

Want More Precise Private Equity Valuations? DLOM Inclusion Provides Good Answer

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Abstract

We aim to provide practical and useful guidance for including discount for lack of marketability (DLOM) as part of precise, true and fair Private Equity (PE) valuation. The present accounting regulation on fair value measurement for PE prescribes use of exit prices (realisable bid-prices) while DLOMs are neglected. The DLOM research area originates from empirical studies on restricted stocks and pre-initial public offerings, followed by theoretical studies using put option models developed as attempts to demonstrate why the DLOM exists and its precise magnitude. The latest two contributions focus on the thinly trading aspect and provide quite precise DLOMs (and thus exit prices). For our analyses, we use relevant and well-defined illustrative examples earlier presented in the DLOM-literature, and we follow two evaluation criteria: The approach must make plausible economic sense, and it should only require minor input that should be either directly available or easily estimable. We contribute to the research area by showing how easy-to-use tools are useful for determining realistic DLOMs (and consequently also bid prices and thus exit prices) as part of the valuation of PE, and the spread of our findings will definitively be helpful when a precise PE value estimation is required.

Keywords

Private Equity, Valuation, DLOM, Fair Value

1. Introduction

According to Investopedia.com, it is without question that selling a nonmarketable ownership interest in a privately held company (Private Equity—PE) is a more costly, uncertain and time-consuming process than liquidating a position in a publicly traded entity. An investment in which the owner can achieve liquidity in

a timely fashion allows a seller to receive a higher price than an investment in which the owner cannot sell the investment quickly. Consequently, privately held companies should sell at a discount compared to the actual intrinsic value because of such additional costs, increased uncertainty and longer time horizons tied to selling non-listed securities.

The detailed regulation for the measurement of fair value for capital investments, including PE, in the annual report is available in IFRS 13 issued by IASB¹ [1]. The fair value is defined in IFRS 13 Appendix A and IFRS 13.2 as the price that would be received to sell an asset in an orderly transaction between market participants at the measurement date under current market conditions (i.e. an exit price at the measurement date from the perspective of a market participant that holds the asset). Further in IFRS 13.11, it is required that accountants present fair value for PE investments which take market participants' concern into consideration, and which reflects intrinsic value in use, i.e. the best estimate of expected future cash flows at present use of asset, discounted at an appropriate discount rate. Although this value can be considered as the company's most realistic ask price, it might be far away from the price a potential buyer would be willing to pay.

IFRS13.53 prescribes that for prices in a bid-ask price span, the most representative price should be applied, and that the use of an ask price or a mid-price is accepted while the use of bid price is permitted. This contrasts with prescribing use of bid price which would be much closer to a realisable exit price in short term, since such a price would reflect the concerns of finding a willing buyer, which for PE might be the most difficult part of a sale.

Unfortunately, for PE the bid price is in general unobservable for which reason other solutions may be considered. [2] defines the discount for lack of marketability (DLOM) as "an amount or percentage deducted from the value of an ownership interest to reflect the relative absence of marketability". Marketability is defined as "the ability to quickly convert property to cash at a minimal cost". The two Glossary definitions can be combined to obtain a more detailed definition of DLOM, as "an amount or percentage deducted from the value of an ownership interest to reflect the relative absence of the ability to convert the property to cash at a minimal cost". Thus, by definition the DLOM is a predictable reduction in obtainable price for the PE that reveal the best bid price. However, since the latter is usually unknown the size of the discount is difficult to estimate, and DLOMs have frequently been the subject of controversy in valuations since applying a DLOM can result in a significant reduction compared with the pro rata share of a business interest. The DLOM magnitude becomes important as the difference between ask price and bid price (usually a percentage calculated as the difference between ask and bid price divided by the ask price), and it reflects both the cost of bearing risk during period in which the asset cannot be sold, and the cost of not

¹ASC 820 issued by FASB provide similar regulation for measuring and disclosing fair value in US-GAAP [3].

having immediate access to capital.

Consequently, our research question is how we should act on these concerns when searching for precise, true and fair PE values for the annual report.

[4] find that although the academic literature on valuation is rigorous, it has provided little useful information for the valuation practitioner seeking fair PE prices. On the other hand, [5] states that CPAs should never simply use a model or a formula for calculating a DLOM in place of professional judgment, since the final price will always be the conclusion of some negotiations. Some of the PE valuation literature recognises the existence of the DLOM, like [6] and [7], although the discussion and concrete guidance on how large an eventual DLOM should be lacks. During the last few decades, practitioners needing support for useful DLOM suggestions have had much influence on the development and implementation of empirical and theoretical models used for estimating realistic PE values. However, there seems not to be a clear and concrete cut as to what to do as evidenced by for instance several court cases on gift and estate tax disputes in the US, as presented by [8] and [9].

The remainder of the paper is structured in the following way: In section 2, we present some of the DLOM background literature and the history behind the development through empirical studies, and later the theoretical put option models. In section 3, we present the two latest theoretical approaches and discuss their usefulness as part of more realistic and precise practical PE valuation, and finally, we conclude the paper in section 4.

2. DLOM-Documentation

Traditionally empirical and theoretical studies have been used to both justify the existence and to measure the DLOM in PE.

2.1. Empirical Restricted Stock Studies and Pre-IPO Studies

In the US, publicly traded companies sometimes raise capital by completing a private placement of equity securities or debt. In an equity private placement, a company can issue either registered stock to general investors or unregistered (i.e. restricted) stock to an accredited investor. Registered stock includes the shares of publicly traded companies that generally can be freely traded in the open market. Unregistered shares of stock are not registered for trading on a stock exchange preventing the buyer from selling the stock freely until after some specific holding period (initially 3 years in 1969, over the years, the restrictions under US SEC Rule 144 have been loosened, like the amendment in 1990 that allowed qualified institutional investors to trade unregistered securities among themselves, which resulted in an increase in the potential buyers of restricted securities) [10]. When publicly traded companies issue restricted (unregistered) stock, the restricted stock is typically sold at a price discount compared to the price of the (registered) publicly traded stock, which is considered to be the DLOM.

In **Table 1** we summarise a number of the studies [11]-[18], when the DLOMs are measured using comparisons of restricted prices with prices of the same company's unrestricted securities eligible for trading on the open market.

Table 1. Summary of restricted stock studies.

Empirical study	Year	Period covered	Trans- actions	Average DLOM	Median DLOM
SEC Overall Average [11]	1971	1966-1969	398	25.8%	24%
Robert E. Moroney [12]	1973	1969-1972	148	35.6%	34%
J. Michael Maher [13]	1976	1969-1973	33	35.4%	33%
Karen H. Wruck [14]	1989	1979-1984	37	14%	12%
Hertzel and Smith [15]	1993	1980-1987	106	20%	13.5%
William L. Silber [16]	1991	1981-1988	69	33.8%	-
Bajaj <i>et al.</i> [17]	2001	1990-1995	88	22%	21%
FMV Opinions Study [18]	2007	1997-2005	475	22.3%	19.5%
Grand total				25%	

The results of these studies have generally shown average discounts between 20 and 35 per cent, and a weighted grand total of 25 per cent. In some cases, the discount was only 12 to 14 per cent, while the discount was higher than 35 per cent in other studies not included in **Table 1**. Many of the studies presented are quite old which is due to the loosening of the restrictions under US SEC Rule 144 over time [10].

The restricted stock studies generally have some severe limitations: the studies are based on quite small sample sizes, they are spread out over several years, companies rarely issue large amounts of restricted stock, the restrictions have been substantially loosened during the last decades, and the standard errors in the observed DLOM estimates are quite big. The wide range in the discounts can be due to several factors that potentially affect the DLOM, like block size, dividends, company size and other characteristics. Additionally, companies using this financial instrument typically bear a higher risk due to the more precarious financial health of companies that regularly trade on a stock market, and consequently, the potential investors often have some relationship with the company for which the discount is made (they are employees or already present shareholders).

Furthermore, restricted shares of public stock may not be traded directly on a stock exchange, but in a short future time, the investor has certainty that the trading restrictions will lapse, which contrasts with the stock of a closely held company that may never be traded on a public stock exchange. Accordingly, the discount in the various studies only reflect the time value, and the potential changes in value during expected holding period. This is in contrary to the PE where a possible time of sales is unknown, and consequently, the studies may not be an appropriate proxy for the DLOM after all?

A pre-Initial Public Offering (IPO) study examines sale transactions in the stock of a closely held company that has subsequently achieved a successful IPO. Since the early 1970s, these studies have been utilised to estimate DLOM by comparing the price of an equity interest in a company prior to IPO as reported in US SEC filings to the trading price of common stock in the same company after the IPO. The underlying assumption is that a stock sold prior to the IPO is nonmarketable while the same stock after the IPO is fully liquid and marketable, and therefore the percentage difference in the two prices must reflect differences in marketability. **Table 2** summarises the two most important pre-IPO studies.

Table 2. Summary of initial public offering studies.

Empirical study	Period covered	Prospectuses reviewed/analysed	Transactions analysed	Weighted Average DLOM	Weighted Median DLOM
Willamette Management Associates Study [19]	1975-2002	605	1071	45.7%	51.7%
Emory [20]	1980-2000	4,088	593	47.0%	47.4%
Grand total				46%	

Willamette study [19]: For 27 years Willamette conducted a series of studies on the prices of private stock transactions relative to those of subsequent offerings of stocks of the same companies. The source documents for the studies were complete US SEC registration statements including disclosure of all private transactions in the stock within three years before the public offering. The studies excluded financial institutions and natural resource companies, but also stock options transactions and main part of sales of stocks to insiders. Everything was heavily validated in an attempt to make sure that only transactions on an arm's-length basis were included. For each transaction for which meaningful earnings data were available in the registration statement as of both the private transaction and public offering dates, the price/earnings multiple of each private transaction was compared with the subsequent public offering price/earnings multiple. Because the private transactions occurred over a period of up to three years prior to the public offering the price/earnings multiples were adjusted for differences in the industry average price/earnings multiple between the time of the private transaction and that of the public offering. The DLOM conclusions showed a mean and median discount over all the studies equal to 46 per cent respectively 52 per cent.

Emory study [20]: The conducted studies saw the differences between prices of private transactions in a company prior to an initial public offering and the subsequent public sales price of that company as DLOMs. Emory reviewed 4088 prospectuses but analysed only 593 transactions from 1980 to 2000, since development-stage companies, companies with operating losses and companies with small IPO prices (below 5\$) were eliminated from the sample. The transactions in the studies were primarily the granting of stock options at the stock's then-fair market value, and the remaining transactions involved sales of stock. In defending

the stock option prices used by the companies in the studies, Emory states that “in most cases the transactions were stated to have been or could reasonably be expected to have been at fair market value. All ultimately would have had to be able to withstand US SEC, IRS or judicial review, particularly considering the subsequent public offering”. However, the transactions were typically with insiders which thus may not be fully reflective of fair market value, but rather smaller and consequently may show unreal larger DLOMs. The mean and median price discounts from the transactions analysed equal 46 per cent and 47 per cent, respectively.

Although the underlying concept of pre-IPO is theoretically appealing, the studies rest upon the fundamental assumption that the change in price entirely reflects improvement in the marketability of the shares. Unlike restricted stock studies, pre-IPO studies suffer from the weakness that the price comparison of the two stocks is non-contemporaneous. As such, changes in pricing may relate to significant changes in the companies’ macroeconomic environment (like the economic conditions and business fundamentals), and more firm specific oriented considerations (like cost of capital and growth), and not just lack of marketability. Consequently, some of the results have later been questioned, since the studies do not consider that the transactions reflect compensation for insider services, are overstated due to self-selection and survivorship bias, and reflect IPO hype and timing with rising earnings.

2.2. Theoretical Put Option Modelling

The theoretical models do not derive DLOM conclusions from transactional data, but as the first, [21] suggested that “if one holds restricted or non-marketable stock and purchases an option to sell those shares at the free market price, the holder has, in effect, purchased marketability for those shares. The price of that put option is the discount for lack of marketability.” [21] used the [22] option pricing model to estimate the put option price and found that the purchase of a restricted stock reasonably replicates the lapsing in accordance with the US SEC rule 144 restrictions [10].

[21] also found that the parameter *volatility* is the most difficult to define in the context of a private company. While the return volatility of the stock market is readily available, and the price volatility of publicly traded stocks is easily calculated from publicly available price data, the price volatility of an interest in a privately owned business is unknown. However, a reasonable estimate is easily made if the appraiser can identify at least one appropriate publicly traded company to use as a benchmark. Alternatively, the practitioner may conclude that an index such as the S&P500 or the VIX would be an appropriate price volatility surrogate, although considerations should be given to tendencies of broad indexes to negate the unsystematic risks of the individual stocks that comprise it, thereby understating the price risks of the underlying stocks and thus the volatility. The average price volatilities of the index constituents may, therefore, be better measures of risk than the index. Since volatility in general is low for large capitalisation, ac-

tively traded issues and trends sharply upward for shares of smaller companies or highly speculative companies, and accordingly the implication is that volatility of shares of a small, privately held company should be at least 60 per cent or maybe rather more than 75 per cent. Among others, [23] show that volatility, both market and stock return volatility, is generally highly persistent, but also that the average stock volatility is remarkably higher trading on the NASDAQ (0.443) compared to the NYSE (0.293).

Between 1993 ([21]) and 2014 ([24]) several plausible and more and more advanced put option models were developed and introduced by several different authors in a process where presentation of one model successively led to critique and presentation of new and probably better models. In **Table 3**, we present the key elements in the different models as well as comparable results using same illustrative example.

Because of their nature, the holding period and volatility factors have the greatest impact on the option prices, and thus such models may understate the DLOM, as they ignore other factors that may reduce the marketability for privately held securities (e.g. contractual transferability restrictions or control over the company, and eventual other relevant factors that also may contribute to lack of marketability). However, the DLOM indicated by a put option pricing model is an appropriate starting point for a DLOM analysis, since, as the restricted stock studies generally indicate, the longer the required holding period, the greater the price discount a buyer demands, while the volatility directly influences the DLOM. When an investor owns a security which is restricted from trading, that investor assumes the risks of both not being able to sell the investment if the value begins to decline, and not being able to sell the investment to reallocate funds to another investment. The first risk factor is affected by highly volatile stocks, since, as volatility increases, the risk of stock price decrease increases along with increases in other risks related to holding a nonmarketable security likewise increases.

Regarding the estimation of DLOMs for a privately held company interest, the results shown are for companies having a range of volatility between 50 per cent and 125 per cent, and within this range of volatility, the Chaffe, Longstaff, Finnerty (modified), and Ghaidarov models, [21]-[30], produce reasonable results since they reflect DLOMs at same level as the discounts implied by the empirical restricted stock and IPO studies.

For comparison our illustrative example presented in **Table 3** is inspired by the articles originally presenting the different put option models. As mentioned above, the volatility for a small privately held company's equity would probably be some 75 per cent if proxied appropriately. The risk-free interest has changed slightly over the years, but a rate of 2.0% may (still) be appropriate. Further, since average time that elapsed from initial offering date to closing for transactions presented in [31] seems to be around an average of 212 days (with some variation, since the standard deviation is around 192 days), we chose a time horizon of one year.

Table 3. DLOM summary for most important put option models. Basic assumptions: $T = 1$ year; $S = X$; $r_f = 2.0\%$; Volatility range = 50% - 125%; no dividend yield; the DLOM is calculated as a percentage.

Author and option model	Mathematical model	DLOM for one-year holding period (Volatility in the range 50% - 125%)	DLOM—at volatility 75%
Chaffé European put option model [21]	$\text{DLOM} = e^{-rT} \times N(-d_2) - N(-d_1)$ where: $d_1 = \frac{T\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$	18.6% - 45.4%	28.0%
Longstaff; Abbott American put (lookback) option model [25]-[27]	$\text{Put value} = \left[\left(2 + \frac{\sigma^2 T}{2} \right) \times N\left(\sqrt{\frac{\sigma^2 T}{2}} \right) + \sqrt{\frac{\sigma^2 T}{2\pi}} e^{-\frac{\sigma^2 T}{8}} - 1 \right]$ Subsequently DLOM = Put value/(1 + Put value)	31.8% - 59.2%	43.0%
Finnerty Asian style (average-strike) put option model [28] [29]	$\text{DLOM} = \left[N\left(\frac{v\sqrt{T}}{2} \right) - N\left(-\frac{v\sqrt{T}}{2} \right) \right]$ where: $v\sqrt{T} = \sqrt{\sigma^2 T + \ln\left[2\left(e^{r\sigma^2} - \sigma^2 T - 1 \right) \right] - 2\ln\left(e^{r\sigma^2} - 1 \right)}$	11.2% - 24.6%	16.3%
Ghaidarov Asian style (average-strike) put option model [30]	$\text{DLOM} = \left[N\left(\frac{V_r}{2} \right) - 1 \right]$ where: $V_r = \sqrt{\ln\left[2\left(e^{r\sigma^2} - \sigma^2 T - 1 \right) \right] - 2\ln\left(\sigma^2 T \right)}$	11.6% - 30.0%	17.5%
Ghaidarov European (forward starting) put option model [24]	$\text{DLOM} = \left[2N\left(\frac{\sigma\sqrt{T}}{2} \right) - 1 \right]$	19.7% - 46.5%	29.3%

The inputs in all above models, and their partial effect on DLOM are as follows:

Variable	Variable description	Effect of change on DLOM
$N(\cdot)$	Cumulative probability function for a standard normal distribution	No effect
$\ln(\cdot)$	Natural logarithm function	No effect
E	Exponential growth constant; the base of the natural logarithm	No effect
π	A mathematical constant	No effect
T	Time to expiration (in years, holding period)	Increases as time increases
r_f	Risk-free interest rate	Increases as interest rate increases
σ	Annualized volatility of the underlying security	Increases as volatility increases

Although an option price and the DLOM are both affected by volatility of the underlying security and time horizon, some analysts think put option pricing models are not applicable for estimating the DLOM, since option pricing models

were originally derived to determine option prices for publicly traded securities, that reflect characteristics very different from those applied to closely held securities. [32] find that only some of the option pricing models work relatively well when certain holding periods and volatilities are used as inputs which they suggest is because option pricing models theoretically have little to do with marketability of the underlying security. In fact, already [33] argued that none of the different put option models quite fit the situation that they are attempting to model, which especially proves right for longer holding periods, i.e. more than three to four years, and thus he suggested an alternative shout put model in those situations. [34] find that option models tend to understate the DLOM since the models ignore other factors than holding period and volatility, that may reduce the marketability for privately held securities. However, despite the differences between how option prices are determined and the factors that affect marketability, the time and price volatility variables work together when determining an appropriate DLOM, and so does option pricing models since they also incorporate the relationship between volatility and a relative shorter time horizon, and thus may provide insight in determining an appropriate DLOM. Relative to immediately marketable investments, the value of illiquid investments must be discounted to reflect the uncertainties of the timing and realisable price of a sale, and assets may be subject to greater illiquidity during periods of market stress calling for an increased DLOM.

Also, [35] as well as [23] and [36] compared the different option models with empirical DLOMs from restricted stock studies and found that models based on [22] did best when comparing with empirical results from restricted stock studies, although such a comparison does not answer the question whether an option model is the most suitable way of describing the DLOM. [37] summarizes and presents some more general criticism on the use of option models, since she finds that it is no secret that the estimation of the DLOM for privately held company ownership interest is possibly the most contentious issue facing valuation analysts. She finds that even though there exists no option market for private shares, option models may still provide insight or an approximation that is useful ([32]). One consequence would be that the option models generate DLOMs that will be too low for privately held companies because 1) securities subject of the studies (and options) are liquid; and 2) owner has not ability to hedge private company stocks. [32] question if a hedging strategy based on stock options is a legitimate way to estimate DLOM for non-marketability securities (see page 27). So, in total, [37] questions whether put option models are useful proxies for what happens, since the put option pricing models were designed to produce results that comport with the discounts of restricted stock studies, and merely proxies for estimating the DLOM based on temporary trading restrictions; they do not reflect all of the marketability issues faced by typical privately held companies, only the illiquidity and not the finding like market-imperfection or non-effectiveness.

In this respect, the two later models take the holding period uncertainty more explicitly into account, trying more realistically and precisely to reflect the DLOM for what it is, and thus also the marketability aspect. The only focus in the option modelling approach is the (fixed) holding period and volatility, but by nature (for private equity), the holding period is unknown. [38] introduce durability to modelling the unknown future trade using existing knowledge. [39] approaches the same problem in his focus on the (thin) trading aspect, that there is not a potential buyer in sight. Thus, the newest two approaches reflect more intuitively what happens, but the problem that remains is how to find useful data in a non-existing market which includes getting the company's stocks marketed, i.e. find a willing buyer and agree on price with him.

3. Latest Modelling—Do They Solve the Problem

[38] states on page 100 that “to (his) knowledge, existing quantitative models are inadequate in addressing the impact of dividend yield on the illiquidity discounts”, and thus we disregard eventual expected dividend payments and other “pre-payments” in the following.

3.1. A Thin Trading Perspective

The starting point for the [39] approach is the recognition that sales of thinly traded assets typically occur at prices far lower than would be the case if there were a liquid public market, which provides a simple framework which is useful for placing an upper bound on the size of the DLOM. In his modelling of the illiquidity, he uses the concept of a stopping rule, i.e. implementing the decision rule to sell an asset at a prespecified date as a restriction which an investor can face in selling the asset. [39] models the illiquidity discount as an exchange option and arrives at the formula: $N(\sqrt{\sigma^2 T}/2) - N(-\sqrt{\sigma^2 T}/2)$, where $N(\cdot)$ is the standard cumulative normal distribution function; where σ is the volatility of the liquid asset; and where T is the length of the illiquidity time horizon. This expression is identical to the [24] formulated discount, shown as the last put option model in our Table 3, because of the symmetry of the standard cumulative normal distribution function.

However, one of the crucial questions is how long the time horizon T really is, and thus the above can be seen as simplification of a more general setting where the time horizon T is random (stochastic), i.e. the asset remains illiquid until the realisation of a random event that could be modelled using a Poisson process with intensity λ . Consequently, the illiquidity horizon is exponentially distributed with density $\lambda \cdot \exp(-\lambda T)$, which implies that the mean time the asset is illiquid is $1/\lambda$. [39] finds that the stochastic pendant to the above discount percentage is (see eventually his Appendix for further details):

$1/\sqrt{1+8\lambda/\sigma^2}$, where λ is the density (as known in the Poisson distribution function), and σ is volatility.

While the classical put option models use “a well-defined restriction period”,

[39] presents evidence that uncertainty over the illiquidity time horizon results in the expected value being higher than the value of the exchange option evaluated at the expected (discrete) length of the illiquidity horizon.

3.2. A Put Option Perspective with Unknown Time Horizon

Like [38] [39] realise that investors in many situations face a random indefinite illiquidity horizon of uncertain length. In the absence of other information, a common choice is to model the random liquidity event as a Poisson process, which in turn, implies that the liquidity horizon is exponentially distributed, exactly like [39]. However, there may be many instances in which the exponential distribution is not an appropriate choice and the distribution of the time frame to a future liquidity event may be unknown or impossible to parametrise. For instance, the intensity rate describing the rate of liquidity (through an exit) per unit of time may not be constant as the company matures through different developmental stages, suggesting that the Weibull distribution might be a more appropriate fit in such cases. As another example, in certain “trendy” sectors there might be a large concentration of early exits for some companies, and significantly delayed exits for other businesses requiring multiple rounds of financing until the development of a commercially viable and scalable product that creates a bimodal distribution of waiting times.

Consequently, [38] make the realistic assumption that the distribution of the random liquidity horizon is unknown since the investor usually does not have sufficient information to estimate the probability of a liquidity event in a future time. Various data aggregators frequently report average exit times for private equity companies across different sectors, but although the expected time frame might be supportable, the choice of a distribution rarely is. Thus, having no other information, [38] assumes that the expected time frame to the liquidity event, L , is known and he makes no specific assumption regarding the probability distribution of exit times. With this construct, he ensures a result which is as general as possible and applicable in a wide variety of scenarios. Further, he let the company’s annual cash flow growth rate g be a constant such that the market value of the investment can be estimated using the Gordon Growth model, and he denote the cash flow duration as $H = 1/(r_i - g)$ where r_i is company i ’s annual risk-adjusted discount rate. Hereafter, the upper bound on the illiquidity discount is given as (consult eventually all details in Appendix of [38]):

$$2N\left(\sigma\sqrt{H\cdot\left(1-e^{-\frac{L}{H}}\right)}\right)-1, \text{ where: } N(\cdot) \text{ is the standard cumulative normal distribution function; } \sigma \text{ is volatility; } L \text{ is known time frame } (1, 2, 3, \dots, \infty); \text{ and } H \text{ is cash flow duration } H=1/(r_i-g).$$

Quite intuitively, this imply that holding everything else equal, long duration equity investments, i.e. the longer the time frame to the liquidity (L), the higher the commanded illiquidity discount is because a larger proportion of the cash flows are expected to be received in the distant future.

3.3. Discussion and Perspectives

It seems that not all the parameters introduced in the [38] example on page 99 are appropriate, since the market risk premium (0.065) as well as both the equity volatility (0.30), and the market volatility (0.15) seem unrealistically small. Further, he makes use of the so-called small stock premium (at size 0.050), which at best is an arbitrary parameter sometimes found in applied asset pricing. Consequently, we eliminate the small stock premium and adjust the other parameter assumptions to slightly more realistic levels for our illustrative purpose in **Table 4** and in correspondence with our assumptions in **Table 3**. It should be noted that the [38] model requires that r_i and g are known, which might be an obstacle, but in situations where the DLOM (upper bound) estimation follows a general valuation process using for instance a DCF-model, the two variables will already be present, and thus available for the DLOM-estimation.

The results in **Table 4** seem at same level as the results in our previous three DLOMs are the result of a more realistic model setting concerning a (presumably)

Table 4. DLOM summary for newer theoretical models. Basic assumptions: $T = 1$ year; $S = X$; $r_F = 2.0\%$; Volatility range = 50% - 125%; no dividend yield; the DLOM is calculated as a percentage.

Author	Mathematical model	DLOM for one-year holding period. (Volatility in the range 50% - 125%)	DLOM—at volatility 75%
Longstaff (2017) [39]	$DLOM = \frac{1}{\sqrt{1 + 8\lambda/\sigma^2}}$ <p>where: λ is Poisson parameter describing the average number of times an event occurs. $1/\lambda$ is average time the asset is illiquid.</p>	17.4% - 40.4%	25.6%
Ghaidarov (2021) [38]	$DLOM = 2N\left(\frac{\sigma\sqrt{H + \left(1 - e^{-\frac{L}{H}}\right)}}{2}\right) - 1$ <p>where: L is 1; 2; 3; ...; ∞. H is $\frac{1}{r_i - g}$.</p> <p>r_i is company i's annual risk-adjusted discount rate. g is company i's annual cash flow growth rate. Adjusting Ghaidarov's assumptions* and including them into our illustrative example, r_i can be estimated at 0.17 and g at 0.03.</p>	19.3% - 44.4%	28.3%

*Following the notation in Exhibit 1, [38]: Long-term growth rate (g) = .02; Risk-free rate (r_F) = 0.02; Correlation (ρ_{im}) = 1.00; Equity volatility (σ_i) = 0.75; Market volatility (σ_m) = 0.40; Stock beta ($\beta = \rho_{im} \times \sigma_i / \sigma_m$) = 1.875; Equity risk premium (erp) = 0.08; Small stock premium (sp) = 0.00; Cost of equity ($r_i = r_F + \beta \times \text{erp} + \text{sp}$) = 0.17. Note a: All other model inputs as in **Table 3**. Note b: The [38] model requires that the r_i and g are known, but in situations where the DLOM problem setting follows a valuation process using for instance a DCF-model, the two variables will presumably be "known", and thus available for the DLOM-estimation.

Tables. Main contribution from **Table 4** numbers is that the shown upper bound unknown time horizon.

If we compare the average of **Table 4** results with the average of **Table 3** results, they are almost identical (26.95 per cent vs. 25.82). This could indicate that the level is plausible, but the results obviously only reflect the assumptions, like for instance in **Table 4** where the Longstaff results are slightly smaller due to Poisson distributed time horizon, i.e. the distribution is assumed known.

Table 5. Effects from partial changes in volatility and time horizon for our example.

Volatility (σ)	Time horizon (T)	DLOM (Longstaff, 2017) [39]	DLOM (Ghaidarov, 2021) [39]
0.75	1	25.6%	28.3%
0.75	212/252 = 0.8413	23.6%	26.2%
0.80	1	27.2%	30.0%
0.80	212/252 = 0.8413	25.1%	27.8%
0.85	1	28.8%	31.7%
0.85	212/252 = 0.8413	26.6%	29.4%

As mentioned earlier, the volatility σ may be underestimated at 0.75 while time horizon T may be overestimated at 1 year, and thus it is important to see how sensible the size of the DLOM is to small changes in the two key parameters. The first line in **Table 5** is identical to the results in **Table 4**, while the next lines show effect on DLOM when volatility changes from 0.75 to 0.80 and 0.85, as well as changes in the time horizon. One calendar year less weekends and public holidays leaves 252 market days per year, and the average time horizon for a sale according to Vianello (2021) is 212 days which combined indicates that $212/252 = 0.8413$ year is a more precise unbiased expected time horizon. The consequences for our calculations due to the changes in volatility and horizon is that the DLOM becomes slightly larger or smaller, but the size level remain essentially the same. Simultaneously decreasing both volatility and time horizon will lead to a decrease in DLOM: an extreme decrease in time horizon to only one day (or one hour) while keeping the volatility at 0.75 would, using [39], lead to a DLOM of only 1.7 per cent (respectively 0.6 per cent), which corresponds to a quite liquid market. At those levels, the DLOM could probably be neglected, but for PE a sale on such a short notice would rarely be the case and thus it would reflect an unrealistic expectation.

Consequently, our results indicate that the fair value at security level should include a DLOM value-reduction for PE, since the DLOM express the difference between the non-marketable value (the best and most realistic bid price) of the stocks and the intrinsic value of the stocks that reflect the company's economic potential, i.e. the net present value of all future cash flows at present use. Successful transmittance of stocks from one owner to another is the result of a negotiation

process, and because of lack of marketability due to a lack of prospective buyers (and heterogeneous preferences among them regarding the value of the asset), the discount is increasing in the thinness of the market and the heterogeneity of investor preferences [40]. Most often the biggest challenge is to find a willing buyer, and during this process, the final price will presumably move closer to uncovered bid prices including a DLOM. While knowledge on the DLOM will be important for concrete sales negotiations for PEs, it is most certainly helpful for the estimation of accounting fair values as exit prices close to bid prices where an anticipated DLOM should be included.

It is without doubt that the DLOM is not as central and important as it should be in financial accounting when aiming for presenting fair values for PE in the annual report (balance sheet). Although the accounting framework, both IASB and FASB ([41] [42]), prescribe prudent valuation, only vague or non-existing incorporation of DLOM can be found in the specific measurement prescriptions in IFRS 13 respectively ASC 820 ([1] [3]). Consequently, the accounting value for such private equity assets is likely to be shown at a too high level in companies' annual reports, since ask prices are likely to be reflected in financial statements, although bid prices are probably far more representative of the prescribed exit prices. Even "mid prices" are better than clean ask prices, since they are fairer and more trustworthy, and thus show more fair representation.

In existing Financial Accounting regulation, we already see several accounting estimates, judgements, and assumptions for which reason it is a bit strange that IASB (and FASB) do not prescribe recognition of DLOM as part of the measuring of fair values for PE interests including appropriate (transparent) disclosure and thus making the values more prudent.

4. Conclusions

There should be no doubt that values of private equity holdings are subject to DLOMs which could be handled more explicitly than now. Although the exact size of a DLOM is unknown until a concrete trade between independent parties has been effectuated, already because of its usual average size on average, it should not be disregarded when PE interests are valued and disclosed in the annual report.

When measuring the DLOM, all facts and circumstances found relevant for estimating a PE ownership interest should be considered, like eventual past and future dividends, contractual and control effects, eventual blockage consequences, etcetera. However, all the theoretical models presented in this paper show that there are two recurring and decisive elements when determining PEs fair value, volatility and time horizon:

- Volatility is unobservable (by nature for a PE), but the volatility for a similar listed company may be used as a proxy.
- Time horizon is unknown and depends on prospects for finding a willing buyer but average experience for similar transactions may be used as proxy.

The two latest approaches to the determination of DLOM, [39] and [38], make it possible to present clearly justified suggestions for largest suitable DLOMs at a minimum of assumptions, and unless special conditions can be demonstrated, the consequence is that a reliable DLOM will be the result. For accounting regulation in accordance with the prudence principle as presented in the accounting conceptual framework, this DLOM should be used for calculating best plausible bid price. Thus, it should have a central position when finding best estimated fair value for recognition, measurement and disclosure of a private equity capital interest in the annual reports, and probably also when selling.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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