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Approximation of the Viswanath's Constant in Closed Form

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Abstract

The calculation of this constant requires advanced numerical methods such as interval mathematics, using specialized software for high accuracy and tight error limits. In this article, we present a simpler method for deriving the Viswanath's constant.

Keywords

Viswanath's Constant, Closed Form, Analytical Approximation, Control Function

1. Introduction

The Fibonacci sequence is a famous mathematical sequence where each number is the sum of the two preceding ones. Fibonacci, born around 1170 and originally known as Leonardo of Pisa, published a large volume in 1202, Liber Abaci, a mathematical book that first mentions this sequence, although it was actually known to Indian mathematicians 200 years before Christ. Fibonacci numbers occur in biology in the branching of trees, the arrangement of plant leaves, the shape of shells, the arrangement of seeds in a pine cone, etc. The ratio of two consecutive Fibonacci numbers approaches the value of the golden ratio 1.61803…, which plays an important role in architecture, anatomy and art. Fibonacci numbers are also used in computer algorithms and graph creation, game theory, coding and even the creation of pseudo-random numbers.

The Random Fibonacci Sequence (RFS) is different; it is a stochastic analogue of the Fibonacci sequence, where numbers are randomly added or subtracted with the same probability of 0.5. Intuitively, we would expect that with chaotic and random growth, negative and positive numbers would cancel each other out and

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the growth rate of this sequence would be zero, but this is not the case. In 1963, Harry Furstemberg proved that the RFS is growing and the rate of this growth is definitely greater than 1. It was not until 1999 that D. Viswanath proved that the rate of this growth is exponential and approaches the value 1.1319882…

Divakar Viswanath received his doctorate from Cornell University in 1998. His main research areas include numerical analysis, nonlinear dynamics, and scientific computing. Since 2016, he has also been involved in mathematical genetics, specifically coalescence theory. He currently serves as a professor of mathematics at the University of Michigan. Viswanath redefined RFS as a product of random matrices. Each Fibonacci number is multiplied by one of the random matrices, rotating its coordinates, and the Fibonacci number acquires a new slope (a line connecting the new coordinates of the point to the origin of the coordinate system) after this rotation. The slope in these formulas can take any value along the real number line, from minus infinity to plus infinity, but not all slopes are equally likely. The key to understanding a random walk—and also to calculating the growth rate of the Fibonacci series—is to identify the probability distribution that determines the probability of each possible slope. Viswanath's main contribution was to find a way to estimate this distribution with any desired degree of accuracy.

The problem is that this distribution is discontinuous and looks like a fractal landscape. Viswanath therefore used a structure called the Stern–Brocot tree, with which he managed to discretely divide the number line of real numbers. Viswanath summarized all paths through the Stern–Brocot tree into a table to a depth of 28 levels, where the tree has more than 50 million nodes. In this way, he was able to calculate the value of the corresponding constant with an accuracy of eight decimal places. His result is $C = 1.13198824$ ... In his honor, this constant was named the Viswanath constant. Very significant research was done in 1999 by researchers M. Embree and L. Trefethen, who investigated the properties of the RFS by multiplying the second Fibonacci number by the beta factor. If this factor was 0.5 or less, the resulting value of the sequence was less than one and decreased sharply. If it was 1, the result was the known value 1.1319882… and the value grew exponentially. At exactly beta = 0.70258, the system neither decreased nor grew, but remained stable. This was a very important result in the field of RFS and also in the Viswanath constant.

Another relevant paper was published by J. Oliveira and L. De Figuereido in 2002. They used interval arithmetic to calculate the constant, avoiding floating-point error analysis. In 2006, researchers E. Makover and J. McGowan provided a more accessible, qualitative argument for the exponential growth of these sequences, avoiding the complex calculations of fractal measures used in the original proof.

Significant research has been done by Karyn McLellan, who developed a simpler method for calculating Viswanath's constant. While Viswanath originally calculated the constant to 8 decimal places using complex upper and lower bounds, McLellan's approach—based on the Kalmár-Nagy formula and Rittaud tree reduction—replicated this accuracy and aimed to improve it further. In her followup work in 2013, she extended Viswanath's ideas by studying sequences where the pattern of addition and subtraction is periodic rather than random.

2. Fibonacci Sequence and Random Fibonacci Sequence

We define the Fibonacci sequence for $n > 1$

$$F_{(n)} = F_{(n-1)} + F_{(n-2)} \tag{1}$$

Using Binet's relation, we get:

$$F_{(n)} = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{2}$$

For the golden ratio and its conjugate, the following applies:

$$\varphi = \frac{1}{2}\left(1+\sqrt{5}\right) = 1.61803\cdots$$

$$\psi = \frac{1}{2}\left(1-\sqrt{5}\right) = -0.61803\cdots \tag{3}$$

A random Fibonacci sequence is given by the recurrence relation $f(n-1) \pm f(n-2)$, where the signs + or – are chosen randomly, with equal probability independent for different n. In 1999, Divakar Vishwanath proved that the growth rate of a random Fibonacci sequence is equal to 1.131988248 [1]. In the same year, it was proved that the sequence (4)

$$f_n = \pm f_{n-1} \pm \beta f_{n-2} \tag{4}$$

also decreases if β is less than or approximately equal to 0.70258 [2]. This value is known as the Embree Trefethen constant. It has also been shown that the asymptotic ratio σ(в) between consecutive terms converges for each value of β with a local minimum approximately equal to 0.36747. σ represents the Lyapunov coefficient. Random Fibonacci sequences (RFS) represent a complex system – causality controlled by chance (probability). Random Hermitian matrices (RHM) are used to study it, which play a role both in the distribution of prime numbers and in determining the quantum energy levels of atoms. They also occur in the Riemann zeta function. Now it is necessary to define the Fibonacci zeta function $\xi_{f(s)}$.

$$\xi_{f(s)} = \sum_{n=1}^{\infty} \left(\frac{1}{F_n}\right)^s \tag{5}$$

s can be both real or complex. For all complex numbers s, with real part greater than zero, the function converges [3]. The value for $s = 1$ is known and this is the reciprocal of the Fibonacci constant with the value 3.359…, which is probably an irrational number. The value for $s = 0.757…$ is also known and this has the value 4.

3. Control Function $g(x)$ and Derivation of Viswanath's Constant

We write the function $g(x)$ as the sum of all integer dimensions in the interval 0 to infinity.

$$g_{(x)} = x^0 + x^1 + x^2 + \cdots + x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (6)$$

Only the interval (−1, 1) is of interest to us. **Figure 1** shows the graph of this function.

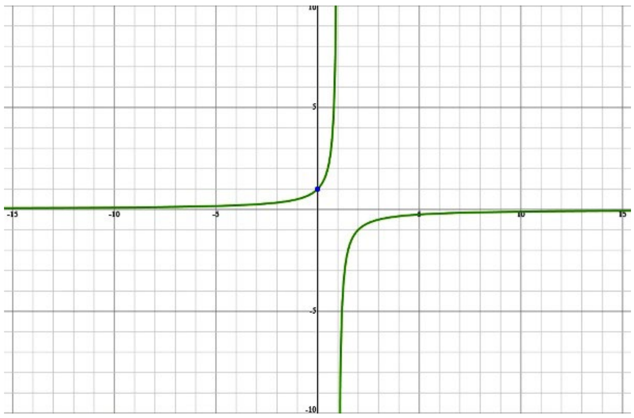

Figure 1. Graph of the function $g(x)$.

This function has the following properties [4] applies:

$$\int_{-1}^{0} g_{(x)}\mathrm{d}x = \int_{0}^{0.5} g_{(x)}\mathrm{d}x = \ln 2 \quad (7)$$

$$\int_{-1}^{0} g_{(x)}^2\mathrm{d}x = \int_{0}^{0.5} g_{(x)}^2\mathrm{d}x = 1 \quad (8)$$

$$\int_{-1}^{0} \sqrt{1+\left(\frac{\mathrm{d}g_{(x)}}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{0}^{0.5} \sqrt{1+\left(\frac{\mathrm{d}g_{(x)}}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 1.13209\cdots \quad (9)$$

Both intervals have the same area, the same normalization (we assume a causal connection there) and the same length of the curve and in the case of the second interval we assume that it is a Viswanath's constant. The increment is related to the initial value of the function $g(x)$ which is equal to one at point zero.

There are several clues to this. The Embree Trefethen constant -see (4) has a value of 0.70258. If we put this value as the x-coordinate in the function $g(x)$ in relation (6), we get a value of 3.362 which is very close to the value of 3.359…, the reciprocal of the Fibonacci constant, which represents the value of the Fibonacci zeta function for $s = 1$.

Furthermore, if we substitute the value of 0.382… into equation (6) for x, we get the value of the golden ratio 1.618… The value of 0.382 is close to the beta value of the local minimum of 0.367… And finally, at $s = 0.757…$, the value of the Fibonacci zeta function is 4, which is close to our function, which has a value of 4.11 at the point 0.757. All this shows that the function $g(x)$ is closely related to the Fibonacci series. Finally, we repeat the derivation of Viswanath's constant.

$$f_{(n)} = f_{(n-2)} \pm f_{(n-1)} \quad (10)$$

$$g_{(x)} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (11)$$

$$\lim_{n\to\infty}\left|f_{(n)}\right|^{\frac{1}{n}}=\int_0^{0.5}\sqrt{1+\left(\frac{\mathrm{d}g_{(x)}}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=1.13209\cdots \tag{12}$$

The control function $g(x)$ has an interesting relationship to the Riemann zeta function (see **Figure 2**)

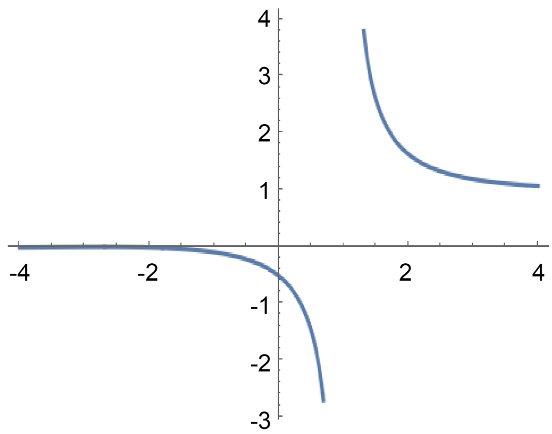

Figure 2. Graph of the Riemann zeta function.

In the interval (0,1) the following holds:

$$g_{(x)}=-\left(\zeta_{(s)}+\sigma_{(s)}\right) \tag{13}$$

The second term on the right is a small deviation depending on s.

There are known relationships between the zeta function, the Moebius function, and the Euler function. Interestingly, the Euler and Moebius functions can be expressed using the golden ratio [5].

We express the golden ratio as t. Then:

$$t=2\cos\left(\frac{\pi}{5}\right)=-\sum_{n=1}^{\infty}\frac{\varphi_{(n)}}{n}\log\left(1-\frac{1}{t^n}\right) \tag{14}$$

$$\frac{1}{t}=2\sin\left(\frac{\pi}{10}\right)=-\sum_{n=1}^{\infty}\frac{\mu_{(n)}}{n}\log\left(1-\frac{1}{t^n}\right) \tag{15}$$

In the article, we derived the value of Viswanath's constant equal to 1.13209…, using the control function $g(x)$. This value lies between the value calculated by Viswanath (1.1319882...) and the value printed by Eric Weisstein (1.13215069...) [6]. Finally it is valid:

$$\lim_{n\to\infty}\left|f_n\right|^{\frac{1}{n}}=\int_0^{0.5}\sqrt{1+\left(\frac{\mathrm{d}\sum_{n=0}^{\infty}x^n}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=1.13209\cdots \tag{16}$$

Viswanath expressed his constant as an exponential integral using a special invariant measure on Stern-Brocot intervals. If its formula is identical to our relation (16) we have Viswanath's constant in the closed form. There is also the possibility that the Viswanath's constant corresponds to some fractal dimension of an as yet unknown shape (for instance a fractal dimension equal to 1.26 is known…).

4. Conclusions

A chaotic system is deterministic, meaning it has its own exact rules, but its behavior is special. Sometimes, when it is governed by nonlinear equations, a sensitive dependence on the initial conditions appears. Chaotic systems are nonlinear and, unlike systems of linear differential equations, cannot be solved analytically. But it is probabilistic and statistical methods, together with powerful computers, that will help us find out a lot about chaotic systems even without knowing the exact solutions. We can only approximate it numerically (solve a complex mathematical problem with successive approximation steps and draw what the solutions will look like). In the case of chaotic systems, this could mean that although we cannot predict them, we know the attractors around which they will move. We do not know exactly what will happen in the long term, but we do know a lot about what can and cannot happen. Even if the typical behavior of the system is irregular, the system remains in the area defined by its attractor. And this is until some external intervention changes the system itself. Such changes are expressed by bifurcation analysis. After adjusting the system parameter, a chaotic system can become a system with periodic behavior. Chaotic systems usually have fractal attractors, because typical chaotic trajectories are aperiodic and must move in such a way that they do not cycle, do not intersect and at the same time remain in a bounded region. As a result, the attractor of the system is a complicated structure that is a fractal.

Finding the attractor of a system that appears chaotic may not be entirely simple. The systems under investigation may also be multidimensional, and thus attractors are defined not only as curves in a surface but also as multidimensional sections through a multidimensional space, for example. If we manage to create an attractor for a system that behaves seemingly randomly, then this system is chaotic (deterministically chaotic). A characteristic feature of chaotic systems is that they occur partly in cycles. Unlike classical cyclic systems, where the attractor is formed by a single closed curve, the attractor curve of chaotic systems does not have to be closed. Nevertheless, using the created attractor, we can predict the development of a chaotic system in the future with some accuracy. This knowledge is used in chaos management. If the dimension of the space in which we are looking for an attractor exceeds a certain number, then we classify the phenomenon under study as stochastic (random) phenomena only because it is practically impossible to determine the attractor here. For these phenomena, we have to make do with statistics for now.

No system is completely deterministic, because there are always factors outside it (e.g. in its structure) that influence the behavior of the system. Randomness, as follows from this axiom of non-isolation, is omnipresent and irreducible. It cannot be eliminated. Mathematicians would like to, but it lurks behind all the axioms that are a necessary condition for every logical and consistent system. After all, it all comes back to them, and in a highly exact way, in the form of Gödel's incompleteness theorem. Undecidable statements clearly exist and defy precise rules. So, chance is all around us, but we manage it well. You can always find a certain determinism and sometimes even almost 100% determinism. Absolute determinism, however, is only in the dreams of mathematicians. Reality is always richer than a mathematical model, even if it describes a reality that we do not yet know.

Determinism is a very strong and attractive concept for physicists and mathematicians, but it is apparently unrealistic. If we assume that the world as a whole is infinite, the idea arises that every system is holistically determined by infinitely many factors, the whole world, of which it is understandable that we can determine only a finite number of factors and we are left with chance. That chance is an infinite number of neglected, from our point of view insignificant factors. However, every chance can be analyzed into deterministic processes, if our abilities are sufficient for this (and therefore the system is not on the knowledge horizon). It is possible to imagine that an infinite whole is absolutely precisely determined inside, that is, every part of it is determined in this way. It is not mechanical determinism, because it always requires a finite number of factors to determine the future, or rather a finite structural level (Laplace's demon), which is impossible. However, an infinite whole, somehow by its definition, has no surroundings, because it is all-encompassing, and therefore is completely random from the outside.

If the whole is absolutely determined and completely random at the same time, it only shows that the categories of determinism and chance cease to apply here. This perfect reconciliation of all opposites in infinity, where we will never see, because all our abilities are only finite. The world known to us is, was and always will be only finite; the all-encompassing world is infinite in everything, or it has no property known to us.

And what about freedom of will? Is it in danger? From the nature of the solution to potentially absolute determinism, of course, yes, but practically much less so, because we have little chance of finding out that free will is not free. Real free will remains with us. This is because our consciousness so far resists understanding about as much as the entire universe or quantum mechanics. Thanks to our inability, we will still think for a long time that free will exists, that it is impossible to go below the level of quantum mechanics and that our universe is all that exists. The reality of our free will is supported by the fact that our entire reality is just a model of the surrounding environment, and finally a model of current infinity, and as follows from the definition of infinity itself, when modeling it we had to make infinitely many simplifications, or the real reality = current infinity, differs from our (scientific) reality in such a way that it actually has nothing to do with it.

Deterministic chaos, in contrast to quantum mechanics, as a potential carrier of absolute randomness, is theoretically falsifiable. It is enough to realize that its quasi-random processes are calculated by precise, repeatable algorithms. In order to produce randomness, they would have to respond in practice to a zero change in the input with a non-zero change in the output. The refutation of randomness in deterministic chaos is surprisingly quantum mechanics, which says that the world is quantized, so there is no such thing as zero change; there is always only a non-zero change. Moreover, it can also be shown mathematically that absolutely zero change is nonsense, because mathematics avoids it and solves it with the concept of a limit approaching zero.

Well, one of the surprising and effective mathematical tools for studying deterministic chaos is the Viswanath constant. It is actually a universal and mathematically exact growth factor for a wide class of random processes. In conclusion, I would like to mention the progress that has been made in recent years in the study of this interesting constant.

One of the papers proved that there is almost certain exponential growth for random Fibonacci sequences with $\{\pm 1\}$ Bernoulli coefficients, thus returning to Viswanath's original problem [7]. Another article studies generalized random Fibonacci sequences with reference to Viswanath's seminal work on the constant [8].

An exceptionally good article is K. McLellan's 2013 paper [9]. The paper addresses the problem of what happens to random Fibonacci sequences when randomness is removed; specifically, coefficients are selected that belong to the set $\{1, -1\}$ and form periodic cycles. By rewriting the recurrences using matrix products, the sequence growth will be analytic and criteria based on eigenvalue, trace, and order will be selected to determine whether a given sequence is bounded, linearly growing, or exponentially growing. Furthermore, an equivalence relation will be introduced on the coefficient cycles so that each equivalence class has a common growth rate and there will be a significant number of such classes for a given cycle length.

In addition, attempts are constantly being made to find a closed form of the Viswanath constant.

Note

There is also a physical interpretation of the function $g(x)$ and its relationship to the Viswanath constant. The function $g(x)$ can be considered as a wave function very well normalized in the interval $(-1, 1)$. The point $x= 0$ represents the present, the interval $(-1, 0)$ the causal past, and the interval $(0, 0.5)$ the causal future. Although chance plays a large role in RFS, causality is always present, due to its the predictable rate of its growth expressed by the Viswanath constant. It is actually the length of the curve bounding the causal region of the function $g(x)$.

Let us consider a system where two options, + and − are decisive. This system has Shannon entropy ln2. If we find a curve that in the interval (a, b) bounds the area ln2 and at the same time this curve is normalized to 1, then we claim that the length of this curve in the interval (a,b) exactly approximates Viswanath's constant.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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