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Optimal Operations of Entrepreneurial Firms under Time-Inconsistent Preferences

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Abstract

While standard models in finance assume that an agent possesses a constant rate of time preferences, there is substantial evidence that people are impatient about choices in the short term but are patient when choosing between long-term alternatives. In this paper, we analyze the impact of the resulting time-inconsistent preferences (TIP) on operations of entrepreneurial firms. We differentiate a naive vs. a sophisticated entrepreneur depending on the different expectations regarding her future time-inconsistent behaviors. Relative to the time-consistent benchmark, a naive entrepreneur assigns a lower firm valuation, derives a lower welfare, liquidates the firm earlier, and in general adopts more conservative firm-level policies. Somewhat surprisingly, a time-inconsistent entrepreneur's sophistication destroys instead of creating values for herself under which she obtains a even lower welfare from the firm. In response, the sophisticated entrepreneur transfers resources from investment to consumption and liquidates the firm even earlier than in the case with a naive entrepreneur.

Keywords

Constant Rate of Time Preferences, Time-Inconsistent Preference, Entrepreneur's Sophistication, Intra-Personal Competition, Stationary Equilibrium

1. Introduction

Economists almost always assumed that agents have a constant rate of time preference which is time-consistent in the following sense: A person's relative preference for rewards at an earlier date over a later date is the same no matter when she is asked. However, virtually every experimental study on time preferences suggests that the assumption of time-consistency is unrealistic (see, for example, Thaler [1]; Ainslie [2]; Loewenstein and Prelec [3]) because it ignores the human tendency to grab immediate rewards and to avoid immediate costs. For example, while an agent acts relatively patiently when two rewards are both far away in time (e.g., she prefers two apples in 101 days, instead of one apple in 100 days), she tends to act more impatiently when the two rewards are imminent (e.g., she prefers one apple today, instead of two apples tomorrow). Such preferences that violate the time-consistency are intuitively termed the "present-biased" preferences by O'Donoghue and Rabin [4].

In this paper, we bring the more realistic time-inconsistent preferences (TIP) to the literature on dynamic corporate finance (e.g., Rampini and Viswanathan [5]; Wang, Wang, and Yang [6]; Bolton, Wang, and Yang [7]; Du [8]). More specifically, we develop and solve a dynamic stochastic model for an entrepreneurial firm when its controlling agent, referred to as the entrepreneur, is endowed with TIP. As is standard in models on time-inconsistent decision making (e.g., Laibson [9]; O'Donoghue and Rabin [4]; Grenadier and Wang (GW) [10]), a time-inconsistent entrepreneur's problems are envisioned as the outcome of an intra-personal game in which the same individual entrepreneur is represented by different players (or selves) at future dates. In particular, the current self of the entrepreneur solves her problem by taking into account the cash flows to be realized and the decisions to be made under the watch of all her future selves.

The literature on TIP proposes two alternative assumptions about the strategies chosen by future selves, both of which are considered in this paper. First, an agent is naive in that she assumes that future selves act according to the preferences of the current self. In other words, a naive agent holds the behavioral belief that current self can commit future selves to act in a time-consistent manner (which indicates her overconfidence in the ability to commit). Second, an agent is sophisticated in that she correctly assumes that future selves choose strategies that are optimal from their own perspectives despite the fact that these strategies are suboptimal from the standpoint of the current self. This rational assumption is in the tradition of subgame perfect game-theoretic equilibrium.

Incorporating TIP into the dynamic corporate finance framework poses several technical challenges. First, the time-inconsistent entrepreneur's problem has to be solved by two interconnected Hamilton-Jacobi-Bellman (HJB) equations: one for the current self's value function and the other for her continuation value function which reflects the values obtained under the watch of her future selves. Second, the entrepreneur's sophistication implies that her current self makes decisions based on current preferences, fully anticipating that all future selves do likewise. In a stationary equilibrium, this means that the continuation value function would load on the optimal decisions made by the current self. Consequently, solving a sophisticated entrepreneur's problem involves the search for a fixed point of the firm's optimal policies. Third, in addition to its capital investment an entrepreneurial firm also accumulates liquid wealth from the financial markets. The resulting HJBs are therefore partial differential equations (PDEs) which in general are very difficult to solve. Fourth, in contrast to just the investment timing aspect of entrepreneurial activity considered in GW [10] who formulate their problem in the real options framework (e.g., McDonald and Siegel [11]), an entrepreneur in the dynamic corporate finance framework simultaneoulsy consider several dimensions of activities: consumption, investment, asset allocation, and costly busines liquidation, which adds to the difficulty of simultaneously solving the multiple HJBs that are interconnected to one another. By tackling these challenges, we provide numerical solutions to the time-inconsistent entrepreneur's problems with accuracy which is deemed as the first contribution of this paper.

As discussed above, an agent with TIP is more impatient when viewed over a short horizon (e.g., today vs. tomorrow) than what she would be when viewed over a long horizon (e.g., in 100 days vs. in 101 days). Laibson [9] models such time-varying impatience with quasi-hyperbolic discount functions in which payoffs in the current period are discounted in the usual way while payoffs in future periods are further discounted by an additional factor $\delta(<1)$. We follow Laibson [9] for our tackling of TIP under which the current self of a time-inconsistent entrepreneur values less the firm-generated cash flows obtained under the watch of her future selves. Consequently, she values the firm less when compared to that in the time-consistent case serving as the benchmark. More interestingly, our numerical solutions show that the extra time-inconsistent discounting, as captured by δ, mainly affects the entrepreneur's welfare with the firm[1] which goes down relative to its benchmark level by a much larger margin than that of the firm's valuation.

Our numerical solutions also suggest that the entrepreneur's sophistication destroys instead of creating values for her, in that it prompts her to assign an even lower valuation to the firm which further depresses her welfare with the firm relative to that in the case under naivety. Intuitively, the extra valuation and welfare losses are attributed to the correctly forecasted decisions made by her future selves which are suboptimal to her current preference. Different than the TIP-discounting effect, this effect due to entrepreneur's sophistication works symmetrically on firm valuation and the implied welfare in that they go down by comparable margins relative to their respective naivety-counterparts.

We next examine the impact of TIP on firm's optimal policies. While the extra TIP-discounting depresses both consumption and investment, the sophistication effect in general transfers resources from investment to consumption. Neither TIP-discounting nor entrepreneur's sophistication affects the firm's asset allocation policy much which in general stays very close to its benchmark levels. Due to the extra δ-discount, a time-inconsistent entrepreneur believes that he has a less valuable business to operate on than a time-consistent entrepreneur does. Consequently, she abandons the firm earlier. The sophistication effect further accelerates the firm's liquidation because the entrepreneur now has the extra incentive

[1]Consistent with the tradition in the dynamic corporate finance literature, the entrepreneur in our setup is risk averse whose welfare with the firm loads on her (certainty equivalent) valuation of the firm but with a critical risk adjustment.

to protect herself against the suboptimal policies made by her future selves.

TIP is one of the most widely studied biases in behavioral economics. Early behavioral researches, however, focused exclusively on biases in individual investors (e.g., overconfidence and loss aversions in Barber and Odean [12]; Lamont and Thaler [13]). Our paper contributes to this literature by bringing behavioral beliefs to the study of entrepreneurial decisions and optimal firm operations. In particular, our analysis with TIP highlights the connections among different types of cognitive limitations. For example, a time-inconsistent entrepreneur's naivety is attributed to her overconfidence (in her ability to commit) while her tendency to avoid costs imposed by future selves, as indicated by the earlier abandonment of the firm, is in spirit similar to loss aversion which causes an agent to take fewer risks.

The rest of the paper is organized as follows. Section II presents the setup of the model. Section III characterizes the benchmark case with time-consistent preferences. Section IV derives and analyzes the problem for a naive entrepreneur while Section V derives and analyzes the problem for a sophisticated entrepreneur. Section VI provides solutions to the entrepreneur's problems that are characterized in Section III-V. Section VII presents the quantitative implications and Section VIII provides some further discussions on empirical evidences and the potential model extensions. Finally, Section IX concludes.

2. Model Setup

2.1. The Opportunity Set to the Firm

An entrepreneurial firm obtains productivity from its capital stock and it simultaneously has the full access to financial trading. The firm's capital stock K_t is accumulated according to

$$dK_t = \left(I_t - \delta_K K_t\right)dt + \sigma_K K_t dZ_t + \epsilon I_t dZ_t^I, \tag{2.1}$$

where I_t is investment; $\delta_K > 0$ is the depreciation rate; σ_K and ϵ are the volatility parameters. Without loss of generality, we decompose the firm-level risk into two orthogonal components: the usual capital depreciation shock driven by dZ_t and an investment-specific shock driven by Z_t^I. The latter specification implies that output fluctuations arise from shocks to the marginal efficiency of investment (Keynes [14]) and it is motivated by the literature that emphasizes the important role of investment-specific technology shocks as a source of aggregate volatility.

The firm's operating profit dY_t over the period $(t, t+dt)$ is given by

$$dY_t = K_t A dt - I_t dt - G\left(I_t, K_t\right)dt, \tag{2.2}$$

where A denotes the (constant) productivity of the firm-held capital; $G(I,K)$ is the adjustment cost. Following Hayashi [15], we assume the following form of $G(I,K)$:

$$G(I,K) = g(i)K = \frac{\theta i^2}{2}K \tag{2.3}$$

which is convex in I and homogeneous of degree one in I and K with $i \equiv I/K$, where the parameter θ measures the degree of adjustment cost.

Besides the capital investment, the firm can also invest in a risk-free asset which pays a constant rate of interest r and the risky market portfolio. Assume that the incremental return dR_t of the market portfolio over the time period dt is i.i.d as follows:

$$dR_t = \mu_R dt + \sigma_R dB_t, \tag{2.4}$$

where B_t is the standard Brownian representing the systematic (or market) risk; μ_R and σ_R are constant mean and volatility parameters of the market portfolio return process. We allow B_t to be correlated with the firm-level risks of Z_t and Z_t^I by ρ and ρ_I, respectively.[2]

Let W and X denote the firm's liquid wealth and the amount invested in the risky asset, respectively. Their difference, $W - X$, is thus invested in the risk-free asset. Out of its liquid asset, the firm pays the investment cost and consumes. Thus, W_t evolves according to

$$dW_t = \left[r\left(W_t - X_t\right) + \mu_R X_t + dY_t - C_t \right] dt + \sigma_R X_t dB_t, \tag{2.5}$$

where dY_t is given by (2.2). Note W can be negative under which firm borrows against its capital stock at the risk-free rate of r.

2.2. Time-Inconsistent Preference (TIP)

From Laibson [9], the essence of TIP is that time can be divided into two periods: the present period and all future periods. Payoffs in the current period are discounted in the usual way with ζ being the (constant) discount rate while payoffs in future periods are further discounted by an additional factor $\delta(<1)$. For its illustration, consider an entrepreneur with TIP who faces the choice between obtaining a payoff of P_{t+1} at time $t+1$ versus obtaining a value of P_{t+2} one period later at $t+2$ where time is discrete and $t \geq 1$. At time 0, this represents a choice between $\delta e^{-\zeta\cdot(t+1)} P_{t+1}$ and $\delta e^{-\zeta\cdot(t+2)} P_{t+2}$ for the entrepreneur. Thus, she would prefer receiving P_{t+1} at time $t+1$ over receiving P_{t+2} at $t+2$ if and only if $P_{t+1} > e^{-\zeta} P_{t+2}$. Therefore, the intertemporal tradeoffs are determined soly by the usual subjective discount factor ζ when viewed over a long time horizon.

Now, consider the same entrepreneur's decision at time t. At that point, the entrepreneur views P_{t+1} as occurring in the current period, *i.e.*, at the end of time t. Thus, the entrepreneur now faces a choice between $e^{-\zeta} P_{t+1}$ and $\delta e^{-2\zeta} P_{t+2}$ because a payoff obtained at the end of the current period is only discounted at the rate ζ while a payoff obtained in future periods are further discounted by δ. Facing such a choice, the entrepreneur would prefer receiving P_{t+1} at time $t+1$ over receiving P_{t+2} at $t+2$ if and only if $P_{t+1} > \delta e^{-\zeta} P_{t+2}$ which is easier to get satisified since $\delta < 1$. In other words, the entrepreneur is more impatient when viewed over a short horizon under which the intertemporal tradeoffs are

[2]We assume that $\rho, \rho_I < 1$ so that firm-level risks cannot be fully hedged away by taking positions in the stock market.

determined by the additional discount factor δ. To summarize, the entrepreneur at time t views the relative choice between these two payoffs to be realized in the near future in a different manner than she does at time 0.

In our setup, we follow Harris and Laibson (HL) [16] and GW [10] by assuming a continuous time where the decision making under TIP is envisioned as the outcome of an intra-personal game in which the same individual entrepreneur is represented by different players (or selves) at future dates. Let t_n be the calendar time of birth for self n so that $T_n = t_{n+1} - t_n$ denotes the lifespan for self n. For simplicity, we model the arrivals of future selves as a Poisson process with intensity λ so that the lifespan is exponentially distributed with parameter λ for all selves. The entrepreneur starts as self 0 which lasts for a random amount of time according to the distribution of $\exp(\lambda)$ before it ends with the birth of self 1. The process then repeats for self 1, 2, 3, ... into the indefinite future.

Given the stochastic arrival process of future selves, the TIP can be conveniently modeled as follows: In addition to the standard discounting at ζ, the current self discounts cash flows realized after the birth of future selves by an additional discounting factor δ. When the agent is risk neutral as assumed in both HL [16] and GW [10], the implied TIP can be summarized by self n's intertemporal discount function $D_n(t,s)$ (for $n = 0,1,2,\cdots$) as follows

$$D_n(t,s) = \begin{cases} e^{-\zeta(s-t)} & \text{if } s \in [t_n, t_{n+1}) \\ \delta e^{-\zeta(s-t)} & \text{if } s \in [t_{n+1}, \infty) \end{cases}, \tag{2.6}$$

where self n uses a heavier discount of $\delta e^{-\zeta(s-t)}$ for cash flows that are realized after t_{n+1}.

2.3. The Entrepreneur's Preferences on Risk and Time

In this paper, we extend HL [16] and GW [10] by allowing a risk-averse entrepreneur who is equipped with a recursive preference (e.g., Epstein and Zin [17]; Dufie and Epstein [18]) by

$$J_t = E_t\left[\int_0^\infty f(C_s, J_s)\,ds\right], \tag{2.7}$$

where J denotes the entrepreneur's utility; $f(C,J)$ is known as the normalized aggregator for consumption C which is given by

$$f(C,J) = \frac{\zeta}{1-1/\psi}\frac{C^{1-1/\psi} - ((1-\gamma)J)^{\chi}}{((1-\gamma)J)^{\chi-1}}, \tag{2.8}$$

where ζ denotes the usual subjective discount rate; $\psi > 0$ measures the elasticity of substitution (EIS); $\gamma > 0$ is the coefficient of relative risk aversion; $\chi \equiv \frac{1-1/\psi}{1-\gamma}$.[3]

To embed TIP into a risk-averse agent's preference in a meanful way, let $J^n(K,W)$ denote the value function for self n ($n = 0,1,2,\cdots$) of the

[3]The widely used constant-relative-risk-averse (CRRA) utility is a special case of the Duffie-Epstein-Zin-Weil recursive utility specification with EIS set to the inverse of γ.

entrepreneur which loads on both the firm's capital stock K and its liquid wealth W. By applying the principle of dynamic programming to J^n, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}0 = \max_{C,X,I} f\left(C_t, J^n\right) &+ \left[rW_t + \eta\sigma_R X_t + AK_t - I_t - G\left(I_t, K_t\right) - C_t\right] J_W^n \\ &+ \left(I - \delta_K K\right) J_K^n + \frac{\epsilon^2 I^2 + \sigma_K^2 K^2}{2} J_{KK}^n + \left(\rho_I \epsilon I + \rho\sigma_K K\right)\sigma_R X J_{KW}^n \\ &+ \frac{\sigma_R^2 X^2}{2} J_{WW}^n + \lambda\left[J^{n+1}\left(\delta K, \delta W\right) - J^n\left(K, W\right)\right],\end{aligned} \tag{2.9}$$

where $J_W^n, J_K^n, J_{KK}^n, J_{KW}^n$ and J_{WW}^n denote the partial derivatives of J^n; we have used (2.1), (2.2), and (2.5); $\eta \equiv \left(\mu_R - r\right)/\sigma_R$ which denotes the market Sharpe ratio. As indicated by the last term of (2.9), J^n transforms into J^{n+1} for self $n+1$ with the intensity λ and TIP manifests itself in terms of the δ-discount applied to both K and W as the arguments for J^{n+1}.[4] Throughout the paper, we conjecture (and verify later) that the entrepreneur's value function J can be generically written as

$$J\left(K, W\right) = \frac{\left(bP\left(K, W\right)\right)^{1-\gamma}}{1-\gamma} = \frac{\left(bKp\left(w\right)\right)^{1-\gamma}}{1-\gamma}, \tag{2.10}$$

where b is some constant; $w \equiv W/K$ which denotes the firm's financial slack. We interpret $P\left(K, W\right) \equiv Kp\left(w\right)$ as the certainty-equivalent (CE) valuation of the firm by the entrepreneur: Financially, it denotes the minimum dollar amount that the entrepreneur would demand to permanently give up the firm.

3. The Time-Consistent Benchmark

As a benchmark, we briefly consider the case where the entrepreneur's preference is time consistent in that $\delta = 1$. In this benchmark case, $J^n\left(K, W\right)$ is invariant to n because all selves face an identical problem when the extra δ-discount is absent. Consequently, the last term of (2.9) cancels out so that the resulting HJB degenerates to

$$\begin{aligned}0 = \max_{C,I,X} f\left(C_t, J^c\right) &+ \left[rW_t + \eta\sigma_R X_t + AK_t - I_t - G\left(I_t, K_t\right) - C_t\right] J_W^c \\ &+ \left(I - \delta_K K\right) J_K^c + \frac{\epsilon^2 I^2 + \sigma_K^2 K^2}{2} J_{KK}^c \\ &+ \left(\rho_I \epsilon I + \rho\sigma_K K\right)\sigma_R X J_{KW}^c + \frac{\sigma_R^2 X^2}{2} J_{WW}^c,\end{aligned} \tag{3.1}$$

[4]One may be tempted to directly apply the δ-discount to J^{n+1} so that the last term in (2.9) becomes $\lambda\left[\delta J^{n+1}\left(K, W\right) - J^n\left(K, W\right)\right]$. However, this way of embedding TIP does not make sense either financially or mechanically. From the financial perpective, directly applying the TIP-discounting of δ to the entrepreneur's value function (which denotes her utility obtained from the firm) does not account for the necessary risk adjustment when the entrepreneur is risk averse. In contrast, the last term in (2.9) applies the TIP-discounting to the arguments of J^{n+1} over which the entrepreneur obtains her utility: This way, her (risk-averse) preference is applied only after the extra δ-discount is applied to the firm-generated cash flows determined by K and W which fits the requirement on risk adjustment. Mechanically and as plotted in Panel B of Figure 2, the implied J under $\gamma > 1$ is negative. Consequently, directly applying δ to $J^{n+1}\left(K, W\right)$ at $\delta < 1$ raises the entrepreneur's utility by alleviating the negativity which runs against the intuition that a further discount on firm assets (or equivalently, the firm-generated cash flows) should reduce an agent's utility.

where we use the superscript "*c*" to denote the time-consistent benchmark case.

To fully characterize J^c, we need boundary conditions at both the upper and the lower end. When $w \to \infty$ which indicates the best possible financial status for the firm, the entrepreneur is no longer concerned about the potential liquidation so that the first-best (FB) is achieved. Consequently,

$$J^c(K,W) \to J^{FB}(K,W) \Leftrightarrow \lim_{w\to\infty} b^c p^c(w) = b^{FB} p^{FB}(w), \tag{3.2}$$

where b^c is the time-consistent version of b introduced in (2.10); the superscript "*FB*" denotes variables under the first-best; we have used our conjectured solution to a generic J for the entrepreneur which is given by (2.10). At the lower end, the firm gets liquidated when W becomes sufficiently negative upon which the entrepreneur receives the firm's terminal payment $W_d^c + lK$ for $l \in (0,1)$ and retires as a Merton consumer (Merton [19]), where W_d^c denote the firm's liquidation boundary under the time-consistent benchmark and lK denotes the residual value of the firm-held capital upon liquidation. Since W_d is optimally chosen, we have the following value matching and smooth-pasting conditions at the lower end:

$$J^c(K,W_d^c) = J^M(W_d^c + lK), \tag{3.3}$$

$$\left.\frac{\partial J^c(K,W)}{\partial W}\right|_{W=W_d^c} = \left.\frac{\partial J^M(W+lK)}{\partial W}\right|_{W=W_d^c}. \tag{3.4}$$

where $J^M(.)$ denotes the value function for a Merton consumer.

4. The Naive Entrepreneur

Turning to TIP, we start with a naive entrepreneur who runs the firm under the false belief that she can commit her future selves to behave according to the current self's preferences. More specifically, the self n of a naive entrepreneur mistakenly believes that all future selves act as if their time preferences remain the same as the current time preference given by (2.6). Of course, there is no actual commitment mechanism in our model and thus the naive entrepreneur's beliefs prove incorrect. Empirically, this type of naivete is discovered on 401(k) investment (Madrian and Shea [20]), task completion (Ariely and Wertenbroch [21]), and health club attendance (DellaVigna and Malmendier [22]).

Under naivety and according to (2.6), the extra δ-discount applies uniformly to cash flows realized under the watch of all her future selves. Therefore, the current self essentially believes that she can commit her future selves to act in a time-consistent manner. In other words, for self n of the naive entrepeneur, all future selves of $n+1$, $n+2$, $\cdots$ are viewed as if they are behaving according to (3.1). This means that the continuation value function for a naive entrepreneur is simply J^c with the modification that the arguments of J^c, K and W, are viewed to be δK and δW in the eyes of the current self (see discussions in footnote 4). Since this false belief applies to any n, we replace J^n in (2.9) with J^N which satisfies the following HJB:

$$0=\max_{C,I,X} f\left(C_t,J^N\right)+\left[rW_t+\eta\sigma_R X_t+AK_t-I_t-G\left(I_t,K_t\right)-C_t\right]J_W^N$$
$$+\left(I-\delta_K K\right)J_K^N+\frac{\epsilon^2 I^2+\sigma_K^2 K^2}{2}J_{KK}^N+\left(\rho_I\epsilon I+\rho\sigma_K K\right)\sigma_R X J_{KW}^N \tag{4.1}$$
$$+\frac{\sigma_R^2 X^2}{2}J_{WW}^N+\lambda\left[J^c\left(\delta K,\delta W\right)-J^N\left(K,W\right)\right],$$

where the superscript "*N*" indicates the entrepreneur's naivete. We emphasize that the irrational expectations of the naive entrepreneur are persistent: While self n truly believes that all future selves' time preferences remain unchanged with the implied continuation value summarized by J^c, her future selves, e.g., self $n+1$, possesses the same belief to which (4.1) also applies. Note that the last term of (4.1) does not cancel out at $\delta<1$ which indicates that the naive entrepreneur still behaves in a time-inconsistent manner.

We next impose boundary conditions on J^N. At the lower end, self n of the naive entrepreneur still retires as a Merton consumer when the firm gets liquidated before the arrival of self $n+1$ ($n=0,1,2,\cdots$). To ensure the internal consistency, the entrepreneur's time and risk preferences, as well as her naivete (or sophistication) remains unchanged after her retirement. In particular, the naive entrepreneur would become a (time-inconsistent) naive Merton consumer and we use $J^{M,N}(W)$ to denote her value function after the firm's liquidation. By the same logic on the firm's optimal liquidation policy that is discussed in last section, we have the following value matching and smooth-pasting conditions for J^N:

$$J^N\left(K,W_d^N\right)=J^{M,N}\left(W_d^N+lK\right), \tag{4.2}$$

$$\left.\frac{\partial J^N\left(K,W\right)}{\partial W}\right|_{W=W_d^N}=\left.\frac{\partial J^{M,N}\left(W+lK\right)}{\partial W}\right|_{W=W_d^N}. \tag{4.3}$$

where W_d^N denotes the firm's liquidation boundary in the case with a naive entrepreneur. By the conjectured solution of (2.10), we can write

$$J^N\left(K,W\right)=\frac{\left(b^N P^N\left(K,W\right)\right)^{1-\gamma}}{1-\gamma}=\frac{\left(b^N K p^N\left(w\right)\right)^{1-\gamma}}{1-\gamma}, \tag{4.4}$$

where $p^N(w)$ is interpreted as the naive entrepeneur's valuation of the firm per unit of capital. At the upper end when $w\to\infty$, Section VI.D shows that

$$\lim_{w\to\infty} p^N\left(w\right)=p^{FB}\left(w\right), \tag{4.5}$$

where $p^{FB}(w)$ denotes the firm valuation per unit of capital under the first-best (FB) which is first introduced by (3.2) for the time-consistent benchmark.

5. The Sophisticated Entrepreneur

Unlike the naive entrepreneur, a sophisticated entrepreneur correctly foresees that her future selves act according to their own preferences. In other words, the entrepreneur's sophistication lies in her correct anticipation that future selves choose strategies that are optimal for future selves, despite being suboptimal from the standpoint of the current self. Thus, in addition to the fear about being preempted by a future self that leads to the extra δ-discount on future cash flows, the current self of a sophisticated entrepreneur also faces the preemption costs that stem from the suboptimal policies to be made by all her future selves. Intuitively, the naive entrepreneur is myopic and worries only about the immediate threat of preemption, whereas the sophisticated entrepreneur is forward-looking and concerned with all future threats of preemption.

Given the correct anticipation by the current self that all her futures selves also behave in a time-inconsistent manner, we can no longer use J^c, the value function in the time-consistent benchmark, as the continuation value function for a sophisticated entrepreneur. To characterize her problem, we make use of the fact that there are infinite number of selves to be born into the indefinite future. Consequently, each self faces the same time-invariant problem that does not load on a particular self *n*. We thus replace the value function J^n in (2.9) with J^S where the superscript "*S*" denotes the entrepreneur's sophistication.

We focus on the stationary characterization of J^S. Essentially, the time-inconsistent entrepreneur is competing against its future selves in an intra-personal game. From the perspective of this intra-personal competition, a stationary equilibrium emerges, which serves as the solution to the sophisticated entrepreneur's problem, when we identify a fixed-point of the firm's optimal policies. Take the firm's liquidation policy as the example. Suppose that all stationary future selves liquidate the firm at W_d^S, then the current self's optimal liquidation boundary is also W_d^S. Of course, a future self may arrive prior to liquidation. In that case, the current self receives the continuation value, denoted by J^{cont}, which summarizes the present value of the payoffs determined by the decisions of all future selves after the necessary risk-adjustment. Taking into account the impact of J^{cont}, J^S satisfies the following HJB:

$$\begin{aligned} 0 = \max_{C,I,X} & f\left(C_t, J^S\right) + \left[rW_t + \eta\sigma_R X_t + AK_t - I_t - G\left(I_t, K_t\right) - C_t\right] J_W^S \\ & + \left(I - \delta_K K\right) J_K^S + \frac{\epsilon^2 I^2 + \sigma_K^2 K^2}{2} J_{KK}^S + \left(\rho_I \epsilon I + \rho\sigma_K K\right)\sigma_R X J_{KW}^S \\ & + \frac{\sigma_R^2 X^2}{2} J_{WW}^S + \lambda\left[J^{cont}\left(\delta K, \delta W\right) - J^S\left(K, W\right)\right], \end{aligned} \tag{5.1}$$

where the two arguments of J^{cont}, *K* and *W*, are once again viewed to be δK and δW by the current self which reflects the extra TIP-discounting.

To characterize J^{cont}, denote by $J^{cont,n+1}$ the continuation value for self *n* where the superscript "$n+1$" indicates that $J^{cont,n+1}$ is relevant only when self $n+1$ is born before self *n* liquidates the firm. Accounting for the fact that self *n*'s continuation value transforms from $J^{cont,n+1}$ into $J^{cont,n+2}$ when self $n+2$ is born before the firm's liquidation, $J^{cont,n+1}$ satisfies the following HJB:

$$\begin{aligned} 0 = & f\left(C_t^S, J^{cont,n+1}\right) + \left[rW_t + \eta\sigma_R X_t^{S,n+1} + AK_t - I_t^{S,n+1} - G\left(I_t^{S,n+1}, K_t\right)\right. \\ & \left. - C_t^{S,n+1}\right] J_W^{cont,n+1} + \left(I_t^{S,n+1} - \delta_K K_t\right) J_K^{cont,n+1} + \frac{\epsilon^2 \left(I_t^{S,n+1}\right)^2 + \sigma_K^2 K_t^2}{2} J_{KK}^{cont,n+1} \\ & + \frac{\sigma_R^2 \left(X_t^{S,n+1}\right)^2}{2} J_{WW}^{cont,n+1} + \left(\rho_I \epsilon I_t^{S,n+1} + \rho\sigma_K K_t\right)\sigma_R X_t^{S,n+1} J_{KW}^{cont,n+1} \\ & + \lambda\left[J^{cont,n+2}\left(K_t, W_t\right) - J^{cont,n+1}\left(K_t, W_t\right)\right], \end{aligned} \tag{5.2}$$

where $C^{S,n+1}$, $I^{S,n+1}$, and $X^{S,n+1}$ denotes the consumption, investment, and asset allocation policies chosen by self $n+1$. (5.2) makes use of the TIP formulation that all cash flows realized after the birth of a future self is uniformly discounted at δ: Consequently, the effect of δ cancels out in the HJB that characterizes $J^{cont,n+1}$ from the perspective of self n.[5] In a stationary equilibrium for the intrapersonal game, optimal policies are independent of n and $J^{cont,n+2} = J^{cont,n+1}$. We can thus write the HJB for $J^{cont}(.)$ as

$$\begin{aligned} 0 = {} & f\left(C_t^S, J^{cont}\right) + \left[rW_t + \eta\sigma_R X_t^S + AK_t - I_t^S - G\left(I_t^S, K_t\right) - C_t^S\right] J_W^{cont} \\ & + \left(I_t^S - \delta_K K_t\right) J_K^{cont} + \frac{\epsilon^2\left(I_t^S\right)^2 + \sigma_K^2 K_t^2}{2} J_{KK}^{cont} \\ & + \left(\rho_I \epsilon I_t^S + \rho\sigma_K K_t\right)\sigma_R X_t^S J_{KW}^{cont} + \frac{\sigma_R^2\left(X_t^S\right)^2}{2} J_{WW}^{cont}, \end{aligned} \tag{5.3}$$

where $\left(C^S, I^S, X^S\right)$, together with W_d^S discussed above, summarize the stationary policies. Section VI.E describe the procedure for identifying a scaled version of $\left(C^S, I^S, X^S, W_d^S\right)$ when we numerically solve the problem for a sophisticated entrepreneur.

To finish the characterizations on the sophisticated entrepreneur's problem, we need to impose necessary boundary conditions for both J^S and J^{cont}. At the upper end when $w \to \infty$, we show in Section VI.D that

$$\lim_{w\to\infty} p^S(w) = p^{FB}(w) \text{ and } \lim_{w\to\infty} p^{cont}(w) = p^{FB}(w) \tag{5.4}$$

where $p^m(w)$ denotes the firm valuation per unit of capital for $m \in \{S, cont, FB\}$. At the lower end, J^S satisfies the following value matching and smooth-pasting conditions:

$$J^S\left(K, W_d^S\right) = J^{M,S}\left(W_d^S + lK\right), \tag{5.5}$$

$$\left.\frac{\partial J^S(K,W)}{\partial W}\right|_{W=W_d^S} = \left.\frac{\partial J^{M,S}(W+lK)}{\partial W}\right|_{W=W_d^S}. \tag{5.6}$$

where $J^{M,S}(.)$ denotes the value function for a time-inconsistent Merton consumer with sophistication. Turning to the boundary condition for J^{cont}, let $J^{M,cont}(W)$ denote the continuation value function for a sophisticated Merton consumer. Then, the value matching condition for J^{cont} implies

$$J^{cont}\left(\delta K, \delta W_d^S\right) = J^{M,cont}\left(\delta\left[W_d^S + lK\right]\right), \tag{5.7}$$

where the extra δ-discount applies to the argument of $J^{M,cont}$ as well because the next self is already born when we consider the characterization of J^{cont}. Note that solving the continuation value function J^{cont} does not involve an optimality decision.

In sum, J^c, J^N, J^S, and J^{cont} denote the value function for a time-consistent entrepreneur, a naive entrepreneur, a sophisticated entrepreneur, and the continuation value for a sophisticated entrepreneur, respectively, while J^M,

[5]This logic is consistent with the fact that the HJB on J^c, which serves as the continuation value function for a naive entrepreneur, is also independent of δ as indicated by (3.1).

$J^{M,N}$, and $J^{M,S}$ denote the value function for a time-consistent, a naive, and a sophisticated Merton consumer which characterize the boundary conditions for J^c, J^N, and J^S respectively. Table 1 summarizes the connections among J^c, J^N, J^S, J^{cont}, J^M, $J^{M,N}$, $J^{M,S}$, as well as J^{FB} which denotes the value function for a time-consistent entrepreneur under the first best.

Table 1. Connections among different value functions.

Panel A: time-		↗	J^{FB} as $w \to \infty$
Consistent	J^c		
Benchmark:		↘	J^M for a time-consistent Merton consumer as $w \to w_d^c$
Panel B: a naive		↗	$\left(\frac{b^N}{b^c}\right)^{1-\gamma} \cdot J^{FB}$ as $w \to \infty$
Entrepreneur:	J^N	→	J^c for a time-consistent entrepreneur
		↘	$J^{M,N}$ for a naive Merton consumer as $w \to w_d^N$
Panel C: a		↗	$\left(\frac{b^S}{b^c}\right)^{1-\gamma} \cdot J^{FB}$ as $w \to \infty$, where $b^S = b^N$
Sophisticated	J^S	→	J^{cont} under stationary equilibrium
Entrepreneur:		↘	$J^{M,S}$ for a sophisticated Merton consumer as $w \to w_d^S$
Panel D: the		↗	J^{FB} as $w \to \infty$
Continuation	J^{cont}	→	J^{cont} under stationary equilibrium
Value for J^S:		↘	J^M for a time-consistent Merton consumer as $w \to w_d^S$

This table summarizes the connections among different value functions that are studied in the present paper, where J^c, J^N, and J^S in Panel A-C denote the value function for a time-consistent entrepreneur (Section III), a naive entrepreneur (Section IV), and a sophisticated entrepreneur (Section V), respectively. The arrows of "↗" and "↘" educe the entrepreneurs' limiting behaviors when the firm's financial status, as measured by w, approaches its upper and lower limit, respectively, where J^{FB} denotes the time-consistent entrepreneur's value function under first-best (see footnote 10 for its formula); J^M, $J^{M,N}$, and $J^{M,S}$ denote the value function for a time-consistent, a naive, and a sophisticated Merton consumer, respectively. The arrow "→" educes the entrepreneurs' continuation value when a future self arrives before the firm's liquidation, where a naive entrepreneur's continuation value is simply given by J^c while a sophisticated entrepreneur's continuation value under the stationary equilibrium is denoted by J^{cont}. Panel D of the table further illustrates the limiting behaviors and the continuation value for J^{cont}.

6. Solutions to the Entrepreneur's Problems

In this Section, we provide solutions to entrepreneur's problems at the different time preferences. We focus on the interconnections among the different problems faced by the entrepreneur which are characterized by the mechanical linkages among the different value functions. Within a given value function, the involved mechanics are largely the same as that for a time-consistent entrepreneur and the readers are referred to Du [8] for details on such derivations.

6.1. Optimal Controls and Reduction to One Dimension

For J^c, J^N, and J^S that are characterized by the HJBs of (3.1), (4.1), and (5.1), respectively, the optimal consumption C, investment I, and asset allocation X are determined by the following first-order conditions (FOCs):

$$[C]: f_C\left(C^m, J^m\right) = J_W^m\left(K, W\right), \tag{6.1}$$

$$[I]: \left[1 + G_I\left(I^m, K\right)\right] J_W^m = J_K^m + \epsilon^2 I^m J_{KK}^m + \rho_I \epsilon \sigma_R X^m J_{KW}^m, \tag{6.2}$$

$$[X]: X^m = -\frac{\mu_R - r}{\sigma_R^2}\frac{J_W^m}{J_{WW}^m} - \frac{\left(\rho\sigma_K K + \rho_I \epsilon I\right)\sigma_R}{\sigma_R^2}\frac{J_{KW}^m}{J_{WW}^m}, \tag{6.3}$$

for $m \in \{c, N, S\}$, where f is given by (2.8); G is given by (2.3). Du [8] provides the financial interpretations of the above FOCs (where J in his paper denotes J^c).

To actually solve the HJBs, we exploit the setup's homogeneity property to reduce the involved problems to one dimension. By the conjectured solution of (2.10), we write

$$J^m\left(K, W\right) = \frac{\left(b^m K p^m\left(w\right)\right)^{1-\gamma}}{1-\gamma}, \tag{6.4}$$

Consistent with the formulation of (6.4), we treat the firm's capital K_t as the scaling factor and use lower case letters to denote the following variables: firm's liquid wealth $w_t = W_t / K_t$, the agent's CE valuation of the firm $p_t = P_t / K_t$, consumption $c_t = c_t / K_t$, investment $i_t = I_t / K_t$, and risky asset allocation $x_t = X_t / K_t$. Under (6.4), we have the following expressions for J^m-derivatives:

$$J_W^m = \left(b^m\right)^{1-\gamma}\left(p^m\left(w\right)K\right)^{-\gamma} p^m\left(w\right), \tag{6.5}$$

$$J_K^m = \left(b^m\right)^{1-\gamma}\left(p^m\left(w\right)K\right)^{-\gamma}\left(p^m\left(w\right) - w\cdot\left(p^m\right)'\left(w\right)\right), \tag{6.6}$$

$$J_{WW}^m = \left(b^m\right)^{1-\gamma}\left(p^m\left(w\right)K\right)^{-\gamma-1}\left[p^m\left(w\right)\cdot\left(p^m\right)''\left(w\right) - \gamma\left(\left(p^m\right)'\left(w\right)\right)^2\right], \tag{6.7}$$

$$J_{KW}^m = \left(b^m\right)^{1-\gamma}\left(p^m\cdot K\right)^{-\gamma-1}\left[-wp^m\cdot\left(p^m\right)'' - \gamma\left(p^m\right)'\left(p^m - w\cdot\left(p^m\right)'\right)\right], \tag{6.8}$$

$$J_{KK}^m = \left(b^m\right)^{1-\gamma}\left(p^m\cdot K\right)^{-\gamma-1}\left[w^2 p^m\cdot\left(p^m\right)'' - \gamma\left(p^m - w\cdot\left(p^m\right)'\right)^2\right], \tag{6.9}$$

Using (6.4) to simplify (2.8) which are then substituted into (6.1), we obtain the following consumption rule after making use of (6.5):

$$c^m\left(w\right) = \zeta^\psi\left(b^m\right)^{1-\psi}\left[\left(p^m\right)'\right]^{-\psi} p^m. \tag{6.10}$$

We next derive the firm's optimal investment and asset allocation policies from the FOCs of (6.2) - (6.3). By substituting (6.5) - (6.9) into these two FOCs and performing necessary manipulations, we obtain

$$i^m\left(w\right) = \frac{p^m - \left(w+1\right)\left(p^m\right)' - \rho_I\epsilon\left(p^m\right)'\dfrac{\gamma p^m - wh}{p^m}\left[\eta\dfrac{p^m}{h} - \rho\sigma_K\left(\dfrac{\gamma p^m}{h^m} - w\right)\right]}{\theta\left(p^m\right)' + \left(1-\rho_I^2\right)\epsilon^2\dfrac{\left(p^m\right)'}{p^m}\dfrac{\left(\gamma p^m - wh^m\right)^2}{h^m} - \gamma\epsilon^2\dfrac{\left(p^m\right)^2\left(p^m\right)''}{h^m\left(p^m\right)'}} \tag{6.11}$$

$$x^m\left(w; i^m\right) = \frac{\eta}{\sigma_R}\frac{p^m}{h^m} - \frac{\rho_I\epsilon i^m + \rho\sigma_K}{\sigma_R}\left(\frac{\gamma p^m}{h^m} - w\right), \tag{6.12}$$

where $m \in \{c, N, S\}$;

$$h^m \equiv \gamma\left(p^m\right) - \frac{p^m\left(p^m\right)''}{\left(p^m\right)'} \tag{6.13}$$

6.2. The Implied Ordinary Differential Equations (ODEs)

For the benchmark case with time-consistent preference, Du [8] shows that $p^c(w)$ satisfies the following second-order nonlinear ODE:

$$\begin{aligned}
0 = {} & \frac{\zeta^{\psi}\left(b^c\right)^{1-\psi}\left(\left(p^c\right)'\right)^{1-\psi} - \zeta\psi}{\psi - 1} p^c + (rw + A)\left(p^c\right)' + \frac{1}{2}\frac{\eta^2 p^c\left(p^c\right)'}{h^c} \\
& - \delta_K\left(p^c - w\cdot\left(p^c\right)'\right) - \frac{\sigma_K^2\left(1-\rho^2\right)}{2}\frac{p^c\left(p^c\right)'}{h^c}\left(\gamma - \frac{wh^c}{p^c}\right)^2 \\
& - \rho\sigma_K\eta\left(p^c\right)'\left(\frac{\gamma p^c}{h^c} - w\right) + \frac{\gamma\sigma_K^2}{2}\frac{\left(p^c\right)^2\left(p^c\right)''}{h^c\left(p^c\right)'} \\
& + \frac{1}{2}\frac{\left[p^c - (w+1)\left(p^c\right)' - \rho_I\epsilon\left(p^c\right)'\frac{\gamma p^c - wh^c}{p^c}\left(\eta\frac{p^c}{h^c} - \rho\sigma_K\left(\frac{\gamma p^c}{h^c} - w\right)\right)\right]^2}{\theta\left(p^c\right)' + \left(1-\rho_I^2\right)\epsilon^2\frac{\left(p^c\right)'}{p^c}\frac{\left(\gamma p^c - wh^c\right)^2}{h^c} - \gamma\epsilon^2\frac{\left(p^c\right)^2\left(p^c\right)''}{h^c\left(p^c\right)'}},
\end{aligned} \tag{6.14}$$

where h^c is defined in (6.13);

$$b^c = \zeta\left[1 + \frac{1-\psi}{\zeta}\left(r - \zeta + \frac{\eta^2}{2\gamma}\right)\right]^{\frac{1}{1-\psi}}. \tag{6.15}$$

Using a similar procedure as outlined in Du [8],[6] we obtain

$$\begin{aligned}
0 = {} & \frac{\zeta^{\psi}\left(b^N\right)^{1-\psi}\left(\left(p^N\right)'\right)^{1-\psi} - \zeta\psi}{\psi - 1} p^N + (rw + A)\left(p^N\right)' + \frac{1}{2}\frac{\eta^2 p^N\left(p^N\right)'}{h^N} \\
& - \delta_K\left(p^N - w\cdot\left(p^N\right)'\right) - \frac{\sigma_K^2\left(1-\rho^2\right)}{2}\frac{p^N\left(p^N\right)'}{h^N}\left(\gamma - \frac{wh^N}{p^N}\right)^2 \\
& - \rho\sigma_K\eta\left(p^N\right)'\left(\frac{\gamma p^N}{h^N} - w\right) + \frac{\gamma\sigma_K^2}{2}\frac{\left(p^N\right)^2\left(p^N\right)''}{h^N\left(p^N\right)'} \\
& + \frac{1}{2}\frac{\left[p^N - (w+1)\left(p^N\right)' - \rho_I\epsilon\left(p^N\right)'\frac{\gamma p^N - wh^N}{p^N}\left(\eta\frac{p^N}{h^N} - \rho\sigma_K\left(\frac{\gamma p^N}{h^N} - w\right)\right)\right]^2}{\theta\left(p^N\right)' + \left(1-\rho_I^2\right)\epsilon^2\frac{\left(p^N\right)'}{p^N}\frac{\left(\gamma p^N - wh^N\right)^2}{h^N} - \gamma\epsilon^2\frac{\left(p^N\right)^2\left(p^N\right)''}{h^N\left(p^N\right)'}} \\
& + \frac{\lambda p^N(w)}{1-\gamma}\left[\delta^{1-\gamma}\left(\frac{b^c\cdot p^c(w)}{b^N\cdot p^N(w)}\right)^{1-\gamma} - 1\right]
\end{aligned} \tag{6.16}$$

[6] More specifically, we substitute (2.8), optimal policies of (6.10) - (6.11), and the expressions of (6.5) - (6.9) into (4.1), make use of the conjectured formulas for J^N and J^c, and perform necessary simplifications. Tedious algebra then gives (6.16).

for the naive entrepreneur, where b^c and $p^c(w)$ in the last term come from the expression of $J^c(K,W)$. To obtain the ODE for the sophisticated entrepreneur, we first conjecture (and verify later) that her continuation value can be written as

$$J^{cont}(K,W)=\frac{\left(b^c K p^{cont}(w)\right)^{1-\gamma}}{1-\gamma}, \tag{6.17}$$

where b^c is given by (6.15). We then apply a similar procedure as that for p^N to obtain

$$\begin{aligned}0=&\frac{\zeta^{\psi}\left(b^S\right)^{1-\psi}\left(\left(p^S\right)'\right)^{1-\psi}-\zeta\psi}{\psi-1}p^S+(rw+A)\left(p^S\right)'+\frac{1}{2}\frac{\eta^2 p^S\left(p^S\right)'}{h^S}\\&-\delta_K\left(p^S-w\cdot\left(p^S\right)'\right)-\frac{\sigma_K^2\left(1-\rho^2\right)}{2}\frac{p^S\left(p^S\right)'}{h^S}\left(\gamma-\frac{wh^S}{p^S}\right)^2\\&-\rho\sigma_K\eta\left(p^S\right)'\left(\frac{\gamma p^S}{h^S}-w\right)+\frac{\gamma\sigma_K^2}{2}\frac{\left(p^S\right)^2\left(p^S\right)''}{h^S\left(p^S\right)'}\\&+\frac{1}{2}\frac{\left[p^S-(w+1)\left(p^S\right)'-\rho_I\epsilon\left(p^S\right)'\frac{\gamma p^S-wh^S}{p^S}\left(\eta\frac{p^S}{hS}-\rho\sigma_K\left(\frac{\gamma p^S}{h^S}-w\right)\right)\right]^2}{\theta\left(p^S\right)'+\left(1-\rho_I^2\right)\epsilon^2\frac{\left(p^S\right)'}{p^S}\frac{\left(\gamma p^S-wh^S\right)^2}{h^S}-\gamma\epsilon^2\frac{\left(p^S\right)^2\left(p^S\right)''}{h^S\left(p^S\right)'}}\\&+\frac{\lambda p^S(w)}{1-\gamma}\left[\delta^{1-\gamma}\left(\frac{b^c\cdot p^{cont}(w)}{b^S\cdot p^S(w)}\right)^{1-\gamma}-1\right],\end{aligned} \tag{6.18}$$

where $p^{cont}(w)$ in the last term comes from the expression of $J^{cont}(K,W)$ given by (6.17). Finally, we substitute (2.8), (6.5) - (6.9), and (6.17) into the HJB of (5.3) on J^{cont} to obtain

$$\begin{aligned}0=&\left[\frac{\zeta}{1-\frac{1}{\psi}}\left(c^S\right)^{1-\frac{1}{\psi}}\left(b^c p^{cont}\right)^{\frac{1}{\psi}-1}-\frac{\zeta\psi}{\psi-1}\right]p^{cont}\\&+\left[rw+x^S\eta\sigma_R+A-i^S-\frac{1}{2}\theta\left(i^S\right)^2-c^S\right]\left(p^{cont}\right)'\\&+\left(i^S-\delta_K\right)\left(p^{cont}-w\cdot\left(p^{cont}\right)'\right)\\&-\frac{\epsilon^2\left(i^S\right)^2+\sigma_K^2}{2}\frac{\gamma p^{cont}\left(p^{cont}-2w\cdot\left(p^{cont}\right)'\right)+w^2\left(p^{cont}\right)'h^{cont}}{p^{cont}}\\&-\left(\rho_I\epsilon i^S+\rho\sigma_K\right)\sigma_R x^S\left(p^{cont}\right)'\left(\gamma-\frac{w\cdot h^{cont}}{p^{cont}}\right)-\frac{\sigma_R^2}{2}\left(x^S\right)^2\frac{\left(p^{cont}\right)'h^{cont}}{p^{cont}},\end{aligned} \tag{6.19}$$

where $\left(c^S, i^S, x^S\right)$ denote the optimal policies chosen by the sophisticated entrepreneur when she reaches the stationary equilibrium for the underlying intra-personal competition; h^{cont} is similarly defined as h^m by (2.8) in that $h^{cont} \equiv \gamma\left(p^{cont}\right) - p^{cont}\left(p^{cont}\right)'' \Big/ \left(p^{cont}\right)'$.

6.3. Simplifying Boundary Conditions at the Lower End

It is well known that the value function for a Merton consumer can be written as

$$J^M\left(W\right) = \frac{\left(b^c W\right)^{1-\gamma}}{1-\gamma} \tag{6.20}$$

where b^c under recursive preference is given by (6.15) which solves

$$\frac{\zeta^{\psi}\left(b^c\right)^{1-\psi} - \zeta\psi}{\psi - 1} + r + \frac{\eta^2}{2\gamma} = 0. \tag{6.21}$$

For a time-inconsistent Merton consumer with naivety, her value function changes to

$$J^{M,N}\left(W\right) = \frac{\left(b^N W\right)^{1-\gamma}}{1-\gamma} \tag{6.22}$$

where b^N solves the following nonlinear equation

$$\frac{\zeta^{\psi}\left(b^N\right)^{1-\psi} - \zeta\psi}{\psi - 1} + r + \frac{\eta^2}{2\gamma} + \frac{\lambda}{1-\gamma}\left[\delta^{1-\gamma}\left(\frac{b^c}{b^N}\right)^{1-\gamma} - 1\right] = 0. \tag{6.23}$$

In particular, the last term of (6.23) is very similar to that of (6.16) for the naive entrepreneur with the following difference: The $p\left(w\right)$-terms, which denote the valuations of firm-held capital, are absent with a Merton consumer. Turning to a sophisticated Merton consumer, we first conjecture that her continuation value function coincides with $J^M\left(W\right)$, *i.e.*,

$$J^{M,cont}\left(W\right) = J^M\left(W\right) = \frac{\left(b^c W\right)^{1-\gamma}}{1-\gamma}. \tag{6.24}$$

On the other hand, $J^{M,cont}$ satisfies the HJB of

$$0 = f\left(C^{M,S}, J^{M,cont}\right) + \left[rW + \left(\mu_R - r\right)X^{M,S} - C^{M,S}\right]J_W^{M,cont} + \frac{\sigma_R^2\left(X^{M,S}\right)^2}{2}J_{WW}^{M,cont}, \tag{6.25}$$

where $\left(C^{M,S}, X^{M,S}\right)$ are optimal policies chosen by the sophisticated Merton consumer. Since the extra δ-discount does not affect the derivation of the implied optimal controls on C and X,[7] $\left(C^{M,S}, X^{M,S}\right) = \left(C^M, X^M\right)$ for the given W which implies that (6.25) coincides the HJB for J^M so that the conjectured

[7]This can be seen by the observation that i) the Merton consumer's problem can be treated as a simplified version of the entrepreneur's problem; and ii) the controls of $\left(C, I, X\right)$ in (5.1) for a sophisticated entrepreneur do not affect the last term of (5.1) through which the extra δ-discount takes effect.

(6.24) is confirmed. Given the characterization of her continuation value $J^{M,cont}$, we next denote the value function for a sophisticated Merton consumer by

$$J^{M,S}(W)=\frac{\left(b^{S}W\right)^{1-\gamma}}{1-\gamma}. \tag{6.26}$$

Since $J^{M,cont}$ is already determined according to (6.24), it is easy to show that b^S implied from (6.26) solves the same nonlinear equation of (6.23) (with b^N replaced with b^S). We thus conclude that $b^S = b^N$ so that the *b*-coefficient remains unchanged for a time-inconsistent agent regardless of the beliefs about her selves' behaviors. By substituting (6.26) and (6.22), together with the expressions of $J^S(K,W)$ and $J^N(K,W)$ jointly given by (6.4), into (4.2) - (4.3) and (5.5) - (5.6) and performing necessary simplifications, we obtain

$$p^m\left(w_d^m\right)=w_d^m+l, \tag{6.27}$$

$$\left(p^m\right)'\left(w_d^m\right)=1 \tag{6.28}$$

where $m\in\{N,S\}$; $w_d\equiv W_d/K$. Finally, by substituting (6.17) and (6.27) into (5.7) and simplyfing, we obtain

$$p^{cont}\left(w_d^S\right)=w_d^S+l \tag{6.29}$$

as the boundary condition for a sophisticated agent's continuation value function.

6.4. The Limiting Behaviors of p^N and p^S at the Upper End

At the upper end when $w\to\infty$, Du [8] shows that

$$\lim_{w\to\infty}p^c(w)=p^{FB}=w+q^{FB} \tag{6.30}$$

for the time-consistent benchmark,[8] where $p^c(w)$ satisfies (6.14); q^{FB} denotes the constant valuation of the firm-held capital under the first-best[9] whose formula is given by Equations (3.30) - (3.31) in his paper.[10] Furthermore, by combining Equations (3.17), (3.27), and (3.30) - (3.31) in Du [8], we see that (6.14) (which is

[8]In view of the boundary condition listed in (3.2), (6.30) implies that $b^c=b^{FB}$ which is also proved in Du [8].

[9]By pursuing its financial interpretation, we can decompose $p(w)$ into $p(w)=w+q(w)$ where w denotes the liquid wealth per unit of capital while $q(w)$ denotes the liquid-wealth valuation of capital. In general, q varies with w which reflects the effect of financial slack on the valuation of capital when the firm faces the liquidation risk. However, the concern about the potential liquidation is gone when first-best (FB) is achieved at $w\to\infty$. Consequently, q under FB becomes a constant which we denote by q^{FB} .

[10]More concretely, $q^{FB}=\frac{1+\theta i^{FB}}{1-\rho_I\epsilon\eta}$ where

$i^{FB}=\frac{r+\rho\sigma_K\eta+\delta_K}{1-\rho_I\epsilon\eta}-\sqrt{\left[\frac{r+\rho\sigma_K\eta+\delta_K}{1-\rho_I\epsilon\eta}\right]^2-\frac{2}{\theta}\left[A-\frac{r+\rho\sigma_K\eta+\delta_K}{1-\rho_I\epsilon\eta}\right]}$. Also recall that the entrepreneur's value function under first-best is given by $J^{FB}(W)=\frac{\left[b^{FB}K\left(w+q^{FB}\right)\right]^{1-\gamma}}{1-\gamma}$ with $b^{FB}=b^c$ (see footnote 8 for an explanation).

simply Equation (3.17) in Du [8] if we replace b^c and p^c with *b* and *p*) at $w \to \infty$ degenerates to

$$0 = \left(\frac{\zeta^{\psi} \left(b^c \right)^{1-\psi} - \psi\zeta}{\psi - 1} + r + \frac{\eta^2}{2\gamma} \right) \left(w + q^{FB} \right). \tag{6.31}$$

It is easy to see that (6.31) is consistent with the expression for b^c given in (6.15).

Based on above results, we now prove (4.5) for the naive entrepreneur by showing that it is consistent with the limiting behavior of $p^N(w)$ that satisfies the HJB of (6.16). By comparing (6.16) with (6.14), we see that they differ in two places: i) the *b*-coefficient is b^N instead of b^c; ii) the extra δ-discount enables an extra term in (6.16) that loads on both p^N and p^c. In view of the degeneration of (6.14) to (6.31), (6.16) at $w \to \infty$ degenerates to

$$\begin{aligned} 0 &= \left(\frac{\zeta^{\psi} \left(b^N \right)^{1-\psi} - \psi\zeta}{\psi - 1} + r + \frac{\eta^2}{2\gamma} \right) \left(w + q^{FB} \right) \\ &+ \frac{\lambda \cdot \lim_{w\to\infty} p^N(w)}{1-\gamma} \left[\delta^{1-\gamma} \left(\frac{b^c}{b^N} \right)^{1-\gamma} \left(\frac{\lim_{w\to\infty} p^c(w)}{\lim_{w\to\infty} p^N(w)} \right)^{1-\gamma} - 1 \right] \end{aligned} \tag{6.32}$$

With the conjectured (4.5) and using (6.30), the above equation further degenerates to

$$0 = \left(\frac{\zeta^{\psi} \left(b^N \right)^{1-\psi} - \psi\zeta}{\psi - 1} + r + \frac{\eta^2}{2\gamma} \right) \left(w + q^{FB} \right) + \frac{\lambda \left(w + q^{FB} \right)}{1-\gamma} \left[\delta^{1-\gamma} \left(\frac{b^c}{b^N} \right)^{1-\gamma} - 1 \right] \tag{6.33}$$

It is easy to see that (6.33) is consistent with the solution to b^N which is given by (6.23). Since the whole logic goes through, we conclude that the conjectured (4.5) is indeed the correct description for the limiting behavior of p^N.

By a similar procedure, we can show for a sophisticated entrepreneur that (6.18) at $w \to \infty$ degenerates to

$$0 = \left(\frac{\zeta^{\psi} \left(b^S \right)^{1-\psi} - \psi\zeta}{\psi - 1} + r + \frac{\eta^2}{2\gamma} \right) \left(w + q^{FB} \right) + \frac{\lambda \left(w + q^{FB} \right)}{1-\gamma} \left[\delta^{1-\gamma} \left(\frac{b^c}{b^S} \right)^{1-\gamma} - 1 \right] \tag{6.34}$$

provided that the limiting behavior of p^S listed in (5.4) holds. (6.34) double confirms that $b^S = b^N$ which we already established in last subsection. To finish the proof on (5.4), we need to show that the limiting behavior of p^{cont}, which is relevant for determining the limiting behavior of p^S as indicated by (6.18), is indeed the same as that of p^c. This is not obvious since the ODE of (6.19) for $p^{cont}(w)$ looks very different than (6.14) for $p^c(w)$. To reconcile the apparent discrepancy, assume $p^{cont}(w) = p^S(w)$. Given that the optimal $\left(c^S, i^S, x^S \right)$ used in (6.19) all load on $p^S(w)$, a substitution of $\left(c^S, i^S, x^S \right)$ from (6.10)-(6.12) into (6.19) under the premise of $p^{cont} = p^S$ would yield

$$\begin{aligned}0=&\frac{\zeta^{\psi}\left(b^{S}\right)^{1-\psi}\left(\left(p^{S}\right)'\right)^{1-\psi}-\zeta\psi}{\psi-1}p^{S}+\left(rw+A\right)\left(p^{S}\right)'+\frac{1}{2}\frac{\eta^{2}p^{S}\left(p^{S}\right)'}{h^{S}}\\&-\delta_{K}\left(p^{S}-w\cdot\left(p^{S}\right)'\right)-\frac{\sigma_{K}^{2}\left(1-\rho^{2}\right)}{2}\frac{p^{S}\left(p^{S}\right)'}{h^{S}}\left(\gamma-\frac{wh^{S}}{p^{S}}\right)^{2}\\&-\rho\sigma_{K}\eta\left(p^{S}\right)'\left(\frac{\gamma p^{S}}{h^{S}}-w\right)+\frac{\gamma\sigma_{K}^{2}}{2}\frac{\left(p^{S}\right)^{2}\left(p^{S}\right)''}{h^{S}\left(p^{S}\right)'}\\&+\frac{1}{2}\frac{\left[p^{S}-\left(w+1\right)\left(p^{S}\right)'-\rho_{I}\epsilon\left(p^{s}\right)'\frac{\gamma p^{S}-wh^{S}}{p^{S}}\left(\eta\frac{p^{S}}{hS}-\rho\sigma_{K}\left(\frac{\gamma p^{S}}{h^{S}}-w\right)\right)\right]^{2}}{\theta\left(p^{S}\right)'+\left(1-\rho_{I}^{2}\right)\epsilon^{2}\frac{\left(p^{S}\right)'}{p^{S}}\frac{\left(\gamma p^{S}-wh^{S}\right)^{2}}{h^{S}}-\gamma\epsilon^{2}\frac{\left(p^{S}\right)^{2}\left(p^{S}\right)''}{h^{S}\left(p^{S}\right)'}},\end{aligned}\tag{6.35}$$

(6.35) is identical to (6.14) when "c" is replaced with "S".[11] Therefore, the implied $p^{cont}(w)$ from (6.35) (recall $p^{cont}=p^{S}$ by our premise) has the same limiting behavior as $p^{c}(w)$ which is stated in (5.4). Of course, the "loophole" of the above logic is that the premise of $p^{cont}=p^{S}$ does not hold in general. So long as (5.4) is true, however, p^{cont} indeed coincides with p^{S} at $w\to\infty$ so that the above logic, which targets on the limiting behaviors, goes through. This in turn confirms that the conjectured (5.4) indeed holds.

6.5. Numerical Procedure and the Implied Solutions

Given $p^{c}(w)$ as the solution in the benchmark case and taking into account the boundary conditions of (4.5) and (6.27) - (6.28), we can directly solve (6.16) to obtain $p^{N}(w)$ for the naive entrepreneur. Solving the sophisticated entrepreneur's problem is more involved because it requires the search for a fixed point of the firm's optimal policies. First, given the conjectured $\left(c^{S},i^{S},x^{S},w_{d}^{S}\right)$ and taking into account the necessary boundary conditions, we can solve (6.19) to obtain $p^{cont}(w)$. Second, given $p^{cont}(w)$ and taking into account the boundary conditions for a sophisticated entrepreneur, we can solve (6.18) to obtain $p^{S}(w)$ and the optimal polices of $\left(c^{S},i^{S},x^{S},w_{d}^{S}\right)$ as conjectured.

We now describe the numerical procedure for identifying the aforementioned fixed point (which by itself identifies the stationary equilibrium for a sophisticated entrepreneur's intra-personal competition). We start by using $\left(c^{N},i^{N},x^{N},w_{d}^{N}\right)$ from the solution to a naive entrepreneur's problem as the initial policies chosen by the sophisticated entrepreneur, which we denote by $\left(c^{S,0},i^{S,0},x^{S,0},w_{d}^{S,0}\right)$. We

[11]It is worth noticing that the obtained (6.35) is only for illustrating the limiting behavior of p^{cont}. In particular, we do not actually solve p^{S} from (6.35) which does not involve the extra δ-discount. Instead, p^{S} for the sophisticated entrepreneur is solved from (6.18) which loads on both δ and p^{cont} as the entrepreneur's continuation value.

then iterate on the two ODEs of (6.19) and (6.18) as follows. In round n, we substitute $\left(c^{S,n-1}, i^{S,n-1}, x^{S,n-1}, w_d^{S,n-1}\right)$ obtained from last round into (6.19) to solve for $p^{cont,n}(w)$ under the two boundary conditions of (6.29) and (5.4). We then substitute the implied $p^{cont,n}$ into (6.18) to solve for $p^{S,n}(w)$ which is subject to the boundary conditions of (5.4) and (6.27) - (6.28). By solving out $p^{S,n}(w)$, we simultaneously obtain $\left(c^{S,n}, i^{S,n}, x^{S,n}, w_d^{S,n}\right)$ for round n. We repeat the iterations until the updated optimal policies no longer change under reasonable criteria, which means that $\left(c^{S,n}, i^{S,n}, x^{S,n}, w_d^{S,n}\right)$ converges to the stationary policies of $\left(c^S, i^S, x^S, w_d^S\right)$. We then use $p^{S,n}(w)$ obtained from the last round as the numerical solution to the sophisticated entrepreneur's problem.

Table 2. Baseline parameterization.

Panel A: Market environment				
$r = 0.046$	$\sigma_R = 0.2$	$\eta = 0.3$	$\rho_K = 0$	$\rho_I = 0.3$
Panel B: Preferences				
$\zeta = 0.046$	$\gamma = 2$	$\psi = 2.2$		
Panel C: Investment and production				
$A = 0.2$	$\theta = 2$	$l = 0.9$	$\delta_K = 0.125$	$\epsilon = 0.5$
Panel D: Related to time-inconsistent preference (TIP)				
$\lambda = 1$		δ: between 0.25 and 1		

This table summarizes the baseline parameterization to our model. Panel A describes the market-related parameters, where r, σ_R, η, ρ_K, and ρ_I denote, respectively, the risk-free rate, the volatility of the market portfolio, market Sharpe ratio, the correlation between the market portfolio returns and capital depreciation shocks, and the correlation between the market portfolio returns and investment-specific shocks. Panel B reports preference parameters, where ζ, γ, and ψ denote the subjective discount rate, the degree of risk aversion, and the elasticity of intertemporal substitution (EIS), respectively. Panel C calibrates firm-related parameters, where A, θ, l, δ_K, and ϵ denote, respectively, the capital's productivity, adjustment cost parameter, capital liquidation price, the rate of capital depreciation, and the volatility of investment-specific shocks. Panel D reports parameters related to time-inconsistent preference (TIP), where λ and δ denote the transition density into the next self and degree of the extra discount, respectively. Parameter values other than λ and δ are taken from Du [8]. Values for λ and δ are taken from GW [10]. All parameters are annualized.

Our numerical procedures are based on (the annualized) parameter values summarized in **Table 2**. parameterization other than those related to TIP are taken from Du [8].[12] For the two TIP-related parameters, we follow GW [10] by setting the Poisson intensity on future selves' arrivals, λ, to one and we allow δ, which reflects the degree of time-inconsistent discounting, to vary between 0.25 and 1. **Figure 1** plots the numerical solutions to the entrepreneur's problems in terms of $p'(w)$ $(= P_W)$ which denotes the firm's marginal value of wealth. Due to the extra benefits of accumulating financial slack when the firm is subject to the costly liquidation, $p'(w)$ generally stays above the norminal gain of one for one unit increase of W. The two exceptions are at $w = w_d$ and at $w \to \infty$ under

[12]Du [8] obtains his baseline parameterizations through the guidelines in the empirical literature. For example, he sets the EIS parameter ψ to 2.2 which is motivated by its recent estimate by Kapoor and Ravi [23].

which $p'(w)=1$ by its boundary conditions. Because of the extra discounting under TIP, the potential liquidation feels less costly to a time-inconsistent entrepreneur. Consequently, she has less needs for liquid wealth when the firm is close to its liquidation which is captured by a lower $p'(w)$ relative to its benchmark levels (dashed line) when w is close to w_d. This TIP-induced effect is more pronounced to a sophisticated entrepreneur who has more concerns about her future selves' behaviors and is thus more eager to abandon the firm. When w drifts away from the w_d so that liquidation is no longer a major concern, however, $p'(w)$ for a naive entrepreneur may rise above its benchmark levels which prompts her to accumulate more of the financial slack as "the hedge" against the potential δ -discount.

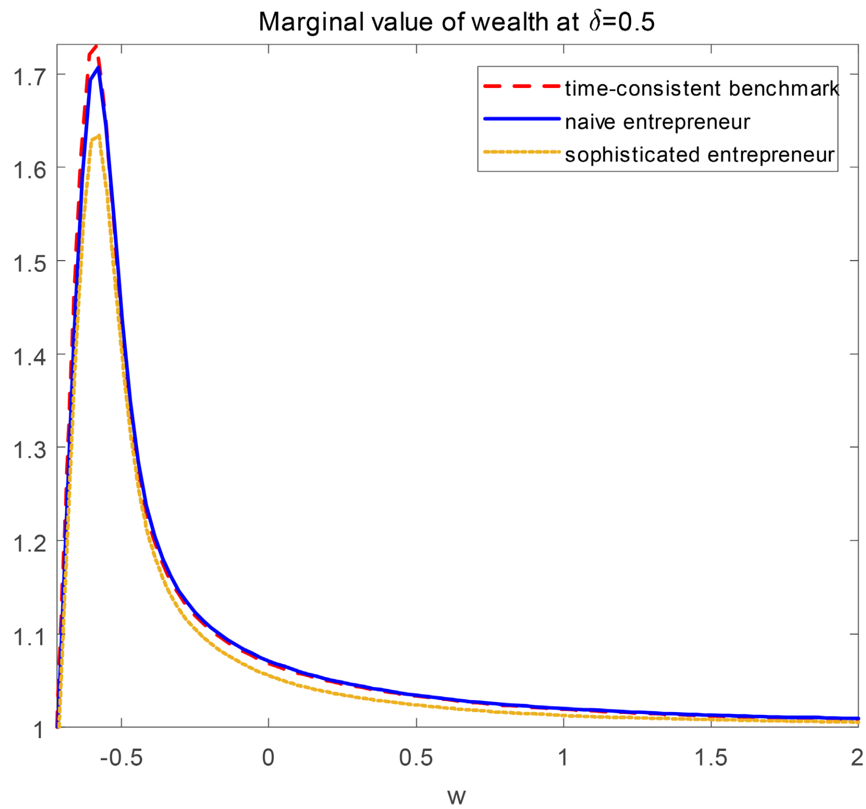

Figure 1. Marginal value of wealth P_W. **Figure 1** plots the numerical solutions to the entrepreneur's problems in terms of $p'(w)$ ($=P_W$) which denotes the firm's marginal value of wealth, where $w \equiv W/K$ which measures the financial slack of the entrepreneurial firm. The dashed, solid, and dotted line plots the $p'(w)$ implied from the time-consistent benchmark, the case with a naive entrepreneur, and the case with a sophisticated entrepreneur, respectively.

7. Quantitative Implications of the Model

Based on the numerical solutions to the entrepreneur's problems, we now derive and discuss the quantitative implications of TIP on the operations of an entrepreneurial firm.

7.1. Valuation and Welfare Effects of δ

Panel A&B of Figure 2 plots, respectively, the valuation of the firm $p(w)$ and the utility that the entrepreneur obtains from the firm $J(1,W)$ as we vary δ, where we normalize the firm's capital stock to one for J without the loss of generality. Furthermore, we set $w = W = 0$ for the plots[13] and we confirm in unreported exercises that the implications remain qualitatively unchanged when w is set at different levels. As expected, $p^c(w)$ and $J^c(1,W)$ for the time-consistent benchmark (dashed lines) are invariant to δ which is relevant only under TIP.

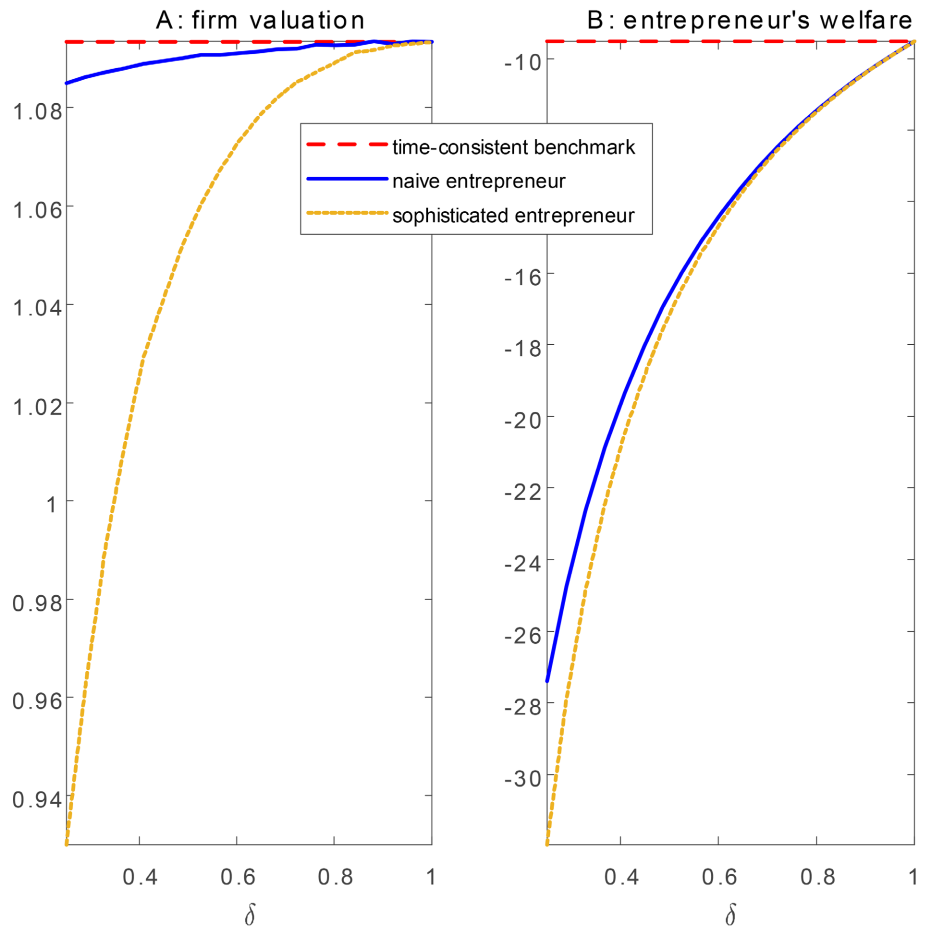

Figure 2. Impact of TIP on firm valuation and the entrepreneur's welfare. Panel A&B of Figure 2 plots, respectively, the valuation of the firm $p(w)$ and the the entrepreneur's utility $J(K,W)$ which measures her welfare with the firm as we vary δ measuring the degree of the TIP-induced discounting. $J(K,W)$ is calculated according to (2.10) where we normalize K to one and set $W = 0$ without the loss of generality. The dashed, solid, and dotted line plots the implications from the time-consistent benchmark (see Section III), the case with a naive entrepreneur (see Section IV), and the case with a sophisticated entrepreneur (see Section V), respectively.

[13]Given that $p(w) = w + q(w)$, $p(w)$ at $w = 0$ is simply the valuation of firm's capital q at the zero degree of financial slack.

7.1.1. Impact of TIP-Induced Discounting

We first study the impact of the extra discounting under TIP by comparing the valuation and welfare implications for a naive entrepreneur (solid lines) to those for the benchmark case (dashed lines). Panel A of Figure 2 shows that p^N falls from the level of p^c and the magnitudes of decline are greater at a lower δ indicating a higher degree of the extra discounting under TIP. Financially, the current self of a naive entrepreneur values less the cash flows obtained from the firm when her future selves are in charge. Consequently, she values the firm less when compared to that in the benchmark case and this effect is naturally stronger at a lower $\delta(<1)$.

Given that the baseline value of γ is greater than one, (6.4) implies that the entrepreneur's utility J^m, which measures her welfare with the firm, is negative for all $m \in \{c, N, S\}$ as indicated by plots in Panel B of Figure 2. By comparison, the extra δ-discount induces a much greater loss in the entrepreneur's welfare than in the firm's valuation. For example, while the percentage discrepancy between p^N at $\delta = 0.25$ and p^c is less than 1% (Panel A), J^N at $\delta = 0.25$ is so much more negative that $\left(J^N - J^c\right) / \left|J^c\right| \times 100\%$ is lower than −180%. Mechanically, the large discrepancy between J^N and J^c is mainly driven by a much smaller b^N, determined by (6.23), which is only 0.0336 at $\delta = 0.25$ as compared to $b^c = 0.0961$ under our baseline parameterization. Financially, the extra time-inconsistent discounting at the presence of risk aversion finds its way mainly in the risk-adjustment component, which leads to a large influence on the entrepreneur's welfare but leaves her valuation of the firm much less affected. This finding adds to the analyses of GW [10] who assume risk neutrality in their setup and therefore cannot differentiate the valuation effect of TIP from its welfare effect.

7.1.2. Impact of the Entrepreneur's Sophistication

Turning to implications for a sophisticated entrepreneur which are plotted in dotted lines in Figure 2, the first observation is that firm valuation (Panel A) and the entrepreneur's welfare (Panel B) are both lower with the entrepreneur's sophistication. While a naive entrepreneur is optimistic in that she incorrectly forecast that her future selves would behave according to her current preference, a sophisticated entrepreneur correctly forecasts that her future selves would make decisions according to their own preferences which are suboptimal to her current preference. The realistic pessimism due to the entrepreneur's sophistication prompts her to assign a even lower valuation to the firm which drives down her welfare even further. To summarize, our study suggests a negative sophistication effect[14] in that a time-inconsistent entrepreneur's sophistication destroys instead of creating values for herself.[15]

[14]The name "sophistication effect" is first used by O'Donoghue and Rabin [4] and they consider a very different setup that does not involve entrepreneurship.

[15]Related to our finding, GW [10] show that the entrepreneur's sophistication erodes the value of their option to wait which further delays investment relative to that of a naive entrepreneur.

Unlike the TIP-induced discounting that works disproportionately on p and J (see discussions in last subsection), the sophistication effect has symmetric valuation and welfare implications. Again use results at $\delta = 0.25$ as the example. The entrepreneur's sophistication depresses her valuation of the firm by 14.3% in that $\left(p^N - p^c\right)/\left|p^c\right| \times 100\% = -14.3\%$ while at the same time $\left(J^N - J^c\right)/\left|J^c\right| \times 100\%$ delivers a comparable percentage reduction in her welfare which is −16.7%.

7.2. Impact of TIP on Firm's Optimal Policies

7.2.1. Optimal Consumption and Investment

Figure 3 plots the firm's optimal policies set by a time-consistent entrepreneur (dashed line), a naive entrepreneur (solid line), and a sophisticated entrepreneur (dotted line), respectively. Without the loss of generality, we set $\delta = 0.25$ for plots in Panel A-C. Compared to that in the time-consistent benchmark, a naive entrepreneur's consumption-capital ratio c (Panel A) and consumption-investment ratio i (Panel B) are both lower which is apparently due to the extra δ-discount. Intuitively, the TIP-induced discounting destroys value which induces the naive entrepreneur to be more conservative in her consumption and investment policies. Except for ws that are close to w_d, the sophistication effect further depresses the entrepreneur's capital investment (dotted line in Panel B) and simultaneously it raises her consumption as plotted in Panel A (dotted line). Intuitively, when facing the costs due to the suboptimal policies (relative to her current preference) made by the future selves, the current self of a sophisticated entrepreneur transfers resources from investment to consumption so as to better obtain utilities from the firm before she is preempted by the suboptimal behaviors of her future selves.

The above logic on a sophisticated entrepreneur only works when w is not too low. This is because when the firm is in debt at $W < 0$, increasing K moves $w \equiv W/K$ from the left towards the origin which relaxes the financial constraints and thus provides more "breathing room" to the firm. This liquidation-relief effect is stronger with a sophisticated entrepreneur because she consumes more and it also gets stronger as w drops towards w_d. When the firm is very close to its liquidation, the liquidation-relief effect can dominate for a sophisticated entrepreneur. In response, she chooses to raise investment to slow down the depreciation of K so as to slow down the fall of w towards w_d.

7.2.2. Optimal Asset Allocation and the Firm's Liquidation

We now examine the firm's optimal asset-allocation-capital ratio x which is plotted in Panel C of Figure 3. Due to the wealth-creation effect from the stock market at a sizable η,[16] an entrepreneur always chooses an aggressive position in the stock market which is largely unaffected by her time preferences. Intuitively, a time-inconsistent entrepreneur also wants to exploit the high equity risk premium and she does by maintaining the firm's time-consistent asset allocation strategy (if one views the time-consistent policies as somehow optimal). In fact, a sophisticated

[16]See Du [8] for detailed discussions on the wealth-creation effect for a time-consistent entrepreneur.

entrepreneur (dotted line), motivated by the liquidation-relief effect, may even raise x above its benchmark levels (dashed line) when the firm is close to its liquidation.

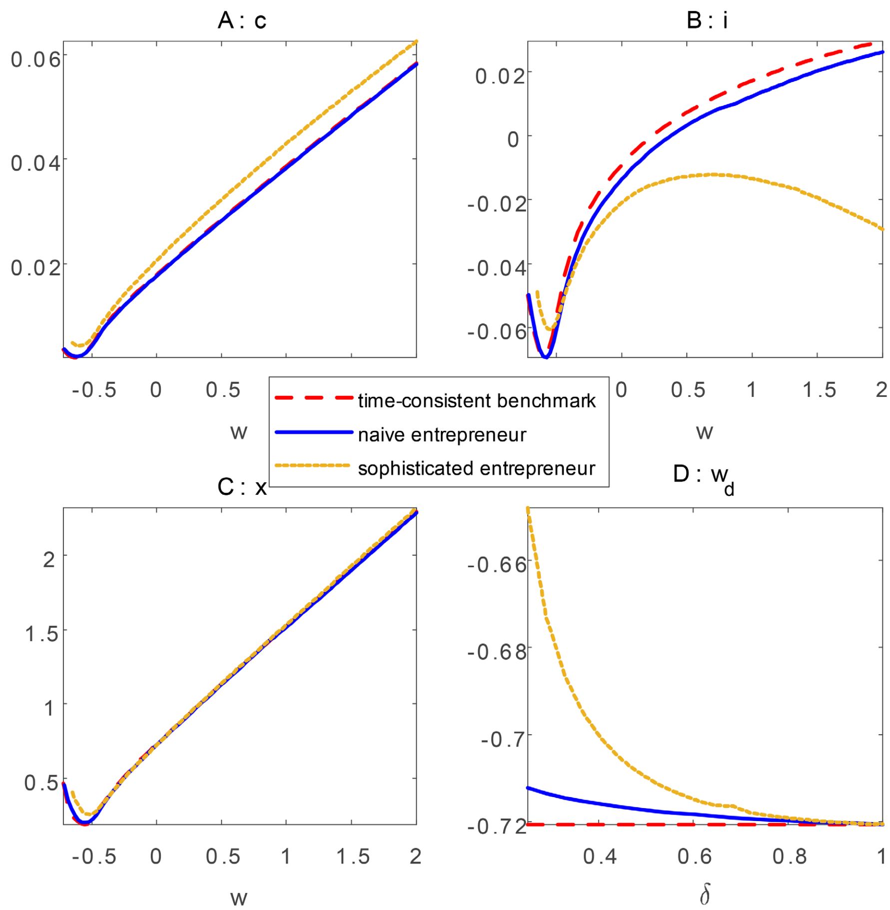

Figure 3. Impact of TIP on firm's optimal policies. **Figure 3** plots the firm's optimal policies set by a time-consistent entrepreneur (dashed line), a naive entrepreneur (solid line), and a sophisticated entrepreneur (dotted line), respectively. Panel A, B, and C plots the consumption-capital ratio c, the investment-capital ratio i, and the asset allocation-capital ratio x, respectively, as the function of the firm's financial slack. Panel D plots the firm's liquidation boundary w_d as the function of δ measuring the degree of TIP-induced discounting.

Due to the TIP-induced discounting, a time-inconsistent entrepreneur believes that she has a less valuable business to operate on than a time-consistent entrepreneur does. Consequently and as plotted in Panel D of Figure 3, she abandons the firm earlier as implied by a higher liquidation boundary which is decreasing in δ since a lower δ indicates a higher degree of the extra discounting. In comparison, a sophisticated entrepreneur chooses to liquidate the firm even earlier because she has the extra incentive to protect herself against the suboptimal policies made by her future selves which exactly tells the "sophistication effect".

8. Discussions

Our paper is focused on the theoretical analysis of TIP when applied to the operations of an entrepreneurial firm. Empirically, which individuals or institutions are more prone to time-inconsistency? It seems reasonable to believe that individual entrepreneurs that run small private firms, as studied in the present paper, are more prone to time-inconsistent behavior than big public firms. Indeed, DellaVigna and Malmendier [24] assume that individuals are time-inconsistent but that big firms with which the individuals contract are rational and time-consistent.[17] As some related but more involved questions, how would industries or cultural contexts affect the entrepreneur's naivety or sophistication? How would such characteristics of an entrepreneur differ across different sectors (e.g., tech startups vs. traditional businesses)? Unfortunately, there is so far little empirical or experimental researches that have been conducted to provide answers to these questions (which of course highlights areas for future research and refinement in behavioral finance).

Based on a survey study in Italy, Panozzo [25] provides the rare direct evidences that support TIP among entrepreneurs. Through an empirical estimation, Angeletos et al. [26] show that TIP helps explain the observed consumption and asset allocation data in the US. Since our model implies a significantly higher liquidation boundary chosen by a time-inconsistent entrepreneur (Panel D of **Figure 3**), it can potentially reconcile the shorter operating period well documented in empirical entrepreneurial finance literature (e.g., Camerer and Lovallo [27]; Lambrecht [28]; among others). Given that TIP also induces lower investment (Panel B of **Figure 3**), our study provides another explaination to the prevalence of underinvestment which is usually attributed to the firm's financial constraints (e.g., Franzoni [29]). Salani and Treich [30] show that lack of the power to commit could well lead to over-savings which is consistent with the more detailed implication from our model that a naive entrepreneur consumes less than her sophisticated counterpart (Panel A of **Figure 3**). At an intuitive level, managerial traits generate heterogeneity among otherwise identical firms. Therefore, our results are also broadly consistent with the empirical evidences that firms similar in terms of fundamentals may choose very different policies.

While the literature on TIP tend to focus on either the complete naivety or the complete sophistication, this convention apparently oversimplifies the spectrum of real-world behaviors as many entrepreneurs might exhibit mixed traits rather than fitting neatly into one category. To account for the partial naivety (or partial sophistication), we may introduce extra parameter on TIP discounting, denoted by $\hat{\delta}$, which captures the current self's belief about the degree of extra discounting to be adopted by her future selves. For a time-consistent entrpreneur,

[17] Presumably, there is "something" about the organization of a big firm and its delegated, professional management that mitigates or removes the time-inconsistency from the firm's decisions. However, there are no researches so far that clearly identifies the underlying economic mechansim for such mitigations (or removals).

apparently $\delta = \hat{\delta} = 1$ since the extra TIP discounting does not exist. For a (completely) sophisticated entrepreneur, $\delta = \hat{\delta} < 1$ [18] while for a completely naive entrepreneur $\hat{\delta} = 1 > \delta$ since she incorrectly believes that her future selves would make decisions according to her current preference. The partially naive entrepreneur can thus be modeled as the one with $\delta < 1$ and $\hat{\delta} \in (\delta, 1)$. Intuitively, she is aware that her futures selves will not behave in a time-consistent manner has future self-control problems so that $\hat{\delta} < 1$, but she underestimates the degree of the future selves' TIP-discounting in that $\hat{\delta} > \delta$. By allowing $\hat{\delta}$ to vary between δ and 1, an extension along this dimension is likely to yield rich implications from an entrepreneur's partial naivety. A full exploitation of such an extension, however, is beyond the scope of the current paper which we leave to future research.

9. Conclusions and Suggestions

This paper extends the dynamic corporate finance framework to account for the more realistic time-inconsistent preferences (TIP). In our setup, entrepreneurs make decisions on consumption, investment, asset allocation, and business liquidation by taking into account their beliefs about the behavior of their future selves. A naive entrepreneur falsely believes that her future selves behave according to her current wishes while a sophisticated entrepreneur correctly anticipates that the future selves act in a manner that deviates from her current preferences. In both cases, TIP induces the extra discounting on payoffs which depresses the firm's valuation and induces more conservative consumption, investment, and business liquidation policies. Somewhat surprisingly, the entrepreneur's sophistication further depresses the firm valuation and her welfare with the firm indicating that self-delusion, as in the case with a naive entrepreneur, is sometimes preferable to true self-awareness. In addition, sophistication prompts the entrepreneur to avoid the rationally anticipated suboptimal policies when her futures selves are in charge which leads to a transfer of resources from investment to consumption and a further acceleration of firm's liquidation.

While our paper focuses on the theoretical study of TIP, we outline a suggested procedure that experimentally investigates TIP as follows.[19] Select a group of young participants who aspire to become an entrepreneur who are then asked to complete some unpleasant tasks over seven participation dates across 6 weeks, with the first date occurring in the laboratory and the later dates requiring logging into a website accessible from any computer. On each date, participants will be asked to state their work preferences, complete the current task (if it is the first

[18] Recall that $\delta < 1$ under TIP. The fact that $\hat{\delta} = \delta$ reflects the entrepreneur's sophistication in that she correctly anticipates that all her future selves will also apply the same degree of the extra TIP discounting to payoffs that are realized after the birth of the even latter selves.

[19] In our theoretical study of TIP, we pursue the perspective that it induces the tendency for an entrepreneur to grab the immediate rewards. In the experimental study, we pursue the perspective that the same TIP also induces the tendency for an entrepreneur to avoid immediate costs (see O'Donoghue and Rabin [4] for a discussion on the equivalence between the two forms of TIP).

date) or tasks from one previous decision (if it is a later date), and make predictions about their future task decisions. Participants receive bonuses that dependent on tasks that they have completed and all bonuses are paid on a fixed date at the end of the experiment. TIP can be identified if task decisions for immediate work appear consistently lower than the decisions for future work. By further exploring how closely predictions on future behavior align with the behavior exhibited for immediate choice, this experiment can also help identify the degree of naivety $\hat{\delta}$ which is introduced in Section VIII.[20]

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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[20]As discussed in Section VIII, $\delta = \hat{\delta} < 1$ under the complete sophistication while $\delta < \hat{\delta} = 1$ under the complete naivety. Thus, a completely sophisticated participant asked to predict immediate-work decisions in the future will predict in line with her immediate-work preferences while a completely naive participant will predict in line with her future-work preferences. If the comparison of predictions with work decisions yields something in between, then this experiment can further help identify the existence of partial naivety in that the implied $\hat{\delta}$ lies between δ and 1.

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