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κ-Deformed Exponentials, Minimal Length and Renormalization Group Flow

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Abstract

The κ-deformed exponential $\exp_\kappa(x) = \exp\left(\frac{1}{\kappa}\operatorname{arcsinh}(\kappa x)\right)$ studied by Kaniadakis [1]-[3] allows to construct "deformed" Lorentz transformations associated with the ordinary velocity boost rapidity parameter ξ and which can be recast in terms of ordinary Lorentz transformations (involving the ordinary exponential) but associated with a κ-deformed (modified) rapidity parameter $\xi_\kappa = \xi f(\kappa\xi)$ given by a ξ-dependent scaling of the original ξ rapidity parameter. It is shown that when *both* the κ parameter and $\xi \to \infty$, and the double scaling limit $\xi\frac{\ln(2\kappa\xi)}{\kappa\xi} = \infty \times 0$ is finite and non-zero, it leads to a *finite* value for the κ-deformed boost rapidity parameter $\xi_{\kappa=\infty} = \xi_\infty \neq \infty$, such that the κ-deformed velocity (in units of $c = 1$) $\tanh(\xi_\kappa) = v_\kappa < 1$ is *less* than the speed of light, and in turn, the Lorentz dilation factor $\gamma(v_\kappa) \neq \infty$ *no* longer *blows* up. Consequently, there is a *lower* bound in the length $L' = \frac{L}{\gamma(v_k)} \neq 0$ due to a *finite* Lorentz length contraction. After imposing that the lower bound L' should not be smaller than the postulated minimum Planck scale one arrives at the length-scale-dependent relation $\operatorname{arccosh}\left(\frac{L}{L_P}\right) = \xi_\infty > 0$ that admits a physical interpretation analogous to the running of the physical couplings and masses with the energy scale in the Renormalization Group program in Quantum Field Theory.

Keywords

Kaniadakis Entropy, κ-Exponentials, Deformed Lorentz Transformations, Minimal Length, Renormalization Group

1. On Kaniadakis κ-Exponentials and Deformed Lorentz Transformations

Relativistic statistical mechanics based on κ-entropy has been shown by Kaniadakis [1]-[3] to preserve the main features of classical statistical mechanics (kinetic theory, molecular chaos hypothesis, maximum entropy principle, thermodynamic stability, H-theorem, and Lesche stability). Many old open problems of relativistic physics, such as how thermodynamic quantities like temperature and entropy vary with the speed of the reference frame, have found answers in the κ-statistical theory described by [1]-[3]. The Boltzmann-Gibbs-Shannon (BGS) entropy is a special case of the more general class of entropic functions

$$S_\kappa = -\sum_i p_i \ln_\kappa\left(p_i\right) \tag{1}$$

involving the function $\ln_\kappa$ that can be regarded as a one-parameter generalization of the ordinary logarithm and where p_i represents the probability that the system is in the microstate i with $\sum_{i=1}^{W} p_i = 1$. The generalized logarithm is an arbitrary strictly increasing function that is negative on the interval $0 < p_i < 1$. Only the standard BGS entropy corresponding to $\ln\left(p_i\right)$ and the κ-entropy S_κ corresponding to the κ-logarithm $\ln_\kappa\left(p_i\right)$ obey the scaling and self-duality axioms simultaneously [1]-[3].

The κ-logarithm was defined as

$$\ln_\kappa\left(p_i\right) = \frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa} = \frac{1}{\kappa}\sinh\left(\kappa \ln p_i\right) \tag{2}$$

The free parameter that appears in the expression of the κ-logarithm varied in the range of $0 < \kappa < 1$ and in the $\kappa \to 0$ limit the κ-logarithm $\ln_\kappa\left(p_i\right)$ reduced to the ordinary logarithm $\ln\left(p_i\right)$.

Given the one-parameter generalized logarithm function of the variable x [1]-[3]

$$\ln_\kappa\left(x\right) \equiv \frac{x^\kappa - x^{-\kappa}}{2\kappa} = \frac{1}{\kappa}\sinh\left(\kappa \ln x\right) \tag{3}$$

its inverse function is the one-parameter generalized exponential

$$\exp_\kappa\left(x\right) \equiv \exp\left(\frac{1}{\kappa}\operatorname{arcsinh}\left(\kappa x\right)\right) \tag{4a}$$

The $\operatorname{arcsinh}\left(\kappa x\right)$ function can also be expressed in terms of the logarithm such that Equation (4a) can be rewritten as

$$\exp_\kappa\left(x\right) = \left(\sqrt{1+\kappa^2 x^2} + \kappa x\right)^{1/\kappa} = \exp\left(\frac{1}{\kappa}\ln\left(\sqrt{1+\kappa^2 x^2} + \kappa x\right)\right) \tag{4b}$$

such that when $\kappa \to 0$ one recovers the ordinary logarithm and exponential functions.

One of the salient features of the $\ln_\kappa\left(x\right)$ function is the asymptotic power law behaviour. When $x \ll 1$, the negative power dominates $-x^{-\kappa}$ so that $\ln_\kappa\left(x\right) < 0$. And when $x \gg 1$, the positive power dominates x^κ. Furthermore, straightforward algebra reveals

$$\ln\left(-y+\sqrt{(-y)^2+1}\right)=\ln\left(\frac{1}{y+\sqrt{y^2+1}}\right)=-\ln\left(y+\sqrt{y^2+1}\right)$$
$$\Rightarrow \operatorname{arcsinh}(-y)=-\operatorname{arcsinh}(y) \tag{4c}$$

From Equations-(4a, 4c) one infers that

$$\exp_{-\kappa}(x)\equiv\exp\left(-\frac{1}{\kappa}\operatorname{arcsinh}(-\kappa x)\right)=\exp\left(\frac{1}{\kappa}\operatorname{arcsinh}(\kappa x)\right)=\exp_\kappa(x) \tag{4d}$$

so the range of the κ parameter can be chosen to be $0\le\kappa\le\infty$.

Given $\sinh(X)=\kappa x$; $\sinh(Y)=\kappa y$, from the addition laws

$$\begin{aligned}&\sinh(X+Y)=\sinh(X)\cosh(Y)+\cosh(X)\sinh(Y)\\&\Rightarrow X+Y=\operatorname{arcsinh}(\kappa x)+\operatorname{arcsinh}(\kappa y)\\&=\operatorname{arcsinh}\left(\sinh(X)\cosh(Y)+\cosh(X)\sinh(Y)\right)\\&=\operatorname{arcsinh}\left(\kappa x\sqrt{1+\kappa^2y^2}+\kappa y\sqrt{1+\kappa^2x^2}\right)\end{aligned} \tag{5}$$

one can derive that the product of two one-parameter generalized exponentials is of the form

$$\exp_\kappa(x)\exp_\kappa(y)=\exp_\kappa(x\oplus_\kappa y)=\exp\left(\frac{1}{\kappa}\operatorname{arcsinh}\left[\kappa(x\oplus_\kappa y)\right]\right) \tag{6}$$

and where the deformed addition law of the variables is given by

$$x\oplus_\kappa y\equiv x\sqrt{1+\kappa^2y^2}+y\sqrt{1+\kappa^2x^2} \tag{7}$$

such that when $\kappa\to0$, the product of the κ exponentials $\exp_\kappa$ reduces to the ordinary exponential product since $x\oplus_\kappa y\to x+y$, and $\operatorname{arcsinh}(z)\to z$ when $z\to0$.

The Lorentz transformations in units of $c=1$ can be written in terms of the velocity boost rapidity parameter ξ as

$$t'=t\cosh(\xi)-x\sinh(\xi);\ \ x'=x\cosh(\xi)-t\sinh(\xi) \tag{8}$$

with

$$\tanh(\xi)=v,\ \ \cosh(\xi)=\gamma(v)=\left(1-v^2\right)^{-\frac{1}{2}},\ \ \sinh(\xi)=v\gamma(v) \tag{9}$$

such that the spacetime interval $t^2-x^2=t'^2-x'^2$ remains invariant and which results from the identity $\cosh^2(\xi)-\sinh^2(\xi)=1$.

The hyperbolic functions $\cosh(x),\sinh(x),\tanh(x)$ are defined in terms of the ordinary exponentials

$$\cosh(x)=\frac{\exp(x)+\exp(-x)}{2},\ \ \sinh(x)=\frac{\exp(x)-\exp(-x)}{2},$$
$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)} \tag{10}$$

In view of the definition of the one-parameter generalized exponential we are going to "deform" the above Lorentz transformations (8, 9) by introducing the "deformed" hyperbolic functions

$$\cosh_\kappa(x) = \frac{\exp_\kappa(x) + \exp_\kappa(-x)}{2},\ \sinh_\kappa(x) = \frac{\exp_\kappa(x) - \exp_\kappa(-x)}{2},$$
$$\tanh_\kappa(x) = \frac{\sinh_\kappa(x)}{\cosh_\kappa(x)} \tag{11}$$

and leading to the "deformed" Lorentz transformations ($c = 1$)

$$t' = t\cosh_\kappa(\xi) - x\sinh_\kappa(\xi);\ x' = x\cosh_\kappa(\xi) - t\sinh_\kappa(\xi) \tag{12}$$

with

$$\tanh_\kappa(\xi) = v_\kappa,\ \cosh_\kappa(\xi) = \gamma(v_\kappa) = \left(1 - v_\kappa^2\right)^{-\frac{1}{2}},\ \sinh_\kappa(\xi) = v_\kappa\gamma(v_\kappa) \tag{13}$$

Due to the relation

$$x \ominus_\kappa x = x \oplus_\kappa (-x) = x\sqrt{1+\kappa^2x^2} - x\sqrt{1+\kappa^2x^2} = 0 \tag{14}$$

one has

$$\exp_\kappa(x)\exp_\kappa(-x) = \exp_\kappa\left(x \oplus_\kappa (-x)\right) = \exp_\kappa(0) = 1 \tag{15}$$

such that the spacetime interval $t^2 - x^2 = t'^2 - x'^2$ still remains *invariant* under the deformed Lorentz transformations (12, 13) resulting also from the similar identity $\cosh_\kappa^2(\xi) - \sinh_\kappa^2(\xi) = 1$.

We shall see below how the "deformed" Lorentz transformations associated with the rapidity parameter ξ, and involving the generalized exponentials (4), can be rewritten in terms of the ordinary Lorentz transformations (involving the ordinary exponential) but associated with a κ-deformed (modified) rapidity parameter ξ_κ which can be defined in terms of ξ after equating

$$\exp_\kappa(\xi) = \exp(\xi_\kappa) \Rightarrow \xi_\kappa \equiv \left(\frac{1}{\kappa}\operatorname{arcsinh}(\kappa\xi)\right) = \xi f(\kappa\xi),\ \kappa \ge 0 \tag{16}$$

In other words, ξ_κ is a scaled version of ξ where the scaling function is defined as

$$f(\kappa\xi) \equiv \left(\frac{1}{\kappa\xi}\operatorname{arcsinh}(\kappa\xi)\right) \tag{17}$$

When $\kappa \to 0 \Rightarrow \xi_\kappa \to \xi$, and the scaling function becomes unity as expected. Given

$$\xi_\kappa \equiv \xi f(\kappa\xi) \to (-\xi)_\kappa = -\xi_\kappa = -\xi f(\kappa\xi),\ \kappa \ge 0 \tag{18}$$

And one arrives at

$$\sinh\left((-\xi)_\kappa\right) = \sinh(-\xi_\kappa) = -\sinh(\xi_\kappa) = -\sinh_\kappa(\xi),\ \kappa \ge 0 \tag{19}$$

$$\cosh\left((-\xi)_\kappa\right) = \cosh(-\xi_\kappa) = \cosh(\xi_\kappa) = \cosh_\kappa(\xi),\ \kappa \ge 0 \tag{20}$$

so that the "deformed" Lorentz transformations (12, 13) can be rewritten in terms of the κ-deformed rapidity parameter as

$$t' = t\cosh(\xi_\kappa) - x\sinh(\xi_\kappa);\ x' = x\cosh(\xi_\kappa) - t\sinh(\xi_\kappa) \tag{21}$$

and, in turn, the *inverse* transformations are obtained by exchanging the primed indices for unprimed ones and reversing the signs of ξ_κ

$$t = t'\cosh(\xi_\kappa) + x'\sinh(\xi_\kappa);\quad x = x'\cosh(\xi_\kappa) + t\sinh(\xi_\kappa) \tag{22}$$

The transformations for the energy-momentum E, p variables are of the same form as those of x, t leading to $E^2 - p^2 = (E')^2 - (p')^2$.

Given the relation between the rapidity parameter and velocity $\tanh(\xi) = v$, the analogous relation between the κ-modified rapidity parameter ξ_κ and the corresponding κ-modified velocity v_κ is

$$\tanh_\kappa(\xi) = \tanh(\xi_\kappa) = \tanh\left(\frac{1}{\kappa}\operatorname{arcsinh}(\kappa\xi)\right) = \tanh\left(\frac{1}{\kappa}\operatorname{arcsinh}\left[\kappa\operatorname{arctanh}(v)\right]\right) \tag{23}$$

and from which one can deduce the explicit algebraic relation between the velocity v and the κ-modified velocity v_κ given by

$$\tanh(\xi_\kappa) = v_k = \tanh\left(\frac{1}{\kappa}\operatorname{arcsinh}\left[\kappa\operatorname{arctanh}(v)\right]\right) \tag{24}$$

When $\kappa \to 0 \Rightarrow v_\kappa \to v$, and $(-v)_\kappa = -v_\kappa$.

Given Equaton-(6) and the definition (24) the deformed addition law of the κ-deformed velocities is

$$\tanh_\kappa(\xi_1 \oplus_\kappa \xi_2) = \frac{\tanh_\kappa(\xi_1) + \tanh_\kappa(\xi_2)}{1 + \tanh_\kappa(\xi_1)\tanh_\kappa(\xi_2)} = \frac{v_{1,\kappa} + v_{2,\kappa}}{1 + v_{1,\kappa}v_{2,\kappa}} \tag{25}$$

and reduces to the standard special relativistic addition law of velocities in the $\kappa \to 0$ limit. Reversing the sign of ξ_2 ($v_{2,\kappa}$) leads to the subtraction law[1].

2. Minimal Length

We turn now to the most important result of this work. In the study of the κ-statistical theory, Kaniadakis restricted the κ parameter to the domain $0 \le \kappa \le 1$. In this work we shall *extend* the domain of the κ-parameter to the *half* of the real line $0 \le \kappa \le \infty$. A careful inspection of the expression for the deformed boost rapidity parameter ξ_κ

$$\xi_\kappa = \left(\frac{1}{\kappa}\operatorname{arcsinh}(\kappa\xi)\right) = \frac{1}{\kappa}\ln\left(\sqrt{1+\kappa^2\xi^2} + \kappa\xi\right),\quad \kappa \ge 0 \tag{26}$$

reveals that in the limiting cases when *both* κ and ξ are infinite

$$\kappa \to \infty;\ \xi \to \infty \Rightarrow v \to 1 \tag{27}$$

one finds that the value

$$\lim_{\kappa\to\infty;\xi\to\infty}\left(\xi_\kappa = \frac{1}{\kappa}\ln\left(\sqrt{1+\kappa^2\xi^2} + \kappa\xi\right)\right) \to \xi\,\frac{\ln(2\kappa\xi)}{\kappa\xi} \to \infty \times 0 \tag{28}$$

is undetermined[2].

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[1] It is important to remark that $\xi_{1,\kappa} + \xi_{2,\kappa} \ne (\xi_1 + \xi_2)_\kappa$. Thus the addition of deformed boost rapidity parameters is not equal to the deformation of the addition of boost rapidity parameters. This is just a consequence of $x \oplus_\kappa y \ne x + y$.

[2] If κ is finite then the value of ξ_κ is ∞. Thus it is essential that $\kappa \to \infty$ as well.

Hence, by choosing the double scaling limit $\xi\frac{\ln(2\kappa\xi)}{\kappa\xi}=\infty\times 0=\text{finite}$ and nonzero, it leads to a *finite* value for the κ-deformed boost rapidity parameter $\xi_{\kappa=\infty}=\xi_\infty\neq\infty$, such that the κ-deformed velocity $\tanh(\xi_\infty)=v_\infty<1$ is *less* than the speed of light, and in turn, the Lorentz dilation factor $\gamma(v_\infty)\neq\infty$ *no* longer *blows* up. Consequently, the limiting cases described by Equation (28) lead to a cutoff in the value of the (deformed) Lorentz dilation factor $\gamma(v_\infty)$, which in turn, furnishes a *lower* bound in the length $L'=\frac{L}{\gamma(v_\infty)}\neq 0$ due to a *finite* Lorentz length contraction.

Concluding, after imposing that the lower bound L' should not be smaller than the postulated minimum Planck scale L_P [4]-[10], it yields

$$\begin{aligned}&L'=\frac{L}{\gamma(v_\infty)}\geq L_P\Rightarrow\frac{L}{L_P}\geq\gamma(v_\infty)=\frac{1}{\sqrt{1-\tanh^2(\xi_\infty)}}=\cosh(\xi_\infty)\\&\Rightarrow\operatorname{arccosh}\left(\frac{L}{L_P}\right)\geq\xi_\infty\end{aligned}\tag{29}$$

The actual equality $\operatorname{arccosh}\left(\frac{L}{L_P}\right)=\xi_\infty$ admits a physical interpretation analogous to the running of the physical couplings and masses with the energy scale in the Renormalization Group program in Quantum Field Theory. Namely, the possible values assigned to the undetermined ratio (∞/∞) depicting ξ_∞[3] in the last term of Equation (28), *flow* with the values of the running length scale L. In other words, one ends up with the L-dependent relation

$$\xi_\infty=\xi_\infty\left(\frac{L}{L_P}\right)=\operatorname{arccosh}\left(\frac{L}{L_P}\right)\tag{30}$$

subjected to the conditions $L>L_P$ and $\xi_\infty>0$ since the arccosh function is double-valued. The $\xi\to-\infty;\kappa\to\infty$ limits leads to $(-\xi)_\kappa=-\xi_\kappa\to-\xi_\infty$ which requires taking the negative values of the doubled-valued $\operatorname{arccosh}(L/L_P)$ function.

The justification why the double scaling limit $\xi_\infty=\xi\frac{\ln(2\kappa\xi)}{\kappa\xi}=\infty\times 0$ can be a function of the running length scale L can be understood as follows. The family of hyperbolas $xy=\lambda$ parametrized by $\lambda>0$, all satisfy the condition $\infty\times 0=\lambda$ for all the positive running values of the $\lambda>0$ parameter. The same result follows for the family of hyperbolas obeying $xy=f(\lambda)>0$. In our case we have $\lambda=\frac{L}{L_P}$, $f(\lambda)=\operatorname{arcosh}(\lambda)$, and $xy=\xi_\infty=\xi\frac{\ln(2\kappa\xi)}{\kappa\xi}=f(\lambda)$, as $x\to\infty$; $y\to 0$, with $x=\xi$; $y=\frac{\ln(2\kappa\xi)}{\kappa\xi}$. Or $x\to 0$; $y\to\infty$ with $y=\xi$;

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[3] $\xi_\infty\to\ln(2\kappa\xi)/\kappa\to\infty/\infty$.

$$x = \frac{\ln(2\kappa\xi)}{\kappa\xi}.$$

One may note the subtle point that if $L = L_P$, from Equation (30) one has $\xi_\infty = 0$ and this value does not belong to the family of hyperbolas since they were required to have $\xi_\infty = f(\lambda) > 0$. Therefore, one must have $L > L_P$ such that $\xi_\infty > 0$ ($f(\lambda) > 0$) if we wish to invoke this picture of the family of hyperbolas in order to understand the double scaling limit.

To sum up, despite that the velocity boost rapidity parameter ξ is infinite when v reaches the speed of light, due to the scaling behavior $\xi_\kappa = \xi f(\kappa\xi) = \text{finite}$ and nonzero as $\xi \to \infty$; $f(\kappa\xi) \to 0$, involving the κ parameter as well, the limiting ∞ value of κ counter-balances the ∞ value of ξ leading to an effective finite Lorentz dilation factor $\cosh(\xi_\infty)$ such that the contracted length $L' = L/\cosh(\xi_\infty)$ is never smaller than the postulated minimal Planck scale L_P. A finite value of κ and $\xi = \infty$ yields $\xi_\kappa = \infty$. A finite value of ξ and $\kappa = \infty$ yields $\xi_\kappa = 0$. Whereas $\kappa = \xi = \infty$ yield a flowing running value of ξ_κ between 0 and ∞. The latter is attained when $L = \infty$.

Recently, a finite time dilation factor in doubly special relativity (DSR) [6]-[8] was obtained in [11] and based on the κ-deformed Poincare symmetry firstly described by [12]-[14]. Caution must be taken *not* to confuse the κ parameter defining the κ-deformed exponential/logarithm and the κ parameter ($\kappa = L_P^{-1}$) of the κ-Poincare algebra corresponding to the symmetry algebra of a non-commutative deformation of Minkowski spacetime, and involving quantum groups, Hopf algebras, noncommutative geometry [15]. A large number of results has been obtained based on the DSR-relativistic frameworks, but the time dilation was never studied. The expression for the finite time dilation factor is fully analytical in the deformation parameter l and it was found to be [11]

$$\gamma_{DSR} = \frac{\cosh(lM) - e^{-lE}}{\sinh(lM)} \tag{31}$$

It reduces to the standard Lorentz factor $\gamma = E/M$ of special relativity in the undeformed limit $l \to 0$. When $l \neq 0$, one finds that as $E \to \infty \Rightarrow \gamma_{DSR} \to (\cosh(lM)/\sinh(lM))$ it leads to a finite result for the dilation factor. One may notice also that γ_{DSR} remains invariant by reversing the signs of l, E, M.

To finalize, one could contemplate the possibility to find two-parameter deformations of the exponential, involving κ and η, such that: 1) it won't be necessary to take $\kappa \to \infty$ in order to attain a finite Lorentz dilation factor as described above. 2) And the second parameter η might be linked to the existence of an maximum upper scale, like it occurs with the Yang algebra of noncommutative spacetime and momentum coordinates in phase space [16], and where both a lower and upper scale are introduced.

A two-parameter generalized exponential $\exp_{\kappa\eta}(y)$ was constructed by a deformation of $\exp_\kappa(y)$ by [1]-[3] as follows

$$\exp_{\kappa\eta}(y) \equiv \frac{1}{\eta}\exp_{\kappa}(ay+b),\ a=\sqrt{1+\kappa^2\left[\ln_{\kappa}(\eta)\right]^2},\ b=\ln_{\kappa}(\eta) \tag{35}$$

When $\eta = 1 \Rightarrow \ln_\kappa(1) = 0 \Rightarrow b = 0; a = 1$ and one recovers $\exp_\kappa(y)$. Unfortunately, the problem with such $\exp_{\kappa\eta}(y)$ is that $\exp_{\kappa\eta}(\xi)\exp_{\kappa\eta}(-\xi) \neq 1$ [4], thus one would not be able to write down deformed Lorentz transformations of the form described by Equations (11) and (12) and preserving the invariance $t^2 - x^2 = t'^2 - x'^2$ because $\cosh^2_{\kappa\eta}(\xi) - \sinh^2_{\kappa\eta}(\xi) \neq 1$.

To conclude, one of the most salient features of this work is that by using the κ-deformed exponential, inspired from the κ-statistical theory described by [1]-[3], one can bypass the machinery of noncommutative spacetime coordinates, quantum groups, Hopf algebras [6]-[8] [12]-[14] [17] and obtain a finite Lorentz contraction factor $\cosh(\xi_\infty)$ in the double scaling limit $\xi_\infty = \xi\frac{\ln(\kappa\xi)}{\kappa\xi} = \operatorname{arcosh}(L/L_P)$, when both $\kappa, \xi \to \infty$, and determined in terms of the ratio L/L_P involving the postulated minimal Planck scale and the running value length scale $L > L_P$.

In physics, the von Neumann entropy $S = -tr(\rho\ln\rho)$, with ρ being the density matrix, [18] is a measure of the statistical uncertainty within a description of a quantum system. It extends the concept of Gibbs entropy from classical statistical mechanics to quantum statistical mechanics, and it is the quantum counterpart of the Shannon entropy from classical information theory [19]. A natural question then is: what are the physical implications of κ-deformations of von Neumann entropy $S_\kappa = -tr(\rho\ln_\kappa(\rho))$ upon replacing the ordinary natural logarithm with the κ-logarithm (2) and which propelled the κ-statistical formulation of Kaniadakis [1]-[3]? This line of research deserves to be investigated.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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[4] Note that $\exp_{\kappa\eta}(\xi)\exp_{\kappa(1/\eta)}(-\xi) = 1$ requires κ, η and $\kappa, (1/\eta)$ for the values of the two parameters, respectively.

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