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Dark Energy from Quintessence Scalar Field in Hybrid Cosmology within *f*(*T*) Theory of Gravity

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Abstract

Current accelerated expansion of our universe, as indicated by number of observations, is addressed in the present work. Like several works in literature, we postulate dark energy as candidate and search for scalar field and $f(T)$ model susceptible to reproducing this negative pressure and dominant component of our universe. By considering hybrid cosmology whose free parameters are recently constrained with observational data, our numerical analysis promotes quintessence-like evolution very close to ΛCDM model as predicted by Ia supernovae observations. Furthermore, the analytical results do not exclude the possibility of falling into phantom-like evolution for suitable choice of the free parameters. The approach followed here directly links the scalar field to the $f(T)$ function and permits obtaining a dark energy-like $f(T)$ model in accordance with observational data.

Keywords

Modified Theory, Quintessence, Hybrid Cosmology, Scalar Factor, Scalar Field, Teleparallel

1. Introduction

Astronomical data show that the expansion of our universe is accelerated at the present epoch [1]. The dark energy (characterized by the cosmological constant Λ) is responsible of the acceleration of its expansion [2] [3]. The ΛCDM [4] model is the most plausible candidate for the description of the expanding universe due to its repulsive nature, which accounts for the contribution of vacuum energy to the curvature of space-time. However, models based on the cosmological constant have faced fine-tuning problems and cosmic coincidence. Several kinds of dark energy are studied in the literature and provide realistic way to distinguish some cosmological models from ΛCDM model. Indeed, expanding universe and dark energy are explored in [5] under the Statefinder diagnostic. The research team examines the Statefinder diagnostic in the light of the proposed SNAP satellite, which is expected to observe about 2000 supernovae per year. They show that the Statefinder is versatile enough to differentiate between dark energy models as varied as the cosmological constant on the one hand, and quintessence, Chaplygin gas and braneworld models, on the other. It is investigated in [6] that the necessary conditions for quintessence to phantom phase transition in quintom model. By studying the behavior of dynamical dark energy fields and Hubble parameter near the transition time, the authors show that the phantom-divide-line $\omega = -1$ is crossed in their considered models.

Moreover, modified theories of gravity, whether it is $f(R)$ [7], $f(G)$ [8], or $f(T)$ [9], are introduced as the equivalent description of dark energy cosmology via different theoretical models [10]-[14]. Furthermore, the gravitational wave astronomy, which recently started with the famous LIGO detections, could be, in principle, fundamental for testing the effective viability of such modified theories of gravity. Such an important and interesting investigation is made in [15], where some differences between different gravity theories can be found in linearized gravity by analyzing gravitational wave polarizations via the interferometric response functions.

The present work uses the modified $f(T)$ theory to construct cosmological models powered by the scalar field in an attempt to explain the current expansion of the universe. It is motivated by recent work on hybrid cosmology whose parameters are constrained by observational data [16] [17]. By dealing with the dynamical characteristics of scalar fields in hybrid cosmology under $f(T)$ theory, we will give more cosmological scope to hybrid cosmology through an approach that will also allow us to reconstruct $f(T)$ models able to reproduce dark energy features in accordance with observational data.

The paper is organized as follows: in Section II, we present the main equations in coupling modified teleparallel theory and scalar field, and apply them to hybrid cosmology in Section III. Numerical analysis and cosmological scope are presented in Section IV before concluding the work in Section V.

2. Main Equations in the Coupling Modified Teleparallel Theory and Scalar Field

The modified $f(T)$ theory of gravity has a solid mathematical foundation. We briefly outline the main points of the Teleparall theory. In general, when formulating theories of gravity, the metric tensor is of paramount importance. It contains the information needed to locally measure distances and thus to make theoretical predictions about experimental findings. Furthermore, the structure of the spacetime can be described by an alternative dynamical variable, the well-known non-trivial tetrad $h^a{}_\mu$ which is a set of four vectors defining a local frame at every point. The tetrads represent the basic entity of the theory of Teleparallel gravity. From their reconstruction arises the Teleparallel theory as a gravitational theory naturally based on the gauge approach of the group of translations. The tetrads are defined from the gauge covariant derivative for a scalar field, as $h^a{}_\mu = \partial_\mu x^a + A^a{}_\mu$ with $A^a{}_\mu$ the translational gauge potential and x^a the tangent-space coordinates [18]. The tetrad $h^a{}_\mu$ and its inverse $h_a{}^\mu$ satisfy the following relations:

$$h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu \qquad h^a{}_\mu h_b{}^\mu = \delta^a_b \tag{1}$$

Another important notion resulting from the establishment of this theory is the condition of absolute parallelism [19], which leads to the Weitzenböck connection seen as the fundamental connection of the theory. It is given by

$$\Gamma^\lambda_{\mu\nu} = h_a{}^\lambda \partial_\nu h^a{}_\mu = -h^a{}_\mu \partial_\nu h_a{}^\lambda \tag{2}$$

We emphasize here that the Latin alphabet ($a, b, c, \cdots = 0,1,2,3$) is used to denote the tangent space indices and the Greek alphabet ($\mu, \nu, \rho, \cdots = 0,1,2,3$) to denote the spacetime indices. The metric and the tetrad are related by

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu, \tag{3}$$

where $\eta_{ab} = \mathrm{diag}(+1,-1,-1,-1)$ is the Minkowski metric of the tangent space. In Teleparallel gravity and due to the no curvature Weitzenböck connection, the effects of gravitation are described by the torsion tensor, while the curvature tensor does not appear. Consequently, the non-vanishing and naturally antisymmetric torsion tensor is expressed via its components by

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu} = h_a{}^\lambda \left(\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu \right) \neq 0. \tag{4}$$

Another important tensor emerging from the use of the Weitzenböck connection is the contortion tensor $K^\lambda_{\mu\nu}$ which shows the difference between the Weitzenböck connection and the Levi-Civita connection [20] according to

$$K^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \tilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}\left(T_\nu{}^\lambda{}_\mu + T_\mu{}^\lambda{}_\nu - T^\lambda{}_{\mu\nu} \right), \tag{5}$$

where $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ are the Christoffel symbols or the coefficient of Levi-Civita connection. We define the action of the $f(T)$ gravity as

$$S = \frac{1}{4\kappa^2} \int \mathrm{d}^4 x h f(T) + \int \mathrm{d}^4 x h \mathcal{L}_M \tag{6}$$

where $h = \left| \det\left(h^a{}_\mu \right) \right|$ is equivalent to $\sqrt{-g}$ in General Relativity, $\kappa^2 = \frac{16\pi G}{C^4}$, $\mathcal{L}_M$ is the Lagrangian of the matter field. Then, the variation of this action with respect to the tetrads $h^a{}_\mu$ gives

$$\frac{1}{h}\partial_\mu\left(hS_a{}^{\mu\nu}\right)f_T(T)-h_a{}^{\lambda}T^{\rho}{}_{\mu\lambda}S_\rho{}^{\mu\nu}f_T(T)+A^{i}{}_{a\mu}S_i{}^{\mu\nu}f_T(T)$$
$$+S_a{}^{\mu\nu}\partial_\mu(T)f_{TT}(T)+\frac{1}{4}h_a{}^{\nu}f(T)=\frac{1}{4\kappa^2}T_a^\nu \tag{7}$$

with $f_T(T)=\frac{\mathrm{d}f(T)}{\mathrm{d}T}$, $f_{TT}(T)=\frac{\mathrm{d}^2f(T)}{\mathrm{d}T^2}$, T_a^ν is the energy-momentum tensor. In this study, we consider a universe geometrically described by the Friedmann-Lemaitre-Robertson-Walker metric given by

$$\mathrm{d}s^2=\mathrm{d}t^2-a^2(t)\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2 \tag{8}$$

where $a(t)$ denotes the scale factor. The scalar torsion related to the metric Equation (8) is given by

$$T=-6H^2(t) \tag{9}$$

$H(t)$ is the Hubble parameter. In the present work, we also suppose that the universe is filled with perfect fluid powered by the scalar field ϕ. In the context of Friedman-Lemaitre-Robertson-Walker universe, Equation (3),

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu-pg_{\mu\nu} \tag{10}$$

where $g_{\mu\nu}$ and u_ν, are the metric tensor and the 4-vector characterizing a co-mobile observer, respectively. Then, ρ and p are the global energy density and the pressure of universe content, respectively. Under these previous considerations, one can extract the Friedmannn-like equations of covariant modified Telleparallel theory

$$\kappa^2\rho=6H^2f_T+\frac{1}{4}f \quad\text{and}\quad \kappa^2p=48\dot{H}H^2f_{TT}-\left(2\dot{H}+6H^2\right)f_T-\frac{1}{4}f \tag{11}$$

The dynamical parameters namely the energy density and the pressure of the scalar field are given by [19]:

$$\kappa^2\rho=6H^2f_T+\frac{1}{4}f''\text{and}\quad \kappa^2p=48\dot{H}H^2f_{TT}-\left(2\dot{H}+6H^2\right)f_T-\frac{1}{4}f \tag{12}$$

where $V(\phi)$ is the scalar field potential. So the Friedman-like equations become

$$\kappa^2\left(\frac{\epsilon}{2}\dot{\phi}^2+V(\phi)\right)\&=6H^2f_T+\frac{1}{4}f \tag{13}$$

$$\kappa^2\left(\frac{\epsilon}{2}\dot{\phi}^2-V(\phi)\right)\&=48\dot{H}H^2f_{TT}-\left(2\dot{H}+6H^2\right)f_T-\frac{1}{4}f \tag{14}$$

The conservation equation $\dot{\rho}+3H(\rho+p)=0$, in the present context, leads to the following equation called Klein-Gordon equation [20]:

$$\epsilon\ddot{\phi}+3H\epsilon\dot{\phi}+V'(\phi)=0 \tag{15}$$

By adding the equations Equation (13) and Equation (14); we have

$$48\dot{H}H^2f_{TT}-2\dot{H}f_T=\kappa^2\epsilon\dot{\phi}^2 \tag{16}$$

In the Brans-Dike construction [21], it is possible to relate the Lagrangian density function $f(T)$ to the scalar field ϕ. Such a construction has already provided an interesting results in early accelerated expansion studying under $f(T)$ theory of gravity [22]. It is also one of the promising approach to build cosmological modified gravity model from the scalar field in the scalar-tensor theory [23]. In the framework of modified teleparallel theory, the Brans-Dike construction is declined through the following equivalence:

$$\phi(T) = f_T - 1 \Leftrightarrow f(T) = \int (\phi(T)+1)\mathrm{d}T + C \tag{17}$$

where C is an integration constant. Under this consideration, the equation Equation (16) becomes:

$$\kappa^2 \epsilon \dot{\phi}^2 + 4H\dot{\phi} + 2\dot{H}(\phi+1) = 0 \tag{18}$$

This equation can be solved either by imposing ϕ or by giving cosmological meaningful expression to H. Our main goal in this work consists to describe the dark energy effect through an emerging cosmological model: Hybrid cosmology.

3. Application to Hybrid Cosmology

The hybrid cosmology is powered by the scalar factor of the type $a(t) = t^{\alpha} \mathrm{e}^{\beta t}$ combining the Sitter expansion and the power-law evolution [16] [17]. The analytical expression of Hybrid parameter is the essential key of this work and will be used to solve the differential equation Equation (18). By making the following approximation $\kappa^2 \epsilon \dot{\phi} \ll 4H$, the solution gives

$$\phi(t) = \frac{c_1 \sqrt{\alpha + \beta t}}{\sqrt{t}} - 1 \tag{19}$$

c_1 is an integration constant. The expression in Equation (19) shows that the solution is valid for $t > 0$ if α and β are all positive parameters. Moreover, if one the of these parameters is negative, the scalar field found in Equation (19) exists if $t > -\frac{\alpha}{\beta}$. The goal of this paper is to explain the current accelerated expansion of the universe. So, hybrid deceleration parameter q must be negative [15]. According to hybrid cosmology scale factor, one has $q(t) = \frac{\alpha}{(\alpha+\beta t)^2} - 1$. If $\alpha > 0$, one has $q(t) < 0$ for $t > \frac{\sqrt{\alpha} - \alpha}{\beta}$. This condition meets those required for the existence of the expression in Equation (19). Furthermore, if $\alpha < 0$, the deceleration parameter is negative even for $t > -\frac{\alpha}{\beta}$. Once again, such exigence goes with the existence of scalar field in Equation (19). As conclusion, the scalar field in Equation (19), obtained under our reconstruction lies with negative deceleration parameter. So, it constitutes a good candidate to the current accelerated expansion of the universe.

Now, we can deal with the potential $V(\phi)$ and the cosmological parameters according to the approach followed in this work. By expressing the cosmic time as function of scalar field, the Klein-Gordon equation Equation (15) is solved and leads to

$$V(\phi) = c_2 - \frac{\epsilon \cdot D}{8\alpha c_1^3(\phi+1)\left(\beta c_1^2 - (\phi+1)^2\right)\sqrt{\frac{\alpha c_1^2}{(\phi+1)^2 - \beta c_1^2}}\sqrt{\frac{\alpha(\phi+1)^2}{(\phi+1)^2 - \beta c_1^2}}} \quad \text{avec}$$

$$\begin{aligned} D = & -\beta^4 c_1^8 + 4\beta c_1^2(\phi+1)^6 - 6\beta^2 c_1^4(\phi+1)^4 \\ & + 2\alpha(\phi+1)^4\left(3\beta^2 c_1^4 - 3\beta c_1^2(\phi+1)^2 + (\phi+1)^4\right) - (\phi+1)^8 \end{aligned} \tag{20}$$

It can also be expressed as cosmic time function. One has

$$V(\phi) = \frac{8\alpha^2 c_2 t^3(\alpha+\beta t) + c_1^2\epsilon\left((2\alpha-1)\alpha^4 + 2(\alpha-2)\beta^4 t^4 + 2(\alpha-2)\alpha\beta^3 t^3 + 2\alpha^4\beta t\right)}{8\alpha^2 t^3(\alpha+\beta t)} \tag{21}$$

By using Equation (19) and Equation (21), the cosmological parameters, the energy density $\rho(\phi)$ and the pressure $p(\phi)$ to the scalar field can be expressed as follows:

$$\rho(\phi) = c_2 + \frac{\epsilon c_1^2}{8}\left(\frac{2(\alpha-2)\beta^3}{\alpha^2} + 4\beta + \frac{\alpha^2\left(2 - \frac{1}{\alpha+\beta t}\right)}{t^3} + \frac{4\alpha}{t}\right) - \frac{\epsilon c_1\sqrt{\alpha+\beta t}}{\sqrt{t}} + \frac{\epsilon}{2} \tag{22}$$

$$p(\phi) = \frac{1}{8}\epsilon\left[c_1^2\left(-\frac{2(\alpha-2)\beta^3}{\alpha^2} + 4\beta + \frac{\alpha^2\left(\frac{1}{\alpha+\beta t} - 2\right)}{t^3} + \frac{4\alpha}{t}\right) - \frac{8c_1\sqrt{\alpha+\beta t}}{\sqrt{t}} + 4\right] - c_2 \tag{23}$$

The EoS parameter $\omega(\phi) = p(\phi)/\rho(\phi)$ is the indicator of the cosmological behavior in such investigation. In order to be really convinced of our model implications through the approach adopted in this work, we opt to a numerical analysis in order to follow the behavior of all the cosmological parameters ongoing study.

4. Numerical Analysis and Cosmological Scope

The numerical analysis is concerned in this work. It consists to depict the evolution of the cosmological parameters for observational constrained values of the parameters α and β.

1) Observational Constraint 1 (ObC1): $H_0 = 70.4 \pm 1.6$, $\alpha = 0.5186 \pm 0.0093$ **and** $\beta = 0.961 \pm 0.040$

These values are provided by applying Markov chain Monte Carlo (MCMC) technique on hybrid cosmology Hubble parameter [16]. Here, $\alpha > 0$ and requires $t > 0$ to meet the accelerated expansion of the universe. So all cosmological parameters will be depicted for $t > 0$ in order to verify if our model should explain the accelerated expansion (see **Figure 1**).

2) Observational Constraint 2 (ObC2): $H_0 = 71.14$ $\alpha = -0.676$ **and** $\beta = 1.77$

The observational constraint 2 on the cosmological parameters that we consider here is provided in [17]. It has been obtained from the Markov Chain Monte

Carlo (MCMC) process based on observational data from Pantheon [24]. As one of the best-fitted values of parameters, the EoS parameter lies in the quintessence era as demonstrated in [16]. Since $\alpha < 0$, according to (14), the accelerated expansion of universe is expected for $t > -\frac{\alpha}{\beta} \approx 0.382$. So basing on the ObC2, we provide the numerical evolution of the cosmological parameters (see **Figure 2**).

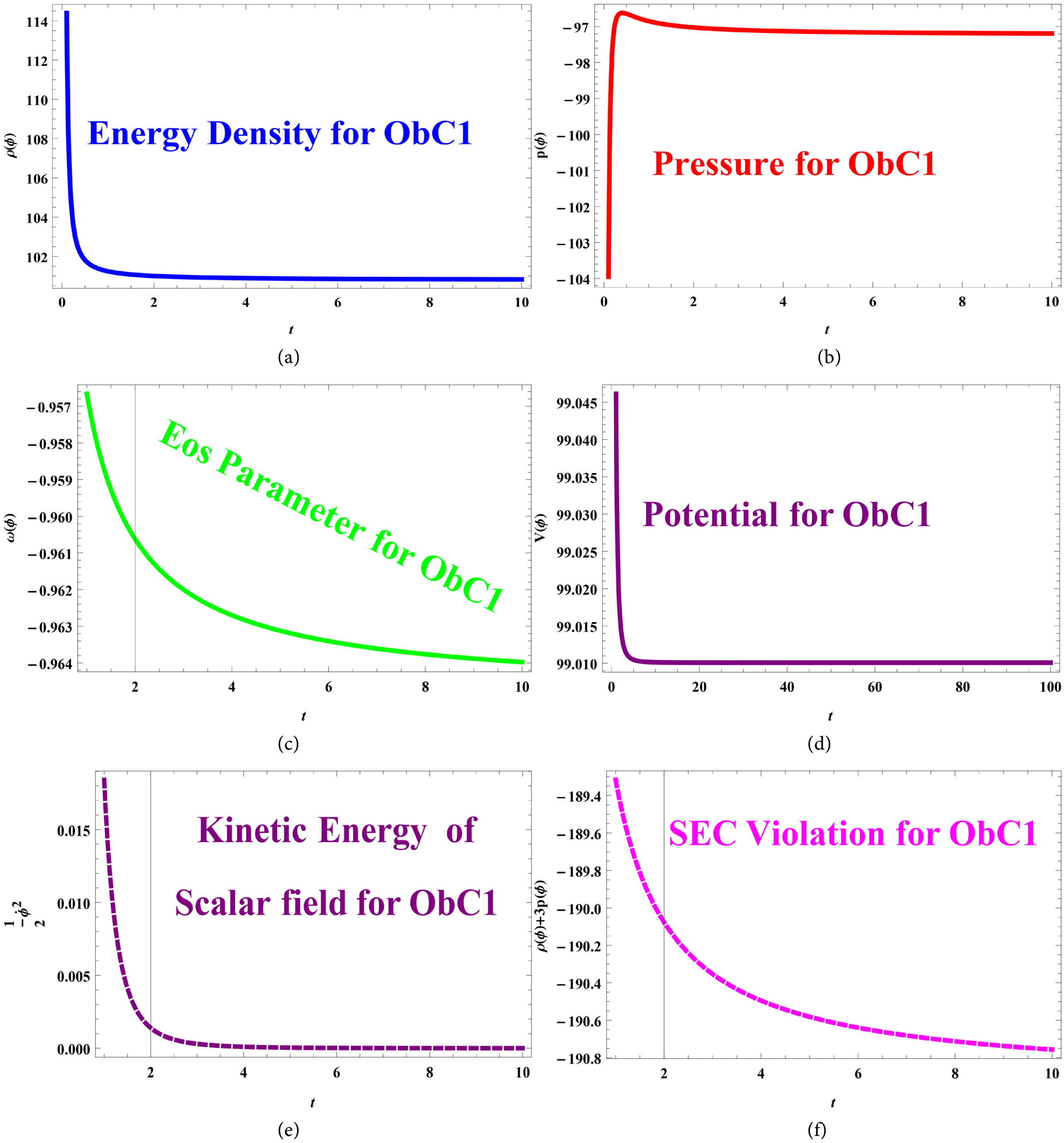

Figure 1. Evolution versus cosmic time of (a) Energy density, (b) Pressure, (c) EoS parameter, (d) Potential, (e) Kinetic energy and (f) SEC violation of the scalar field for ObC1. The curves are obtained for $\epsilon = 1$, $c_1 = -0.9$ and $\sigma = 100$.

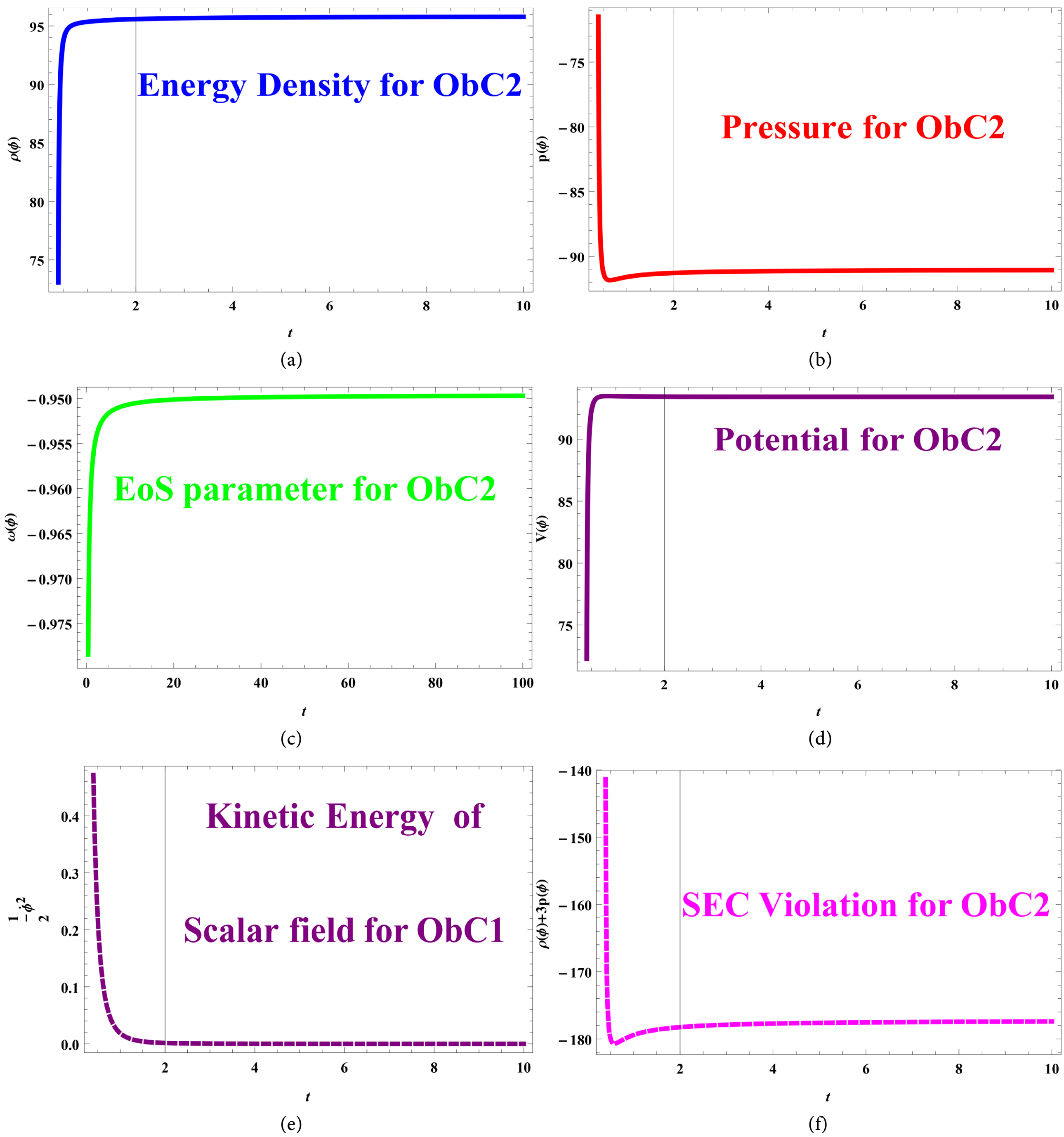

Figure 2. Evolution versus cosmic time of (a) Energy density, (b) Pressure, (c) EoS parameter, (d) Potential, (e) Kinetic energy and (f) SEC violation of the scalar field for ObC2. The curves are obtained for $\epsilon = 1$, $c_1 = -0.9$ and $\sigma = 100$.

3) Physical implications and reconstruction method

Figure 1 and Figure 2 show the behaviors of the cosmological parameters and the Second Energy Condition (SEC) violation when the universe is in an accelerated expansion phase for two different observational constraints on free parameters. In the two studied case, the scalar field potential is positive during the evolution, which means that the studied model is associated to stable configuration.

Furthermore, during the evolution, the energy density and the pressure of the scalar field are positive and negative respectively and led to negative Eos parameter $\omega(\phi)$ which by satisfying the condition $-1<\omega(\phi)<0$ confirms the quintessence nature of the studied scalar field. The kinetic energy is also positive as it expected in quintessence evolution [17]. The two observational constraints employed in this analysis describe a quintessence scalar field driven with negative pressure and positive potential under two different scenarios. The ObC1 gives results that align with those found by [16] where, before becoming constants, the energy density and the potential decrease while the pressure increases (see **Figure 1**), unlike those observed with the ObC2 (see **Figure 2**). In both cases, the EoS parameter varies weakly throughout the evolution and is closed to −1 as predicted by the current supernovae data [25]. It is also showed in [25] that the most probable equation of state parameters for galaxy cluster dispersion, are between −0.7 and −1.0. Furthermore, like Ia supernovae observations [26], the quintessential scalar field model moving closer to the ΛCDM model like our present case, is confirmed by several cosmological measurements, such as the CMB Radiation [27] and Large Scale Structure formation [28]. Finally, under quintessential tachyon scalar field [14], the condition of accelerated expansion is satisfied if $\dot{\phi}^2<2/3$ (see **Figure 1(e)** and **Figure 2(e)** for confirmation here) and in the same time, one has $-1<\omega(\phi)<-1/3$ which is also qualified as quintessence (see **Figure 1(c)** and **Figure 2(c)** for confirmation here).

Moreover, the violation of Strong Energy Condition ($\rho(\phi)+3p(\phi)>0$) is also numerically provided in this analysis through the figures **Figure 1(f)** and **Figure 2(f)**. This violation has been previously discussed in the context of supernovae observations and energy conditions in [29]. The results show that the SEC is validated by our model because the corresponding curve lies in negative values throughout the evolution. In conclusion, the scalar field in Equation (19) leads to accelerated expansion and to evidence of exotic matter in the cosmos [30].

As it is aimed in this work, we have to provide the $f(T)$ model associated to quintessence scalar field in Equation (19). From the hybrid cosmology Hubble parameter, it is possible to express the cosmic time in term of scalar torsion. So, by using the result and the relations, Equation (17) and Equation (19), one has

$$f(T)=-\frac{2^{7/4}c_1}{5\sqrt[4]{3}}(-T)^{\frac{5}{4}}+C \tag{24}$$

Here, C is an integration constant. So, we have provided cosmological $f(T)$ that can mimic the dark energy feature and consequently should explain the current acceleration of universe expansion. As predicted by EoS parameter, in order to have a model very closed to ΛCDM one ($f(T)=T+\Lambda$, and supported by observational data, we realize the power series expansion for Equation (24) about the point $T_0=-6H_0^2$, to the first order. One has

$$f(T)=\left(\frac{24}{5}c_1\left(H_0^2\right)^{\frac{5}{4}}+C\right)-c_1\sqrt[4]{H_0^2}\left(6H_0^2+T\right)+O\left[T+6H_0^2\right]^2 \tag{25}$$

$$\approx C-\frac{6c_1}{5}H_0^{5/4}-c_1H_0^{1/2}T \tag{26}$$

$$\approx T+\Lambda \to c_1=-H_0^{1/2},\ C=\Lambda-\frac{6}{5}H_0^{3/4} \tag{27}$$

H_0 is the current value of Hubble parameter and Λ is the cosmological constant whose present value will permit us to constrain fully the free parameter C with observational data. The parameter c_1 is calculated from ObC1 and ObC2.

5. Conclusions

In the present work, we deal with current accelerated expansion of the universe in modified teleparallel theory. The metric adopted in this work is those of Friedman-Robertson-Walker, whereas the universe content is supposed to be powered by the scalar field. It is important to recall here that the energy density and the cosmic pressure of the scalar are defined such that for $\epsilon=1$ and $\epsilon=-1$, one has quintessence evolution and phantom evolution respectively. By explicitly linking the scalar field to $f(T)$ function (Brans-Dike construction), we obtain from the Friedman-like equations, a differential equation depending on the Hubble parameter and scalar field and leading to the possibility to explore the hybrid cosmology. Through the differential equation and the Klein-Gordon equation resolution, we obtain the scalar field expression and its potential. Both functions are used to describe the dynamic quantities such as the energy density, the cosmic pressure, and the EoS parameter. To address the dynamical features of the obtained scalar field, two different observational constraints (ObC1 and ObC2) on hybrid cosmology are used and all led to quintessence-like evolution with EoS parameter very near −1 (ΛCDM case). In these two cases, the scalar field potential is positive traducing stable configuration, the pressure is negative, making the concerned scalar field a candidate of dark energy and the violation of the Strong Energy Condition confirms that the reconstructed scalar field can explain the accelerated expansion of universe. Our results are supported by several observational data and theoretical results on quintessence in literature.

Finally, it is important to clarify here that the reconstructed scalar field in Equation (19) can develop phantom behavior for $\epsilon=-1$ and for suitable choice of the free parameters. But, the problem is that the scalar field in Equation (19) can not lead to $\dot{\phi}^2<0$ when $\epsilon=-1$ as it is found in [16] and [17] on hybrid cosmology. The reason why we did not provide more details on phantom evolution is that this can be due to the approximation made when solving the differential Equation (18). To overcome this limit, a numerical resolution of this differential equation with an appropriate initial condition could allow for extending the analyses to the phantom evolution, where all the conditions should be satisfied. Unfortunately, it will be difficult to reconstruct the corresponding $f(T)$ model, which is one of the key goals of the approach followed in this work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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