

Modeling a Yield Curve under G2++

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Abstract

Yield curve modeling is a fundamental concept in finance, playing a crucial role in understanding the relationship between interest rates and time. G2++ is a popular interest rate model used for this purpose, an improved version of the original Hull-White model. In this work, we describe the two Gaussian interest rate models (G2++) where the instantaneous short rate “r” is the sum of two correlated stochastic processes plus a deterministic function. We assume that each of these processes has a Gaussian distribution with time-dependent volatility. The deterministic function is determined by exact fitting to observed term structures. We test the model through various numerical experiments to assess its goodness of fit to yield curves with different maturities quoted in the market. Additionally, we analyze the errors between the model and the initial yield after a time lapse. Overall, implementing G2++ for yield curve modeling can provide a powerful tool for analyzing and managing interest rate risk in financial markets.

Keywords

Yield Curve, Short Rate, G1++ Model, G2++ Model, Period, Zero Coupon Yield, Maturity, Error, Horizon

1. Introduction

The yield curve plays a vital role in today’s financial markets. It represents the relationship between the interest rates and the maturity dates of different bonds or securities. Yield curve modelling involves analyzing the shape and movements of the yield curve to make predictions about future interest rate movements and economic conditions. It is essential for investors and financial professionals to predict future interest rate movements and assess the risk-return profiles of different investments.

Since the 1970s, there has been a significant amount of literature on yield curve modeling. Initially, most of the research focused on the well-established

financial markets. However, in recent years, the unusual behavior exhibited by the yield curve in various countries recently has garnered significant attention regarding the identification and potential prediction of the factors influencing the forward and spot rates based on their maturity. Central banks, in response to economic changes, aim to uphold price and macroeconomic stability, while investors in financial markets must have timely knowledge of the shocks driving yield movements to act effectively.

Modeling yield curve involves constructing mathematical representations of historical yield curve data. This process is essential for understanding past interest rate dynamics, analyzing historical trends, and calibrating yield curve models for forecasting future interest rate movements.

In this paper, we introduce a comprehensive exogenous model, the G2++ model, characterized by the sum of two correlated Gaussian processes and a deterministic function, representing the instantaneous short rate. In the literature, a model in which the instantaneous short rate is given by one Gaussian process plus a deterministic function is known as the G1++ model. Despite the good tractability of the one-factor Gaussian model (G1++), it has two major drawbacks. Firstly, the model is not capable of satisfactorily reproducing a large volatility surface due to the lack of free calibration parameters. Secondly, the G1++ has been criticized for generating a significant number of highly negative interest rates. In contrast, the G2++ model is much more flexible in terms of fitting volatility surfaces due to the introduction of a second factor and its ability to handle negative rates more effectively.

The remainder of this paper is structured as follows: Section 2 provides a brief literature review on the theoretical models of the yield curve. Section 3 discusses the one Gaussian model, G1++. This is followed by Section 4, which explores the two Gaussian model, G2++, and reports on its findings. In this section, we present Proposition 1, which outlines the future variable states related to the short rate under G2++, and Theorem 2, which demonstrates the value of the future zero coupon price. Additionally, we delve into applications of zero-coupon yield under the G2++ model in Section 5, concluding with Section 6.

2. Yield Curve

2.1. Overview of Yield Curve

A yield curve, also known as a term structure, is the representation of the function which, at a given date, associates the corresponding interest rate level with each maturity. In practice, there is not just one but several yield curves on the markets. We can, however, distinguish two families:

- 1) **Market curves:** these are constructed from quotations observed on the financial markets, they are:
 - **Government bond yield curve:** constructed from bonds issued by the government in its own currency. It is the risk-free yield curve for G71 countries, since it is assumed that these yields do not present a default risk.

- **Swap rate curve:** this is an interbank curve which includes a risk premium linked to the risk of bank default (the average rating of the banks considered is between AA and A). By definition, they are higher than government rates, since they include the bank's credit risk spread.

2) Implicit curves: these are constructed indirectly from observed market data, such as bond and swap prices. They are mainly zero-coupon yield curves and forwards (instantaneous) rates.

Yield curves can take the form of three shapes:

- **Normal (upward sloping curve):** shows yields on longer-term bonds continuing to increase, pointing to an economic expansion.
- **Inverted (downward sloping curve):** shows that short-term interest rates exceed long-term interest rates, pointing to an economic recession.
- **Flat:** the yields are similar across all maturities, indicating an unpredictable economic situation.

A shift in the yield curve will occur for a number of reasons, connected not just with the market's view on interest rates but also factors such as liquidity and supply and demand. The main types of shifts in the yield curve are one of the following:

- Upward and downward parallel shifts;
- Flattening and steepening yield curve twists;
- Changes in the humped shape of the curve, sometimes called butterfly twists.

2.2. Choice of the Stochastic Rate Models

In practice, we use stochastic models for the shift of the yield curve for two key reasons:

1) Assessment and coverage of interest-rate products delivering random future flows (caplets) are essential. The option seller must be able to provide a price for the product they sell, but more importantly, they must cover the option they sell because they are exposed to unlimited loss. These models are primarily used in trading contexts within market rooms and in risk departments.

2) A portfolio manager makes predictions on the interest rate curve when implementing a strategy. They need to anticipate the scenario of the yield shift. To do this, they use a tool that enables them to consider all possible scenarios of interest rate curve shifts. Since they are not certain that their scenario will come true, they also need to measure the risk they take if that scenario does not occur.

The choice of the perfect model is based on some properties as follows:

1) A rate model shall take into account the empirical properties of the previously identified rate curve.

2) A rate model must be well constructed in the sense that the input of the model is observable on the market or easily estimable, and also frequently re-adjustable.

3) A rate model must be compatible with the market prices of vanilla products (caplets...).

4) Simple enough to allow quick calculations.

- 5) A model that can be easily manipulated by the user (Traders, Risk Controller).
- 6) A model that does not forget the risk factor.
- 7) A model that satisfies the theoretical assumption of the absence of opportunity for arbitrage.
- 8) A model that offers an effective coverage method that allows the seller to cover his product throughout the life of the product.

2.3. Some Models

Various models have been proposed to capture the stochastic dynamics of interest rates.

1) Vasicek Model (1977): This was one of the pioneering models for modeling interest rates, specifically the short rate. It assumes that the short rate follows a Gaussian distribution and is mean-reverting. (See [1])

2) Cox-Ingersoll-Ross (CIR) Model (1985): Building on the Vasicek model, CIR introduced a non-negative short rate by modeling the rate as a square root process. This model is widely used in fixed income and interest rate modeling due to its ability to capture the volatility of interest rates. (See [2])

3) Ho-Lee Model (1986): This model assumes that the short rate is driven by a stochastic process, typically calibrated to match observed term structure data. (See [3])

4) Hull-White Model (1990): This model extends the Vasicek model by adding a time-varying volatility term. It allows for more flexibility in capturing interest rate dynamics and is commonly used in interest rate derivatives pricing; it is equivalent to the G1++ model. (See [4] [5])

5) Black-Derman-Toy (BDT) Model (1990): Used primarily in the context of interest rate derivatives, this model is a tree-based approach that incorporates mean reversion and stochastic volatility. (See [6])

6) Black-Karasinski Model (1991): Another tree-based model that extends the CIR model by introducing a stochastic volatility component. These models have different assumptions about interest rate behavior and are used in various financial applications, such as pricing derivatives, risk management, and understanding yield curve dynamics. (See [7])

Each of these models has its assumptions and characteristics, making them suitable for different applications and market conditions.

A prominent candidate model is the 2-Additive-Factor Gaussian Model (G2++ model)—alternatively known as the 2-Factor Hull-White model) as in [4] [8] [9]. It possesses functional qualities required for various practical purposes such as Asset Liability Management and Trading of interest rate derivatives. The G2++ model is not only considered as a Vasicek two-factor model, but its main strength lies in its ability to precisely fit the market zero-coupon at any given initial value. Such a perfect fit is a primary requirement for practitioners, especially from the front office perspective. Before introducing the G2++ model (an exten-

sion of the one-factor model), let us first introduce the one-factor G1++ model as follows.

3. The G1++ Model or One Factor Hull White

In the G1++ model, the dynamics of the instantaneous short-rate r_t process under the risk-neutral measure \mathbb{Q} is given by:

$$r_t = x_t + \varphi(t) \tag{1}$$

x_t may be viewed as state variables $\{x_t; t \geq 0\}$ whose the dynamics are assumed to be given by

$$dx_t = -\kappa x_t dt + \sigma_t dW_t \tag{2}$$

$$x_0 = 0 \tag{3}$$

where

- $t \mapsto \varphi(t)$ is deterministic function and is given by an exact fitting to the term structure of discount factor observed in the market.
- κ is a positive constant.
- σ_t is a deterministic function of time that is regular enough to ensure the existence and uniqueness of a solution.
- $dW(t)$ is a Wiener process (Brownian motion¹).

Integrating Equation (2) having $s \leq t$, then the Equation (1) is given by

$$r_t = x_s e^{-\kappa(t-s)} + \int_s^t \sigma_u e^{-\kappa(t-u)} dW(u) + \varphi(t) \tag{4}$$

and supposing that r_t have some statistical restrictions:

$r(t)$ conditional to F_s , the σ -field representing the information available in the market up to time s is normally distributed with mean and variance given by,

$$\begin{aligned} E[r_t | F_s] &= x_s e^{-\kappa(t-s)} + \varphi(t) \\ Var[r_t | F_s] &= \int_s^t \sigma_u^2 e^{-2\kappa(t-u)} du \end{aligned} \tag{5}$$

where E and VAR denote the mean and the variance under the measure \mathbb{Q} , respectively. Having defined r_t as in Equation (4), (dependent on x_t , denote by $P(t, T)$ the price at time t of a zero-coupon bond maturing at T with unit face value, that is:

$$P(t, T) = E \left[e^{-\int_t^T r_s ds} \right] \tag{6}$$

and

$$P^{mkt}(0, T) = e^{-\int_0^T f^{mkt}(0, t) dt} \tag{7}$$

where the market instantaneous forward rate $f^{mkt}(0, t)$. In other words, is the rate of return for an infinitesimal amount of time dt measured at some date t for a particular start-value date.

¹In mathematical terms, Brownian motion is described as a continuous-time stochastic process $W(t)$, where t represents time, and $W(t)$ represents the displacement of the particle from its initial position at time t .

In the case $\sigma_t = \sigma$ is a positive constant function and denoting

$$\nu \equiv (\kappa, \sigma). \tag{8}$$

so the model perfectly fits the market term structure of the discount factor if and only if for $(T > 0)$ and we have:

$$\varphi(T) = f^{mkt}(0, T) + \frac{\sigma^2}{2\kappa^2} \left(\frac{1 - e^{-\kappa T}}{\kappa} \right)^2 \tag{9}$$

So the zero coupon price at time t of maturity T is given by:

$$P(t, T) = A(t, T) e^{-B(t, T)x_t} \tag{10}$$

where,

$$A(t, T) = \frac{P^{mkt}(0, T)}{P^{mkt}(0, t)} e^{\frac{1}{2}[V(t, T; \nu) - V(0, T; \nu) + V(0, t; \nu)]}$$

$$B(t, T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}$$

$$V(t, T; \nu) = \frac{\sigma^2}{\kappa^2} \left(T - t - 2 \frac{1 - e^{-\kappa(T-t)}}{\kappa} + \frac{1 - e^{-2\kappa(T-t)}}{2\kappa} \right)$$

Finally, the future price of the zero coupon is given by:

$$P(t, T) = \frac{P^{mkt}(0, T)}{P^{mkt}(0, t)} e^{\left\{ -B(t, T)x_t + \frac{1}{2}[V(t, T; \nu) - V(0, T; \nu) + V(0, t; \nu)] \right\}} \tag{11}$$

This model fits the current observed term structure of the discount factor if, for each maturity T , the discount factor $P(0, T)$ is equal to the one observed in the market $P^{mkt}(0, T)$.

$$P^{mkt}(0, T) = P(0, T) \tag{12}$$

where the market instantaneous forward rate $f^{mkt}(0, t)$.

Note that the price distribution of a zero-coupon bond is log-normal. The G1++ model is perfectly analogous to the one proposed by Hull and White in [4] and known as the Hull-White model.

4. The G2++ Model

The two-factor Gaussian model was introduced by Brigo and Mercurio [10]. According to them, Hull and White’s two-factor model is equivalent to the additive two-factor Gaussian model (G2++). It is an extension of the previous one-factor model discussed in Section 3. Widely utilized in interest rate modeling, especially for pricing interest rate derivatives, this model is characterized by its theoretical foundations, mathematical formulation, assumptions, and connections to other interest rate models.

4.1. Mathematical Formulation

4.1.1. Zero Coupon under G2++

In G2++, the short-rate process is represented by the sum of two correlated

Ornstein-Uhlenbeck processes (denoted as $x_{t;1}$ and $x_{t;2}$, referred to as stochastic factors that are normally distributed to simplify the computation of bond prices and other derivatives.) along with a deterministic shift $\varphi(t)$ added to precisely match the initial zero-coupon curve established using the most liquid market data. The instantaneous short-rate process under the risk-neutral measure \mathbb{Q} is given by:

$$r_t = x_{t;1} + x_{t;2} + \varphi(t) \tag{13}$$

where $t \mapsto \varphi(t)$ is a (deterministic) function which allows the model to fit the current observed interest rates.

$x_{t;1}$ and $x_{t;2}$ may be viewed as state variables $\{x_{t;l}; t \geq 0\}$ whose the dynamics are assumed to be given by

$$dx_{t;1} = -\kappa_1 x_{t;1} dt + \sigma_{t;1} dW_{t;1} \quad x_{0;1} = 0 \tag{14}$$

and

$$dx_{t;2} = -\kappa_2 x_{t;2} dt + \sigma_{t;2} dW_{t;2} \quad x_{0;2} = 0 \tag{15}$$

In (14) and (15), all of these dynamics are given under the risk-adjusted risk-neutral measure \mathbb{Q} . Here

- $W_{t;1}(\cdot)$ and $W_{t;2}(\cdot)$ are two correlated standard Brownian motions with a (constant) correlation $\rho \equiv \rho_{1,2}$, with $-1 < \rho < 1$.

The correlation of these two Brownian motions is capturing the possible correlation between different movements in the interest rate.

- κ_1, κ_2 are nonnegative real numbers which represent the model parameters.
- $\sigma_{t;1}$ and $\sigma_{t;2}$ are a deterministic function of time that is regular enough to ensure the existence and uniqueness of a solution.

Integrating Equations (14) and (15) having $s \leq t$, then the Equation (13) is given

$$r_t = x_{s;1} e^{-\kappa_1(t-s)} + x_{s;2} e^{-\kappa_2(t-s)} + \int_s^t \left\{ \sigma_{u;1} e^{-\kappa_1(t-u)} dW_1(u) + \sigma_{u;2} e^{-\kappa_2(t-u)} dW_2(u) \right\} + \varphi(t) \tag{16}$$

and supposing that r_t have some statistical restrictions (by considering the probability measure \mathbb{Q} that is conditional on the filtration F_s), it is possible to define the expected value and the variance of the interest rate r_t by:

$$E[r_t | F_s] = x_{s;1} e^{-\kappa_1(t-s)} + x_{s;2} e^{-\kappa_2(t-s)} + \varphi(t)$$

$$Var[r_t | F_s] = \int_s^t \left(\sigma_{u;1}^2 e^{-2\kappa_1(t-u)} + \sigma_{u;2}^2 e^{-2\kappa_2(t-u)} + \sigma_{u;1} \sigma_{u;2} e^{-\kappa_1(t-u)} e^{-\kappa_2(t-u)} \right) du \tag{17}$$

Having defined r_t as in Equation (16), dependent on $x_{t;1}$ and $x_{t;2}$, denoted by $P(t, T)$ the price at time t of a zero-coupon bond maturing at $T > 0$ with unit face value, that is:

$$P(t, T) = E \left[e^{-\int_t^T r(s) ds} \right] \tag{18}$$

and

$$P^{mkt}(0, T) = e^{-\int_0^T f^{mkt}(0, t) dt} \tag{19}$$

where $f^{mkt}(0, t)$ is the market instantaneous forward. In other words, is the rate of return for an infinitesimal amount of time dt measured as at some date (t) for a particular start-value date.

After some statistical work and conditions, and supposing that $\sigma_{t;1} = \sigma_1$, $\sigma_{t;2} = \sigma_2$ positive constants and denoting

$$Y \equiv (\kappa_1, \kappa_2, \sigma_1, \sigma_2, \rho). \tag{20}$$

So the model perfectly fits the market term structure of the discount factor if and only if for ($T > 0$) and we have:

$$\varphi(T) = f^{mkt}(0, T) + \frac{1}{2} \sum_{i,j=1}^2 \frac{\sigma_i \sigma_j}{\kappa_i \kappa_j} (1 - e^{-\kappa_i T}) (1 - e^{-\kappa_j T}) \tag{21}$$

then the zero coupon price at time t of maturity T is given by:

$$P(t, T) = A(t, T) e^{-B_1(t, T)x_{t;1} - B_2(t, T)x_{t;2}} \tag{22}$$

where, $A(t, T) = \frac{P^{mkt}(0, T)}{P^{mkt}(0, t)} e^{\frac{1}{2}[V(t, T; Y) - V(0, T; Y) + V(0, t; Y)]}$

$$B_1(t, T) = \frac{1 - e^{-\kappa_1(T-t)}}{\kappa_1}, \quad B_2(t, T) = \frac{1 - e^{-\kappa_2(T-t)}}{\kappa_2} \tag{23}$$

$$V(t, T; Y) = \sum_{i,j=1}^2 \rho_{ij} \frac{\sigma_i \sigma_j}{\kappa_i \kappa_j} \left(T - t - \frac{1 - e^{-\kappa_i(T-t)}}{\kappa_i} - \frac{1 - e^{-\kappa_j(T-t)}}{\kappa_j} + \frac{1 - e^{-(\kappa_i + \kappa_j)(T-t)}}{\kappa_i + \kappa_j} \right) \tag{24}$$

Finally

$$P(t, T) = \frac{P^{mkt}(0, T)}{P^{mkt}(0, t)} e^{\{-B_1(t, T)x_{t;1} - B_2(t, T)x_{t;2} + \frac{1}{2}[V(t, T; Y) - V(0, T; Y) + V(0, t; Y)]\}} \tag{25}$$

The zero-coupon market prices $P^{mkt}(0, t)$ and $P^{mkt}(0, T)$ are assumed to be known.

Putting $t = 0$ in (25), and taking into account that $P^{mkt}(0, 0) = 1$ and $V(0, 0; Y) = 0$ then one obtains

$$P(0, T) = P^{mkt}(0, T) e^{\{-B_1(0, T)x_{0;1} - B_2(0, T)x_{0;2}\}}.$$

It means that

$$P(0, T) = P^{mkt}(0, T) \text{ under the initial choice } x_{0;1} = 0 \text{ and } x_{0;2} = 0. \tag{26}$$

One appealing aspect of using the G2++ model is its ability to accurately fit the currently observed term structure of market zero-coupon prices at the initial time 0.

At any (future) time t , with $0 < t$, resulting from (25), the time- t yield-to-maturity for a maturity T , is readily given by

$$y(t, T) \equiv \frac{B_1(t, T)x_{t;1} + B_2(t, T)x_{t;2} - c(t, T; Y)}{T - t} \tag{27}$$

where

$$c(t, T; Y) \equiv \ln \left[\frac{P^{mkt}(0, T)}{P^{mkt}(0, t)} \right] + \frac{1}{2} (V(t, T; Y) - V(0, T; Y) + V(0, t; Y)) \quad (28)$$

Therefore, Equations (25) and (27) depend on the values of the state variables $x_{t;1}$ and $x_{t;2}$, which are essentially unobservable but are theoretically assumed to be described by Equations (14) and (15).

The G2++ model is only practically meaningful when the yield-to-maturity defined in Equation (27) satisfies $0 \leq y(t, T)$, which is equivalent to saying $P(t, T) \geq 1$. This implies that the time- t values $x_{t;1}$ and $x_{t;2}$ of the state variables should satisfy

$$c(t, T; Y) \leq B_1(t, T)x_{t;1} + B_2(t, T)x_{t;2} \quad \text{for all } 0 < t < T. \quad (29)$$

At $t = 0$ with the initial values states in (26), then the constraint (29) is reduced to $c(0, T; Y) \leq 0$. This is always true since $c(0, T; Y) = \ln[P^{mkt}(0, T)]$ and $V(0, 0; Y) = 0$.

When considering $t > 0$ as a future time, then the state variable values may be considered as given by the random variables $x_{t;1}$ and $x_{t;2}$ therefore, it is logical to ask about the probability that yield is negative. And consequently it makes sense to ask about the probability that $y(t, T) \leq 0$. Academic literature [11], has granted very few attention to the computation of the probability that the instantaneous short rate r_t is negative. Generally, people admitted that this probability is nonnegative but it stays to be little, such that this inconvenience is essentially neglected face to the various benefits connected with the use of the model G2 itself.

4.1.2. Zero Coupon Future Price under G2++

In this subsection, our contribution is to provide an accurate analysis of the future price of the zero-coupon bond, without resorting to the dynamics of the bond itself, which is typically obtained by applying the two-dimensional Ito's formula as

$$dP = \left(\frac{\partial P}{\partial t} + \frac{1}{2} \frac{\partial^2 P}{\partial x_1^2} + \frac{1}{2} \frac{\partial^2 P}{\partial x_2^2} \right) dt + \frac{\partial P}{\partial x_1} dx_1 + \frac{\partial P}{\partial x_2} dx_2. \quad (30)$$

In the following sections, we assume that

$$0 < t < T,$$

where t represents a future time horizon with respect to the initial time 0.

From time 0, the zero-coupon price at the horizon t is unknown and may be considered as given by a random variable, such that if the price model (25) remains consistent then one should have

$$P(t, T) = P(t, T; P^{mkt}(0, t), P^{mkt}(0, T); x_{t;1}, x_{t;2}; Y). \quad (31)$$

For various practical purposes (such as valuing positions, measuring risks, managing positions, etc.), it is very beneficial to have a measure of the change in the zero-coupon price,

$$P(t, T) - P(0, T).$$

For example, t may represent one or more days, such as $t = 10$ days or $t = 90$. In any case, there is no reason to consider t as an infinitesimal number since, from a practical perspective, it has a given and well-defined size. From a theoretical point of view, it is tempting to utilize the dynamics followed by the zero-coupon bond price as in Equation (30) and then discretize the corresponding equation to obtain an approximate value of $P(t, T) - P(0, T)$. However, this approach may have limited practical utility since t should be small in size due to the discretization constraint.

Proposition 1. Under the G2++ model, let us consider the uncertainty risk/opportunity factors $x_{s;1}$ and $x_{s;2}$ which follow the stochastic differential Equations (14) and (15). Then the future time- t values of these state variables, conditionally on their current values $x_{0;1}$ and $x_{0;2}$, are given by

$$x_{t;1}(\cdot) = \mathcal{E}(t; \kappa_1)x_{0;1} + \sigma_1 \mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) \varepsilon_1(\cdot) \tag{32}$$

and

$$x_{t;2}(\cdot) = \mathcal{E}(t; \kappa_2)x_{0;2} + \sigma_2 \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \left\{ \omega \varepsilon_1(\cdot) + \sqrt{1 - \omega^2} \varepsilon_2(\cdot) \right\} \tag{33}$$

where $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are two independent standard normal random variables,

$$\omega \equiv \omega(t; \rho, \kappa_1, \kappa_2) = \rho \frac{\mathcal{F}(t; \kappa_1, \kappa_2)}{\mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2)} \tag{34}$$

$$\mathcal{E}(u; k) = \exp[-ku] \tag{35}$$

and the quantity $\mathcal{F}(u; k_i, k_j)$ is defined by

$$\mathcal{F}(u; \kappa_i, \kappa_j) \equiv \frac{1}{\kappa_i + \kappa_j} \left\{ 1 - \exp[-(\kappa_i + \kappa_j)u] \right\} \tag{36}$$

Though it is not readily useful in practical simulations, when making reference to our previous model (14) and (15), it may be seen that $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are defined such that

$$\begin{aligned} \varepsilon_1(\cdot) &\equiv \varepsilon_1(\cdot; t, \kappa_1) \\ &= \exp[-\kappa_1 t] \mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) \int_0^t \exp[\kappa_1 s] dW_{s;1}(\cdot) \end{aligned} \tag{37}$$

and

$$\begin{aligned} \varepsilon_2(\cdot) &\equiv \varepsilon_2(\cdot; t, \kappa_1, \kappa_2, \rho) \\ &= \frac{1}{\sqrt{1 - \omega^2}} \left\{ -\omega \exp[-\kappa_1 t] \mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) \int_0^t \exp[\kappa_1 u] dW_{u;1}(\cdot) \right. \\ &\quad \left. + \exp[-\kappa_2 t] \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \int_0^t \exp[\kappa_2 u] dW_{u;2}(\cdot) \right\} \end{aligned} \tag{38}$$

It may be noted that $x_{t;2}(\cdot)$, given in (33), is well defined whenever $|\omega| < 1$, that is

$$|\omega(t, \rho, \kappa_1, \kappa_2)| = |\rho| \frac{\mathcal{F}(t; \kappa_1, \kappa_2)}{\mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2)} < 1. \tag{39}$$

Simulations of the state variables $x_{t,1}(\cdot)$ and $x_{t,2}(\cdot)$, at the future time t can be performed using identities (32) and (33). It is worth noting, for example that people often only use Euler discretization of the Stochastic Differential Equations (SDE) governing the state variables. However, for simulation purposes, such discretization (which naturally leads to some errors) is not really necessary since we have completely solved the SDE as presented in identities (32) and (33). Therefore, it is sufficient to generate independent realizations ε_1 and ε_2 of the standard Gaussian distribution and then apply these identities. With the corresponding realizations of the state variables, a realization of the zero-coupon bond price $P(t, T)$ at the future time horizon t can be obtained by using the dependence relations (25), (32) and (33).

Actually with the state variables $x_{t,1}(\cdot)$ and $x_{t,2}(\cdot)$ values, as given in (32) and (33), then the following quantitative expression for the zero coupon price $P(t, T)(\cdot)$ at the future time-horizon t and for the remaining maturity $T - t$ is quantitatively described by the following result.

Theorem 2. Let us consider the G2++ model as defined in (13), (14) and (15). Let us consider a future horizon time t such that $\omega \equiv \omega(t, \rho, \kappa_1, \kappa_2)$ satisfies the condition (39). Then there are $\varepsilon_1(\cdot) \equiv \varepsilon_1(\cdot; t; \kappa_1)$ and $\varepsilon_2(\cdot) \equiv \varepsilon_2(\cdot; t; \kappa_1, \kappa_2, \rho)$ realizations of two independent standard Gaussian random variables such that the future time- t value of any zero-coupon with the maturity T , with $t < T$, conditionally on the values $x_{0,1}$ and $x_{0,2}$ of the state variables, is given by

$$P(t, T)(\cdot) = \Theta \exp[-(\lambda_1 \varepsilon_1(\cdot) + \lambda_2 \varepsilon_2(\cdot))]. \tag{40}$$

The terms Θ , λ_1 and λ_2 in (40) are defined as follows

$$\begin{aligned} \Theta &\equiv \Theta(t, T; P^{mkt}(0, t), P^{mkt}(0, T); x_{0,1}, x_{0,2}; Y) \\ &= \exp[-\{B_1(t, T)\mathcal{E}(t; \kappa_1)x_{0,1} + B_2(t, T)\mathcal{E}(t; \kappa_2)x_{0,2} - c(t, T; Y)\}] \end{aligned} \tag{41}$$

$$\lambda_1 \equiv \lambda_1(t, T; Y) = \left\{ \sigma_1 B_1(t, T) \mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) + \omega \sigma_2 B_2(t, T) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \right\} \tag{42}$$

$$\lambda_2 \equiv \lambda_2(t, T; \kappa_2, \sigma_2, \rho) = \left\{ \sqrt{1 - \omega^2} \sigma_2 B_2(t, T) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \right\} \tag{43}$$

and where $\omega \equiv \omega(t; \rho, \kappa_1, \kappa_2)$ is defined as in (34).

At any (future) time t , with $0 < t$, resulting from (25), the time- t yield-to-maturity for a maturity T , is readily given by

$$\begin{aligned} y(t, T) &\equiv \frac{1}{T - t} \{ B_1(t, T) \mathcal{E}(t; \kappa_1)x_{0,1} + B_2(t, T) \mathcal{E}(t; \kappa_2)x_{0,2} \\ &\quad - c(t, T; Y) + \lambda_1 \varepsilon_1(\cdot) + \lambda_2 \varepsilon_2(\cdot) \} \end{aligned} \tag{44}$$

where

Technically, the future zero-coupon price $P(t, T)$, with $0 < t < T$, generated

by the model as written in (40) makes only a practical sense whenever

$$0 \leq \lambda_1 \varepsilon_1(\cdot) + \lambda_2 \varepsilon_2(\cdot) - \ln(\Theta) \tag{45}$$

or equivalently

$$\begin{aligned} & \frac{1}{T-t} c(t, T; Y) - \{B_1(t, T) \mathcal{E}(t; \kappa_1) x_{0,1} + B_2(t, T) \mathcal{E}(t; \kappa_2) x_{0,2}\} \\ & \leq \left\{ \sigma_1 B_1(t, T) \mathcal{F}^{\frac{1}{2}}(t; \kappa_1, \kappa_1) + \omega \sigma_2 B_2(t, T) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \right\} \varepsilon_1(\cdot) \\ & \quad + \left\{ \sqrt{1 - \omega^2} \sigma_2 B_2(t, T) \mathcal{F}^{\frac{1}{2}}(t; \kappa_2, \kappa_2) \right\} \varepsilon_2(\cdot). \end{aligned} \tag{46}$$

This corresponds to making restrictions on the values of $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$. That is, arbitrary shocks are not always suitable to run meaningful simulation. We should be aware that a correct use of the model G2++ implies to introduce the restrictions (29) for the time-0 (which are always satisfied with the assumption $x_{0,1} = x_{0,2} = 0$) and the new restrictions (46) for the admissible shocks simulations.

Recently, we are faced with market regime changes in which negative interest rates for sovereign bonds and negative spreads are regularly observed. The G2++ model is not designed to take into account such pathological situations. Since there is no prominent model that can properly deal with negative interest rates, a suitable solution would be to use displaced filtered historical simulation. For more information about this aspect of negative yields, refer to [9].

As mentioned in [11], the two models 2 HW and G2++ are actually equivalent. Below in Section 4.2, we discuss the relationship between G2++ and some other interest rate models.

4.2. Relationship to Other Interest Rate Models

1) Vasicek Model: The G2++ model extends the Vasicek model by introducing two factors instead of one. In contrast to the Vasicek model, which characterizes the short rate using a single mean-reverting process, the G2++ model includes an extra factor to account for more intricate term structure dynamics.

2) Hull-White Model: The Hull-White model with one factor is a specific instance of the G2++ model, which focuses on considering only one factor. By incorporating a second factor, the G2++ model expands the Hull-White framework, allowing for enhanced adaptability in modeling the term structure and capturing the changes in interest rates.

3) Heath-Jarrow-Morton (HJM) Framework: The G2++ model is a derivation of the HJM framework, where the HJM framework directly models the complete forward rate curve, while the G2++ model concentrates on the short rate and employs a two-factor method to explain its behavior. It serves as a practical application of the HJM framework, incorporating certain assumptions regarding the underlying factors.

4) Affine Term Structure Models: The G2++ model is categorized as an af-

fine term structure model (ATSM), which means that the bond prices and yields can be represented as exponential-affine functions of the state variables. This characteristic simplifies the calculation of bond prices and other derivatives, leading to the widespread use of the G2++ model in practical applications.

4.3. Calibration of the Model

Various processes can be used to calibrate the G2++ model, but the basic idea remains the same: to ensure that observed market prices (such as zero coupon prices) closely match the corresponding prices generated by the model as in [12]-[14].

Here is a comprehensive guide on enhancing transparency in this process, covering the key aspects: methodology, data, algorithm details, software tools, code, and validation.

Calibration Methodology

1) Objective Function: The objective function is defined as the minimization of the sum of squared differences between the observed zero coupon prices or yields and the model-predicted ones.

$$\text{Objective Function} = \sum (P_{obs_i} - P_{model_i})^2$$

where P_{obs_i} are market bond prices and P_{model_i} are model-predicted bond prices.

2) Optimization Algorithm: We can use algorithms like Levenberg-Marquardt, Nelder-Mead simplex, or gradient descent techniques to effectively reduce the objective function. Establish convergence criteria, such as a threshold for the alteration in the objective function or parameter values between iterations.

3) Initial Parameters: The optimization process can be greatly impacted by the initial guesses. Using historical data or findings from past studies can guide the selection of initial parameters.

4) Software Tools: Specify the software and libraries used for calibration, such as MATLAB, R, or Python (e.g., SciPy for optimization).

5) Simulation of Yield Curves: Describe the procedure for simulating yield curves, including the discretization methods used (e.g., Euler-Maruyama). Analyze how changes in model parameters affect the simulated yield curves and include visualizations.

6) Assessment of Goodness-of-Fit: Define and calculate metrics such as SSE, RMSE, MAE, and MAPE.

Provide visualizations and residual plots to identify patterns or biases.

7) Validation and Testing: Validate the calibrated model against historical data to ensure accuracy. Test the model on new data not used in calibration to assess its predictive power.

To achieve accurate parameter estimation, a systematic approach is required for calibrating the G2++ model. By utilizing historical data, market prices, and robust optimization techniques, practitioners can ensure that the model accu-

rately captures current market dynamics. This, in turn, enables reliable insights for interest rate hedging, derivative pricing, and portfolio optimization. It is essential to regularly validate and recalibrate the model to maintain its effectiveness in financial risk management.

4.4. Practical Applications

The G2++ model plays a crucial role in pricing interest rate derivatives like swaps, caps, floors, and swaptions due to its accurate modeling of the term structure of interest rates. This feature makes it an essential tool for pricing purposes. For instance, when pricing European swaptions, the G2++ model enables us to calculate closed-form solutions for swaption prices, taking advantage of the model's flexibility and precision.

In the case of more intricate derivatives such as Bermudan swaptions or interest rate caps and floors with embedded options, numerical methods are often employed for pricing. Methods like Monte Carlo simulations and finite difference methods help us address the path dependency and optionality aspects of these derivatives. Additionally, the G2++ model's capability to capture correlations between different factors proves to be advantageous in accurately pricing multi-currency derivatives and managing associated risks.

For **Interest Rate Derivatives Pricing**, the G2++ model is employed to determine the prices of interest rate derivatives, including caps, floors, swaptions, and exotic interest rate products. This model effectively captures the fluctuations in the yield curve, encompassing both short-term rate changes and long-term trends. As a result, it is highly useful in pricing intricate derivative structures.

For **Risk management**, financial institutions utilize the G2++ framework to effectively manage risks, such as evaluating exposure to interest rate fluctuations and implementing hedging strategies. Through the simulation of various interest rate scenarios using the model's parameters, risk managers can assess the level of risk within their portfolios and enhance returns by optimizing risk-adjusted strategies.

For **Term Structure Modeling** the G2++ framework is utilized in term structure modeling, encompassing the anticipation of forthcoming yield curve patterns and term premia. This is imperative for the valuation of bonds, asset-liability management, and the projection of interest rate fluctuations in diverse economic scenarios.

For **Asset Allocation and Portfolio Management**, portfolio managers rely on the G2++ model to make well-informed asset allocation choices by considering interest rate projections. By integrating yield curve dynamics into asset pricing models, they can effectively enhance portfolio returns while effectively managing risk exposures.

For **the Market Risk Analysis**, G2++ model is utilized by traders and risk analysts to examine market risk factors associated with interest rates. This encompasses evaluating the influence of yield curve shifts, changes in volatility, and correlation dynamics on investment portfolios and trading strategies.

As a conclusion G2++ yield curve model is of utmost importance in financial markets as it offers a sophisticated framework for interest rate modeling, pricing, risk management, and product innovation. Its capacity to comprehend intricate yield curve dynamics renders it an invaluable tool for financial institutions, investors, and analysts.

5. Numerical Applications for ZC under G2++

To better visualize our above results on the Zero-coupon bond modeled by G2++ stated in Section 4, it is useful to consider numerical illustrations corresponding to a given calibration.

At the present time 0, it is assumed here that

$$x_{0,1} = x_{0,2} = 0.$$

We denote by $d = \text{day}$, $m = \text{month}$ and $y = \text{year}$. One yield curve is used in this Subsection as displayed in **Table 1**.

Table 1. Initial yield curve.

$T(y)$	1	2	3	4	5	6	7	8	9	10
Yield (%)	5.98	6.32	6.57	6.75	6.88	6.98	7.06	7.12	7.17	7.21

The reconstitution of the yield curve is a necessary operation because there are not enough zero-coupon bonds listed on the market. Consequently, we cannot obtain zero-coupon rates for a continuum of maturities. Therefore, it is necessary to interpolate the points to obtain a continuous curve. Subsequently, this zero-coupon rate curve will need to be extrapolated to obtain rates for maturities that are later than those observed today in the financial market. In our work, we will distinguish between the methods of Nelson-Siegel and Svensson to derive the yield curve, as shown in **Figure 1**.

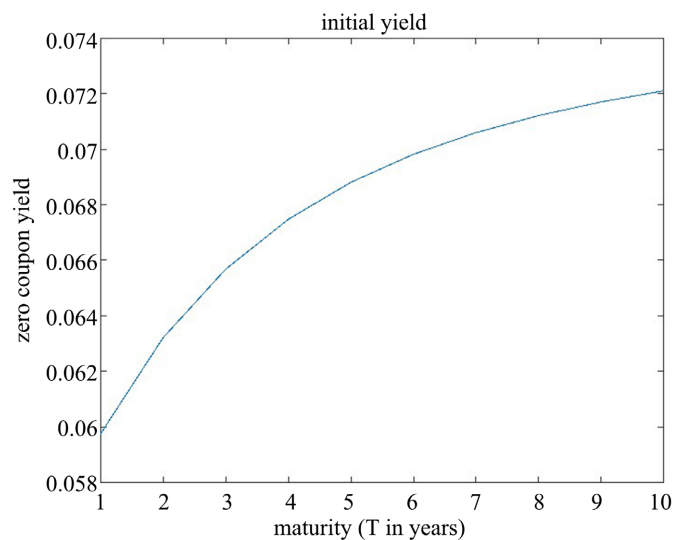


Figure 1. Initial yield.

It should be noted that the set calibration (**Table 2**) we use is taken directly from [11]. We applied our work using MATLAB codes and generated figures. By minimizing the squared gap between the model price and the market price of the zero coupon using MATLAB, we can obtain a good fit of the model parameters.

Table 2. Set calibration.

κ_1	77.35%
κ_2	8.20%
σ_1	2.23%
σ_2	1.04%
ρ	-70.19%

Figure 2 shows that there is no difference between the calibrated model and the initial yield at $t = 0$. Our aim is to observe the expectation of the future yield modeled by G2++ for a future time $t > 0$. We assume that $t = 10$ days, so the modeled yields and the initial yield are plotted in **Figure 3**.

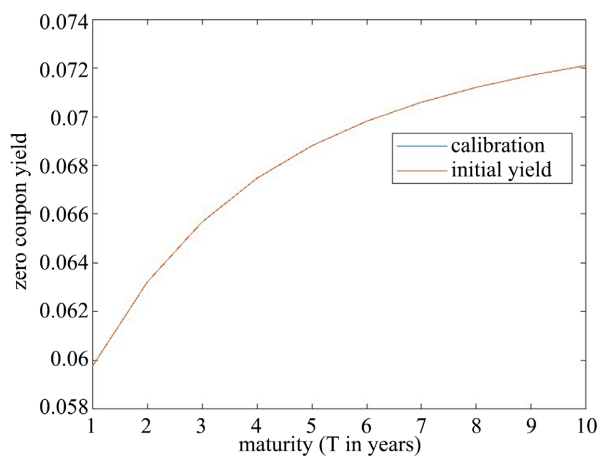


Figure 2. Yields modeled and initial yield.

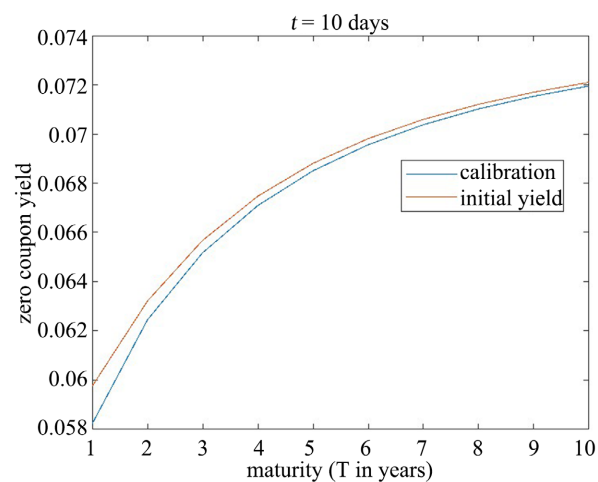


Figure 3. Yield modeled after 10 days and initial yield.

Figure 4 plots the errors between the modeled yield and the initial yield, illustrating the difference after 10 days.

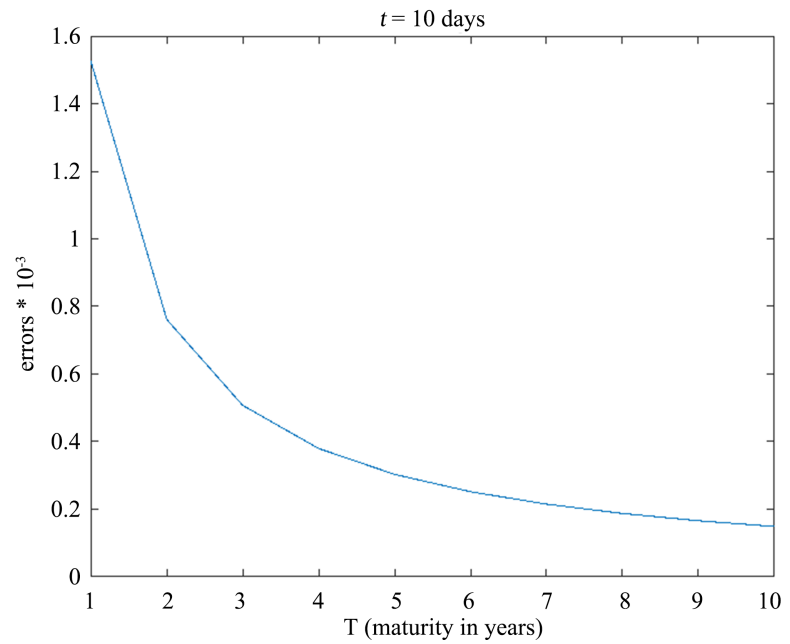


Figure 4. Errors between initial yield and yield modeled after 10 days.

Supposing that $t = 90$ days, so the yield modeled and the initial yield are plotted in **Figure 5**.

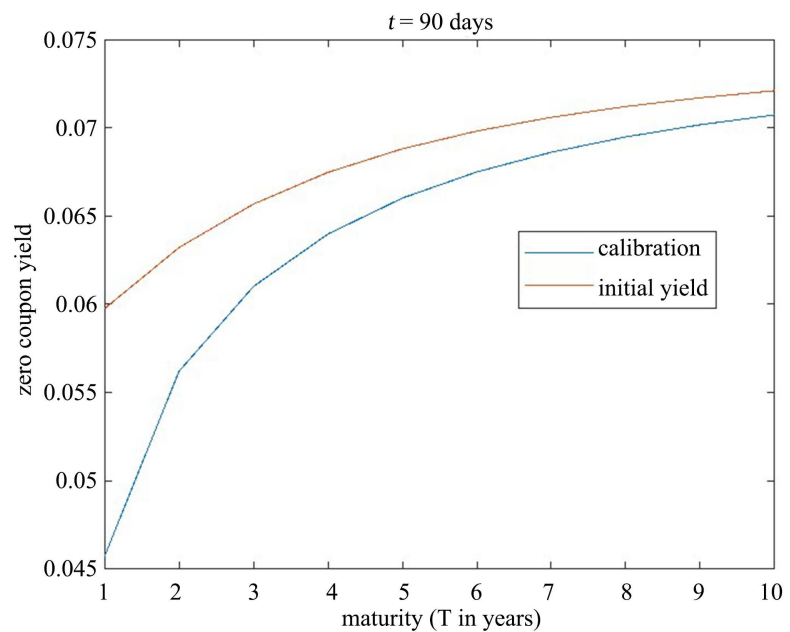


Figure 5. Yield modeled after 90 days and initial yield.

By plotting the errors between the modeled yield and the initial yield in **Figure 6**, we remark that the errors increase with the horizon time $t = 90$ days.

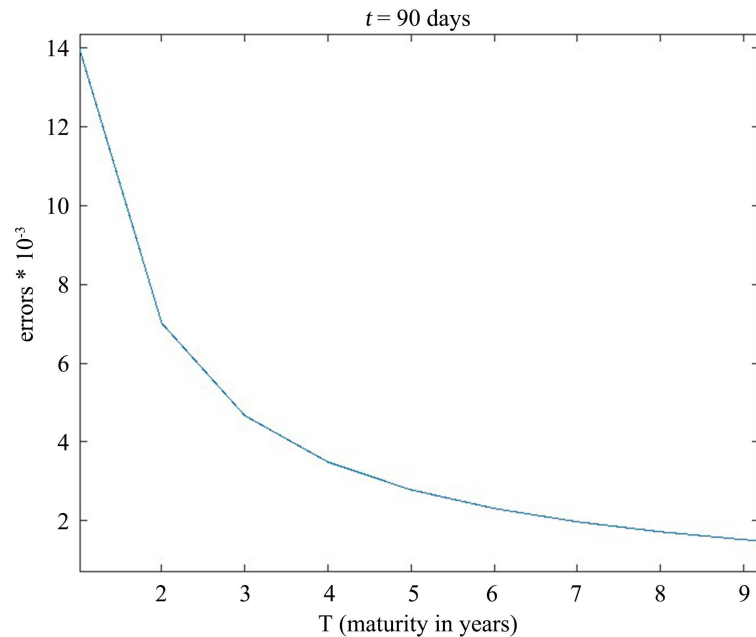


Figure 6. Errors between initial yield and yield modeled after 90 days.

We observe that as the time horizon t increases, the future zero-coupon yield shifts in accordance with the initial yield. It is noteworthy that as t (the horizon) increases, the errors also increase. This result can assist us in estimating the precise change in the zero-coupon price and subsequently estimating the exact changes in different financial products such as bonds or swaps. Once we calculate these exact changes, we can gain insight into future hedging positions for various financial products.

6. Conclusions

The study and implementation of rate models are crucial for valuing financial products. We were particularly interested in the Gaussian G2++ two-factor model as a modeling method. Recognizing the limitations of single-factor models, we found that the two-factor model enhances the modeling capacity of rate models. Our approach began with simulating short rates and calculating zero-coupon yields under G2++. We then examined the errors between initial and calibrated yields as the horizon (t) increased. This approach is enhanced by considering the passage of time (horizon) and checking the yield shift.

The results of the G2++ model are highly acceptable and slightly better than those obtained by single-factor models. This model appears robust enough to accurately model short rates. However, various extensions and questions arise from our work:

- 1) The G2++ model serves as the underlying model of our approach. Exploring situations where alternative interest rate models with more uncertainty factors, such as the Arbitrage Free Dynamic Nelson-Siegel model introduced in [15], could be interesting.

2) Extending the uncertainty risk factors approach, like the G2++ model, to account for risks related to both interest rates and credit is an important area for further exploration.

3) Studying products such as bonds, swaps, and contingent interest rate derivatives under the G2++ model of the yield curve remains a challenging yet promising avenue for future research.

4) The weakness of the G2++ model lies in its calibration, which is often done improperly by not discarding zero-coupon prices higher than 1. Addressing this issue raises questions: What happens if the calibration is done in a consistent manner? How can we modify the standard G2++ model to handle interest rates near zero or below?

In conclusion, the approach and ideas introduced in this paper appear to be useful for rolling hedging positions under the G2++ model. Further empirical illustrations are needed to assess the effectiveness of our approach over longer time periods.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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