frac{\partial \text{Sparsity}}{\partial s} = -\frac{2\lambda}{\lambda(N-1)} < 0$ as $\lambda > 1$. Therefore, the number of core agents s can serve as an inverse measure of sparsity in the network.

Proposition 3. $\boldsymbol{W}$ is a block-diagonal matrix, where each block $\boldsymbol{W}_i$ represents an incomplete core-periphery component of size $2p$. Each $\boldsymbol{W}_i$ consists of two interconnected sets of core agents, each connecting to a separate set of periphery agents. The structure of $\boldsymbol{W}_i$ is as follows:

$$\boldsymbol{W}_i = \begin{pmatrix} \mathbf{1}_{\mathbf{0}_{2s\times 2s}} & \boldsymbol{R}_{\mathbf{0}_{2s\times 2(p-s)}} \\ \boldsymbol{R}'_{\mathbf{0}_{2(p-s)\times 2s}} & \mathbf{0}_{2(p-s)\times 2(p-s)} \end{pmatrix}_{2p\times 2p} \tag{12}$$

where

$$\boldsymbol{R}_{\mathbf{0}_{2s\times 2(p-s)}} = \begin{pmatrix} \mathbf{1}_{s\times(p-s)} & \mathbf{0}_{s\times(p-s)} \\ \mathbf{0}_{s\times(p-s)} & \mathbf{1}_{s\times(p-s)} \end{pmatrix}_{2s\times 2(p-s)}.$$

If $S \overset{\text{def}}{=} (I-\rho W)^{-1}$ then $\boldsymbol{S}$ is also a block-diagonal matrix with submatrix $\boldsymbol{S}_i$ of the following form:

$$\begin{aligned} \boldsymbol{S}_i &= \begin{pmatrix} A_{2s\times 2s} & B_{2s\times 2(p-s)} \\ B'_{2(p-s)\times 2s} & D_{2(p-s)\times 2(p-s)} \end{pmatrix}_{p\times p} \\ &= \begin{pmatrix} M_{s\times s} & F_{s\times s} & J_{s\times(p-s)} & H_{s\times(p-s)} \\ F_{s\times s} & M_{s\times s} & H_{s\times(p-s)} & J_{s\times(p-s)} \\ J'_{(p-s)\times s} & H'_{(p-s)\times s} & K_{(p-s)\times(p-s)} & G_{(p-s)\times(p-s)} \\ H'_{(p-s)\times s} & J'_{(p-s)\times s} & G_{(p-s)\times(p-s)} & K_{(p-s)\times(p-s)} \end{pmatrix} \end{aligned} \tag{13}$$

where

$$A_{2s\times 2s} = \begin{pmatrix} M_{s\times s} & F_{s\times s} \\ F_{s\times s} & M_{s\times s} \end{pmatrix}$$

$$B_{2s\times 2(p-s)} = \begin{pmatrix} J_{s\times(p-s)} & H_{s\times(p-s)} \\ H_{s\times(p-s)} & J_{s\times(p-s)} \end{pmatrix}$$

$$D_{2(p-s)\times 2(p-s)} = \begin{pmatrix} K_{(p-s)\times(p-s)} & G_{(p-s)\times(p-s)} \\ G_{(p-s)\times(p-s)} & K_{(p-s)\times(p-s)} \end{pmatrix}$$

$$M_{s\times s} = a\boldsymbol{I}_{s\times s} + c\mathbf{1}_{\mathbf{0}_{s\times s}}$$

$$F_{s\times s} = f\mathbf{1}_{s\times s}$$

$$K_{(p-s)\times(p-s)} = b\boldsymbol{I}_{(p-s)\times(p-s)} + d\mathbf{1}_{\mathbf{0}_{(p-s)\times(p-s)}}$$

$$G_{(p-s)\times(p-s)} = g\mathbf{1}_{(p-s)\times(p-s)}$$

$$J_{s\times(p-s)} = e\mathbf{1}_{s\times(p-s)}$$

$$H_{s\times(p-s)} = h\mathbf{1}_{s\times(p-s)}$$

$\boldsymbol{I}$ is an identity matrix and $\mathbf{1}_{\mathbf{0}}$ consists of all ones except that the diagonal terms are zeros.

Detailed formulae of a to h can be found in the Appendix A.2.

From Proposition 3, we can calculate network impacts based on definition 2 in Section 2. We can prove the following proposition in the Appendix A.2.

Proposition 4. Given the network size N, if a merger of incomplete core-periphery components that leads to a decrease in network sparsity, then the average total impact of the network increases.

Since both derivatives of direct and indirect impact with respect to s are complicated, we employ a simulation method to determine their signs. We also verify the total impact to support Proposition 4. The simulation results presented in **Table 2** indicate that if the number of core agents s increases due to the merger of incomplete core-periphery components, then network sparsity decreases and all impact measures increase. **Figure 4** illustrates the inverse relationship between sparsity and network impacts, further validating Proposition 4 and leading to the following conclusion.

Conclusion 2. Given the network size N, if a merger of incomplete core-periphery components that leads to a reduction of network sparsity, then both average direct and indirect impacts of the network increase.

4. Sparse Network with Random Core-Periphery Components

This section further relaxes the assumption of the complete core-periphery component studied in Jie and Ma [6], and specific incomplete core-periphery component investigated in previous Section 3. Instead, we consider sparse networks with randomly generated core-periphery components. While the network maintains symmetric and homogeneous components structural properties, each component has links among core and periphery agents that are assigned randomly. It also

Table 2. Simulation results of merging incomplete core-periphery components. Panel A to Panel C report simulation results of the derivatives of network impacts with respect to s (the number of core agents), where s is an inverse measure of the network sparsity. The component sample size p is assigned to 10, 50, 100, 500, 10,000, 50,000. 100,000, and 200,000, respectively. $\lambda = 2$ is fixed, *i.e.*, two components are merged, and $s = p/2$. ρ is chosen as three quartiles of $\left(0, \frac{1}{200000-1}\right)$, *i.e.*, $\frac{1}{200000-1} \times \frac{1}{4}$, $\frac{1}{200000-1} \times \frac{1}{2}$, and $\frac{1}{200000-1} \times \frac{3}{4}$, respectively. This will make ρ within the consistent and nonsingular range. Note that our network matrix W is an unnormalized $N \times N$ matrix, hence the ρ is unnormalized and is relatively small. If we normalize W, say, by dividing all its elements by $N-1$, then the normalized $\hat{\rho} = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively (see Jie and Ma [6], Corollary 1 in Appendix A.1). In Panel A, all numbers are multiplied by 10^12. In Panel B and Panel C, all numbers are multiplied by 10^6. Panel D reports the simulation results of the relationship between sparsity and network impacts. $N = 200000$ is fixed. ρ is the median of $\left(0, \frac{1}{N-1}\right)$, *i.e.*, $\rho = \frac{1}{200000-1} \times \frac{1}{2}$. The formula for sparsity is $1 - \frac{2\lambda s - 1}{\lambda(N-1)}$.

Panel A. Derivatives of direct impact with respect to s (the number of core agents).

p	B	s	ρ		
			$1/4(N-1)$	$1/2(N-1)$	$3/4(N-1)$
10	20,000	5	2.8127	11.2518	25.3183
50	4000	25	3.0637	12.2593	27.5936
100	2000	50	3.0961	12.3937	27.9066
500	400	250	3.1305	12.5689	28.3861
1000	200	500	3.1454	12.6761	28.7352
5000	40	2500	3.2434	13.4649	31.4368
10,000	20	5000	3.3666	14.4967	35.0891
50,000	4	25,000	4.5050	25.7416	83.1339
100,000	2	50,000	6.4355	54.1684	310.7771

Panel B. Derivatives of indirect impact with respect to s (the number of core agents).

p	B	s	ρ		
			$1/4(N-1)$	$1/2(N-1)$	$3/4(N-1)$
10	20,000	5	2.2501	4.5004	6.7508
50	4,000	25	2.4505	4.9020	7.3544
100	2,000	50	2.4760	4.9540	7.4340
500	400	250	2.5001	5.0103	7.5308
1000	200	500	2.5077	5.0358	7.5846
5000	40	2500	2.5511	5.2085	7.9774
10,000	20	5000	2.6045	5.4322	8.5052
50,000	4	25,000	3.0983	7.8901	15.6092
100,000	2	50,000	3.9451	14.3943	52.5393

Panel C. Derivatives of total impact with respect to s (the number of core agents).

p	B	s	ρ		
			$1/4(N-1)$	$1/2(N-1)$	$3/4(N-1)$
10	20,000	5	2.2501	4.5004	6.7508
50	4,000	25	2.4505	4.9020	7.3545
100	2,000	50	2.4760	4.9540	7.4340
500	400	250	2.5001	5.0103	7.5308
1000	200	500	2.5077	5.0358	7.5847
5000	40	2500	2.5511	5.2085	7.9774
10,000	20	5000	2.6045	5.4322	8.5052
50,000	4	25,000	3.0983	7.8902	15.6093
100,000	2	50,000	3.9451	14.3944	52.5397

Panel D. Relationship between sparsity and network impacts.

p	B	s	Sparsity	Direct impact	Indirect impact	Total impact
10	20,000	5	0.999952500	1.000000000	0.000045002	1.000045002
50	4000	25	0.999752499	1.000000001	0.000245053	1.000245054
100	2000	50	0.999502498	1.000000001	0.000495216	1.000495218
500	400	250	0.997502488	1.000000006	0.002500468	1.002500474
1000	200	500	0.995002475	1.000000013	0.005016944	1.005016957
5000	40	2500	0.975002375	1.000000064	0.025553722	1.025553787
10,000	20	5000	0.950002250	1.000000133	0.052280174	1.052280307
50,000	4	25,000	0.750001250	1.000000875	0.319581142	1.319582017
100,000	2	50,000	0.500000000	1.000002629	0.882344499	1.882347128

retains the core-periphery structure that (1) all core agents are interconnected with each other and (2) each core agent is connected to at least one periphery agent and vice versa, fulfilling the row- and column-regular matrix conditions for block $\boldsymbol{R}$ in definition 1 of Section 2.

Two basic algorithms are available to generate random graphs. The original random graph model developed by Erdős and Rényi [18] has a *fixed* number of links, and the algorithm places these links at random positions of the network. Gilbert [19] relaxes their assumption of fixed number of links and allows *random* number of links to be placed at random positions with equal probability. Therefore, we choose the algorithm proposed by Nobari, Lu, Karras, and Bressan [20] that implemented the original random graph of Gilbert [19], which is more appropriate for generating realistic financial networks.

Specifically, we employ Algorithm 1, proposed by Nobari, Lu, Karras, and Bressan [20], to generate the block $R(0)$ uniformly at random with dimension of $s \times (p - s)$,

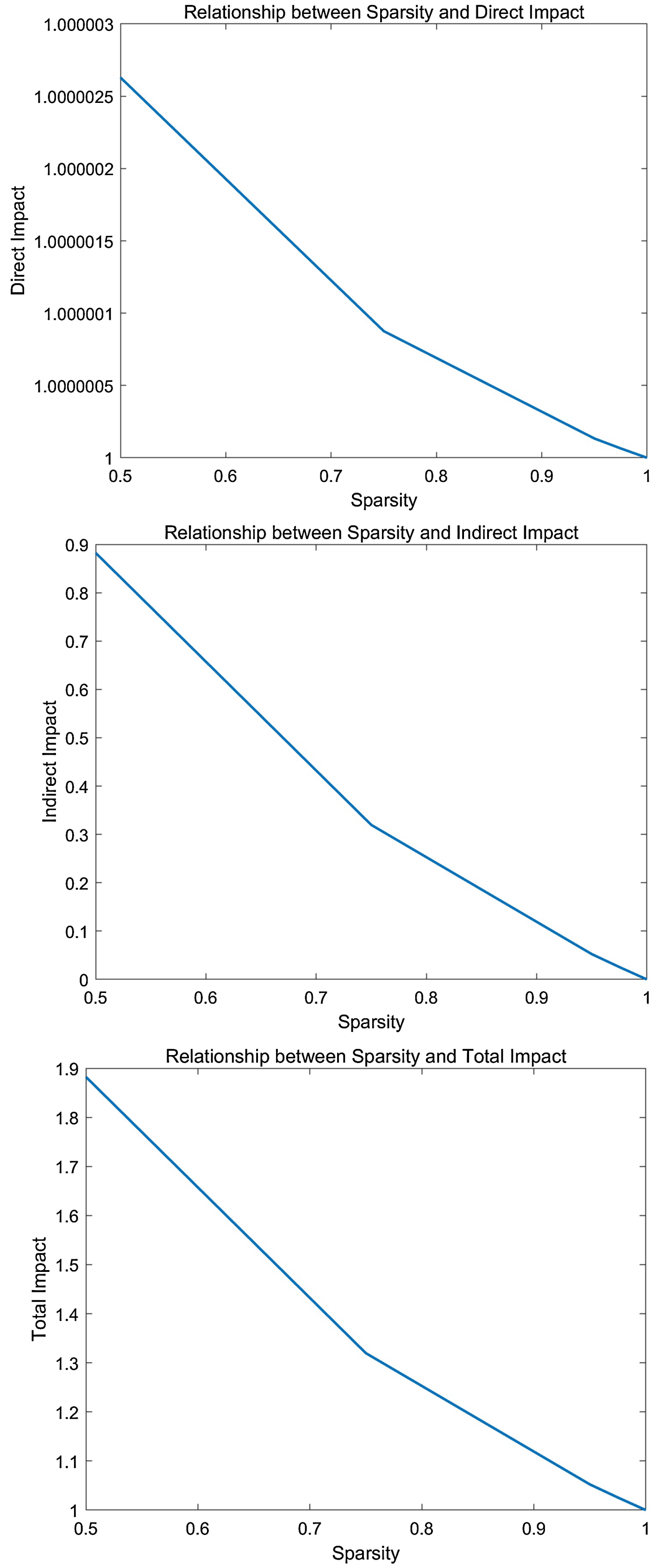

Figure 4. Sparsity and its impact on networks with incomplete components.

where s and $p-s$ are numbers of core and periphery agents respectively. Each link of $R(0)$ is randomly formed independently with probability q. To ensure that the resulting random block $R(0)$ is both row-regular and column-regular, Algorithm 1 repeatedly draws random $R(0)$ until a $R(0)$ meets both regular conditions. During the simulations, we find Algorithm 1 is very fast to locate such a $R(0)$. Then we use this random block $R(0)$ to form the matrix $\boldsymbol{W}_i$ of a random core-periphery component according to definition 1 of Section 2. Finally, we assemble the whole network matrix W by random core-periphery components $\boldsymbol{W}_i$ ($i=1,2,\cdots,B$) along its diagonal, denoted as $G(0)$.

Algorithm 1

Input: Empty link graph with the numbers of row s and column $p-s$
Output: $R(0)$: the block CP of a random core-periphery graph
while $R(0)$ is not row- and column-regular **do**
Generate a uniform random number $q\in[0,1)$
for i=1 to s **do**
for j=1 to $p-s$ **do**
Generate a uniform random number $\theta_1\in[0,1)$
if $\theta_1<q$ **then**
$R\leftarrow(i,j)$
end for
end for
end while
Fill in zeros for remaining cells
Note: q is the inclusion probability to form a random link, see Nobari, Lu, Karras, and Bressan [20], Algorithm 1.

Algorithm 2

Input: $R(0)$: the block CP of a random core-periphery graph with numbers of row s and column $p-s$
q : inclusion probability of link in $R(0)$, $q\in[0,1)$
Output: $R(1)$: a more connected graph than $R(0)$
while no new link is added to $R(0)$ **do**
for i=1 to s **do**
for j=1 to $p-s$ **do**
Generate a uniform random number $\theta_2\in[0,1)$
if (i, j) has no link and $\theta_2<\min\left(1,\frac{q}{1-q}\right)$ **then**
Add link: $G(1)=G(0)\cup\{(i,j)\}$
Accept $G(1)$ and $G(0) = G(1)$
end if
end for
end for
end while
Note: The acceptance probability of adding a new link to $G(0)$ is $q/(1-q)$, see Gray, Mitchell, and Roughan [21], p. 9.

To randomly reduce the sparsity of $R(0)$ while preserving the core-periphery structure of $\boldsymbol{W}_i$, we implement Algorithm 2 to add new random links to block $R(0)$ with the Metropolis-Hastings acceptance ratio of min(1, $q/(1-q)$), which is the acceptance probability of adding new links to $G(0)$. Algorithm 2 is introduced by Gray, Mitchell, and Roughan [21] and can generate a more connected random block $R(1)$. Alternative algorithms such as "double edge swaps" have been analyzed by Nishimura [22] and turned out to have undesirable properties to maintain the conditional ensemble of the underlying network structure. Gray, Mitchell, and Roughan [21] theoretically proved and validated by numerical simulations that their algorithm can produce more connected random networks while maintaining the underlying network probabilistic characteristics after adding random links to the original network.

Similar to the network formation process based on $G(0)$, $R(1)$ produced from Algorithm 2 is then used to form denser core-periphery component $\boldsymbol{W}_i$ according to definition 1 of Section 2, which in turn is utilized to construct the full network W, denoted as $G(1)$. Finally, we compare the network impact differences between the two random core-periphery networks of $G(0)$ and $G(1)$.

We run simulations to investigate the relationship between sparsity and network impacts. We always keep the whole network size at $N = 10{,}000$. To generate seven random numbers p of the random core-periphery sizes, we first generate seven random numbers from seven ranges [1, 10), [10, 50), [50, 100), [100, 500), [500, 1000), [1000, 5000), and [5000, 10,000). Then, the seven p's are chosen as seven factors of N that are closest to each seven random numbers respectively. This will ensure that the number of components $B = N/p$ to be an integer in seven networks. For each p, we choose two numbers of core agents s randomly from two intervals $(1, p/2)$ and $(p/2, p-1)$ respectively. This gives a total of $7 \times 2 = 14$ random network configurations in total. For each configuration, we run Algorithms 1 and 2 to generate a pair of random core-periphery networks $G(0)$ and $G(1)$ respectively.

The simulation results of the differences in average direct/indirect/total impacts between each pair of $G(1)$ and $G(0)$ are presented in Panel A of **Table 3**. It shows that all the differences are positive. Since the sparsity of $G(1)$ is lower than that of $G(0)$, it indicates that a reduction in sparsity leads to an increase in network impacts.

Panel B of **Table 3** further analyzes the relationship between sparsity and network impacts by keeping core agents $s = 175$, number of agents in each random component $p = 400$, number of components $B = 25$, and total number of agents in the network $N = Bp = 10{,}000$. Starting with a random core-periphery component $G(0)$ generated by Algorithm 1, we apply Algorithm 2 to add random number of links to $G(0)$ and generate a denser $G(1)$. $G(1)$ is then utilized as an input graph to Algorithm 2 to generate even denser $G(2)$ and subsequent networks. Then the network impacts are calculated and presented in Panel B of **Table 3**.

Figure 5 visualizes the simulation results in Panel B. It shows an inverse relationship between sparsity and average direct/indirect/total impacts. As additional random links are introduced between core-periphery agents, sparsity decreases, leading to an increase in all network impacts. Consistent with the findings of Jie and Ma [6] and the previous section on respective complete and incomplete

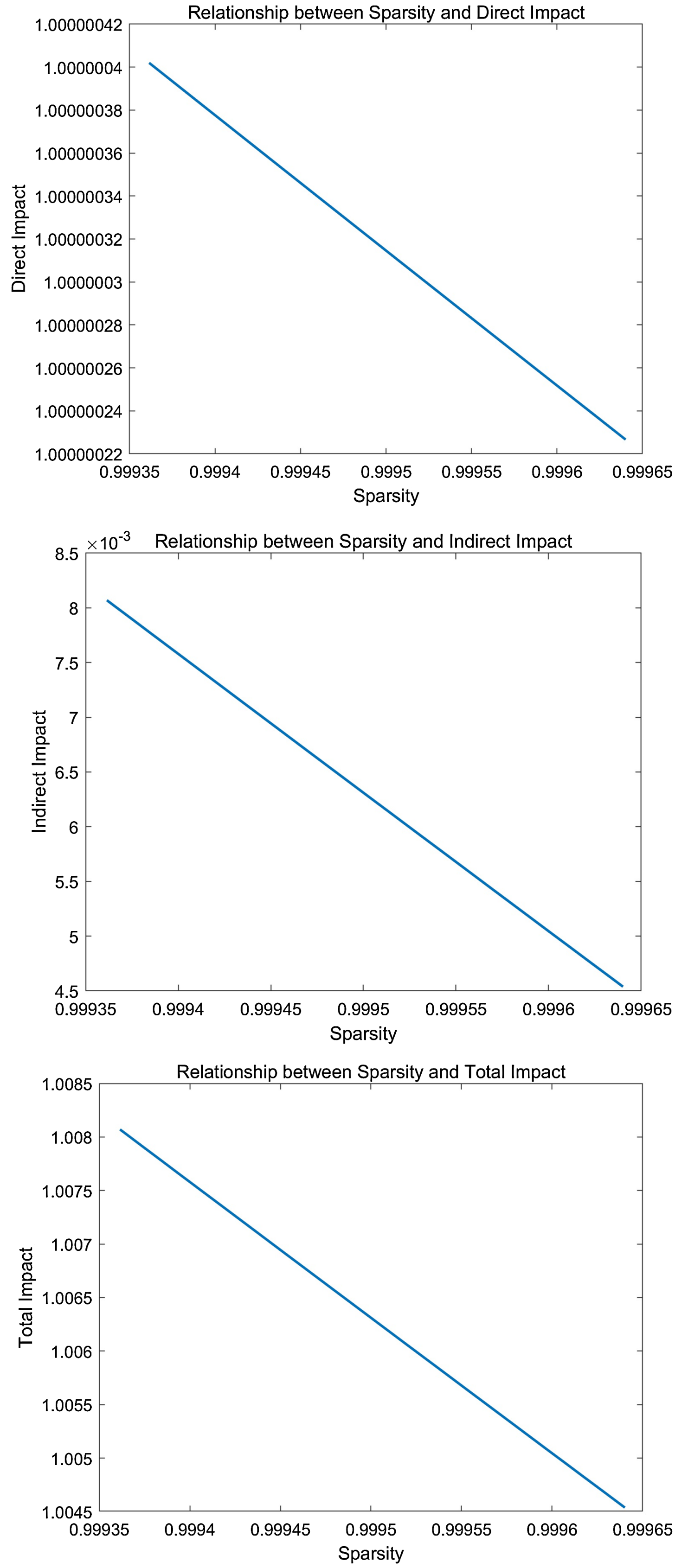

Figure 5. Sparsity and its impact on random core-periphery networks.

core-periphery components, we observe that the random network also exhibits a similar characteristic. Whilst the differences of indirect impacts significantly influence that of total impacts (Panel A of Table 3), the levels of indirect impacts make relatively weaker contributions to the levels of total impacts (Panel B of Table 3). This underscores the importance of analyzing both direct and indirect impacts to achieve a thorough apprehension of the total network effects. We finally arrive at the following conclusion.

Table 3. Simulation results of random core-periphery components. Panel A reports simulation results of the differences in network impacts between $G(1)$ and $G(0)$, where $G(0)$ is a sparser network than $G(1)$. $N = 10000$. To obtain the random number p, we first choose 7 random numbers within range of $[1,10)$, $[10,50)$, $[50,100)$, $[100,500)$, $[500,1000)$, $[1000,5000)$, $[5000,10000)$ respectively. p is then chosen from a factor of N closest to each of the random numbers, which will ensure the number of network blocks $B = N/p$ to be an integer. The number of core agents s is randomly chosen from $\left(1,\frac{p}{2}\right)$ and $\left(\frac{p}{2}, p-1\right)$ respectively after choosing p. Panel B reports simulation results of the relationship between sparsity and network impacts, where $N = 10000$, $p = 400$, $s = 175$, and $B = 25$ are fixed. In both Panel A and B, $\rho = \frac{1}{10000-1} \times \frac{1}{2}$. This will make ρ within the consistent and nonsingular range. Note that our network matrix W is an unnormalized $N \times N$ matrix, hence the ρ is unnormalized and is relatively small. If we normalize W, say, by dividing all its elements by $N-1$, then the normalized $\hat{\rho} = \frac{1}{2}$ (see Jie and Ma [6], Corollary 1 in Appendix A.1). The formula for sparsity is $1 - l/N(N-1)$, where l is the number of random links in a random network.

Panel A. Differences of network impacts between $G(1)$ and $G(0)$, where $G(0)$ is a sparser network than $G(1)$. $N = 10000$.

Range	[1, 10)	[10, 50)	[50, 100)	[100, 500)	[500, 1000)	[1000, 5000)	[5000, N)
Random Number	7	14	58	379	553	3027	5526
p	8	16	50	400	500	2500	5000
B	1250	625	200	25	20	4	2
Random $s \in (1, p/2)$	4	7	10	175	224	354	629
Differences of Direct Impact	6.2545E−10	3.1275E−10	1.0011E−10	1.2524E−11	1.0132E−11	2.1179E−12	1.0754E−12
Differences of Indirect Impact	1.2506E−05	6.2556E−06	2.0042E−06	2.5257E−07	2.0426E−07	4.5902E−08	2.4371E−08
Differences of Total Impact	1.2507E−05	6.2559E−06	2.0043E−06	2.5259E−07	2.0427E−07	4.5904E−08	2.4372E−08
Random $s \in (p/2, p-1)$	6	13	45	364	282	1796	4539
Differences of Direct Impact	6.2543E−10	3.1310E−10	1.0043E−10	1.2762E−11	1.0026E−11	2.0846E−12	1.6749E−12
Differences of Indirect Impact	1.2507E−05	6.2581E−06	2.0074E−06	2.5657E−07	2.0323E−07	4.4772E−08	3.1755E−08
Differences of Total Impact	1.2508E−05	6.2584E−06	2.0075E−06	2.5658E−07	2.0324E−07	4.4774E−08	3.1757E−08

Panel B. Relationship between sparsity and network impacts, where $G(k)$ is a sparser network than $G(k+1)$, $k = 0, 1, 2, \ldots, 6$.

	Number of Links	Sparsity	Direct impact	Indirect impact	Total impact
$G(0)$	35,982	0.999640144	1.000000227	0.004537980	1.004538207
$G(1)$	40,754	0.999592419	1.000000257	0.005140989	1.005141246
$G(2)$	45,244	0.999547515	1.000000285	0.005709031	1.005709316
$G(3)$	49,464	0.999505311	1.000000311	0.006243507	1.006243818
$G(4)$	53,422	0.999465727	1.000000336	0.006745313	1.006745649
$G(5)$	57,216	0.999427783	1.000000360	0.007226805	1.007227165
$G(6)$	60,550	0.999394439	1.000000381	0.007650302	1.007650683
$G(7)$	63,850	0.999361436	1.000000402	0.008069830	1.008070232

Conclusion 3. Given a random size p of a random core-periphery component, if a random number of links are added among core-periphery agents that leads to a decrease in network sparsity, then the average direct, indirect, and total impacts of the network all increase.

5. Conclusions

This paper examines the theoretical relationship between sparsity and its impact on financial networks with core-periphery structure. We begin by analyzing networks with incomplete core-periphery components, exploring two strategies to reduce network sparsity: increasing the number of core agents within components and merging multiple components. Under this analytical framework, we derive closed-form solutions that quantify the relationship between sparsity and its impact on financial networks with incomplete core-periphery structure. Complemented with numerical simulations, we demonstrate that network impact increases as sparsity decreases.

Furthermore, we generalize our findings beyond structured core-periphery networks by introducing random core-periphery components. Through random network generation and simulation, we confirm the inverse relationship between sparsity and network impact, reinforcing the robustness of our main findings.

This research has focused on a tractable linear financial network model, analyzing the sparsity of the network and its associated impacts. However, this approach has limitations in capturing the complexities of real-world financial networks, which often exhibit nonlinear relationships. For instance, di Giovanni and Hale [16] employed a nonlinear spatial autoregressive network model, derived from a macroeconomic general equilibrium framework, to uncover the global stock market transmission mechanism under U.S. monetary policy shocks. Future research could build on our work by incorporating such a nonlinear framework to better understand the relationship between sparsity and its network impact under these complex environments.

Another limitation of our study is its reliance on the network impact measures developed by LeSage and Pace [10], while recent developments in network research have been advanced toward exploring feedback loops within networks. For example, Sharifkhani and Simutin [23] utilized a classic algorithm of Dijkstra [24] to analyze the strength of feedback loops, showing how an industry shock can proliferate through a network and ultimately feed back to the originating industry via network channels. Extending our analytical framework to examine how sparsity influences feedback loops and their broader network impacts would be a compelling direction for future research.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

A1. Proofs for Section 3.1

Proof for Proposition 1:

The adjacency matrix W_i of each core-periphery component i is as follows:

$$\boldsymbol{W}_i = \begin{pmatrix} CC & CP \\ PC & PP \end{pmatrix} = \begin{pmatrix} \mathbf{1_0} & \boldsymbol{R} \\ \boldsymbol{R}' & \mathbf{0} \end{pmatrix} \tag{1'}$$

In this case, $CC = \mathbf{1_0}$ is a $2s \times 2s$ matrix with all ones except the diagonal terms. PP is a $(2p-2s)\times(2p-2s)$ zero matrix. CP is a $2s \times 2(p-s)$ matrix with all ones. PC is the transpose matrix of CP. Thus, we have

$$\boldsymbol{I} - \rho \boldsymbol{W}_i = \begin{pmatrix} \boldsymbol{I} - \rho \mathbf{1_0} & -\rho \boldsymbol{R} \\ -\rho \boldsymbol{R}' & \boldsymbol{I} \end{pmatrix} \tag{2'}$$

$$\boldsymbol{S}_i \stackrel{\text{def}}{=} (\boldsymbol{I} - \rho \boldsymbol{W}_i)^{-1} = \begin{pmatrix} \boldsymbol{I} - \rho \mathbf{1_0} & -\rho \boldsymbol{R} \\ -\rho \boldsymbol{R}' & \boldsymbol{I} \end{pmatrix}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3'}$$

where

$$A = \left(\boldsymbol{I} - \rho \mathbf{1_0} - \rho^2 \boldsymbol{R}\boldsymbol{R}'\right)^{-1} = \begin{pmatrix} a & c & \cdots & c & f & f & \cdots & f \\ c & a & \cdots & c & f & f & \cdots & f \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c & c & \cdots & a & f & f & \cdots & f \\ f & f & \cdots & f & a & c & \cdots & c \\ f & f & \cdots & f & c & a & \cdots & c \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f & f & \cdots & f & c & c & \cdots & a \end{pmatrix},$$

$$a = \frac{1}{2(1+\rho)}\left\{2 + \frac{\rho^2(-p+s)}{-1+\rho\left[-1+\rho(p-s)s\right]} + \frac{\rho(-2-p\rho+\rho s)}{-1+\rho\left[-1+s(2+p\rho-\rho s)\right]}\right\}$$

$$c = \frac{\rho\left(1+\rho+p\rho+p\rho^2-\rho s-\rho^2 s-2p\rho^2 s-p^2\rho^3 s+2\rho^2 s^2+2p\rho^3 s^2-\rho^3 s^3\right)}{(1+\rho)\left(1+\rho-p\rho^2 s+\rho^2 s^2\right)\left(1+\rho-2\rho s-p\rho^2 s+\rho^2 s^2\right)}$$

$$f = \frac{\rho}{\left(1+\rho-p\rho^2 s+\rho^2 s^2\right)\left(1+\rho-2\rho s-p\rho^2 s+\rho^2 s^2\right)}$$

$$B = (\boldsymbol{I} - \rho \mathbf{1_0})^{-1} \rho \boldsymbol{R}\left[\boldsymbol{I} - \rho^2 \boldsymbol{R}'(\boldsymbol{I} - \rho \mathbf{1_0})^{-1} \boldsymbol{R}\right]^{-1}$$

$$= \begin{pmatrix} e & e & \cdots & e & h & h & \cdots & h \\ e & e & \cdots & e & h & h & \cdots & h \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e & e & \cdots & e & h & h & \cdots & h \\ h & h & \cdots & h & e & e & \cdots & e \\ h & h & \cdots & h & e & e & \cdots & e \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ h & h & \cdots & h & e & e & \cdots & e \end{pmatrix}$$

$$e=\frac{\rho\left(1+\rho-\rho s-p\rho^2 s+\rho^2 s^2\right)}{\left(1+\rho-p\rho^2 s+\rho^2 s^2\right)\left(1+\rho-2\rho s-p\rho^2 s+\rho^2 s^2\right)}$$

$$h=\frac{\rho^2 s}{\left(1+\rho-p\rho^2 s+\rho^2 s^2\right)\left(1+\rho-2\rho s-p\rho^2 s+\rho^2 s^2\right)}$$

$$C=\rho\boldsymbol{R}'\left(\boldsymbol{I}-\rho\mathbf{1_0}-\rho^2\boldsymbol{RR}'\right)^{-1}=B'$$

$$D=\left[\boldsymbol{I}-\rho^2\boldsymbol{R}'\left(\boldsymbol{I}-\rho\mathbf{1_0}\right)^{-1}\boldsymbol{R}\right]^{-1}=\begin{pmatrix} b & d & \cdots & d & g & g & \cdots & g \\ d & b & \cdots & d & g & g & \cdots & g \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ d & d & \cdots & b & g & g & \cdots & g \\ g & g & \cdots & g & b & d & \cdots & d \\ g & g & \cdots & g & d & b & \cdots & d \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ g & g & \cdots & g & d & d & \cdots & b \end{pmatrix}$$

$$b=1+\frac{1}{2}\rho^2 s\left(\frac{1}{1+\rho+\rho^2 s(-p+s)}+\frac{1}{1+\rho+\rho s(-2-p\rho+\rho s)}\right)$$

$$d=\frac{\rho^2 s\left(1+\rho-\rho s-p\rho^2 s+\rho^2 s^2\right)}{\left(1+\rho-p\rho^2 s+\rho^2 s^2\right)\left(1+\rho-2\rho s-p\rho^2 s+\rho^2 s^2\right)}$$

$$g=\frac{\rho^3 s^2}{\left(1+\rho-p\rho^2 s+\rho^2 s^2\right)\left(1+\rho-2\rho s-p\rho^2 s+\rho^2 s^2\right)}$$

QED

Proof for Proposition 2:

Based on definition 2 in Section 2, we can calculate the average direct/indirect/total impacts with normalized $\beta_k=1$ as follows. However, as will be shown, we can only provide a theoretical conclusion regarding the average total impact of the network. The formulae of average direct and indirect impacts are too complicated to obtain a theoretical result. We conduct numerical simulations and present the results in **Table 1**.

First, we have average total impact with normalized $\beta_k=1$:

$$\text{Average total impact}=\frac{p+p\rho-\rho s}{p\left(-1-\rho+2\rho s+p\rho^2 s-\rho^2 s^2\right)} \tag{4'}$$

We take the derivative of total impact with respect to s:

$$\begin{aligned}&\frac{\partial\text{Average Total Impact}}{\partial s}\\&=\frac{\rho\left(-1+2p-\rho+2p\rho+p^2\rho+p^2\rho^2-2p\rho s-2p\rho^2 s+\rho^2 s^2\right)}{p\left(-1-\rho+2\rho s+p\rho^2 s-\rho^2 s^2\right)^2}\\&=\frac{\rho\left(2p-1+(2p-1)\rho+p\rho(p-2s)+\rho^2(p-s)^2\right)}{p\left(-1-\rho+2\rho s+p\rho^2 s-\rho^2 s^2\right)^2}\end{aligned} \tag{5'}$$

Since $2p-1+p\rho(p-2s)>2p-1+p\rho(s-2s)=2p-1-p\rho s$

$$=p(2-\rho s)-1\geq p-1\geq 0\text{, as } 0<\rho<\frac{1}{N-1}<\frac{1}{s}$$

(see Jie and Ma [6], Corollary 1 of Appendix A.1).

$$\text{Thus } \frac{\partial \text{Average Total Impact}}{\partial s}>0. \tag{6'}$$

Next, we have the following two complicated formulae for average direct and indirect impacts with normalized $\beta_k=1$. Their numerical simulations results are presented in Table 1.

$$\begin{aligned}\text{Average direct impact} &= \left(\frac{1}{N}\sum_{i=1}^{N}s_{ii}\right)\beta=\frac{1}{p}\left[sa+(p-s)b\right]\\ &=\frac{1}{2p}\left\{-2+2p-\frac{2+2\rho s}{1+\rho}+\frac{2+\rho}{1+\rho+\rho^2 s(-p+s)}\right.\\ &\quad\left.+\frac{2+\rho-2\rho s}{1+\rho+\rho s(-2-p\rho+\rho s)}\right\}\end{aligned} \tag{7'}$$

$$\begin{aligned}\text{Average indirect impact} &= \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j\neq i}s_{ij}\right)\\ &=\frac{1}{p}\left[(p-s)(p-s-1)d+2s(p-s)(e+h)+(p-s)^2 g+s^2 f+s(s-1)c\right]\\ &=\frac{1}{2p}\left\{2-2p+\frac{2+2\rho s}{1+\rho}-\frac{2+\rho}{1+\rho+\rho^2 s(-p+s)}\right.\\ &\quad\left.-\frac{2+\rho-2p(1+\rho)}{1+\rho+\rho s(-2-p\rho+\rho s)}\right\}\end{aligned} \tag{8'}$$

QED

A2. Proofs for Section 3.2

Proof for Proposition 3:

The adjacency matrix W_i of each core-periphery component i is as follows:

$$\boldsymbol{W}_i=\begin{pmatrix} CC & CP\\ PC & PP\end{pmatrix}=\begin{pmatrix}\boldsymbol{1_0} & \boldsymbol{R}\\ \boldsymbol{R'} & \boldsymbol{0}\end{pmatrix} \tag{9'}$$

In this case, $CC=\boldsymbol{1_0}$ is a $2s\times 2s$ matrix with all ones except the diagonal terms. PP is a $(2p-2s)\times(2p-2s)$ zero matrix. CP is a $2s\times 2(p-s)$ matrix with all ones. PC is the transpose matrix of CP. Suppose the number of cores in each component in the initial adjacency matrix is $2s_0$ and the size of component in the initial adjacency matrix is $2p_0$, the number of cores in each component is $2s=2\mu s_0$ after merging μ small components into one big component with the size of $2p=2\mu p_0$. That is, $p=\frac{p_0}{s_0}s=\lambda s$ where $\lambda=\frac{p_0}{s_0}>1$ is fixed. In this circumstance, p has a linear relationship with s. Thus, we have

$$\boldsymbol{I}-\rho\boldsymbol{W}_i=\begin{pmatrix}\boldsymbol{I}-\rho\mathbf{1}_0 & -\rho\boldsymbol{R}\\ -\rho\boldsymbol{R}' & \boldsymbol{I}\end{pmatrix} \tag{10'}$$

$$\boldsymbol{S}_i\overset{\text{def}}{=}\left(\boldsymbol{I}-\rho\boldsymbol{W}_i\right)^{-1}=\begin{pmatrix}\boldsymbol{I}-\rho\mathbf{1}_0 & -\rho\boldsymbol{R}\\ -\rho\boldsymbol{R}' & \boldsymbol{I}\end{pmatrix}^{-1}=\begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{11'}$$

where

$$A=\left(\boldsymbol{I}-\rho\mathbf{1}_0-\rho^2\boldsymbol{R}\boldsymbol{R}'\right)^{-1}=\begin{pmatrix}a & c & \cdots & c & f & f & \cdots & f\\ c & a & \cdots & c & f & f & \cdots & f\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ c & c & \cdots & a & f & f & \cdots & f\\ f & f & \cdots & f & a & c & \cdots & c\\ f & f & \cdots & f & c & a & \cdots & c\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ f & f & \cdots & f & c & c & \cdots & a\end{pmatrix}$$

$$a=\frac{1}{2(1+\rho)}\left\{2+\frac{\rho^2(1-\lambda)s}{-1+\rho\left[-1+\rho(\lambda-1)s^2\right]}+\frac{\rho\left(-2+(1-\lambda)\rho s\right)}{-1+\rho\left[-1+s\left(2+(\lambda-1)\rho s\right)\right]}\right\}$$

$$c=\frac{\rho\left(1+\rho+\lambda s\rho+\lambda s\rho^2-\rho s-\rho^2 s-2\lambda\rho^2 s^2-\lambda^2\rho^3 s^3+2\rho^2 s^2+2\lambda\rho^3 s^3-\rho^3 s^3\right)}{(1+\rho)\left(1+\rho-\lambda\rho^2 s^2+\rho^2 s^2\right)\left(1+\rho-2\rho s-\lambda\rho^2 s^2+\rho^2 s^2\right)}$$

$$f=\frac{\rho}{\left(1+\rho+\rho^2 s^2-\lambda\rho^2 s^2\right)\left(1+\rho-2\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}$$

$$B=\left(\boldsymbol{I}-\rho\mathbf{1}_0\right)^{-1}\rho\boldsymbol{R}\left[\boldsymbol{I}-\rho^2\boldsymbol{R}'\left(\boldsymbol{I}-\rho\mathbf{1}_0\right)^{-1}\boldsymbol{R}\right]^{-1}$$

$$=\begin{pmatrix}e & e & \cdots & e & h & h & \cdots & h\\ e & e & \cdots & e & h & h & \cdots & h\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ e & e & \cdots & e & h & h & \cdots & h\\ h & h & \cdots & h & e & e & \cdots & e\\ h & h & \cdots & h & e & e & \cdots & e\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ h & h & \cdots & h & e & e & \cdots & e\end{pmatrix}$$

$$e=\frac{\rho\left(1+\rho-\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}{\left(1+\rho+\rho^2 s^2-\lambda\rho^2 s^2\right)\left(1+\rho-2\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}$$

$$h=\frac{\rho^2 s}{\left(1+\rho+\rho^2 s^2-\lambda\rho^2 s^2\right)\left(1+\rho-2\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}$$

$$C=\rho\boldsymbol{R}'\left(\boldsymbol{I}-\rho\mathbf{1}_0-\rho^2\boldsymbol{R}\boldsymbol{R}'\right)^{-1}=B'$$

$$D = \left[\boldsymbol{I} - \rho^2 \boldsymbol{R}' \left(\boldsymbol{I} - \rho \mathbf{1_0} \right)^{-1} \boldsymbol{R} \right]^{-1} = \begin{pmatrix} b & d & \cdots & d & g & g & \cdots & g \\ d & b & \cdots & d & g & g & \cdots & g \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ d & d & \cdots & b & g & g & \cdots & g \\ g & g & \cdots & g & b & d & \cdots & d \\ g & g & \cdots & g & d & b & \cdots & d \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ g & g & \cdots & g & d & d & \cdots & b \end{pmatrix}$$

$$b = 1 + \frac{1}{2}\rho^2 s \left(\frac{1}{1+\rho-(\lambda-1)\rho^2 s^2} - \frac{1}{-1+\rho\left(-1+s\left(2+(\lambda-1)\rho s\right)\right)} \right)$$

$$d = \frac{\rho^2 s\left(1+\rho-\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}{\left(1+\rho+\rho^2 s^2-\lambda\rho^2 s^2\right)\left(1+\rho-2\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}$$

$$g = \frac{\rho^3 s^2}{\left(1+\rho+\rho^2 s^2-\lambda\rho^2 s^2\right)\left(1+\rho-2\rho s+\rho^2 s^2-\lambda\rho^2 s^2\right)}$$

QED

Proof for Proposition 4:

Based on definition 2 in Section 2, we can calculate the average direct/indirect/total impacts with normalized $\beta_k = 1$ as follows. However, as will be shown, we can only provide a theoretical conclusion regarding the average total impact of the network. The formulae of average direct and indirect impacts are too complicated to obtain a theoretical result, leading their numerical simulations to be conducted in Table 2.

First, we have average total impact with normalized $\beta_k = 1$:

$$\begin{aligned} &\text{Average total impact} \\ &= \text{Average direct impact} + \text{Average indirect impact} \\ &= \frac{\lambda + \lambda\rho - \rho}{\lambda\left(1+\rho-2\rho s+\rho^2 s^2(1-\lambda)\right)} \end{aligned} \tag{12'}$$

We take the derivative of average total impact with respect to s:

$$\begin{aligned} &\frac{\partial \text{Average Total Impact}}{\partial s} \\ &= \frac{2\rho(\lambda-\rho+\lambda\rho)(1-\rho s+\lambda\rho s)}{\lambda\left(1+\rho-2\rho s+\rho^2 s^2(1-\lambda)\right)^2} > 0 \end{aligned} \tag{13'}$$

as $\lambda - 1 > 0$. Thus $\frac{\partial \text{Average Total Impact}}{\partial s} > 0$.

Next, we have the following two complicated formulae for average direct and indirect impacts with normalized $\beta_k = 1$. Simulations results of them are presented in Table 2.

$$\text{Average direct impact} = \left(\frac{1}{N}\sum_{i=1}^{N} s_{ii}\right)\beta = \frac{1}{p}\left[sa + (p-s)b\right]$$
$$= \frac{1}{2\lambda s}\left\{-2 + 2\lambda s - \frac{2\rho s + 2}{1+\rho} + \frac{2+\rho}{1+\rho+\rho^2 s(-\lambda s + s)} + \frac{2+\rho-2\rho s}{1+\rho+\rho s(-2-\lambda s\rho+\rho s)}\right\} \tag{14'}$$

$$\text{Average indirect impact} = \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j\neq i} s_{ij}\right)$$
$$= \frac{1}{p}\left[(p-s)(p-s-1)d + 2s(p-s)(e+h) + (p-s)^2 g + s^2 f + s(s-1)c\right]$$
$$= \frac{1}{2\lambda s}\left\{2 - 2\lambda s + \frac{2+2\rho s}{1+\rho} - \frac{2+\rho}{1+\rho+\rho^2 s(-\lambda s + s)} - \frac{2+\rho-2\lambda s(1+\rho)}{1+\rho+\rho s(-2-\lambda s\rho+\rho s)}\right\} \tag{15'}$$

QED