Scientific Research Publishing

Padé Approximant for the Equation of Motion of a Supernova Remnant

Lorenzo Zaninetti

Physics Department, via P. Giuria 1, Turin, Italy
Email: zaninetti@ph.unito.it

Abstract

In this paper we derive three equations of motion for a supernova remnant (SNR) in the framework of the thin layer approximation using the Padé approximant. The circumstellar medium is assumed to follow a density profile of either an exponential type, a Gaussian type, or a Lane-Emden ($n = 5$) type. The three equations of motion are applied to four SNRs: Tycho, Cas A, Cygnus loop, and SN 1006. The percentage error of the Padé approximated solution is always less than 10%. The theoretical decrease of the velocity over ten years for SNRs is evaluated.

Keywords

Supernovae: General, ISM: Supernova Remnants, Supernovae: Individual (Tycho), Supernovae: Individual (Cas A), Supernovae: Individual (Cygnus Loop), Supernovae: Individual (SN 1006)

1. Introduction

The equation of motion for a supernova remnant (SNR) can be modeled by a single law of motion or multiple laws of motion when the appropriate boundary conditions are provided. Examples of a single law of motion are: The Sedov expansion in the presence of a circumstellar medium (CSM) with constant density where the radius, r, scales as $r \propto t^{0.4}$, see [1], and the momentum conservation in the framework of the thin layer approximation with CSM at constant density where $R \propto t^{0.25}$, see [2]. Examples of piece-wise solutions for an SNR can be found in [3]: A first energy conserving phase, $r \propto t^{0.4}$ followed by a second adiabatic phase where $r \propto t^{0.285}$. At the same time it has been shown that in the first ten years of SN 1993J 1993j $r \propto t^{0.82}$, which means an observed exponent larger than the previously suggested exponents, see [4]. The previous analysis allows posing a basic question: "Is it possible to find an analytical solution for SNRs given the three observable astronomical parameters, age, radius and velocity ?". In order to answer the above question, Section 2 introduces three profiles

for the CSM, Section 3 derives three Padé approximated laws of motion for SNRs, and Section 4 closes the derived equations of motion for four SNRs.

2. Profiles of Density

This section introduces three density profiles for the CSM: An exponential profile, a Gaussian profile, and a self-gravitating profile of Lane-Emden type.

2.1. The Exponential Profile

This density is assumed to have the following exponential dependence on r in spherical coordinates:

$$\rho(r; r_0, b, \rho_0) = \rho_0 \exp\left(-\frac{(r - r_0)}{b}\right), \tag{1}$$

where b represents the scale. The piece-wise density is

$$\rho(r; r_0, b, \rho_0) = \begin{cases} \rho_0 & \text{if } r \leq r_0 \\ \rho_0 \exp\left(-\frac{(r - r_0)}{b}\right) & \text{if } r > r_0 \end{cases} \tag{2}$$

The total mass swept, $M(r; r_0, b, \rho_0)$, in the interval $[0, r]$ is

$$M(r; r_0, b, \rho_0) = \frac{4}{3}\rho_0 \pi r_0^3 - 4b\left(2b^2 + 2br + r^2\right)\rho_0 \mathrm{e}^{\frac{r_0 - r}{b}} \pi \\ + 4b\left(2b^2 + 2br_0 + r_0^2\right)\rho_0 \pi. \tag{3}$$

2.2. The Gaussian Profile

This density has the Gaussian dependence

$$\rho(r; r_0, b, \rho_0) = \rho_0 \exp\left(-\frac{1}{2}\frac{r^2}{b^2}\right), \tag{4}$$

and the piece-wise density is

$$\rho(r; r_0, b, \rho_0) = \begin{cases} \rho_0 & \text{if } r \leq r_0 \\ \rho_0 \exp\left(-\frac{1}{2}\frac{r^2}{b^2}\right) & \text{if } r > r_0 \end{cases} \tag{5}$$

The total mass swept, $M(r; r_0, b, \rho_0)$, in the interval $[0, r]$ is

$$M(r; r_0, b, \rho_0) = \frac{4}{3}\rho_0 \pi r_0^3 + 4\rho_0 \pi \left(-\mathrm{e}^{-\frac{1}{2}\frac{r^2}{b^2}} r b^2 + \frac{1}{2}b^3\sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}r}{b}\right)\right) \\ - 4\rho_0 \pi \left(-\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}} r_0 b^2 + \frac{1}{2}b^3\sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}r_0}{b}\right)\right), \tag{6}$$

where erf is the error function, see [5].

2.3. The Lane-Emden Profile

The Lane-Emden profile when $n = 5$, after [6] [7], is

$$\rho\left(r;r_0,b,\rho_0\right)=\rho_0\frac{1}{\left(1+\dfrac{r^2}{3b^2}\right)^{\frac{5}{2}}}, \tag{7}$$

$$\rho\left(r;r_0,b,\rho_0\right)=\begin{cases}\rho_0 & \text{if } r\leq r_0 \\[2mm] \rho_0\dfrac{1}{\left(1+\dfrac{r^2}{3b^2}\right)^{\frac{5}{2}}} & \text{if } r>r_0\end{cases} \tag{8}$$

The total mass swept, $M\left(r;r_0,b,\rho_0\right)$, in the interval $\left[0,r\right]$ is

$$M\left(r;r_0,b,\rho_0\right)=\frac{4}{3}\rho_0\pi r_0^3+4\frac{b^3r^3\rho_0\sqrt{3}\pi}{\left(3b^2+r^2\right)^{\frac{3}{2}}}-4\frac{b^3r_0^3\rho_0\sqrt{3}\pi}{\left(3b^2+r_0^2\right)^{\frac{3}{2}}}. \tag{9}$$

3. The Equation of Motion

The conservation of the momentum in spherical coordinates in the framework of the thin layer approximation states that

$$M_0\left(r_0\right)v_0=M\left(r\right)v, \tag{10}$$

where $M_0\left(r_0\right)$ and $M\left(r\right)$ are the masses swept at r_0 and r, and v_0 and v are the velocities of the thin layer at r_0 and r.

3.1. Motion with Exponential Profile

Assuming an exponential profile as given by Equation (2) the velocity is

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{NE}{DE}, \tag{11}$$

where

$$NE=-r_0^3v_0,$$

and

$$DE=6\mathrm{e}^{\frac{r_0-r}{b}}b^3+6\mathrm{e}^{\frac{r_0-r}{b}}b^2r+3\mathrm{e}^{\frac{r_0-r}{b}}br^2-r_0^3-3r_0^2b-6r_0b^2-6b^3.$$

In the above differential equation of the first order in r, the variables can be separated and integration gives the following non-linear equation:

$$\frac{1}{r_0^3v_0}\left(18\mathrm{e}^{\frac{r_0-r}{b}}b^4+12\mathrm{e}^{\frac{r_0-r}{b}}b^3r+3\mathrm{e}^{\frac{r_0-r}{b}}b^2r^2-r_0^4-3r_0^3b\right.$$

$$\left.+r_0^3r-9r_0^2b^2+3r_0^2br-18b^3r_0+6r_0b^2r-18b^4+6b^3r\right) \tag{12}$$

$$=\left(t-t_0\right).$$

In this case is not possible to find an analytical solution for the radius, r, as a function of time. We therefore apply the Padé rational polynomial approximation of degree 2 in the numerator and degree 1 in the denominator about the point $r=r_0$ to the left-hand side of Equation (12):

$$\frac{-\left(r_0-r\right)\left(-5br-br_0-2rr_0+2r_0^2\right)}{2v_0\left(2br-5br_0-rr_0+r_0^2\right)}=t-t_0. \tag{13}$$

The resulting Padé approximant for the radius $r_{2,1}$ is

$$r_{2,1} = \frac{1}{2r_0 + 5b}\left(r_0 t v_0 - r_0 t_0 v_0 - 2btv_0 + 2bt_0 v_0 + 2r_0^2 + 2r_0 b \right.$$
$$+ \left(4b^2 t^2 v_0^2 - 8b^2 tt_0 v_0^2 + 4b^2 t_0^2 v_0^2 - 4bt^2 r_0 v_0^2 + 8btt_0 r_0 v_0^2 \right.$$
$$- 4bt_0^2 r_0 v_0^2 + t^2 r_0^2 v_0^2 - 2tt_0 r_0^2 v_0^2 + t_0^2 r_0^2 v_0^2 + 42b^2 tr_0 v_0$$
$$\left. \left. - 42b^2 t_0 r_0 v_0 + 6btr_0^2 v_0 - 6bt_0 r_0^2 v_0 + 9r_0^2 b^2 \right)^{\frac{1}{2}} \right), \tag{14}$$

and the velocity is

$$v_{2,1} = \frac{\mathrm{d}r_{2,1}}{\mathrm{d}t} = \frac{NVE}{DVE}, \tag{15}$$

$$NVE = 4v_0 \left\{ \left(-b/2 + 1/4\, r_0 \right) \times \sqrt{ 4\left(b - \frac{1}{2}r_0 \right)^2 (t-t_0)^2 v_0^2 + 42\left(b + 1/7\, r_0 \right)(t-t_0)br_0 v_0 + 9r_0^2 b^2 } \right.$$
$$\left. + \left(3/4\, b + \left(t/4 - 1/4t_0 \right)v_0 \right)r_0^2 + \frac{21r_0 b}{4}\left(v_0\left(-\frac{4t}{21} + \frac{4t_0}{21} \right) + b \right) + b^2 v_0 (t - t_0) \right\}, \tag{16}$$

and

$$DVE = \sqrt{ 4\left(b - \frac{1}{2}r_0 \right)^2 (t-t_0)^2 v_0^2 + 42\left(b + 1/7\, r_0 \right)(t-t_0)br_0 v_0 + 9r_0^2 b^2 } \times \left(2r_0 + 5b \right). \tag{17}$$

3.2. Motion with Gaussian Profile

Assuming a Gaussian profile as given by Equation (4) the velocity is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{NG}{DG}, \tag{18}$$

where

$$NG = -2r_0^3 v_0 \tag{19}$$

and

$$DG = -3b^3 \sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}r}{b} \right) + 3b^3 \sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}r_0}{b} \right)$$
$$+ 6\mathrm{e}^{-\frac{1}{2}\frac{r^2}{b^2}} rb^2 - 6\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}} r_0 b^2 - 2r_0^3. \tag{20}$$

The appropriate non-linear equation is

$$\frac{1}{2r_0^3 v_0}\left(\left(-12b^4 + 6r_0\left(r - r_0 \right)b^2 \right)\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}} + 12b^4 \mathrm{e}^{-\frac{1}{2}\frac{r^2}{b^2}} \right.$$
$$\left. -3\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}r_0}{b} \right)\sqrt{2}b^3 r + 3b^3 \sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}r}{b} \right)r + 2r_0^3\left(r - r_0 \right) \right) \tag{21}$$
$$= t - t_0.$$

The Padé rational polynomial approximation of degree 2 in the numerator and degree 1 in the denominator about $r = r_0$ for the left-hand side of the above equation gives

$$\frac{1}{2v_0\left(2b^2r - 5r_0b^2 - rr_0^2 + r_0^3\right)}\left(-(r - r_0)\left(9\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}}b^2r - 9\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^2 - 4b^2r + 10r_0b^2 + 2rr_0^2 - 2r_0^3\right)\right) = t - t_0. \quad (22)$$

The resulting Padé approximant for the radius $r_{2,1}$ is

$$r_{2,1} = \frac{1}{9\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}}b^2 + 2r_0^2 - 4b^2}\left\{9\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^2 - 2b^2tv_0 + 2b^2t_0v_0 + r_0^2tv_0 - r_0^2t_0v_0 - 7r_0b^2 + 2r_0^3 \right.$$
$$\left. + \left[54b^4r_0v_0\left(t - t_0\right)\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}} + 4\left(\left(t - t_0\right)\left(b^2 - \frac{1}{2}r_0^2\right)v_0 - \frac{3}{2}r_0b^2\right)^2\right]^{\frac{1}{2}}\right\}, \quad (23)$$

and the velocity is

$$v_{2,1} = \frac{\mathrm{d}r_{2,1}}{\mathrm{d}t} = \frac{NVG}{DVG}, \quad (24)$$

$$NVG = -\left(-27\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}}r_0b^4 + \left(2b^2 - r_0^2\right)v_0\left(t - t_0\right)r_0^2 + 3r_0b^2 - 2v_0b^2\left(t - t_0\right) \right.$$
$$\left. + \left\{54b^4r_0v_0\left(t - t_0\right)\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}} + 4\left(\left(t - t_0\right)\left(b^2 - \frac{1}{2}r_0^2\right)v_0 - 3/2r_0b^2\right)^2\right\}^{\frac{1}{2}}\right)v_0, \quad (25)$$

and

$$DVG = \left\{54b^4r_0v_0\left(t - t_0\right)\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}} + 4\left(\left(t - t_0\right)\left(b^2 - \frac{1}{2}r_0^2\right)v_0 - 3/2r_0b^2\right)^2\right\}^{\frac{1}{2}}\left(9\mathrm{e}^{-\frac{1}{2}\frac{r_0^2}{b^2}}b^2 + 2r_0^2 - 4b^2\right). \quad (26)$$

3.3. Motion with the Lane-Emden Profile

Assuming a Lane-Emden profile, $n = 5$, as given by Equation (7), the velocity is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{NL}{DL}, \quad (27)$$

where

$$NL = r_0^3v_0\left(3b^2 + r^2\right)^{\frac{3}{2}}\left(3b^2 + r_0^2\right)^{\frac{3}{2}} \quad (28)$$

and

$$DL = -3\left(3b^2 + r^2\right)^{\frac{3}{2}}\sqrt{3}r_0^3b^3 + 3\left(3b^2 + r_0^2\right)^{\frac{3}{2}}\sqrt{3}b^3r^3 + \left(3b^2 + r^2\right)^{\frac{3}{2}}\left(3b^2 + r_0^2\right)^{\frac{3}{2}}r_0^3. \quad (29)$$

The connected non-linear equation is

$$\frac{1}{r_0^3v_0\left(3b^2 + r_0^2\right)^{\frac{3}{2}}\sqrt{3b^2 + r^2}} \times \left(54\left(b^2 + \frac{1}{3}r_0^2\right)\left(\frac{1}{18}r_0^3\left(r - r_0\right)\sqrt{3b^2 + r^2}\right.\right.$$
$$\left.\left. + b^3\sqrt{3}\left(b^2 + \frac{1}{6}r^2\right)\right)\sqrt{3b^2 + r_0^2} - 54\sqrt{3b^2 + r^2}\sqrt{3}b^3\left(b^4 + \frac{1}{2}b^2r_0^2 + \frac{1}{18}rr_0^3\right)\right) = t - t_0.$$

The Padé rational polynomial approximation of degree 2 in the numerator and degree 1 in the denominator for the left-hand side of the above equation gives

$$\frac{NP}{2\left(3b^2 + r_0^2\right)^{\frac{3}{2}} v_0 \left(2rb^2 - 5b^2 r_0 - rr_0^2\right)} = t - t_0, \tag{30}$$

where

$$PN = -27\left(r - r_0\right)\left(-\frac{4}{9}\left(rb^2 - \frac{5}{2}b^2 r_0 - \frac{1}{2}rr_0^2\right) \times \left(b^2 + \frac{1}{3}r_0^2\right)\sqrt{3b^2 + r_0^2} + b^5\sqrt{3}\left(r - r_0\right)\right). \tag{31}$$

The Padé approximant for the radius is

$$r_{2,1} = \frac{NR}{DR} \tag{32}$$

where

$$
\begin{aligned}
NR = -18&\left(b^2 + \frac{1}{3}r_0^2\right)^2 b^2\left(-\frac{1}{2}r_0^3 - \frac{1}{2}v_0\left(t - t_0\right)r_0^2\right.\\
&+\frac{7}{2}b^2 r_0 + b^2 v_0\left(t - t_0\right)\right)\sqrt{3b^2 + r_0^2} + \left(81b^9 r_0 + 27b^7 r_0^3\right)\sqrt{3}\\
&+\sqrt{972}\left(\left(b^2 + \frac{1}{3}r_0^2\right)^4 b^4\left(\frac{9}{2}\sqrt{3}r_0 b^5 v_0\left(t - t_0\right)\sqrt{3b^2 + r_0^2}\right.\right.\\
&\left.\left.+\left(-\frac{1}{2}r_0^3 - \frac{1}{2}v_0\left(t - t_0\right)r_0^2 - \frac{3}{2}b^2 r_0 + b^2 v_0\left(t - t_0\right)\right)^2\left(b^2 + \frac{1}{3}r_0^2\right)\right)\right)^{\frac{1}{2}},
\end{aligned} \tag{33}
$$

and

$$DR = b^2\left(3b^2 + r_0^2\right)\left(27b^5\sqrt{3} - 12b^4\sqrt{3b^2 + r_0^2} + 2b^2 r_0^2\sqrt{3b^2 + r_0^2} + 2r_0^4\sqrt{3b^2 + r_0^2}\right), \tag{34}$$

and the velocity is

$$v_{2,1} = \frac{\mathrm{d}r_{2,1}}{\mathrm{d}t} = \frac{NVL}{DVL}, \tag{35}$$

where

$$
\begin{aligned}
NVL = -18\sqrt{3}&\left(3b^2 + r_0^2\right)v_0\left(-243\left(b^2 + \frac{1}{3}r_0^2\right)^2 b^7 r_0\sqrt{3}\right.\\
&+\sqrt{972}\left\{\left(b^2 + \frac{1}{3}r_0^2\right)^4 b^4\left(9/2\sqrt{3}r_0 b^5 v_0\left(t - t_0\right)\sqrt{3b^2 + r_0^2}\right.\right.\\
&\left.\left.+\left(b^2 + \frac{1}{3}r_0^2\right)\left(-\frac{1}{2}r_0^3 - \frac{1}{2}v_0\left(t - t_0\right)r_0^2 - 3/2b^2 r_0 + b^2 v_0\left(t - t_0\right)\right)^2\right)\right\}^{\frac{1}{2}}\\
&\times\left(2b^2 - r_0^2\right)\right)\sqrt{3b^2 + r_0^2} - 108\left(b^2 + \frac{1}{3}r_0^2\right)^3 b^2\left(-1/2r_0^3 - \frac{1}{2}v_0\left(t - t_0\right)r_0^2\right.\\
&\left.-3/2b^2 r_0 + b^2 v_0\left(t - t_0\right)\right)\left(b^2 - \frac{1}{2}r_0^2\right),
\end{aligned} \tag{36}
$$

and

$$DVL = 18\sqrt{972}\sqrt{3}\left\{\left(b^2 + \frac{1}{3}r_0^2\right)^4 b^4 \left(9/2\sqrt{3}r_0 b^5 v_0\left(t - t_0\right)\sqrt{3b^2 + r_0^2}\right.\right.$$

$$\left.\left.+\left(b^2 + \frac{1}{3}r_0^2\right)\left(-\frac{1}{2}r_0^3 - \frac{1}{2}v_0\left(t - t_0\right)r_0^2 - 3/2b^2 r_0 + b^2 v_0\left(t - t_0\right)\right)^2\right)\right\}^{\frac{1}{2}} \quad (37)$$

$$\times\left(\left(-12b^4 + 2b^2 r_0^2 + 2r_0^4\right)\sqrt{3b^2 + r_0^2} + 27b^5\sqrt{3}\right).$$

4. Astrophysical Applications

In the previous section, we derived three equations of motion in the form of non-linear equations and three Padé approximated equations of motion. We now check the reliability of the numerical and approximated solutions on four SNRs: Tycho, see [8], Cas A, see [9], Cygnus loop, see [10], and SN 1006, see [11]. The three astronomical measurable parameters are the time since the explosion in years, t, the actual observed radius in pc, r, and the present velocity of expansion in km·s^{-1}, see **Table 1**. The astrophysical units have not yet been specified: pc for length and yr for time are the units most commonly used by astronomers. With these units, the initial velocity is $v_0\left(\text{km}\cdot\text{s}^{-1}\right) = 9.7968\times10^5 v_0\left(\text{pc}\cdot\text{yr}^{-1}\right)$. The determination of the four unknown parameters, which are t_0, r_0, v_0 and b, can be obtained by equating the observed astronomical velocities and radius with those obtained with the Padé rational polynomial, *i.e.*

$$r_{2,1} = \text{Radius}\left(pc\right), \quad (38)$$

$$v_{2,1} = \text{Velocity}\left(\text{km}\cdot\text{s}^{-1}\right). \quad (39)$$

In order to reduce the unknown parameters from four to two, we fix v_0 and t_0. The two parameters b and r_0 are found by solving the two non-linear Equations (38) and (39). The results for the three types of profiles here adopted are reported in **Tables 2-4**.

Table 1. Observed astronomical parameters of SNRs.

Name	Age (yr)	Radius (pc)	Velocity (km·s^{-1})	References
Tycho	442	3.7	5300	Williams *et al.* 2016
Cas A	328	2.5	4700	Patnaude and Fesen 2009
Cygnus loop	17,000	24.25	250	Chiad *et al.* 2015
SN 1006	1000	10.19	3100	Uchida *et al.* 2013

Table 2. Theoretical parameters of SNRs for the padé approximated equation of motion with an exponential profile.

Name	t_0 (yr)	r_0 (pc)	$v_0\left(\text{km}\cdot\text{s}^{-1}\right)$	b(pc)	δ (%)	$\Delta v\left(\text{km}\cdot\text{s}^{-1}\right)$
Tycho	0.1	1.203	8000	0.113	5.893	−1.35
Cas A	1	0.819	8000	0.1	6.668	−3.29
Cygnus loop	10	12.27	3000	45.79	6.12	−0.155
SN 1006	1	5.49	3100	2.332	1.455	−12.34

Table 3. Theoretical parameters of SNRs for the Padé approximated equation of motion with a gaussian profile.

Name	t_0 (yr)	r_0 (pc)	$v_0\left(\mathrm{km}\cdot\mathrm{s}^{-1}\right)$	b(pc)	δ (%)	$\Delta v\left(\mathrm{km}\cdot\mathrm{s}^{-1}\right)$
Tycho	0.1	1.022	8000	0.561	8.517	−10.469
Cas A	1	0.741	7000	0.406	7.571	−13.16
Cygnus loop	10	11.92	3000	21.803	7.875	−0.161
SN 1006	1	5.049	10000	4.311	4.568	−18.58

Table 4. Theoretical parameters of SNRs for the Padé approximated equation of motion with a Lane-Emden profile.

Name	t_0 (yr)	r_0 (pc)	$v_0\left(\mathrm{km}\cdot\mathrm{s}^{-1}\right)$	b(pc)	δ (%)	$\Delta v\left(\mathrm{km}\cdot\mathrm{s}^{-1}\right)$
Tycho	0.1	0.971	8000	0.502	3.27	−14.83
Cas A	1	0.635	8000	0.35	4.769	−23.454
Cygnus loop	10	11.91	3000	27.203	7.731	−0.162
SN 1006	1	5	10000	4.85	3.297	−19.334

The goodness of the approximation is evaluated through the percentage error, δ, which is

$$\delta = \frac{\left|r_{2,1} - r_E\right|}{r_E} \times 100, \tag{40}$$

where $r_{2,1}$ is the Padé approximated radius and r_E is the exact solution which is obtained by solving numerically the non-linear equation of motion, as an example Equation (12) in the exponential case. The numerical values of δ are reported in column 6 of Tables 2-4. Another useful astrophysical variable is the predicted decrease in velocity on the basis of the Padé approximated velocity, $v_{2,1}$, in 10 years, see column 7 of Tables 2-4.

5. Conclusion

The expansion of an SNR can be modeled by the conservation of momentum in the presence of a decreasing density: here we analysed an exponential, a Gaussian and a Lane-Emden profile. The three equations of motion have complicated left-hand sides but simple left-hand sides, viz., $\left(t - t_0\right)$. The application of the Padé approximant to the left-hand side of the complicated equation of motion allows finding three approximate laws of motion, see Equations (14, 23, 32), and three approximate velocities, see Equations (15, 24, 35). The astrophysical test is performed on four spherical SNRs assumed to be spherical and the four sets of parameters are reported in Tables 2-4. The percentage of error of the Padé approximated solutions for the radius is always less than 10% with respect to the numerical exact solution, see column 6 of the three last tables. In order to produce an astrophysical prediction, the theoretical decrease in velocity for the four SNRs here analysed is evaluated, see column 7 of Tables 2-4.

References

[1] Sedov, L.I. (1959) Similarity and Dimensional Methods in Mechanics. Academic Press, New York.

[2] Dyson, J.E. and Williams, D.A. (1997) The Physics of the Interstellar Medium. Institute of Physics Publishing, Bristol. https://doi.org/10.1887/075030460X

[3] McCray, R.A. (1987) Coronal Interstellar Gas and Supernova Remnants. In: Dalgarno, A. and Layzer, D., Eds., *Spectroscopy of Astrophysical Plasmas*, Cambridge University Press, Cambridge, 255-278. https://doi.org/10.1017/CBO9780511564659.011

[4] Zaninetti, L. (2011) Time-Dependent Models for a Decade of SN 1993J. *Astrophysics and Space Science*, **333**, 99. https://doi.org/10.1007/s10509-011-0609-x

[5] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (2010) NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge.

[6] Lane, H.J. (1870) On the Theoretical Temperature of the Sun, under the Hypothesis of a Gaseous Mass Maintaining Its Volume by Its Internal Heat, and Depending on the Laws of Gases as Known to Terrestrial Experiment *American Journal of Science*, **148**, 57. https://doi.org/10.2475/ajs.s2-50.148.57

[7] Emden, R. (1907) Gaskugeln: Anwendungen der mechanischen warmetheorie auf kosmologische und meteorologische Probleme. B. Teubner, Berlin.

[8] Williams, B.J., Chomiuk, L., Hewitt, J.W., Blondin, J.M., Borkowski, K.J., Ghavamian, P., Petre, R. and Reynolds, S.P. (2016) An X-Ray and Radio Study of the Varying Expansion Velocities in Tycho Supernova Remnant. *ApJ,* **823**, L32.

[9] Patnaude, D.J. and Fesen, R.A. (2009) Proper Motions and Brightness Variations of Nonthermal X-Ray Filaments in the Cassiopeia A Supernova Remnant. *ApJ,* **697**, 535.

[10] Chiad, B.T., Ali, L.T. and Hassani, A.S. (2015) Determination of Velocity and Radius of Supernova Remnant after 1000 yrs of Explosion. *International Journal of Astronomy and Astrophysics,* **5**, 125.

[11] Uchida, H., Yamaguchi, H. and Koyama, K. (2013) Asymmetric Ejecta Distribution in SN 1006. *ApJ,* **771**, 56.