

Closed Classes of Binary Complete Decision Tables with Many-Valued Decisions

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Abstract

A binary complete decision table with many-valued decisions is a table with n attributes and 2^n pairwise distinct rows filled with numbers from the set $\{0,1\}$. Each row of this table is labeled with a nonempty finite set of decisions. For a given row of the table, the task is to find a decision from the set of decisions attached to the row. Such tables are generalizations of Boolean functions. They can also be viewed as representations of various problems related to systems of decision rules. In this paper, we consider three types of classes of binary complete decision tables with many-valued decisions, closed with respect to removal of columns and changing of decisions. For tables from these classes, we study the relationships between the minimum weighted depth of deterministic, nondeterministic, and (for one type of classes) strongly nondeterministic decision trees and the total weight of attributes attached to columns. Note that nondeterministic decision trees and strongly nondeterministic decision trees for decision tables can be interpreted as a way of representing the two types of systems of decision rules for these tables.

Keywords

Binary Complete Decision Table, Closed Class, Deterministic Decision Tree, Nondeterministic Decision Tree, Strongly Nondeterministic Decision Tree

1. Introduction

In this paper, we continue to study the relationships between deterministic decision trees and decision rule systems (represented as nondeterministic or strongly nondeterministic decision trees) for three types of classes of decision tables with many-valued decisions closed relative to removal of columns and changing of decisions. A class of tables is closed relative to the considered operations if, for

any table in the class, we can remove any number of columns (attributes) from that table and change the decisions attached to rows, and the resulting table will still be in the class.

The main results we have obtained in this direction earlier are collected in our book [1], in which arbitrary closed classes and arbitrary bounded complexity measures are investigated. The goal of this work is a more in-depth study of special types of closed classes and complexity measures.

Decision trees [2]-[6] and decision rule systems [7]-[11] are widely used for knowledge representation, classification, and problem solving in areas such as fault diagnosis and combinatorial optimization. These models are among the most interpretable classification and knowledge representation methods [12]. Understanding the relationships between decision trees and decision rules is an important area of research in computer science.

Decision tables with many-valued decisions often arise in data analysis, where they are called multi-label decision tables [13]-[15]. They are also widely used in areas such as combinatorial optimization, computational geometry, and fault diagnosis to represent and analyze problems [10] [16].

Various classes of objects closed under certain operations have been studied extensively. Well-known examples are the classes of Boolean functions closed under superposition [17], and the minor-closed classes of graphs [18]. Decision tables with many-valued decisions are an interesting mathematical object that deserves further study, especially in the context of studying closed classes of decision tables. The most natural examples of closed classes of decision tables are those generated by information systems [19], where each problem in the system is transformed into a decision table [1]. However, the family of all closed classes is significantly larger than the family of closed classes generated by information systems.

In this paper, we consider three types of classes of binary complete decision tables with many-valued decisions closed under operations of removal of columns and changing of decisions. For tables from these classes, we study the relationships between the minimum weighted depth of deterministic, nondeterministic, and (for one type of classes) strongly nondeterministic decision trees and the total weight of attributes attached to the columns. We study weighted depth of decision trees instead of standard depth because the complexity of computing values for different attributes can vary greatly.

A binary complete decision table with many-valued decisions is a rectangular table with n columns labeled with pairwise distinct attributes and 2^n pairwise distinct rows filled with numbers from the set $\{0,1\}$. Each row of this table is labeled with a nonempty finite set of decisions. The rows are interpreted as tuples of attribute values. We also consider two special kinds of binary complete decision tables with many-valued decisions: conventional tables in which each decision set contains exactly one decisions and tables with 0-1-decisions in which each decision set is either $\{0\}$ or $\{1\}$.

For a given row of the decision table, we must find a decision from the set of

decisions attached to the row. To do this, we can use the following queries: we can select an attribute and ask what the value of that attribute is in the row in question.

We study three types of algorithms based on these queries: deterministic, nondeterministic and strongly nondeterministic decision trees. The latter are used only for tables with 0-1-decisions. One can interpret nondeterministic decision trees for a decision table as a way of representing an arbitrary system of true decision rules for this table that cover all rows. One can interpret strongly nondeterministic decision trees for a decision table with 0-1-decisions as a way of representing an arbitrary system of true decision rules for this table that cover all rows labeled with the set $\{1\}$. To characterize the time complexity of decision trees, we consider their weighted depth, which is equal to the maximum total weight of attributes on a path from the root to a terminal node of the tree.

Binary complete decision tables with many-valued decisions can be viewed as representations of various problems involving decision rule systems with binary attributes. Consider three examples:

- The problem of finding, for a given tuple of attribute values, a rule with a true left-hand side for this tuple can be represented as a table with many-valued decisions.
- The problem of finding, for a given tuple of attribute values, the set of rules with a true left-hand side for this tuple can be represented as a conventional table.
- The problem of the existence of a rule with a true left-hand side for a given tuple of attribute values can be represented as a table with 0-1-decisions.

The notion of a binary complete decision table with 0-1-decisions actually coincides with the notion of a Boolean function. Binary complete decision tables with many-valued decisions are an interesting generalization of Boolean functions. The properties of these tables may differ significantly from the properties of Boolean functions. For example, for any Boolean function, the minimum depth of a deterministic decision tree does not exceed the square of the minimum depth of a nondeterministic decision tree [20]-[22] (see also [23]). In Chap. 7 of the book [1], we constructed an infinite sequence of binary complete decision tables with many-valued decisions for which the minimum depth of deterministic decision trees is not bounded from above by a constant, and the minimum depth of nondeterministic decision trees does not exceed 3.

In this paper, for classes of each type, we study the functions characterizing the worst-case growth of the minimum weighted depth of deterministic and nondeterministic decision trees for decision tables from the class with increasing total weight of attributes attached to the table columns. For classes of decision tables with 0-1-decisions, we study the function characterizing the worst-case growth of the minimum weighted depth of strongly nondeterministic decision trees for decision tables from the class with increasing total weight of attributes attached to the table columns. We prove that each of these functions is either bounded from above by a constant or grows almost linearly. The main novelty is that we found

simple criteria for the behavior of these functions: it is enough to recognize whether the total weight of attributes for tables from the class is bounded from above by a constant. In general, we need to study the behavior of the number of rows in tables in a class and the behavior of the minimum complexity of the set of attributes that separate a given row from all other rows of the table.

For classes of each type, a function is studied that characterizes the growth in the worst case of the minimum weighted depth of a deterministic decision tree for a decision table from a class with the growth of the minimum weighted depth of a nondeterministic decision tree for the table. For classes of decision tables with many-valued decisions, a condition is specified under which the function is defined everywhere. For the function that is defined everywhere, its behavior is studied. In particular, a condition is specified under which the function under consideration, depending on n , is bounded from above by a polynomial in n . For classes of conventional decision tables and tables with 0-1-decisions, it is proved that the function under consideration is defined everywhere and does not exceed n^2 . The main novelty for the classes of decision tables with many-valued decisions is that it is sufficient to consider how the maximum number of attributes with a weight of no more than m in a table from a class (if the maximum exists) grows with m . The criteria for the general case considered in [1] are significantly more complex. The main novelty for classes of conventional decision tables and decision tables with 0-1-decisions is that it was possible to generalize the previously obtained results for the depth of decision trees for Boolean functions [20]-[22] to the case of weighted depth of decision trees for such tables.

For classes of decision tables with 0-1-decisions, a function is also studied that characterizes the growth in the worst case of the minimum weighted depth of a deterministic decision tree for a decision table from a class with the growth of the minimum weighted depth of a strongly nondeterministic decision tree for the table. A condition is specified under which the function is defined everywhere. For the function that is defined everywhere, its behavior is studied. In particular, a condition is specified under which the function under consideration is bounded from above by a polynomial in n . The main novelty is that it suffices to consider how the maximum number of attributes with a weight no greater than m in a table from a class grows with m (if the maximum exists). The criteria for the general case, considered in [1], are more complex.

The proofs of the new statements are based, in particular, on the results obtained in [1]. This explains the fact that in our work we use many definitions and notation from the book [1] with minimal changes.

This paper is an extension of two conference papers: paper [24], which contains results for classes of tables with many-valued decisions, and paper [25], whose proof we adapted to show upper bounds on n^2 for the function describing the relationship between deterministic and nondeterministic decision trees.

The paper consists of six sections. Section 2 discusses the basic definitions and notation. Sections 3-5 are devoted to studying classes of tables with many-valued

decisions, classes of conventional tables, and classes of tables with 0-1-decisions, respectively. Section 6 contains brief conclusions.

2. Main Definitions and Notation

In this section, when considering the main definitions and notations, we either quote verbatim or very closely to the text from our book [1].

Denote $\omega = \{0, 1, 2, \dots\}$ and $E_2 = \{0, 1\}$. Let $P = \{f_i : i \in \omega\}$ be the set of *attributes* (really, names of attributes). Two attributes $f_i, f_j \in P$ are considered *different* if $i \neq j$.

2.1. Decision Tables and Their Closed Classes

We now define three families of nonempty finite subsets of the set ω . Let \mathcal{P}^∞ be the family of all nonempty finite subsets of the set ω , $\mathcal{P} = \{\{d\} : d \in \omega\}$, and $\mathcal{P}^{0-1} = \{\{0\}, \{1\}\}$.

Definition 1 Denote by \mathcal{M}_2^∞ the set of rectangular tables filled with numbers from E_2 in each of which rows are pairwise different, each row is labeled with a set from \mathcal{P}^∞ (set of decisions), and columns are labeled with pairwise different attributes from P . Rows are interpreted as tuples of values of these attributes. Empty tables without rows belong also to the set \mathcal{M}_2^∞ . Tables from \mathcal{M}_2^∞ will be called *decision tables with many-valued decisions*.

Definition 2 Denote by \mathcal{M}_2 the set of tables from \mathcal{M}_2^∞ in which each row is labeled with a set from \mathcal{P} . Empty tables without rows belong also to the set \mathcal{M}_2 . Tables from \mathcal{M}_2 will be called *conventional decision tables*.

Definition 3 Denote by \mathcal{M}_2^{0-1} the set of tables from \mathcal{M}_2^∞ in which each row is labeled with a set from \mathcal{P}^{0-1} . Empty tables without rows belong also to the set \mathcal{M}_2^{0-1} . Tables from \mathcal{M}_2^{0-1} will be called *decision tables with 0-1-decisions*.

It is clear that $\mathcal{M}_2^{0-1} \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_2^\infty$.

For a table $T \in \mathcal{M}_2^\infty$, we denote by $\Pi(T)$ the intersection of sets of decisions attached to rows of T . Decisions from $\Pi(T)$ are called *common decisions* for the table T .

Let T be a nonempty table from \mathcal{M}_2^∞ . Denote by $\text{At}(T)$ the set of attributes attached to columns of the table T . We denote by $\Omega_2(T)$ the set of finite words over the alphabet $\{(f_i, \delta) : f_i \in \text{At}(T), \delta \in E_2\}$ including the empty word λ . For any $\alpha \in \Omega_2(T)$, we now define a subtable $T\alpha$ of the table T . If $\alpha = \lambda$, then $T\alpha = T$. If $\alpha \neq \lambda$ and $\alpha = (f_{i_1}, \delta_1) \cdots (f_{i_m}, \delta_m)$, then $T\alpha$ is the table obtained from T by removal of all rows that do not satisfy the following condition: in columns labeled with attributes f_{i_1}, \dots, f_{i_m} , the row has numbers $\delta_1, \dots, \delta_m$, respectively.

We now define four operations on decision tables: removal of columns and changing of decisions. Let $T \in \mathcal{M}_2^\infty$.

Definition 4 *Removal of columns.* We can remove from T any columns. In each group of rows equal on the remaining columns, we keep the first one.

Definition 5 *\mathcal{P}^∞ -Changing of decisions.* We can change sets of decisions

attached to rows of T to arbitrary sets from \mathcal{P}^∞ .

Definition 6 \mathcal{P} -Changing of decisions. We can change sets of decisions attached to rows of T to arbitrary sets from \mathcal{P} .

Definition 7 \mathcal{P}^{0-1} -Changing of decisions. We can change sets of decisions attached to rows of T to arbitrary sets from \mathcal{P}^{0-1} .

Definition 8 Let $A \subseteq \mathcal{M}_2^\infty$ and $A \neq \emptyset$. The class (the set) of decision tables A will be called a closed class of decision tables from \mathcal{M}_2^∞ if it is closed under operations of removal of columns and \mathcal{P}^∞ -changing of decisions.

Definition 9 Let $A \subseteq \mathcal{M}_2$ and $A \neq \emptyset$. The class (the set) of decision tables A will be called a closed class of decision tables from \mathcal{M}_2 if it is closed under operations of removal of columns and \mathcal{P} -changing of decisions.

Definition 10 Let $A \subseteq \mathcal{M}_2^{0-1}$ and $A \neq \emptyset$. The class (the set) of decision tables A will be called a closed class of decision tables from \mathcal{M}_2^{0-1} if it is closed under operations of removal of columns and \mathcal{P}^{0-1} -changing of decisions.

A closed class of decision tables will be called *nontrivial* if it contains nonempty decision tables.

Definition 11 A nonempty decision table $T \in \mathcal{M}_2^\infty$ is called *complete* if T contains $2^{|\text{At}(T)|}$ rows. Empty table Λ without rows and columns is complete by definition.

Later we will consider only nontrivial closed classes of complete decision tables from \mathcal{M}_2^∞ , from \mathcal{M}_2 or from \mathcal{M}_2^{0-1} . It is clear that each such class contains the empty table Λ .

2.2. Deterministic, Nondeterministic, and Strongly Nondeterministic Decision Trees

A *finite tree with root* is a finite directed tree in which exactly one node called the *root* has no entering edges. The nodes without leaving edges are called *terminal* nodes.

Definition 12 A 2-decision tree is a finite tree with root, which has at least two nodes and in which

- The root and edges leaving the root are not labeled.
- Each terminal node is labeled with a decision from the set ω .
- Each node, which is neither the root nor a terminal node (such nodes will be called *working*), is labeled with an attribute from the set P . Each edge leaving working node is labeled with a number from the set E_2 .

We denote by \mathcal{T}_2 the set of all 2-decision trees. Let $\Gamma \in \mathcal{T}_2$. We denote by $\text{At}(\Gamma)$ the set of attributes attached to working nodes of Γ . A *complete path* of Γ is a sequence $\tau = v_1, d_1, \dots, v_m, d_m, v_{m+1}$ of nodes and edges of Γ in which v_1 is the root of Γ , v_{m+1} is a terminal node of Γ and, for $j = 1, \dots, m$, the edge d_j leaves the node v_j and enters the node v_{j+1} . We denote by $d(\tau)$ the decision attached to the terminal node of τ .

Let T be a nonempty table from \mathcal{M}_2^∞ . If $\text{At}(\Gamma) \subseteq \text{At}(T)$, then we correspond to the table T and the complete path τ a word $\pi(\tau) \in \Omega_2(T)$. If $m = 1$, then

$\pi(\tau) = \lambda$. If $m > 1$ and, for $j = 2, \dots, m$, the node v_j is labeled with the attribute f_{i_j} and the edge d_j is labeled with the number δ_j , then $\pi(\tau) = (f_{i_2}, \delta_2) \cdots (f_{i_m}, \delta_m)$. Denote $T(\tau) = T\pi(\tau)$.

Definition 13 Let T be a nonempty table from \mathcal{M}_2^∞ . A deterministic decision tree for the table T is a 2-decision tree Γ satisfying the following conditions:

- Only one edge leaves the root of Γ .
- For any working node, edges leaving this node are labeled with pairwise different numbers.
- $\text{At}(\Gamma) \subseteq \text{At}(T)$.
- For any row of T , there exists a complete path τ of Γ such that the considered row belongs to the table $T(\tau)$.
- For any complete path τ of Γ , either $T(\tau)$ is empty or the decision $d(\tau)$ attached to the terminal node of τ is a common decision for the table $T(\tau)$.

Definition 14 Let T be a nonempty table from \mathcal{M}_2^∞ . A nondeterministic decision tree for the table T is a 2-decision tree Γ satisfying the following conditions:

- $\text{At}(\Gamma) \subseteq \text{At}(T)$.
- For any row of T , there exists a complete path τ of Γ such that the considered row belongs to the table $T(\tau)$.
- For any complete path τ of Γ , either $T(\tau)$ is empty or the decision $d(\tau)$ attached to the terminal node of τ is a common decision for the table $T(\tau)$.

Definition 15 Let T be a nonempty table from \mathcal{M}_2^{0-1} without common decisions. A strongly nondeterministic decision tree for the table T is a 2-decision tree Γ satisfying the following conditions:

- Each terminal node is labeled with the decision 1.
- $\text{At}(\Gamma) \subseteq \text{At}(T)$.
- For any row of T labeled with the set of decisions $\{1\}$, there exists a complete path τ of Γ such that the considered row belongs to the table $T(\tau)$.
- For any complete path τ of Γ , either $T(\tau)$ is empty or 1 is the common decision for the table $T(\tau)$.

2.3. Weighted Depth and Complexity Parameters

Denote by P^* the set of all finite words over the alphabet $P = \{f_i : i \in \omega\}$, which contains the empty word λ .

Definition 16 A weighted depth is an arbitrary function $\psi : P^* \rightarrow \omega$ such that $\psi(\lambda) = 0$, $\psi(f_i) > 0$ for any $f_i \in P$ and $\psi(f_{i_1} \cdots f_{i_m}) = \sum_{j=1}^m \psi(f_{i_j})$ for any nonempty word $f_{i_1} \cdots f_{i_m} \in P^*$. The weighted depth ψ such that $\psi(f_i) = 1$ for any $f_i \in P$ is called the depth and is denoted h .

Definition 17 Let ψ be a weighted dept. We extend it to the set of all finite subsets of the set P . Let D be a finite subset of the set P . If $D = \emptyset$, then $\psi(D) = 0$. If $D \neq \emptyset$ and $D = \{f_{i_1}, \dots, f_{i_m}\}$, then $\psi(D) = \psi(f_{i_1} \cdots f_{i_m})$.

Definition 18 Let ψ be a weighted depth. We extend it to the set of finite words Ω_2 over the alphabet $\{(f_i, \delta) : f_i \in P, \delta \in E_2\}$ including the empty word λ . Let

$\alpha \in \Omega_2$. If $\alpha = \lambda$, then $\psi(\alpha) = 0$. If $\alpha \neq \lambda$ and $\alpha = (f_{i_1}, \delta_1) \cdots (f_{i_m}, \delta_m)$, then $\psi(\alpha) = \psi(f_{i_1} \cdots f_{i_m})$.

Definition 19 Let ψ be a weighted depth. We extend it to the set \mathcal{T}_2 . Let $\Gamma \in \mathcal{T}_2$. Then $\psi(\Gamma) = \max\{\psi(\pi(\tau))\}$, where the maximum is taken over all complete paths τ of the decision tree Γ . For a given weighted depth ψ , the value $\psi(\Gamma)$ will be called the weighted depth of the decision tree Γ . The value $h(\Gamma)$ will be called the depth of the decision tree Γ .

Let ψ be a weighted depth. We now describe the functions ψ^d , ψ^a , m_ψ , W_ψ , and S_ψ defined on the set \mathcal{M}_2^∞ . By definition, the value of each of these functions for an empty table is equal 0. Let T be a nonempty table from \mathcal{M}_2^∞ . Then

- $\psi^d(T) = \min\{\psi(\Gamma)\}$, where the minimum is taken over all deterministic decision trees Γ for the table T .
- $\psi^a(T) = \min\{\psi(\Gamma)\}$, where the minimum is taken over all nondeterministic decision trees Γ for the table T .
- $m_\psi(T) = \max\{\psi(f_i) : f_i \in \text{At}(T)\}$.
- $W_\psi(T) = \psi(\text{At}(T))$. This is the *total weight of attributes attached to columns of the table T* . It is clear that $W_h(T) = |\text{At}(T)|$.
- Let $\bar{\delta}$ be a row of the table T . Denote $S_\psi(T, \bar{\delta}) = \min\{\psi(D)\}$, where the minimum is taken over all subsets D of the set $\text{At}(T)$ such that in the set of columns of T labeled with attributes from D the row $\bar{\delta}$ is different from all other rows of the table T . Then $S_\psi(T) = \max\{S_\psi(T, \bar{\delta})\}$, where the maximum is taken over all rows $\bar{\delta}$ of the table T .

Let ψ be a weighted depth. We now describe the function ψ^s defined on the set \mathcal{M}_2^{0-1} . By definition, the value of this function for an empty table or a table with a common decision is equal 0. Let T be a nonempty table from \mathcal{M}_2^{0-1} without common decisions. Then

- $\psi^s(T) = \min\{\psi(\Gamma)\}$, where the minimum is taken over all strongly nondeterministic decision trees Γ for the table T .

3. Closed Classes of Complete Decision Tables with Many-Valued Decisions

In this section, building on Chapter 7 of our book [1], we obtain more in-depth results for a special case—weighted depth and closed classes of binary complete decision tables with many-valued decisions. To make reading easier, we try to follow the text of the book [1] as closely as possible.

In this section, we consider results obtained in the paper for the functions $\mathcal{F}_{\psi,A}^\infty$, $\mathcal{G}_{\psi,A}^\infty$, and $\mathcal{H}_{\psi,A}^\infty$.

3.1. Function $\mathcal{F}_{\psi,A}^\infty$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ . We now define a function $\mathcal{F}_{\psi,A}^\infty : \omega \rightarrow \omega$. Let $n \in \omega$. Then

$$\mathcal{F}_{\psi,A}^{\infty}(n) = \max \{ \psi^d(T) : T \in A, W_{\psi}(T) \leq n \}.$$

The function $\mathcal{F}_{\psi,A}^{\infty}$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Let $D = \{n_i : i \in \omega\}$ be an infinite subset of the set ω in which, for any $i \in \omega$, $n_i < n_{i+1}$. We now define a function $H_D : \omega \rightarrow \omega$. Let $n \in \omega$. If $n < n_0$, then $H_D(n) = 0$. If, for some $i \in \omega$, $n_i \leq n < n_{i+1}$, then $H_D(n) = n_i$.

Theorem 1 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{∞} . Then $\mathcal{F}_{\psi,A}^{\infty}$ is an everywhere defined nondecreasing function such that $\mathcal{F}_{\psi,A}^{\infty}(n) \leq n$ for any $n \in \omega$ and $\mathcal{F}_{\psi,A}^{\infty}(0) = 0$. For this function, one of the following statements holds:

(a) If the function W_{ψ} is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{F}_{\psi,A}^{\infty}(n) \leq c$ for any $n \in \omega$.

(b) If the function W_{ψ} is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{F}_{\psi,A}^{\infty}(n)$ for any $n \in \omega$.

For the depth h , the bound from the statement (b) of Theorem 1 can be improved.

Theorem 2 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{∞} for which the function W_h is not bounded from above. Then $\mathcal{F}_{h,A}^{\infty}(n) = n$ for any $n \in \omega$.

We now consider two auxiliary statements. The next lemma follows directly from Lemma 7.1 [1].

Lemma 3 For any weighted depth ψ and any complete decision table T from \mathcal{M}_2^{∞} ,

$$\psi^d(T) \leq W_{\psi}(T).$$

Lemma 4 For any weighted depth ψ and any complete decision table T from \mathcal{M}_2^{∞} ,

$$S_{\psi}(T) = W_{\psi}(T).$$

Proof. Let $T = \Lambda$. Then $S_{\psi}(T) = W_{\psi}(T) = 0$ by definition. Let T be a nonempty complete table, $\bar{\delta}$ be a row of the table T , and D be a subset of the set $\text{At}(T)$ such that in the set of columns of T labeled with attributes from D the row $\bar{\delta}$ is different from all other rows of the table T . Since, for any column of T , there exists a row of T different from $\bar{\delta}$ only in this column, $\text{At}(T) \subseteq D$. Therefore $S_{\psi}(T, \bar{\delta}) \geq \psi(\text{At}(T)) = W_{\psi}(T)$. Since rows of T are pairwise different, the row $\bar{\delta}$ is different from all other rows of the table T in the set of all columns of T . Therefore $S_{\psi}(T, \bar{\delta}) \leq \psi(\text{At}(T)) = W_{\psi}(T)$. Thus, $S_{\psi}(T, \bar{\delta}) = W_{\psi}(T)$ and $S_{\psi}(T) = W_{\psi}(T)$. \square

Proof of Theorem 1. From Theorem 7.1 [1] it follows that $\mathcal{F}_{\psi,A}^{\infty}$ is an everywhere defined nondecreasing function such that $\mathcal{F}_{\psi,A}^{\infty}(n) \leq n$ for any $n \in \omega$ and $\mathcal{F}_{\psi,A}^{\infty}(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive constant c . Using Lemma 3, we obtain that the function ψ^d is bounded from above on the class A by c . Therefore $\mathcal{F}_{\psi,A}^\infty(n) \leq c$ for any $n \in \omega$.

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 4, we obtain that the function S_ψ is not bounded from above on the class A . From here and from Theorem 7.1 [1] it follows that there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{F}_{\psi,A}^\infty(n)$ for any $n \in \omega$. \square

Proof of Theorem 2. Using Lemma 4, we obtain that the function S_h is not bounded from above on the class A . From here and from Theorem 7.2 [1] it follows that $\mathcal{F}_{h,A}^\infty(n) = n$ for any $n \in \omega$. \square

3.2. Function $\mathcal{G}_{\psi,A}^\infty$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ . We now define a function $\mathcal{G}_{\psi,A}^\infty$. Let $n \in \omega$. Then

$$\mathcal{G}_{\psi,A}^\infty(n) = \max \{ \psi^a(T) : T \in A, W_\psi(T) \leq n \}.$$

The function $\mathcal{G}_{\psi,A}^\infty$ characterizes the growth in the worst case of the minimum weighted depth of nondeterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Theorem 5 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ . Then $\mathcal{G}_{\psi,A}^\infty$ is an everywhere defined nondecreasing function such that $\mathcal{G}_{\psi,A}^\infty(n) \leq n$ for any $n \in \omega$ and $\mathcal{G}_{\psi,A}^\infty(0) = 0$. For this function, one of the following statements holds:

(a) If the function W_ψ is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{G}_{\psi,A}^\infty(n) \leq c$ for any $n \in \omega$.

(b) If the function W_ψ is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{G}_{\psi,A}^\infty(n)$ for any $n \in \omega$.

For the depth h , the bound from the statement (b) of Theorem 5 can be improved.

Theorem 6 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ for which the function W_h is not bounded from above. Then $\mathcal{G}_{h,A}^\infty(n) = n$ for any $n \in \omega$.

The next statement follows directly from Lemma 7.6 [1].

Lemma 7 For any weighted depth ψ and any complete table T from \mathcal{M}_2^∞ ,

$$\psi^a(T) \leq \psi^d(T).$$

Proof of Theorem 5. From Theorem 7.3 [1] it follows that $\mathcal{G}_{\psi,A}^\infty$ is an everywhere defined nondecreasing function such that $\mathcal{G}_{\psi,A}^\infty(n) \leq n$ for any $n \in \omega$ and $\mathcal{G}_{\psi,A}^\infty(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive constant c . Using Lemmas 3 and 7, we obtain that the function ψ^a is bounded

from above on the class A by c . Therefore $\mathcal{G}_{\psi,A}^\infty(n) \leq c$ for any $n \in \omega$.

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 4, we obtain that the function S_ψ is not bounded from above on the class A . From here and from Theorem 7.3 [1] it follows that there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{G}_{\psi,A}^\infty(n)$ for any $n \in \omega$. \square

Proof of Theorem 6. Using Lemma 4, we obtain that the function S_h is not bounded from above on the class A . From here and from Theorem 7.4 [1] it follows that $\mathcal{G}_{h,A}^\infty(n) = n$ for any $n \in \omega$. \square

3.3. Function $\mathcal{H}_{\psi,A}^\infty$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ . We now define possibly partial function $\mathcal{H}_{\psi,A}^\infty : \omega \rightarrow \omega$. Let $n \in \omega$. If the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$ is infinite, then the value $\mathcal{H}_{\psi,A}^\infty(n)$ is undefined. Otherwise, $\mathcal{H}_{\psi,A}^\infty(n) = \max\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$. This definition is correct since $\Lambda \in A$, $\psi^a(\Lambda) = 0$ and therefore the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$ is nonempty.

The function $\mathcal{H}_{\psi,A}^\infty$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the minimum weighted depth of nondeterministic decision trees for these tables.

We now define possibly partial function $\Phi_{\psi,A} : \omega \rightarrow \omega$. Let $n \in \omega$. If the set $\{|\text{At}(T)| : T \in A, m_\psi(T) \leq n\}$ is infinite, then the value $\Phi_{\psi,A}(n)$ is undefined. Otherwise, $\Phi_{\psi,A}(n) = \max\{|\text{At}(T)| : T \in A, m_\psi(T) \leq n\}$. This definition is correct since $\Lambda \in A$, $m_\psi(\Lambda) = 0$ and therefore the set $\{|\text{At}(T)| : T \in A, m_\psi(T) \leq n\}$ is nonempty. It is clear that $\{T : T \in A, m_\psi(T) \leq 0\} = \{\Lambda\}$. Therefore $\Phi_{\psi,A}(0) = 0$.

The following statement describes a criterion for the function $\mathcal{H}_{\psi,A}^\infty$ to be everywhere defined.

Theorem 8 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ . Then the function $\mathcal{H}_{\psi,A}^\infty$ is everywhere defined if and only if the function $\Phi_{\psi,A}$ is everywhere defined.

The next statement clarifies the behavior of everywhere defined function $\mathcal{H}_{\psi,A}^\infty$.

Theorem 9 Let ψ be a weighted depth, A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ , and the function $\mathcal{H}_{\psi,A}^\infty$ be everywhere defined. Then

(a) For any $n \in \omega$, $\mathcal{H}_{\psi,A}^\infty(3n) \geq \sqrt{2\Phi_{\psi,A}(n)} - 3$ and $\mathcal{H}_{\psi,A}^\infty(n) \leq n\Phi_{\psi,A}(n)$.

(b) A polynomial p such that $\mathcal{H}_{\psi,A}^\infty(n) \leq p(n)$ for any $n \in \omega$ exists if and only if there exists a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$.

We now consider two auxiliary statements. The first one follows from Lemmas 7.13 and 7.15 [1].

Lemma 10 For any $k \in \omega \setminus \{0\}$, there exists a complete decision table $T \in \mathcal{M}_2^\infty$ such that $|\text{At}(T)| = k(k+1)/2$, $h^a(T) \leq 3$ and $h^d(T) \geq k-1$.

Lemma 11 Let ψ be a weighted depth, A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^∞ , and the function $\Phi_{\psi,A}$ be everywhere defined.

Then the function $\mathcal{H}_{\psi,A}^\infty$ is everywhere defined and $\mathcal{H}_{\psi,A}^\infty(n) \leq n\Phi_{\psi,A}(n)$ for any $n \in \omega$.

Proof. Let $n \in \omega$ and T be a table from the class A for which $\psi^a(T) \leq n$. Let us show that $\psi^d(T) \leq n\Phi_{\psi,A}(n)$. If T is empty or has a common decision, then, as it is easy to show, $\psi^d(T) = 0$ and the considered inequality holds.

Let T be a nonempty table without common decisions and Γ be a nondeterministic decision tree for T such that $\psi(\Gamma) = \psi^a(T)$. It is clear that $\text{At}(\Gamma) \neq \emptyset$. Let $\text{At}(\Gamma) = \{f_{i_1}, \dots, f_{i_m}\}$. We now describe a deterministic decision tree G for the table T . This tree sequentially computes values of the attributes f_{i_1}, \dots, f_{i_m} . The set of words $\pi(\tau)$ corresponding to complete paths τ of G coincides with the set $\{(f_{i_1}, \delta_1) \dots (f_{i_m}, \delta_m) : \delta_1, \dots, \delta_m \in E_2\}$. It is clear that, for any row of T , there exists a complete path τ of G such that the considered row belongs to the subtable $T(\tau)$. Let us consider an arbitrary complete path τ of G with $\pi(\tau) = (f_{i_1}, \delta_1) \dots (f_{i_m}, \delta_m)$. It is clear that the subtable $T\pi(\tau) = T(\tau)$ is nonempty. Let $\bar{\delta}$ be a row of $T\pi(\tau)$. Then in Γ there exists a complete path ξ such that $\bar{\delta}$ is a row of subtable $T\pi(\xi) = T(\xi)$ and the subtable $T\pi(\xi)$ has a common decision d . It is clear that the set of letters of the word $\pi(\xi)$ is a subset of the set of letters of the word $\pi(\tau)$. Therefore d is a common decision of the subtable $T\pi(\tau)$. The minimum common decision for the subtable $T\pi(\tau)$ is attached to the terminal node of the path τ . It is easy to check that the tree G is indeed a deterministic decision tree for the table T .

Since $\psi(\Gamma) = \psi^a(T) \leq n$, for any $f_{i_j} \in \text{At}(\Gamma)$, $\psi(f_{i_j}) \leq n$. It is clear that $\text{At}(\Gamma) \subseteq \text{At}(T)$. Using the fact that A is a closed class, we obtain $|\text{At}(\Gamma)| \leq \Phi_{\psi,A}(n)$. Therefore $\psi(G) \leq n\Phi_{\psi,A}(n)$ and $\psi^d(T) \leq n\Phi_{\psi,A}(n)$. Thus, for any table $T \in A$ such that $\psi^a(T) \leq n$, the inequality $\psi^d(T) \leq n\Phi_{\psi,A}(n)$ holds. As a result, we obtain that the value $\mathcal{H}_{\psi,A}^\infty(n)$ is defined and $\mathcal{H}_{\psi,A}^\infty(n) \leq n\Phi_{\psi,A}(n)$. □

Proof of Theorem 8. Let the function $\Phi_{\psi,A}$ be everywhere defined. Then, by Lemma 11, the function $\mathcal{H}_{\psi,A}^\infty$ is everywhere defined.

Let the function $\Phi_{\psi,A}$ be not everywhere defined. Then there exists a number $n \in \omega \setminus \{0\}$ for which the value $\Phi_{\psi,A}(n)$ is undefined. From here and from Lemma 10 it follows that, for any $k \in \omega \setminus \{0\}$, there exists a complete decision table $Q \in A$ such that $m_\psi(Q) \leq n$, $|\text{At}(Q)| = k(k+1)/2$, $h^a(Q) \leq 3$ and $h^d(Q) \geq k-1$. Since $m_\psi(Q) \leq n$ and $h^a(Q) \leq 3$, $\psi^a(Q) \leq 3n$. It is clear that $\psi^d(Q) \geq k-1$. As a result, we obtain that the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq 3n\}$ is infinite and the value $\mathcal{H}_{\psi,A}^\infty(3n)$ is undefined. □

Proof of Theorem 9. (a) Using Theorem 8, we obtain that the function $\Phi_{\psi,A}$ is everywhere defined. From here and from Lemma 11 it follows that

$$\mathcal{H}_{\psi,A}^\infty(n) \leq n\Phi_{\psi,A}(n) \text{ for any } n \in \omega.$$

We now show that $\mathcal{H}_{\psi,A}^\infty(3n) \geq \sqrt{2\Phi_{\psi,A}(n)} - 3$ for any $n \in \omega$. If $\Phi_{\psi,A}(n) = 0$, then, evidently, $\mathcal{H}_{\psi,A}^\infty(3n) \geq \sqrt{\Phi_{\psi,A}(n)} - 3$. Let $\Phi_{\psi,A}(n) > 0$ and k be the maximum number from ω such that $\frac{k(k+1)}{2} \leq \Phi_{\psi,A}(n)$. From

Lemma 10 it follows that there exists a complete decision table $Q \in A$ such that $m_\psi(Q) \leq n$, $|\text{At}(Q)| = k(k+1)/2$, $h^a(Q) \leq 3$ and $h^d(Q) \geq k-1$. Since $m_\psi(Q) \leq n$ and $h^a(Q) \leq 3$, $\psi^a(Q) \leq 3n$. It is clear that $\psi^d(Q) \geq k-1$.

Evidently, $\frac{(k+1)(k+2)}{2} > \Phi_{\psi,A}(n)$. Therefore $(k+2)^2 > 2\Phi_{\psi,A}(n)$ and

$$k > \sqrt{2\Phi_{\psi,A}(n)} - 2. \text{ Thus, } \mathcal{H}_{\psi,A}^\infty(3n) \geq \sqrt{2\Phi_{\psi,A}(n)} - 3.$$

(b) Let there exist a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$. From part (a) of the theorem statement it follows that $\mathcal{H}_{\psi,A}^\infty(n) \leq nq(n)$ for any $n \in \omega$. Therefore there exists polynomial p such that $\mathcal{H}_{\psi,A}^\infty(n) \leq p(n)$ for any $n \in \omega$.

Let there be no a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$. We now show that there is no a polynomial p such that $\mathcal{H}_{\psi,A}^\infty(n) \leq p(n)$ for any $n \in \omega$. Assume the contrary: there exists a polynomial p such that $\mathcal{H}_{\psi,A}^\infty(n) \leq p(n)$ for any $n \in \omega$. From part (a) of the theorem statement it follows that $\mathcal{H}_{\psi,A}^\infty(3n) \geq \sqrt{2\Phi_{\psi,A}(n)} - 3$ for any $n \in \omega$. Therefore $(p(3n)+3)^2 \geq \Phi_{\psi,A}(n)$ for any $n \in \omega$. Thus, there exists polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$, but this is impossible. \square

4. Closed Classes of Complete Conventional Decision Tables

This section is based on Chapter 8 of our book [1] and paper [25], the proof of which we adapted to demonstrate upper bounds on n^2 for the function describing the relationship between deterministic and nondeterministic decision trees. Our goal is to obtain deeper results for the special case of weighted depth and closed classes of conventional binary complete decision tables. For readability, we try to follow the text of [1] as closely as possible.

In this section, we consider results obtained in the paper for the functions $\mathcal{F}_{\psi,A}$, $\mathcal{G}_{\psi,A}$, and $\mathcal{H}_{\psi,A}$.

4.1. Function $\mathcal{F}_{\psi,A}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 . We now define a function $\mathcal{F}_{\psi,A} : \omega \rightarrow \omega$. Let $n \in \omega$. Then

$$\mathcal{F}_{\psi,A}(n) = \max \{ \psi^d(T) : T \in A, W_\psi(T) \leq n \}.$$

The function $\mathcal{F}_{\psi,A}$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Theorem 12 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 . Then $\mathcal{F}_{\psi,A}$ is an everywhere defined nondecreasing function such that $\mathcal{F}_{\psi,A}(n) \leq n$ for any $n \in \omega$ and $\mathcal{F}_{\psi,A}(0) = 0$. For this function, one of the following statements holds:

(a) If the function W_ψ is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{F}_{\psi,A}(n) \leq c$ for any $n \in \omega$.

(b) If the function W_ψ is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{F}_{\psi,A}(n)$ for any $n \in \omega$.

Proof. From Theorem 8.1 [1] it follows that $\mathcal{F}_{\psi,A}$ is an everywhere defined nondecreasing function such that $\mathcal{F}_{\psi,A}(n) \leq n$ for any $n \in \omega$ and $\mathcal{F}_{\psi,A}(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive constant c . Using Lemma 3, we obtain that the function ψ^d is bounded from above on the class A by c . Therefore $\mathcal{F}_{\psi,A}(n) \leq c$ for any $n \in \omega$.

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 4, we obtain that the function S_ψ is not bounded from above on the class A . From here and from Theorem 8.1 [1] it follows that there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{F}_{\psi,A}(n)$ for any $n \in \omega$. \square

For the depth h , the bound from the statement (b) of Theorem 12 can be improved.

Theorem 13 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 for which the function W_h is not bounded from above. Then $\mathcal{F}_{h,A}(n) = n$ for any $n \in \omega$.

Proof. Using Lemma 4, we obtain that the function S_h is not bounded from above on the class A . From here and from Theorem 8.2 [1] it follows that $\mathcal{F}_{h,A}(n) = n$ for any $n \in \omega$. \square

4.2. Function $\mathcal{G}_{\psi,A}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 . We now define a function $\mathcal{G}_{\psi,A}$. Let $n \in \omega$. Then

$$\mathcal{G}_{\psi,A}(n) = \max \{ \psi^a(T) : T \in A, W_\psi(T) \leq n \}.$$

The function $\mathcal{G}_{\psi,A}$ characterizes the growth in the worst case of the minimum weighted depth of nondeterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Theorem 14 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 . Then $\mathcal{G}_{\psi,A}$ is an everywhere defined nondecreasing function such that $\mathcal{G}_{\psi,A}(n) \leq n$ for any $n \in \omega$ and $\mathcal{G}_{\psi,A}(0) = 0$. For this function, one of the following statements holds:

(a) If the function W_ψ is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{G}_{\psi,A}(n) \leq c$ for any $n \in \omega$.

(b) If the function W_ψ is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{G}_{\psi,A}(n)$ for any $n \in \omega$.

Proof. From Theorem 8.3 [1] it follows that $\mathcal{G}_{\psi,A}$ is an everywhere defined nondecreasing function such that $\mathcal{G}_{\psi,A}(n) \leq n$ for any $n \in \omega$ and $\mathcal{G}_{\psi,A}(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive

constant c . Using Lemmas 3 and 7, we obtain that the function ψ^a is bounded from above on the class A by c . Therefore $\mathcal{G}_{\psi^a, A}(n) \leq c$ for any $n \in \omega$.

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 4, we obtain that the function S_ψ is not bounded from above on the class A . From here and from Theorem 8.3 [1] it follows that there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{G}_{\psi^a, A}(n)$ for any $n \in \omega$. \square

For the depth h , the bound from the statement (b) of Theorem 14 can be improved.

Theorem 15 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 for which the function W_h is not bounded from above. Then $\mathcal{G}_{h, A}(n) = n$ for any $n \in \omega$.

Proof. Using Lemma 4, we obtain that the function S_h is not bounded from above on the class A . From here and from Theorem 8.4 [1] it follows that $\mathcal{G}_{h, A}(n) = n$ for any $n \in \omega$. \square

4.3. Function $\mathcal{H}_{\psi, A}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 . We now define possibly partial function $\mathcal{H}_{\psi, A} : \omega \rightarrow \omega$. Let $n \in \omega$. If the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$ is infinite, then the value $\mathcal{H}_{\psi, A}(n)$ is undefined. Otherwise, $\mathcal{H}_{\psi, A}(n) = \max\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$. This definition is correct since $\Lambda \in A$, $\psi^a(\Lambda) = 0$ and therefore the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$ is nonempty.

The function $\mathcal{H}_{\psi, A}$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the minimum weighted depth of nondeterministic decision trees for these tables.

Theorem 16 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2 . Then the function $\mathcal{H}_{\psi, A}$ is everywhere defined and $\mathcal{H}_{\psi, A}(n) \leq n^2$ for any $n \in \omega$.

First we prove the following statement.

Lemma 17 Let ψ be a weighted depth and T be a complete decision table from \mathcal{M}_2 . Then $\psi^d(T) \leq \psi^a(T)^2$.

Proof. If $T = \Lambda$ or T has a common decision, then as it is not difficult to show, $\psi^d(T) = \psi^a(T) = 0$ and the considered inequality holds. Indeed, the equalities $\psi^d(T) = \psi^a(T) = 0$ follow in the case of $T = \Lambda$ from the definitions, and in the case when T has a common decision they follow from the fact that we do not need to calculate the values of attributes to return this common decision. Let T be a nonempty decision table without common decisions. Let, for the definiteness, T have n columns labeled with attributes f_1, \dots, f_n . The set of rows of the table T coincides with the set of n -tuples from E_2^n . For any row $\bar{\delta} \in E_2^n$, we denote by $dec(\bar{\delta})$ the only decision from the set of decisions attached to the row $\bar{\delta}$ in the table T .

We will say that a word $\alpha \in \Omega_2(T)$ is *inconsistent* if it contains two letters of

the kind (f_i, δ) and (f_i, σ) such that $\delta \neq \sigma$. If the word α is not inconsistent it will be called *consistent*. It is easy to show that the table $T\alpha$ is empty if and only if the word α is inconsistent.

Let Γ be a nondeterministic decision tree for the table T such that $\psi(\Gamma) = \psi^a(T)$. We denote by $CP^+(\Gamma)$ the set of complete paths ξ from $CP(\Gamma)$ for which the word $\pi(\xi)$ is consistent. Let's assign each path from $CP^+(\Gamma)$ a number from ω so that different paths have different numbers. Two complete paths $\xi_1, \xi_2 \in CP^+(\Gamma)$ will be called *equivalent* if $d(\xi_1) = d(\xi_2)$, i.e., their terminal nodes are labeled with the same number. The considered equivalence relation partitions the set $CP^+(\Gamma)$ into equivalence classes C_1, \dots, C_t .

We now show that, for any complete paths $\xi_1, \xi_2 \in CP^+(\Gamma)$ that are not equivalent, the word $\pi(\xi_1)\pi(\xi_2)$ is inconsistent. Let us assume the contrary. Then there is a row $\bar{\sigma} \in E_2^n$ of the table T , which belongs to both subtables $T(\xi_1) = T\pi(\xi_1)$ and $T(\xi_2) = T\pi(\xi_2)$, but this is impossible since $d(\xi_1) \neq d(\xi_2)$.

Let us consider a row $\bar{\delta} = (\delta_1, \dots, \delta_n) \in E_2^n$ of the table T . We now describe the work on the row $\bar{\delta}$ of a deterministic decision tree G for the table T . As a result, we obtain the description of a complete path $\rho_{\bar{\delta}}$ in the decision tree G such that the row $\bar{\delta}$ belongs to the subtable $T(\rho_{\bar{\delta}})$. The set of complete paths of the decision tree G coincides with the set $\{\rho_{\bar{\delta}} : \bar{\delta} \in E_2^n\}$.

Step 1. Set $\Xi := CP^+(\Gamma)$ and, for each $\xi \in CP^+(\Gamma)$, set $R(\xi) := \pi(\xi)$. Move on to Step 2.

Step 2. This step consists of the three phases: (a), (b), and (c).

(a) If there is a complete path $\xi \in \Xi$ for which the word $R(\xi)$ is empty, then the decision tree G finishes its work and returns the decision $d(\rho_{\bar{\delta}}) = d(\xi)$, which will be attached to the terminal node of the path $\rho_{\bar{\delta}}$. Otherwise, move on to (b).

(b) Set i_0 the minimum index i from the set $\{1, \dots, t\}$ for which $C_i \cap \Xi \neq \emptyset$. Choose a complete path $\xi \in C_{i_0} \cap \Xi$ with the minimum number. If $C_{i_0} \cap \Xi = \Xi$, then the tree G finishes its work and returns the decision $d(\rho_{\bar{\delta}}) = d(\xi)$, which will be attached to the terminal node of the path $\rho_{\bar{\delta}}$. Otherwise, move on to (c).

(c) Let $\{f_{i_1}, \dots, f_{i_m}\}$ be all attributes from the letters of the word $R(\xi)$. The decision tree G finds the values of attributes f_{i_1}, \dots, f_{i_m} and obtains that $f_{i_1} = \delta_{i_1}, \dots, f_{i_m} = \delta_{i_m}$. Set $\alpha = (f_{i_1}, \delta_{i_1}) \dots (f_{i_m}, \delta_{i_m})$. For each complete path $\xi \in \Xi$, we do the following. If the word $\alpha R(\xi)$ is inconsistent, then set $\Xi := \Xi \setminus \{\xi\}$. Otherwise, set $R(\xi) := R(\xi) \setminus \alpha$ where $R(\xi) \setminus \alpha$ denotes the word obtained from $R(\xi)$ by removal of all letters from the word α . Move on to Step 2.

It is clear that the row $\bar{\delta}$ belongs to the subtable $T(\rho_{\bar{\delta}})$. We now show that the decision $d(\rho_{\bar{\delta}})$ attached to the terminal node of this path is a common decision for the subtable $T(\rho_{\bar{\delta}})$. The two variants of finishing the work of the decision tree G are described in phases (a) and (b) of Step 2.

(a) There is a complete path $\xi \in \Xi$ for which the word $R(\xi)$ is empty. In this case, the tree G finishes its work and returns the decision $d(\rho_{\bar{\delta}}) = d(\xi)$,

which will be attached to the terminal node of the path $\rho_{\bar{\delta}}$. It is clear that the set of letters of the word $\pi(\xi)$ is a subset of the set of letters of the word $\pi(\rho_{\bar{\delta}})$. Therefore the set of rows of the subtable $T(\rho_{\bar{\delta}})$ is a subset of the set of rows of the subtable $T(\xi)$. From here it follows that the row $\bar{\delta}$ belongs to the subtable $T(\xi)$. Since Γ is a nondeterministic decision tree for T and the subtable $T(\xi)$ is nonempty, the decision $d(\xi)$ is a common decision of the subtable $T(\xi)$. Thus, the decision $d(\rho_{\bar{\delta}})$ is a common decision for the subtable $T(\rho_{\bar{\delta}})$.

(b) There is no a complete path $\xi \in \Xi$ for which the word $R(\xi)$ is empty and there is $i_0 \in \{1, \dots, t\}$ for which $C_{i_0} \cap \Xi \neq \emptyset$ and $C_{i_0} \cap \Xi = \Xi$. Let ξ be a complete path from the set $C_{i_0} \cap \Xi$ with the minimum number. Then the tree G finishes its work and returns the decision $d(\rho_{\bar{\delta}}) = d(\xi)$, which will be attached to the terminal node of the path $\rho_{\bar{\delta}}$.

Since Γ is a nondeterministic decision tree for the table T , there is a complete path $\zeta \in CP^+(\Gamma)$ such that the row $\bar{\delta}$ belongs to the subtable $T(\zeta)$. For this path, $d(\zeta) = dec(\bar{\delta})$. It is clear that the word $\pi(\zeta)\pi(\rho_{\bar{\delta}})$ is consistent. Hence the path ζ belongs to the set Ξ and therefore, to the set C_{i_0} . Thus, terminal nodes of the paths from C_{i_0} are labeled with the decision $dec(\bar{\delta})$. In particular, $d(\xi) = dec(\bar{\delta})$.

We now show that $dec(\bar{\delta})$ is a common decision of the subtable $T(\rho_{\bar{\delta}})$. Assume the contrary. Then there exists a row $\bar{\sigma} \in E_2^n$, which belongs to the subtable $T(\rho_{\bar{\delta}})$ and for which $dec(\bar{\sigma}) \neq dec(\bar{\delta})$. Since Γ is a nondeterministic decision tree for the table T , there is a complete path $\theta \in CP^+(\Gamma)$ such that row $\bar{\sigma}$ belongs to the subtable $T(\theta)$. It is clear that $d(\theta) = dec(\bar{\sigma})$ and the word $\pi(\theta)\pi(\rho_{\bar{\delta}})$ is consistent. Hence, the path θ belongs to the set Ξ and therefore, to the set C_{i_0} , but this is impossible since terminal nodes of all paths from C_{i_0} are labeled with the decision $dec(\bar{\delta})$ and $dec(\bar{\sigma}) \neq dec(\bar{\delta})$.

As a result, we obtain that G is a deterministic decision tree for the table T .

It is obvious that during each complete repetition of Step 2, which includes phase (c), the decision tree G computes the values of attributes, which are attached to some working nodes of a complete path of Γ . Therefore the total weight of these attributes is at most $\psi(\Gamma)$. Now we show that at most $h(\Gamma)$ complete repetitions of Step 2 are performed.

We know that, for any complete paths $\xi_1, \xi_2 \in CP^+(\Gamma)$ that are not equivalent, the word $\pi(\xi_1)\pi(\xi_2)$ is inconsistent. Using this fact, one can show that after each complete repetition of Step 2, for each complete path $\xi \in \Xi \setminus C_{i_0}$, this path will be removed from the set Ξ or the length of the word $R(\xi)$ will decrease by at least 1. Evidently, for each complete path $\xi \in CP^+(\Gamma)$, the length of the word $\pi(\xi)$ is at most $h(\Gamma)$. Therefore, after $h(\Gamma)$ complete repetitions of Step 2, we will find a complete path $\xi \in \Xi$ for which the word $R(\xi)$ is empty or we will have $\Xi = \Xi \cap C_{i_0}$. In both cases, the decision tree G will finish its work.

As a result, we obtain that, in the complete path $\rho(\bar{\delta})$, we find values of a group of attributes, which total weight is at most $\psi(\Gamma)$, at most $h(\Gamma)$ times. It is clear that $h(\Gamma) \leq \psi(\Gamma)$. Taking into account that we considered an arbitrary

complete path in the decision tree G and $\psi(\Gamma) = \psi^a(T)$, we obtain $\psi(G) \leq \psi^a(T)^2$. Since G is a deterministic decision tree for the table T , we have $\psi^d(T) \leq \psi^a(T)^2$. □

Proof of Theorem 16. The statement of theorem follows from Lemma 17. □

5. Closed Classes of Complete Decision Tables with 0-1-Decisions

This section is based on Chapter 9 of the book [1] and the previous section of this paper. We obtain deeper results for the special case of weighted depth and closed classes of binary complete decision tables with 0-1-decisions. For the convenience of the reader, we try to follow the text of [1] as closely as possible.

In this section, we consider results obtained in the paper for the functions $\mathcal{F}_{\psi,A}^{0-1}$, $\mathcal{G}_{\psi,A}^{0-1}$, $\mathcal{H}_{\psi,A}^{0-1}$, $\mathcal{S}_{\psi,A}^{0-1}$, and $\mathcal{K}_{\psi,A}^{0-1}$.

5.1. Function $\mathcal{F}_{\psi,A}^{0-1}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . We now define a function $\mathcal{F}_{\psi,A}^{0-1} : \omega \rightarrow \omega$. Let $n \in \omega$. Then

$$\mathcal{F}_{\psi,A}^{0-1}(n) = \max \{ \psi^d(T) : T \in A, W_\psi(T) \leq n \}.$$

The function $\mathcal{F}_{\psi,A}^{0-1}$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Theorem 18 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . Then $\mathcal{F}_{\psi,A}^{0-1}$ is an everywhere defined nondecreasing function such that $\mathcal{F}_{\psi,A}^{0-1}(n) \leq n$ for any $n \in \omega$ and $\mathcal{F}_{\psi,A}^{0-1}(0) = 0$. For this function, one of the following statements holds:

- (a) If the function W_ψ is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{F}_{\psi,A}^{0-1}(n) \leq c$ for any $n \in \omega$.
- (b) If the function W_ψ is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{F}_{\psi,A}^{0-1}(n)$ for any $n \in \omega$.

Proof. From Theorem 9.1 [1] it follows that $\mathcal{F}_{\psi,A}^{0-1}$ is an everywhere defined nondecreasing function such that $\mathcal{F}_{\psi,A}^{0-1}(n) \leq n$ for any $n \in \omega$ and $\mathcal{F}_{\psi,A}^{0-1}(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive constant c . Using Lemma 3, we obtain that the function ψ^d is bounded from above on the class A by c . Therefore $\mathcal{F}_{\psi,A}^{0-1}(n) \leq c$ for any $n \in \omega$. □

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 4, we obtain that the function S_ψ is not bounded from above on the class A . From here and from Theorem 9.1 [1] it follows that there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{F}_{\psi,A}^{0-1}(n)$ for any $n \in \omega$. □

For the depth h , the bound from the statement (b) of Theorem 18 can be improved.

Theorem 19 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} for which the function W_h is not bounded from above. Then $\mathcal{F}_{h,A}^{0-1}(n) = n$ for any $n \in \omega$.

Proof. Using Lemma 4, we obtain that the function S_h is not bounded from above on the class A . From here and from Theorem 9.2 [1] it follows that $\mathcal{F}_{h,A}^{0-1}(n) = n$ for any $n \in \omega$. \square

5.2. Function $\mathcal{G}_{\psi,A}^{0-1}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . We now define a function $\mathcal{G}_{\psi,A}^{0-1}$. Let $n \in \omega$. Then

$$\mathcal{G}_{\psi,A}^{0-1}(n) = \max\{\psi^a(T) : T \in A, W_\psi(T) \leq n\}.$$

The function $\mathcal{G}_{\psi,A}^{0-1}$ characterizes the growth in the worst case of the minimum weighted depth of nondeterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Theorem 20 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . Then $\mathcal{G}_{\psi,A}^{0-1}$ is an everywhere defined nondecreasing function such that $\mathcal{G}_{\psi,A}^{0-1}(n) \leq n$ for any $n \in \omega$ and $\mathcal{G}_{\psi,A}^{0-1}(0) = 0$. For this function, one of the following statements holds:

- (a) If the function W_ψ is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{G}_{\psi,A}^{0-1}(n) \leq c$ for any $n \in \omega$.
- (b) If the function W_ψ is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{G}_{\psi,A}^{0-1}(n)$ for any $n \in \omega$.

First, we prove the following auxiliary statement.

Lemma 21 Let T be a nonempty complete decision table from \mathcal{M}_2^{0-1} . Then there exists a complete decision table T' from \mathcal{M}_2^{0-1} , which is obtained from T by \mathcal{P}^{0-1} -changing of decisions and for which $\psi^a(T') = W_\psi(T') = W_\psi(T)$.

Proof. From Lemma 4 it follows that $S_\psi(T) = W_\psi(T)$. Let $\bar{\delta}$ be a row of the table T for which $S_\psi(T) = S_\psi(T, \bar{\delta})$. We denote by T' a decision table obtained from T by changing of the sets of decisions attached to rows of T such that the row $\bar{\delta}$ is labeled with the set $\{1\}$ and all other rows are labeled with the set $\{0\}$.

It is clear that $W_\psi(T') = W_\psi(T)$. We now show that $\psi^a(T') = W_\psi(T')$. Let Γ be a nondeterministic decision tree for the table T' such that $\psi(\Gamma) = \psi^a(T')$, τ be a complete path of Γ such that the row $\bar{\delta}$ belongs to the subtable $T'(\tau)$ and f_{i_1}, \dots, f_{i_k} be attributes attached to working nodes of τ . It is clear that $\bar{\delta}$ is the only row of the subtable $T'(\tau)$. Therefore in the set of columns of T labeled with attributes from the set $\{f_{i_1}, \dots, f_{i_k}\}$ the row $\bar{\delta}$ is different from all other rows of the table T . Thus, $\psi(\{f_{i_1}, \dots, f_{i_k}\}) \geq S_\psi(T, \bar{\delta}) = S_\psi(T) = W_\psi(T)$ and $\psi(\Gamma) \geq W_\psi(T)$. As a result, we obtain $\psi^a(T') \geq W_\psi(T')$. By Lemmas 3 and 7, $\psi^a(T') = W_\psi(T')$. \square

Proof of Theorem 20. Since A is a closed class, $\Lambda \in A$. By definition,

$W_\psi(\Lambda) = 0$. Using this fact and Lemmas 3 and 7, we obtain that $\mathcal{G}_{\psi,A}^{0-1}$ is an everywhere defined function and $\mathcal{G}_{\psi,A}(n) \leq n$ for any $n \in \omega$. Evidently, $\mathcal{G}_{\psi,A}$ is a nondecreasing function. Let $T \in A$ and $W_\psi(T) \leq 0$. It is clear that $T = \Lambda$. By definition, $\psi^a(\Lambda) = 0$. Therefore $\mathcal{G}_{\psi,A}^{0-1}(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive constant c . Using Lemmas 3 and 7, we obtain that the function ψ^a is bounded from above on the class A by c . Therefore $\mathcal{G}_{\psi,A}^{0-1}(n) \leq c$ for any $n \in \omega$.

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 12, we obtain that the set $D = \{W_\psi(T) : T \in A, \psi^a(T) = W_\psi(T)\}$ is infinite. Since the class A is closed, $\Lambda \in A$ and, since $\psi^a(\Lambda) = W_\psi(\Lambda) = 0$, $0 \in D$. Evidently, for any $n \in D$, $\mathcal{G}_{\psi,A}^{0-1}(n) \geq n$. Taking into account that $\mathcal{G}_{\psi,A}^{0-1}$ is a nondecreasing function, we obtain that $\mathcal{G}_{\psi,A}(n) \geq H_D(n)$ for any $n \in \omega \setminus \{0\}$. \square

For the depth h , the bound from the statement (b) of Theorem 20 can be improved.

Theorem 22 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} for which the function W_h is not bounded from above. Then $\mathcal{G}_{h,A}^{0-1}(n) = n$ for any $n \in \omega$.

Proof. Since A is a closed class of complete decision tables from \mathcal{M}_2^{0-1} for which the function W_h is not bounded from above, the class A contains for each $n \in \omega$ a complete table T_n such that $W_h(T_n) = n$. Using Lemma 12, we obtain that the class A contains for each $n \in \omega$ a complete table T'_n such that $h^a(T'_n) = W_h(T'_n) = n$. Therefore $\mathcal{G}_{h,A}^{0-1}(n) \geq n$ for any $n \in \omega$. Using Theorem 20, we obtain that $\mathcal{G}_{h,A}^{0-1}(n) = n$ for any $n \in \omega$. \square

5.3. Function $\mathcal{H}_{\psi,A}^{0-1}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . We now define possibly partial function $\mathcal{H}_{\psi,A}^{0-1} : \omega \rightarrow \omega$. Let $n \in \omega$. If the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$ is infinite, then the value $\mathcal{H}_{\psi,A}(n)$ is undefined. Otherwise, $\mathcal{H}_{\psi,A}^{0-1}(n) = \max\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$. This definition is correct since $\Lambda \in A$, $\psi^a(\Lambda) = 0$ and therefore the set $\{\psi^d(T) : T \in A, \psi^a(T) \leq n\}$ is nonempty.

The function $\mathcal{H}_{\psi,A}^{0-1}$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the minimum weighted depth of nondeterministic decision trees for these tables.

Theorem 23 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . Then the function $\mathcal{H}_{\psi,A}^{0-1}$ is everywhere defined and $\mathcal{H}_{\psi,A}^{0-1}(n) \leq n^2$ for any $n \in \omega$.

Proof. The statement of theorem follows from Lemma 17. \square

5.4. Function $\mathcal{S}_{\psi,A}^{0-1}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete

decision tables from \mathcal{M}_2^{0-1} . We now define a function $\mathcal{S}_{\psi,A}^{0-1}$. Let $n \in \omega$. Then

$$\mathcal{S}_{\psi,A}^{0-1}(n) = \max \{ \psi^s(T) : T \in A, W_\psi(T) \leq n \}.$$

The function $\mathcal{S}_{\psi,A}^{0-1}$ characterizes the growth in the worst case of the minimum weighted depth of strongly nondeterministic decision trees for decision tables from A with the growth of the total weight of attributes attached to columns of these tables.

Theorem 24 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . Then $\mathcal{S}_{\psi,A}^{0-1}$ is an everywhere defined nondecreasing function such that $\mathcal{S}_{\psi,A}^{0-1}(n) \leq n$ for any $n \in \omega$ and $\mathcal{S}_{\psi,A}^{0-1}(0) = 0$. For this function, one of the following statements holds:

(a) If the function W_ψ is bounded from above on the class A , then there exists a positive constant c such that $\mathcal{S}_{\psi,A}^{0-1}(n) \leq c$ for any $n \in \omega$.

(b) If the function W_ψ is not bounded from above on the class A , then there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{S}_{\psi,A}^{0-1}(n)$ for any $n \in \omega$.

The next statement follows directly from Lemma 9.8 [1].

Lemma 25 For any weighted depth ψ and any complete table T from \mathcal{M}_2^{0-1} ,

$$\psi^s(T) \leq \psi^d(T).$$

Proof of Theorem 24. From Theorem 9.7 [1] it follows that $\mathcal{S}_{\psi,A}^{0-1}$ is an everywhere defined nondecreasing function such that $\mathcal{S}_{\psi,A}^{0-1}(n) \leq n$ for any $n \in \omega$ and $\mathcal{S}_{\psi,A}^{0-1}(0) = 0$.

(a) Let the function W_ψ be bounded from above on the class A by a positive constant c . Using Lemmas 3 and 25, we obtain that the function ψ^s is bounded from above on the class A by c . Therefore $\mathcal{S}_{\psi,A}^{0-1}(n) \leq c$ for any $n \in \omega$.

(b) Let the function W_ψ be not bounded from above on the class A . Using Lemma 4, we obtain that the function S_ψ is not bounded from above on the class A . From here and from Theorem 9.7 [1] it follows that there exists an infinite subset D of the set ω such that $H_D(n) \leq \mathcal{S}_{\psi,A}^{0-1}(n)$ for any $n \in \omega$. \square

For the depth h , the bound from the statement (b) of Theorem 24 can be improved.

Theorem 26 Let A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} for which the function W_h is not bounded from above. Then $\mathcal{S}_{h,A}^{0-1}(n) = n$ for any $n \in \omega$.

Proof. Using Lemma 4, we obtain that the function S_h is not bounded from above on the class A . From here and from Theorem 9.8 [1] it follows that $\mathcal{S}_{h,A}^{0-1}(n) = n$ for any $n \in \omega$. \square

5.5. Function $\mathcal{K}_{\psi,A}^{0-1}$

Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . We now define possibly partial function $\mathcal{K}_{\psi,A}^{0-1} : \omega \rightarrow \omega$. Let $n \in \omega$. If the set $\{ \psi^d(T) : T \in A, \psi^s(T) \leq n \}$ is infinite, then the value

$\mathcal{K}_{\psi,A}^{0-1}(n)$ is undefined. Otherwise, $\mathcal{K}_{\psi,A}^{0-1}(n) = \max \{ \psi^d(T) : T \in A, \psi^s(T) \leq n \}$. This definition is correct since $\Lambda \in A$, $\psi^s(\Lambda) = 0$ and therefore the set $\{ \psi^d(T) : T \in A, \psi^s(T) \leq n \}$ is nonempty. From here it follows that if the value $\mathcal{K}_{\psi,A}^{0-1}(n)$ is defined, then $\mathcal{K}_{\psi,A}^{0-1}(n) \geq 0$.

The function $\mathcal{K}_{\psi,A}^{0-1}$ characterizes the growth in the worst case of the minimum weighted depth of deterministic decision trees for decision tables from A with the growth of the minimum weighted depth of strongly nondeterministic decision trees for these tables.

We now remind the definition of possibly partial function $\Phi_{\psi,A} : \omega \rightarrow \omega$. Let $n \in \omega$. If the set $\{ |At(T)| : T \in A, m_\psi(T) \leq n \}$ is infinite, then the value $\Phi_{\psi,A}(n)$ is undefined. Otherwise, $\Phi_{\psi,A}(n) = \max \{ |At(T)| : T \in A, m_\psi(T) \leq n \}$. This definition is correct since $\Lambda \in A$, $m_\psi(\Lambda) = 0$ and therefore the set $\{ |At(T)| : T \in A, m_\psi(T) \leq n \}$ is nonempty. It is clear that $\{ T : T \in A, m_\psi(T) \leq 0 \} = \{ \Lambda \}$. Therefore $\Phi_{\psi,A}(0) = 0$.

The following statement describes a criterion for the function $\mathcal{K}_{\psi,A}^{0-1}$ to be everywhere defined.

Theorem 27 Let ψ be a weighted depth and A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} . Then the function $\mathcal{K}_{\psi,A}^{0-1}$ is everywhere defined if and only if the function $\Phi_{\psi,A}$ is everywhere defined.

First, we prove two auxiliary statements.

Lemma 28 Let ψ be a weighted depth, A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} , and the function $\Phi_{\psi,A}$ be everywhere defined. Then the function $\mathcal{K}_{\psi,A}^{0-1}$ is everywhere defined and $\mathcal{K}_{\psi,A}^{0-1}(n) \leq n\Phi_{\psi,A}(n)$ for any $n \in \omega$.

Proof. Let $n \in \omega$ and T be a table from the class A for which $\psi^s(T) \leq n$. Let us show that $\psi^d(T) \leq n\Phi_{\psi,A}(n)$. If T is empty or has a common decision, then, as it is easy to show, $\psi^d(T) = 0$ and the considered inequality holds.

Let T be a nonempty table without common decisions and Γ be a strongly nondeterministic decision tree for T such that $\psi(\Gamma) = \psi^s(T)$. It is clear that $At(\Gamma) \neq \emptyset$. Let $At(\Gamma) = \{ f_{i_1}, \dots, f_{i_m} \}$. We now define a deterministic decision tree G for the table T . This tree sequentially computes values of the attributes f_{i_1}, \dots, f_{i_m} . The set of words $\pi(\tau)$ corresponding to complete paths τ of G coincides with the set $\{ (f_{i_1}, \delta_1) \dots (f_{i_m}, \delta_m) : \delta_1, \dots, \delta_m \in E_2 \}$. It is clear that, for any row of T , there exists a complete path τ of G such that this row belongs to the subtable $T(\tau)$. Let us consider an arbitrary complete path τ of G with $\pi(\tau) = (f_{i_1}, \delta_1) \dots (f_{i_m}, \delta_m)$. If there exists a complete path ξ of Γ such that the word $\pi(\tau)\pi(\xi)$ is consistent, then the terminal node of τ is labeled with the decision 1. Otherwise, this node is labeled with the decision 0. It is easy to check that the subtable $T(\tau)$ is nonempty and the decision attached to the terminal node of τ is the common decision for this subtable. Thus, the tree G is a deterministic decision tree for the table T .

Since $\psi(\Gamma) = \psi^s(T) \leq n$, for any $f_{i_j} \in At(\Gamma)$, $\psi(f_{i_j}) \leq n$. Using the fact that A is a closed class, we obtain $|At(\Gamma)| \leq \Phi_{\psi,A}(n)$. Therefore $\psi(G) \leq n\Phi_{\psi,A}(n)$

and $\psi^d(T) \leq n\Phi_{\psi,A}(n)$. Thus, for any table $T \in A$ such that $\psi^s(T) \leq n$, the inequality $\psi^d(T) \leq n\Phi_{\psi,A}(n)$ holds. As a result, we obtain that the value $\mathcal{K}_{\psi,A}^{0-1}(n)$ is defined and $\mathcal{K}_{\psi,A}^{0-1}(n) \leq n\Phi_{\psi,A}(n)$. \square

Lemma 29 For any $t \in \omega \setminus \{0\}$, there exists a complete decision table T from \mathcal{M}_2^{0-1} such that $|\text{At}(T)|=t$, $h^s(T) \leq 1$ and $h^d(T) \geq t$.

Proof. Let $t \in \omega \setminus \{0\}$. We denote by T a complete decision table from \mathcal{M}_2^{0-1} with t columns labeled with the attributes f_1, \dots, f_t in which the row $(0, \dots, 0) \in E_2^t$ is labeled with the set of decisions $\{0\}$ and all other rows are labeled with the set of decisions $\{1\}$.

We denote by Γ a strongly nondeterministic decision tree for T in which the set of words $\pi(\tau)$ for complete paths τ of Γ coincides with the set $\{(f_i, 1) : i = 1, \dots, t\}$ and the terminal node of each complete path is labeled with the decision 1. One can show that, for each row $\bar{\delta}$ of the table T labeled with the set of decisions $\{1\}$, there exists a complete path τ of Γ , for which the row $\bar{\delta}$ belongs to the subtable $T(\tau)$, and, for each complete path τ of Γ , 1 is the common decision for the subtable $T(\tau)$. It is clear that $h(\Gamma) = 1$. Therefore $h^s(T) \leq 1$.

Let G be a deterministic decision tree for the table T for which $h(G) = h^d(T)$. Then G contains a complete path ξ such that the row $(0, \dots, 0)$ of the table T belongs to the subtable $T(\xi)$. Then the terminal node of ξ is labeled with the decision 0 and 0 is the common decision for the table $T(\xi)$. We now show that the set of letters of the word $\pi(\xi)$ coincides with the set $\{(f_i, 0) : i = 1, \dots, t\}$. Assume the contrary: a letter $(f_i, 0)$ for some $i \in \{1, \dots, t\}$ does not belong to the set of letters of the word $\pi(\xi)$. Then, evidently, the row $(0, \dots, 1, \dots, 0)$ containing only one unit in the i th digit belongs to the subtable $T(\xi)$ but this is impossible. Therefore the length of the word $\pi(\xi)$ is at least t , $h(G) \geq t$ and $h^d(T) \geq t$. \square

Proof of Theorem 27. Let the function $\Phi_{\psi,A}$ be everywhere defined. Then, by Lemma 28, the function $\mathcal{K}_{\psi,A}^{0-1}$ is everywhere defined.

Let the function $\Phi_{\psi,A}$ be not everywhere defined. Then there exists a number $n \in \omega \setminus \{0\}$ for which the value $\Phi_{\psi,A}(n)$ is undefined. From here and from Lemma 29 it follows that, for any $t \in \omega \setminus \{0\}$, there exists a complete decision table $Q \in A$ such that $m_\psi(Q) \leq n$, $|\text{At}(Q)|=t$, $h^s(Q) \leq 1$ and $h^d(Q) \geq t$. Since $m_\psi(Q) \leq n$ and $h^s(Q) \leq 1$, $\psi^s(Q) \leq n$. It is clear that $\psi^d(Q) \geq t$. As a result, we obtain that the set $\{\psi^d(T) : T \in A, \psi^s(T) \leq n\}$ is infinite and the value $\mathcal{K}_{\psi,A}^{0-1}(n)$ is undefined. \square

The next statement clarifies the behavior of everywhere defined function $\mathcal{K}_{\psi,A}^{0-1}$.

Theorem 30 Let ψ be a weighted depth, A be a nontrivial closed class of complete decision tables from \mathcal{M}_2^{0-1} , and the function $\mathcal{K}_{\psi,A}^{0-1}$ be everywhere defined. Then

- (a) For any $n \in \omega$, $\Phi_{\psi,A}(n) \leq \mathcal{K}_{\psi,A}^{0-1}(n) \leq n\Phi_{\psi,A}(n)$.
- (b) A polynomial p such that $\mathcal{K}_{\psi,A}^{0-1}(n) \leq p(n)$ for any $n \in \omega$ exists if and only if there exists a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$.

Proof. (a) Using Theorem 27, we obtain that the function $\Phi_{\psi,A}$ is everywhere defined. From here and from Lemma 28 it follows that $\mathcal{K}_{\psi,A}^{0-1}(n) \leq n\Phi_{\psi,A}(n)$ for any $n \in \omega$.

We now show that $\mathcal{K}_{\psi,A}^{0-1}(n) \geq \Phi_{\psi,A}(n)$ for any $n \in \omega$. If $\Phi_{\psi,A}(n) = 0$, then, evidently, $\mathcal{K}_{\psi,A}^{0-1}(n) \geq \Phi_{\psi,A}(n)$. Let $\Phi_{\psi,A}(n) > 0$ and $t = \Phi_{\psi,A}(n)$. From Lemma 29 it follows that there exists a complete decision table $Q \in A$ such that $m_{\psi}(Q) \leq n$, $|\text{At}(Q)| = t$, $h^s(Q) \leq 1$ and $h^d(Q) \geq t$. Since $m_{\psi}(Q) \leq n$ and $h^s(Q) \leq 1$, $\psi^s(Q) \leq n$. It is clear that $\psi^d(Q) \geq t$. Therefore $\mathcal{K}_{\psi,A}^{0-1}(n) \geq t = \Phi_{\psi,A}(n)$.

(b) Let there exist a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$. From part (a) of the theorem statement it follows that $\mathcal{K}_{\psi,A}^{0-1}(n) \leq nq(n)$ for any $n \in \omega$. Therefore there exists a polynomial p such that $\mathcal{K}_{\psi,A}^{0-1}(n) \leq p(n)$ for any $n \in \omega$.

Let there be no a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$. We now show that there is no a polynomial p such that $\mathcal{K}_{\psi,A}^{0-1}(n) \leq p(n)$ for any $n \in \omega$. Assume the contrary: there exists a polynomial p such that $\mathcal{K}_{\psi,A}^{0-1}(n) \leq p(n)$ for any $n \in \omega$. From part (a) of the theorem statement it follows that $\mathcal{K}_{\psi,A}^{0-1}(n) \geq \Phi_{\psi,A}(n)$ for any $n \in \omega$. Therefore $p(n) \geq \Phi_{\psi,A}(n)$ for any $n \in \omega$. Thus, there exists a polynomial q such that $\Phi_{\psi,A}(n) \leq q(n)$ for any $n \in \omega$, but this is impossible. \square

6. Conclusion

In this paper, we considered three types of closed classes of binary complete decision tables with many-valued decisions and studied for tables from these classes functions describing relationships among the minimum weighted depth of deterministic decision trees, the minimum weighted depth of nondeterministic decision trees, the minimum weighted depth of strongly nondeterministic decision trees, and the total weight of attributes attached to columns of the tables. Our goal was to simplify the criteria for the behavior of these functions compared to the criteria considered in the book [1] and to obtain new results for relationships between deterministic and nondeterministic decision trees. The obtained results can be useful for the study of various problems of transforming decision rule systems with binary attributes into decision trees [1]. In the future, we plan to conduct similar studies for closed classes of k -valued complete decision tables, where $k > 2$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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