Journal of High Energy Physics, Gravitation and Cosmology, 2024, 10, 1450-1466
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

From Dirac's Aether to the Dirac Equation

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How to cite this paper: Bateson, R.D. (2024) From Dirac's Aether to the Dirac Equation. *Journal of High Energy Physics, Gravitation and Cosmology*, **10**, 1450-1466.
https://doi.org/10.4236/jhepgc.2024.104081

Received: May 31, 2024
Accepted: September 24, 2024
Published: September 27, 2024

Abstract

In 1951, Dirac proposed a formalism for a Lorentz invariant Aether with a vacuum state that contains all possible velocity states at each space-time point. Dirac showed no explicit path from the Aether towards the Quantum Mechanics. In this paper, we demonstrate that Dirac's proposed Aether can be described by a lattice of possible events in space-time built in the local Lorentz frame. The idealised case of single velocity state leads to the famous Dirac equation for a plane wave state and is compatible with quantum statistics. On the lattice, possible space-time events are connected by the Dirac spinors which provide the probability of observing an event. The inertial mass of a particle is shown to be equivalent to the density of possible events on the lattice. Variation of the lattice density of events modifies the metric and provides a space-time curvature leading to the Hilbert action associated with general relativity. In classical limit, the perturbation in the density of possible events of the Aether is proportional to the Newtonian gravitational potential.

Keywords

Dirac Aether, Lorentz Invariance, Dirac Equation, Quantum Mechanics, Space-Time Lattice, Dirac Spinors, Inertial Mass, Metric Modification, Space-Time Curvature, General Relativity

1. Introduction

Numerous attempts over the last couple of hundred years have been made to introduce the concept of an Aether but Einstein's discovery of the principle of relativity in 1905 and its apparent incompatibility with a directional Aether led to its abandonment. In 1951, Dirac proposed [1] a formalism for a Lorentz invariant Aether:

"Let us assume the four components v_i of the velocity of the aether at any point of space-time commute with one another. Then we can set up a representation with the wave functions involving the v's. The four v's can be pictured as defining a point on a three-dimensional hyperboloid in a four dimensional space, with the equation:

$$v_0^2 - v_1^2 - v_2^2 - v_3^2 = 1 \qquad v_0 > 0 \tag{1}$$

A wave function which represents a state for which all aether velocities are equally probable must be independent of the *v*'s, so it is a constant over the hyperboloid (1). If we form the square of the modulus of this wave function and integrate over the three-dimensional surface (1) in a Lorentz-invariant manner, which means attaching equal weights to elements of the surface which can be transformed into one another by a Lorentz transformation, the result will be infinite. Thus this wave function cannot be normalized."

In Dirac's formulation at a point in space-time, the velocity of the Aether is subject to quantum uncertainty and is potentially multi-valued. The velocity is badly defined but may be described by a probability distribution. Dirac assumed that in a Lorentz invariant manner the components of the 4-velocity $v_\mu = \gamma(c, \mathbf{v})$, where c is the speed of light and γ the Lorentz, factor commute with one another. This idealised state represents a perfect vacuum and is not attainable in practice. Indeed, the pure isotropic empty vacuum state is not measurable.

Also in 1951, Dirac published [2] a new classical theory of electrons which first introduced the concept of the Aether. In Dirac's theory, the electron and the electromagnetic field are intimately related and co-exist together and one is not required to consider the structure of the electron or treat it as a point particle. His theory was largely ignored and superseded by the successful quantum perturbation techniques of QED which Dirac himself considered ugly and cumbersome, primarily due to the renormalisation techniques required to deal with various infinities arising in calculations. To briefly summarise the theory, Maxwell's equations for an electromagnetic field can be commonly written as

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0. \tag{2}$$

The field variables $F_{\mu\nu}$ can be expressed as electromagnetic vector potentials A_μ

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}. \tag{3}$$

Now Dirac decided to ignore the commonly applied Lorentz gauge condition and instead destroy the gauge transformations by the subsidiary condition

$$A^\mu A_\mu = k^2 = \left(mc^2/e\right)^2. \tag{4}$$

This condition leads to classical electrons "appearing" in his theory. The electron velocity or the local velocity of the Aether is then defined as

$$v_\mu = k^{-1} A_\mu, \tag{5}$$

and the electron momentum provided in a straight forward manner.

$$eA_\mu = p_\mu c. \tag{6}$$

Thus in Dirac's theory the electron velocity at different points in space-time is directly linked to the local electromagnetic potential. There is no momentum without a field and *vice versa*. The free variables in the theory have been used to define the ratio (e/m) and the electron of Dirac's theory cannot be considered apart from the electromagnetic field and indeed the problem of the electron interacting with itself cleverly does not arise. The velocity of the Aether v_μ can be interpreted as the velocity with which a small charge placed in a region of no charge flows. Dirac showed that a simple iterative summation process could be used to calculate the electromagnetic potential and resultant velocities if a series of charges were present.

Notably, Dirac himself showed no explicit path from the Aether towards the Quantum Mechanics or his Dirac equation describing a fermion [3]. A consequence of Dirac's Aether is that physical quantities reliant upon the fluctuating velocity at each point in the space-time Aether become based on statistical averages. Petroni and Vigier [4] interpreted Dirac's Aether as a chaotic subquantal random motion that "assumes the particles embedded in it undergo random jumps at the velocity of light". They described a lattice model of the Aether and derived a Klein-Gordon equation for particle motion. Their approach was reminiscent of the famous "Feynman chequerboard" calculation [5] for an electron where a luminal velocity massive particle is viewed as "zig-zagging" diagonally forwards through space-time in a similar manner to a bishop in chess.

The Aether lattice approach adopted here differs in that a principle of simultaneity is used where equivalent time for particles is defined as the proper time experienced by the particles and particles move at a subluminal velocity. The lattice of events in space-time is built on the primitives of a set of "possible point events" and relations between them. The lattice is constructed in the local Lorentz frame and simultaneity on the space-time lattice is defined for events lying on hyperplanes of equivalent proper time. The lattice space-time discretisation method exactly derives the Dirac equation [3] and provides all the common fermion features, such as spin, negative energy states, action of a potential and summation of paths [5]. The most basic lattice describes a plane wave solution with the space axis aligned along the direction of momentum. The probabilities connecting possible events on the lattice are directly related to the Dirac spinor components. The lattice is consistent with Dirac's gauge which describes the existence and motion of a classical electron with no self interaction [2].

In this paper extension of the Aether lattice to curved space-time is also explored. Curved space-time is mapped by the lattice, as a grid of geodesics. The lattice Dirac equation in the local Lorentz frame is unchanged in these geodesic coordinates and the geometry and probabilities linking events are preserved. The curvature of space-time is produced by varying the space-time density of events leading to a modified coordinate system for observers. This model is shown to be consistent with both Einstein's General Relativity and Newtonian gravitation.

2. A Lattice Model of the Aether

Dirac's Aether as represented by Equation (1) contains all possible velocities at each point or region of space-time. To simplify we will first consider the simple idealised case of a particle plane wave state with one velocity or momentum state. Associated with each event the 4-vector velocity $v_\mu = \gamma(c, \boldsymbol{v})$ provides a commuting invariant relation and Equation (1) reduces to

$$v^\mu v_\mu = \gamma^2 \left(c^2 - \boldsymbol{v} \cdot \boldsymbol{v}\right) = c^2 \tag{7}$$

This invariant relation defines Dirac's Lorentz invariant Aether. In Dirac's formulation at each point in space-time the velocity of the Aether is subject to quantum uncertainty and is potentially multi-valued. Here we first analyse the case of one velocity state which can be later generalised to any velocity state or a distribution of velocities.

To construct the lattice for a particle motion in space-time, consider a 1 dimensional space aligned with the direction of particle motion, and embedded in 3 + 1 dimensional space. In this 1 dimensional space the simplest lattice that satisfies the definition of simultaneity is a 1 + 1 dimensional "diamond" lattice with links connecting the lattice points as in **Figure 1**. Each connection is defined by a connecting arrow giving a definite lineal order and an associated probability. Each vertex on the lattice represents a possible event—meaning a possible observation of the particle—and has two incoming and two outgoing connections. Starting at a vertex and following an outgoing arrow at random at each subsequent vertex describes a "causal chain" or a series of possible events.

Measurement or observation at a vertex or a region of the lattice provides, through Bayesian statistics, a re-evaluation of these probabilities after a measurement. A lattice of possible events thus constitutes a simple Bayesian network. This is illustrated in **Figure 1** where an event at $\mathcal{A}$ is more likely to have been caused by an event at $\mathcal{B}$ than $\mathcal{C}$ and $\mathcal{D}$ is an impossibility due to zero connectivity between the paths. Bayes theorem provides a way of translating this common sense concept into a formal probabilistic context since $P(\mathcal{A} \mid \mathcal{B}) > P(\mathcal{A} \mid \mathcal{C}) > P(\mathcal{A} \mid \mathcal{D})$. On measurement of an event this Bayesian re-evaluation and reassessment of probabilities is analogous to the well known "collapse of the wavefunction" in other interpretations of quantum mechanics.

Consider the simple case of representing a particle randomly diffusing on the 1 + 1 dimension lattice shown in **Figure 1**. The lattice is then made up of elementary triangles labelled with $(\Delta x, c\Delta t, c\Delta\tau)$ as shown in **Figure 2**. On the discrete lattice we define the observed velocity in terms of finite differences. The definition adopted is $v = \Delta x / \Delta t$ which we equate to the expectation of the velocity on the lattice. To guarantee invariance of causality on the lattice we impose c as the speed of light [6]. Since, from geometry, $\Delta x / c\Delta t = \sin\theta \le 1$, then we identify Δt as relativistic time intervals in an observer frame and $\Delta\tau$ as the particle proper time interval. The proper time interval is the actual time experienced by a particle or a local Lorentz observer moving between the two events and simultaneity on the lattice is then defined for events lying on hyperplanes of equivalent proper time. The lattice geometry guarantees the invariant space-time interval and in accordance with Equation (7) we can write

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2. \tag{8}$$

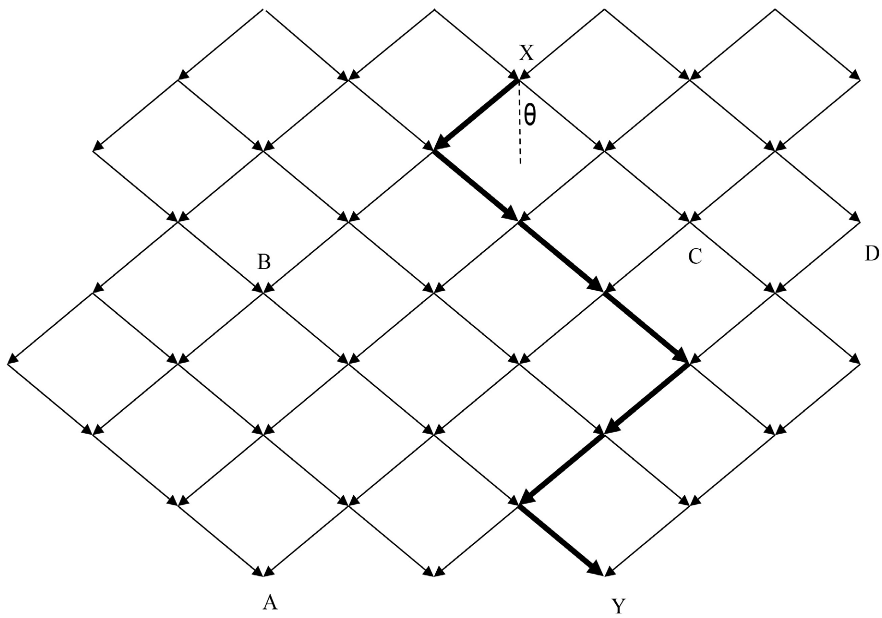

Figure 1. Space-time lattice showing path from X to Y. The lattice angle θ is shown.

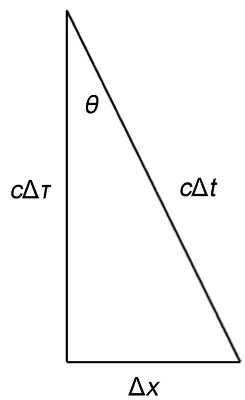

Figure 2. The elementary space-time "riangle" for the lattice built in the local Lorentz frame.

The two time intervals are then related by $\Delta\tau = \Delta t/\gamma$ where $\gamma = 1/\sqrt{1-v^2/c^2}$ is the Lorentz factor specifying the lattice angle θ and

$$\cos\theta = 1/\gamma \qquad \sin\theta = v/c. \tag{9}$$

The lattice is built in the local Lorentz frame for each velocity state. The proper time is then the time experienced by the particle in the local Lorentz frame travelling between two events in a similar manner to the famous "travelling twins" paradox in special relativity [7]. The lattice can be Lorentz boosted to any observer reference frame and the space-time coordinates of each event and linking probabilities will be modified. The observer imposes basis vectors and coordinates in his own inertial frame. In the local Lorentz frame the chosen orthogonal basis for measurement $(\hat{e}_\tau, \hat{e}_x)$ provides $\hat{e}_\tau \cdot \hat{e}_x = 0$ but in an observer frame with basis $(\hat{e}_t, \hat{e}_{\hat{x}})$ it will satisfy $\hat{e}_{\hat{x}} \cdot \hat{e}_t = 0$ with different space coordinates $\hat{x}$. In 1 + 1 dimensions, the lattice triangle for any observer inertial frame can be constructed by simply multiplying the scale of the local Lorentz triangle by γ so that the time and space axes are orthogonal $\Delta t = \gamma \Delta \tau$ and $\Delta \hat{x} = \gamma \Delta x$.

Clearly Equation (7) and thus the lattice can be scaled by a factor. If we identify this with the particle rest mass m then from Equation (7) we then have the relativistic dispersion relation

$$E^2 = p^2c^2 + m^2c^4, \tag{10}$$

where E is the particle energy $E = \gamma mc^2$ and $p = \gamma mv$ the momentum. We can further rearrange to derive a second useful invariant relation

$$-mc^2 \Delta \tau = p\Delta x - E\Delta t, \tag{11}$$

and a third, the Lagrangian for a free particle

$$L = -mc^2/\gamma = pv - E = pv - H, \tag{12}$$

where H is the Hamiltonian.

If we identify the lattice as representing an electron with charge e and velocity v moving in an electromagnetic potential (A_0, A_1) with

$$eA_0 = E \qquad eA_1 = pc. \tag{13}$$

then the lattice automatically describes the movement of an electron with the dispersion relation

$$-m^2c^4 = e^2A_1^2 - e^2A_0^2. \tag{14}$$

The above 1 + 1 dimension case is for the field component A_1 aligned along the lattice x space axis. In 3 + 1 dimensions this corresponds to Dirac's gauge condition [2]

$$A^\mu A_\mu = k^2 = \left(mc^2/e\right)^2, \tag{15}$$

where the electron velocity at different points in space-time is directly linked to the local electromagnetic potential. In Dirac's classical theory there is no electron without momentum and without a field and *vice versa*. The gauge connects directly with Maxwell's equations for the electromagnetic field as Equation (2).

3. Quantisation of the Aether

From our definition of simultaneity and the geometry of the lattice the sum $\sum p\Delta x$ is the same on the lattice for all paths between two events which implies that $p\Delta x$ is a constant. Identifying the lattice constant with Planck's constant h provides the de Broglie relation [8] with $\Delta x = \lambda/2$

$$\lambda p = h, \tag{16}$$

and a Heisenberg like relation [9] [10]

$$\Delta p \Delta x \sim h/2. \tag{17}$$

The discrete nature of the lattice automatically entails a de Broglie relation and an uncertainty principle. The spatial separation of events for a given proper time is equal to the de Broglie wavelength.

Starting with the relation Equation (11) we can consider the path sum of this relation over n events between two non adjacent distant events which leads to several action principles. The Maupertuis action S_M, that does not explicitly involve the time taken between events in space, can be written as a function of the number of steps between events n and h

$$S_M = \sum_i^n p\Delta x = nh. \tag{18}$$

We can also derive a more general action principle S_τ based on proper time and related to the Lagrangian

$$S_\tau = -\sum_i^n mc^2 \Delta\tau = -mc^2 n \Delta\tau = -\sum_i^n L\Delta t, \tag{19}$$

which can approximated for large n with an integral to acquire the well known result

$$S = -\int L\mathrm{d}t = -\int mc^2 \mathrm{d}\tau. \tag{20}$$

4. From Dirac's Aether to the Dirac Equation

The indeterminism on the lattice is governed by Equation (7). Thus a particle in its own local Lorentz frame in a proper time interval $\Delta\tau$ moving at a speed $|v|$ can move to a position $\pm\Delta x$ over time Δt in an inertial observer frame. This produces a random trajectory in space-time as in Figure 1 and branching probabilities can be associated with this movement. Consider an individual vertex on the lattice and label the incoming probabilities on row 1 P_{11} and P_{21} and outgoing probabilities on row 2 P_{12} and P_{22} as shown in Figure 3. Probability is conserved at the vertex and the total probability at a vertex is given by $P_T = P_{11} + P_{21}$. For a plane wave state we must consider a lattice with infinite extent and no boundary conditions. If the average velocity measured on the lattice is uniform then

$$P_{11} = P_{22} \qquad P_{12} = P_{21}. \tag{21}$$

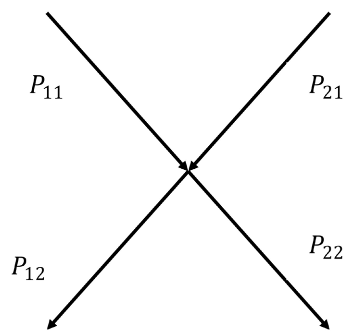

Figure 3. A vertex (1, 2) on the lattice with associated probabilities.

If we consider normalised branching probabilities at the vertex defined as $\hat{P}_{11}+\hat{P}_{21}=1$ then since expected velocity $\langle v\rangle$ at the vertex is defined to be v we have

$$\langle v\rangle=\frac{\Delta x}{\Delta\tau}\left[\hat{P}_{11}-\hat{P}_{21}\right]=v. \tag{22}$$

The branching probabilities are then given by

$$\hat{P}_{11}=\frac{E+mc^2}{2E} \qquad \hat{P}_{21}=\frac{E-mc^2}{2E}. \tag{23}$$

From this we can see that in the low velocity limit $|v|\to 0$ then $\hat{P}_{11}\to 1$ and $\hat{P}_{21}\to 0$ and in the high velocity limit $|v|\to c$ then $\hat{P}_{11}\to\hat{P}_{21}\to 1/2$. At higher velocities the paths followed becomes more random and the trajectory exhibits "Zitterbewegung". The probabilities can also be written simply in terms of lattice angles

$$\hat{P}_{11}=\cos^2(\theta/2) \qquad \hat{P}_{21}=\sin^2(\theta/2). \tag{24}$$

The probabilities can also be expressed in the form of a Lorentz boost ω where $\tanh\omega=v/c=\sin\theta$

$$\hat{P}_{11}=\frac{\cosh^2(\omega)}{\gamma} \qquad \hat{P}_{21}=\frac{\sinh^2(\omega)}{\gamma}. \tag{25}$$

Each real probability can be formed by combining complex probability amplitudes $P_{ij}=\phi_{ij}\cdot\phi_{ij}^*$ and for positive roots

$$\phi_{ij}=\sqrt{P_{ij}}\,\mathrm{e}^{-imc^2\tau/\hbar}=\sqrt{P_{ij}}\,\mathrm{e}^{i(px-Et)/\hbar}, \tag{26}$$

which depend on the proper time τ at the lattice vertices and x and t are defined at the discrete lattice vertices. The probability amplitudes at each vertex on the lattice can be expressed in terms of a unique non trivial transfer matrix M

$$\Phi=\begin{pmatrix}\phi_{22}\\ \phi_{12}\end{pmatrix}=\begin{pmatrix}\phi_{11}\\ \phi_{21}\end{pmatrix}=M\begin{pmatrix}\phi_{11}\\ \phi_{21}\end{pmatrix}, \tag{27}$$

defined as

$$M=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}=\begin{pmatrix}1/\gamma & v/c\\ v/c & -1/\gamma\end{pmatrix}=\frac{1}{E}\begin{pmatrix}mc^2 & pc\\ pc & -mc^2\end{pmatrix}=\frac{H_D}{E}. \tag{28}$$

Here we recognise H_D as the Dirac Hamiltonian for a free particle [11] [12] with defined momentum p. To connect with the complete quantum mechanics we note that Equation (28) can be put in the conventional form [12] by assuming that space-time is locally differentiable at the vertex, allowing us to use the usual momentum operator $\hat{p}$ to replace the momentum eigenvalues p. We can write

$$\begin{pmatrix}mc^2 & c\hat{p}\\ c\hat{p} & -mc^2\end{pmatrix}\Psi=E\Psi=i\hbar\frac{\partial\Psi}{\partial t}, \tag{29}$$

where we have replaced the probability amplitudes Φ with the familiar 2 component Dirac spinor Ψ for the free particle [12].

To extend to the general 3 + 1 dimension case we must consider transformations of the lattice that leave it invariant under spatial direction of velocity $\boldsymbol{v}$. Using polar coordinates then for momentum $\boldsymbol{p}=|\boldsymbol{p}|(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)$ the wavefunction components become dependent on the coordinates (ϑ,φ) so $\sqrt{P_{ij}}$ becomes $\sqrt{P_{ij}}\chi(\vartheta,\varphi)$ where $\chi(\vartheta,\varphi)$ is a multiplicative function. Following Dirac's convention [3] we can replace the 1 dimension momentum operator $\hat{p}$ with the 3 dimensional momentum operator $(\boldsymbol{\sigma}\cdot\boldsymbol{p})$, formed from Pauli matrices $\sigma_k\ (k=1,2,3)$. By definition this momentum operator provides the relation $(\boldsymbol{\sigma}\cdot\boldsymbol{p})\chi_\pm=|\boldsymbol{p}|\chi_\pm$ with two eigenvectors

$$\chi_+=\left(\cos\vartheta/2,\mathrm{e}^{i\varphi}\sin\vartheta/2\right)\qquad\chi_-=\left(-\mathrm{e}^{-i\varphi}\sin\vartheta/2,\cos\vartheta/2\right).\tag{30}$$

The general solutions for the wavefunction then become four 4-component orthogonal vectors corresponding to up and down spin $S=\pm1/2$ with positive and negative energies $\epsilon=\pm1$. Omitting the phase factors and normalisation constant these are

$$\Psi_{\substack{\epsilon=+1\\S=+1/2}}=\begin{pmatrix}\sqrt{P_{11}}\chi_+\\\sqrt{P_{21}}\chi_+\end{pmatrix}\qquad\Psi_{\substack{\epsilon=+1\\S=-1/2}}=\begin{pmatrix}\sqrt{P_{11}}\chi_-\\-\sqrt{P_{21}}\chi_-\end{pmatrix},\tag{31}$$

and

$$\Psi_{\substack{\epsilon=-1\\S=+1/2}}=\begin{pmatrix}-\sqrt{P_{21}}\chi_+\\\sqrt{P_{11}}\chi_+\end{pmatrix}\qquad\Psi_{\substack{\epsilon=-1\\S=-1/2}}=\begin{pmatrix}\sqrt{P_{21}}\chi_-\\\sqrt{P_{11}}\chi_-\end{pmatrix}.\tag{32}$$

These are the common solutions to the conventional 3 + 1 dimension Dirac equation [3] [12]

$$\begin{pmatrix}mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\\c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2\end{pmatrix}\Psi=E\Psi=i\hbar\frac{\partial\Psi}{\partial t}.\tag{33}$$

5. Inertial Mass in the Aether Lattice Model

The lattice and the separation of events in space and time leads naturally to the concept of mass as being the density of possible events in space-time. Since we have fixed the lattice constant as Planck's constant *h*, the absolute size or scale of the lattice in time and space can thus be evaluated as

$$\Delta\tau=\frac{h}{2mc^2\left(\gamma^2-1\right)}\qquad\Delta x=\frac{h}{2mc\sqrt{\gamma^2-1}},\tag{34}$$

with the scaling relation

$$\Delta\tau=\frac{2mc}{h}(\Delta x)^2.\tag{35}$$

Thus the lattice decreases in size with increasing velocity or energy as the de Broglie wavelength decreases. Also the scale of the lattice is also inversely proportional to the mass. Particles of higher mass will have more finely resolved lattices. Importantly, objects that are large relative to the lattice size will transition to a more classical behaviour for observers and this resolution effect in the lattice approach provides a *correspondence principle* between the classical and quantum domains.

From the distance between events on the lattice we can see that for equivalent velocity, and hence similar lattice triangles, the lattice size scales inversely with mass. This is shown schematically in **Figure 4**. For two masses m_1 and m_2 we have

$$\frac{m_1}{m_2} = \frac{\Delta x_2}{\Delta x_1} = \frac{\Delta t_2}{\Delta t_1}. \tag{36}$$

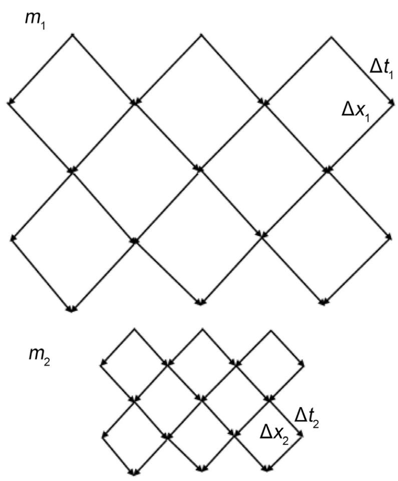

Figure 4. Lattice paths for same velocity and hence lattice angle for mass m_1 and m_2 where $m_2 > m_1$.

The number density of events on the lattice with distance or time is thus proportional to the particle mass. If σ_1 and σ_2 are the density of events in space or time their ratio is given by

$$\frac{\sigma_1}{\sigma_2} = \frac{m_1}{m_2}. \tag{37}$$

Thus, in the Aether lattice model the concept of inertial mass is related to the density of possible events.

We can see that the concept of mass on a lattice and simultaneity in proper time provides a link to Mach's principle, whereby it was postulated distant events or far away nebula can influence physics on a local scale. For paths with relativistic velocities, the scale of the lattice is reduced and the lattice angle $\theta = \arcsin(v/c)$ increases towards $\pi/2$. For a free particle, events over a finite proper time τ but that are at very large cosmic distances (since $x = c\tau \tan\theta$) can have a influence on local events whilst remaining simultaneous with closer events. However, due to the smaller lattice size such spatially distant lattices must traverse many intermediate events on their passage and their probability of influencing local events is much lower than for nearby events as in **Figure 5**.

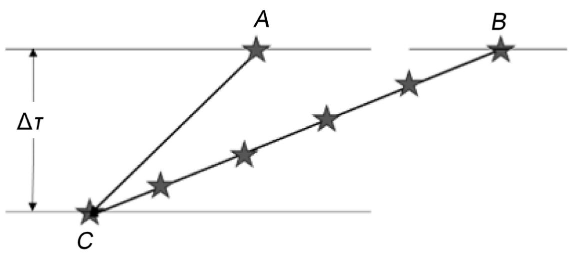

Figure 5. An event at B may be much further away in space and time than event A from event C but may be simultaneous in proper time and have an influence on C despite traversing many events in the path.

6. The Aether Lattice Model for Curved Space-Time

We shall assume that space-time is Reimann as assumed in the general relativity of Einstein. In the language of general relativity the space-time interval is given by the metric $g_{\mu\nu}$ so $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}$. Previously, we have considered the special case of the Minkowski metric $\eta_{\mu\nu}$ for flat space-time but general relativity considers Riemann spaces that have quadratic metric equations and are characterised as locally flat. The lattice is built in the local Lorentz frame with geodesics connecting possible events. A hypothetical observer moving along the path between two events experiences motion along a straight line with constant velocity. We can construct a lattice for Minkowski space-time and then embed the lattice into curved space-time with new coordinates. The coordinates of events in any curved space-time will thus change for any observer not in the geodesic local Lorentz frame. The curved space-time will then be criss-crossed by a grid of geodesics linking events.

The lattice for each velocity is built using a so-called Riemann normal coordinate system [13] with the following characteristics close to an event at $\mathcal{P}$

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \tag{38}$$

$$g_{\alpha\beta,\mu}(\mathcal{P}) = 0. \tag{39}$$

Thus close to any point or event the Riemann space-time is effectively Minkowski and described by the Minkowski metric. The lattice of events and the probabilities connecting events, and hence the spinor components will be preserved in moving from flat to curved space-time since there exists local Lorentz invariance in the vicinity of every possible event in Reimann space. This means that at each event space-time is Euclidean $g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta}$ and the angle θ of the lattice is preserved for a given geodesic. At each event the branching probabilities are preserved for the plane wave state and hence the Dirac equation is maintained in its simple, elegant form in the local Lorentz frame.

As for the Minkowski metric, for simultaneity we must consider hyperplanes between events that have constant proper time, that is the time experienced travelling along the geodesic between events is the same. A path of extremal τ is straight with constant velocity in every local Lorentz frame and is a geodesic of space-time. To illustrate how the metric $g_{\alpha\beta}$ influences the coordinate system

x^{α} for the lattice, we return to the 1 + 1 dimension lattice case and assume that observers can chose a orthogonal local Lorentz coordinate system to diagonalise the metric. This diagonalisation separates the time and space components of the metric. For flat space-time we have

$$-(c\Delta\tau)^2 = \eta_{00}(c\Delta t)^2 + \eta_{11}(\Delta x)^2, \tag{40}$$

and for curved space-time in 1 + 1 dimension with time and space coordinates $\hat{t}$ and $\hat{x}$,

$$-(c\Delta\tau)^2 = g_{00}(c\Delta\hat{t})^2 + g_{11}(\Delta\hat{x})^2. \tag{41}$$

If simultaneity between events on the net is determined, as before, by their equivalent separation in proper time $\Delta\tau$, these equations can be rearranged to give

$$\gamma^2\left(c^2 + \frac{\eta_{11}(\Delta x)^2}{\eta_{00}(\Delta t)^2}\right) = \gamma^2\left(c^2 + \frac{g_{11}(\Delta\hat{x})^2}{g_{00}(\Delta\hat{t})^2}\right) = v^{\mu}v_{\mu} = c^2 \tag{42}$$

which is Dirac's Aether invariant condition. Thus if the lattice 3-velocity is equivalent to the ratio

$$\frac{\eta_{11}(\Delta x)}{\eta_{00}(\Delta t)} = \frac{g_{11}(\Delta\hat{x})}{g_{00}(\Delta\hat{t})} = -v, \tag{43}$$

the lattice paths remain geodesics of constant velocity. Any geodesic trajectory through curved space-time can be mapped piecewise to a Minkowski lattice built in the local Lorentz frame with lattice triangles as in Figure 6.

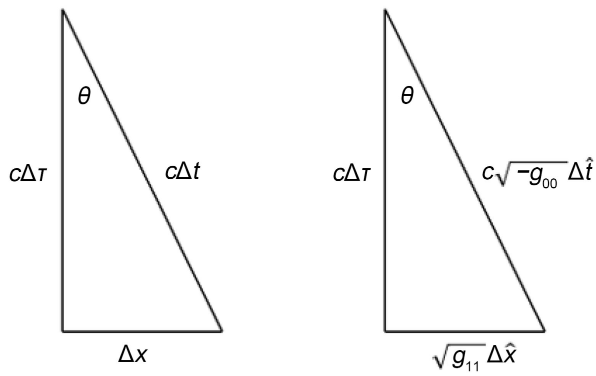

Figure 6. The lattice triangles are equivalent in the local Lorentz frame (left) and the observer frame (right) when scaled by the space-time metric. Curved space-time for geodesics can be mapped to a Minkowski lattice.

This generalises to the traditional general relativity formulation geodesic relation for the observer frame in 3 + 1 dimensions if time and space can be separated. The use of orthogonal local Lorentz coordinate systems allows time and space to be separated in obeying simultaneity. This results in the lattice obeying the relativistic energy dispersion relations in all measurable reference frames. Experimentally, measurements of the universality of the relativistic energy dispersion relations, such as those of electrons in the Crab Nebula [14] have placed tight constraints on any deviations from the well known energy dispersion relations and their invariant quantities. Conventionally, in general relativity by standard deformation and maximising or extremal lapse of proper time [7] [13], we have the geodesic equation in general coordinates at an event $\mathcal{P}$ at x^α as

$$\frac{\partial^2 x^\alpha}{\partial \tau^2} = -\Gamma^\alpha_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau} = 0. \tag{44}$$

At the event $\mathcal{P}$ we have $-\Gamma^\alpha_{\mu\nu} = 0$ leading to a secondary relation analogous to the discrete space-time lattice

$$g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \tau} = -c^2. \tag{45}$$

The above consideration shows that if a lattice is constructed using geodesics to separate neighbouring events with space-time separations Δx^α and if a metric is applied to curve space then new coordinates $\Delta \hat{x}^\alpha$ must be applied to each event on the lattice. This effectively "bends" the lattice in space-time for observers. For the 1 + 1 dimension case, the distance between events changes such that

$$\Delta \hat{t} = \frac{\Delta t}{\sqrt{-g_{00}}} \qquad \Delta \hat{x} = \frac{\Delta x}{\sqrt{g_{11}}}, \tag{46}$$

and the $\Delta \hat{t}$ and $\Delta \hat{x}$ no longer have a Pythagorean relationship in constructing the lattice but require a metric scaling factor as in **Figure 6**.

The metric is provided by the density of events along each path. Consider the path between two events of total proper time τ for a particle of mass m_1 which comprises n_1 events in the local Lorentz frame. We can write in 1 + 1 dimensions for a lattice

$$-(\tau c)^2 = -(n_1 c \Delta \tau_1)^2 = \eta_{00} (n_1 c \Delta t_1)^2 + \eta_{11} (n_1 \Delta x_1)^2. \tag{47}$$

The absolute number of events n_1 can be written as

$$n_1 = \frac{\tau}{\Delta \tau_1} = \frac{2 m_1 c^2 \tan^2 \theta}{h} \tau. \tag{48}$$

For Minkowski space-time events are distributed homogeneously but the more general case might be where the density of events varies differently in time σ_t and space σ_x. The path can now be written in the new coordinates $\hat{t}$ and $\hat{x}$ as the ratio of density of events between the curved and flat space-time

$$-(\tau c)^2 = -(n_1 c \Delta \tau_1)^2 = \eta_{00} \left(n_1 \frac{\hat{\sigma}_{t1}}{\sigma_1} c \Delta \hat{t}_1 \right)^2 + \eta_{11} \left(n_1 \frac{\hat{\sigma}_{x1}}{\sigma_1} \Delta \hat{x}_1 \right)^2. \tag{49}$$

The metric then becomes $g_{00_1} = \eta_{00} \left(\frac{\hat{\sigma}_{t1}}{\sigma_1} \right)^2$ and $g_{11_1} = \eta_{11} \left(\frac{\hat{\sigma}_{x1}}{\sigma_1} \right)^2$ for 1 + 1 dimensions. The number of events n_1 is the same over the proper time interval τ but the distance in space and time between them scales with the density of events such that

$$\frac{\sqrt{-\hat{g}_{00_1}}}{\sqrt{-\eta_{00_1}}} = \frac{\Delta t_1}{\Delta \hat{t}_1} = \frac{\hat{\sigma}_{t1}}{\sigma_1} \qquad \frac{\sqrt{\hat{g}_{11_1}}}{\sqrt{\eta_{11_1}}} = \frac{\Delta x_1}{\Delta \hat{x}_1} = \frac{\hat{\sigma}_{x1}}{\sigma_1}. \tag{50}$$

The above results are similar to the well known continuous time definition for the metric based on differentials [7]

$$g_{rs} = \delta_{mn} \frac{\partial x^m}{\partial \hat{x}^r} \frac{\partial x^n}{\partial \hat{x}^s}, \tag{51}$$

but the lattice result assumes a choice of inertial frame coordinates that diagonalises the metric and is based on finite differences.

This methodology extends to 3 + 1 dimensions and we can write the proper time between events for the curved space-time as

$$-(\tau c)^2 = -(n_1 c \Delta\tau)^2 = \hat{g}_{00_1} (c\Delta\hat{t}_1)^2 + \sum \hat{g}_{ii_1} (\Delta\hat{x}_{ii_1})^2, \tag{52}$$

where

$$\frac{\sqrt{-\hat{g}_{00_1}}}{\sqrt{-\eta_{00_1}}} = \frac{\Delta t_1}{\Delta\hat{t}_1} = \frac{\hat{\sigma}_{t1}}{\sigma_1} \qquad \frac{\sqrt{\hat{g}_{ii_1}}}{\sqrt{\eta_{ii_1}}} = \frac{\Delta x_{i_1}}{\Delta\hat{x}_{i_1}} = \frac{\hat{\sigma}_{i_1}}{\sigma_{i_1}}. \tag{53}$$

This leads to the common connection between the proper volume in the local Lorentz frame and the observer frame coordinates $\mathrm{d}^4 x = \sqrt{-g}\mathrm{d}^4\hat{x}$.

In most physical situations the metric changes in space and time and this is equivalent to the density of events varying. We have assumed the density of events is constant for all the n_1 events in the path for simplicity of presentation. However, any changing metric must be accommodated with a sum or approximated with integrals. In general, for curved space-time we can conventionally write the proper time interval between events $\mathcal{A}$ and $\mathcal{B}$ as the integral

$$\tau = \int_{\mathcal{B}}^{\mathcal{A}} \mathrm{d}\tau = -\int_{\mathcal{B}}^{\mathcal{A}} \left(g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right)^{1/2}. \tag{54}$$

We can see from our simple 1 + 1 dimension model that the change in density of events modifies the action along the path (**Figure 7**). The additional action δS due to a change in the density of events $\delta\sigma$ in the path can be approximated as a volume integral over an appropriate energy density ρc^2

$$\delta S = -m\left(\frac{\delta\sigma}{\sigma}\right)c^2\tau \approx -\int \rho c^2 \mathrm{d}^3 x \mathrm{d}\tau. \tag{55}$$

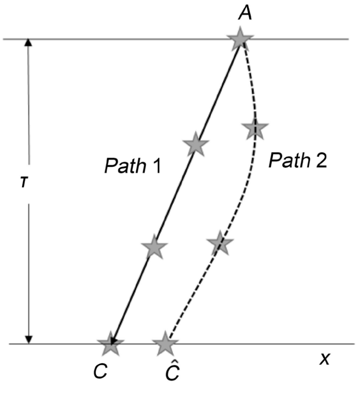

Figure 7. In curved space-time the density of events varies and this leads to an effective metric if simultaneity in proper time is preserved for different paths. The number of events in each path (Path 1 and Path 2) is equal for a particular particle mass but the space-time separation between them varies.

This provides a link to the gravitational action

$$\delta S = -\int \rho c^2 \mathrm{d}^3 x \mathrm{d}\tau = \int T \mathrm{d}^4 x = -\int \frac{R}{K} \mathrm{d}^4 x. \tag{56}$$

Here we have used the trace of the Einstein field equation

$$G = KT = -R, \tag{57}$$

and its relation to the Reimann curvature R and the constant K. The Einstein-Hilbert action [7] can be then written in the coordinate system $\hat{x}$ as

$$S_g = \int T\sqrt{-g}\,\mathrm{d}^4\hat{x} = -\frac{1}{K}\int R\sqrt{-g}\,\mathrm{d}^4\hat{x}. \tag{58}$$

Variation of the Einstein-Hilbert action recovers the complete Einstein field equations in a standard way [7] [13]. This demonstrates that the Aether lattice model based on possible events is compatible with Einstein's general theory of relativity if the density of possible events represents the gravitational energy density.

7. The Aether Lattice Model and Newtonian Gravitation

The spherically symmetric Schwarzchild metric in general relativity is an important metric that can be used to represent the gravitational field of planets, stars and black holes [7]. In spherical coordinates at a distance of r it is conventionally given as

$$-c^2\mathrm{d}\tau^2 = -\left(1-\frac{R_S}{r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{R_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{59}$$

where $\mathrm{d}\Omega^2 = d\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the metric on a unit two sphere and $R_S = 2GM/c^2$ the Schwarzchild radius with metric mass M. To reconstruct for the lattice we can discretise and relabel the coordinates as observer coordinates $\left(\Delta\hat{t}, \Delta\hat{r}, \Delta\hat{\Omega}\right)$ and considering a purely radial trajectory then the metric components can be written

$$g_{00} = -\alpha \qquad g_{11} = \alpha^{-1}. \tag{60}$$

with $\alpha = 1-\frac{R_S}{\hat{r}}$. The coordinates can be mapped to a Minkowski lattice in the local Lorentz frame as

$$\Delta t = \alpha^{1/2}\Delta\hat{t} \qquad \Delta r = \alpha^{-1/2}\Delta\hat{r}. \tag{61}$$

In the limit of large $\hat{r}$ the Schwarzchild solution approaches the Minkwoski lattice. Moving towards the Schwarzchild radius $\alpha \to 0$ so $\Delta\hat{t} \to \infty$ and $\Delta\hat{r} \to 0$ and the observed time to reach the radius from the outside becomes infinity and this result is consistent with theories of black holes [7]. Since in the lattice model the metric is related to the density of events the relative density of lattice events in the time and radial directions can be written as

$$\frac{\hat{\sigma}_t^2}{\sigma^2} = \alpha \qquad \frac{\hat{\sigma}_r^2}{\sigma^2} = \alpha^{-1}. \tag{62}$$

In the non relativistic limit we can expand for a perturbation in the density of events so

$$\frac{\delta\hat{\sigma}_t}{\sigma} = -\frac{\delta\hat{\sigma}_r}{\sigma} \approx -\frac{GM}{\hat{r}c^2} = -\frac{\Phi}{c^2}, \tag{63}$$

where Φ is the Newtonian gravitational potential. Thus approximately, the extra density of possible events, above the Minkowski case, is proportional to the classical Newtonian gravitational potential. Using the action definition in the non relativistic limit reconciles with the classical Lagrangian for a particle in a gravitational field

$$L = T - V = \frac{1}{2}mv^2 - m\Phi, \tag{64}$$

ignoring the rest mass term [7]. The Poisson's equation for the classical gravitational potential can be tentatively viewed as a steady state diffusion equation of extra possible events with the mass M of density ρ_M as the source

$$\nabla^2\Phi = -4\pi G\rho_M = -c^2\nabla^2\left(\frac{\delta\hat{\sigma}_r}{\sigma}\right). \tag{65}$$

8. Conclusions

Dirac in 1951 outlined a Lorentz invariant Aether based on the commuting invariant relation of the 4-velocity. Also in the same year, he published his new classical theory of the electron using the Dirac gauge. However, Dirac proposed no route from his Aether to quantum mechanics and his Dirac equation. In this paper, we outline a lattice approach to modelling the Aether, where the lattice is constructed in the local Lorentz frame to model the Dirac equation, the plane wave state and the associated fermion qualities such as spin and interaction with electromagnetic potentials. The discrete lattice formalism describes an uncertainty principle, the de Broglie relation and quantum mechanical statistics consistent with the quantum path integral approach. Inertial mass in the lattice model is provided by the lattice scaling or density of events in space-time. Arguably, a fermion can be viewed as an emergent quasi-particle of the lattice describing the Aether, in a similar fashion to quasi-particles in solid state physics.

The lattice approach can be extended to curved space-time to provide an analogue to gravitation. The density of events in space-time enters into the metric for the lattice, even for the Minkowski flat metric, which suggests an equivalence between inertial and gravitational mass. For curved space-time the paths of the lattice are geodesics and the Dirac equation in the local Lorentz frame is unchanged in these geodesic coordinates. In a Reimann space-time the vicinity of an event is locally flat and the lattice angle and probabilities linking events are preserved. The curvature of space-time in the Aether lattice model is produced by variation of the density of possible events in the path. If relativistic simultaneity is maintained, this leads to changes in the metric. The density of events modifies the action and provides a route to the Einstein-Hilbert action and general relativity. Evaluating the Schwarzchild metric in a classical limit shows that the perturbation in the relative density of possible events from the Minkowski case is proportional to the Newtonian gravitational potential. Thus, the introduction of possible extra events into the Aether lattice paths creates a gravitational potential under Newton's gravitation and an effective curvature under Einstein's relativity.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Dirac, P.A.M. (1951) Is There an Æther? *Nature*, **168**, 906-907. https://doi.org/10.1038/168906a0

[2] Dirac, P.A.M. (1951) A New Classical Theory of Electrons. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **209**, 291-296. https://doi.org/10.1098/rspa.1951.0204

[3] Dirac, P.A.M. (1928) The Quantum Theory of the Electron. *Proceedings of the Royal Society A*, **117**, 778.

[4] Petroni, N.C. and Vigier, J.P. (1983) Dirac's Aether in Relativistic Quantum Mechanics. *Foundations of Physics*, **13**, 253-286. https://doi.org/10.1007/bf01889484

[5] Feynman, R.P. and Hibbs, A.R. (1965) Quantum Mechanics and Path Integrals. McGraw-Hill.

[6] Rindler, W. (1982) Introduction to Special Relativity. Wiley.

[7] Carroll, S.M. (2019) Spacetime and Geometry. Cambridge University Press. https://doi.org/10.1017/9781108770385

[8] De Broglie, L. (1925) Recherches sur la théorie des Quanta. *Annales de Physique*, **10**, 22-128. https://doi.org/10.1051/anphys/192510030022

[9] Heisenberg, W. (1927) Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, **43**, 172-198. https://doi.org/10.1007/bf01397280

[10] Schürmann, T. and Hoffmann, I. (2009) A Closer Look at the Uncertainty Relation of Position and Momentum. *Foundations of Physics*, **39**, 958-963. https://doi.org/10.1007/s10701-009-9310-0

[11] Coulter, B. and Adler, C. (1971) The Relativistic One-Dimensional Square Potential. *AJP*, **39**, 305.

[12] Davydov, A.S. (1965) Quantum Mechanics. Pergamon Press.

[13] Misner, C.W., Thorne, K.S. and Wheeler, J.A. (2017) Gravitation. Princeton.

[14] Li, C. and Ma, B. (2022) Testing Lorentz Invariance of Electrons with LHAASO Observations of PeV γ-Rays from the Crab Nebula. *Physics Letters B*, **829**, Article ID: 137034. https://doi.org/10.1016/j.physletb.2022.137034