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The Equation for the CP Violating Phase for Quarks: The Rule for the Sum of Quark Oscillation Probabilities

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Abstract

By applying the rules for the sum of quark oscillation probabilities for the original CKM matrix and for Wolfenstein's parameterization, equations were derived in which the CP violating phase for quarks appears as an unknown quantity. Quark oscillations occur in spaces that are on the femtometer scale and they are unmeasurable from the point of view of experiments. However, the consequence of those oscillations is the CP violating phase for quarks, which is measured through unitary triangles in Wolfenstein's parameterization. Through the mathematical model presented in this paper, the equation in Wolfenstein's parameterization was derived, the root of which is consistent with measurements in today's quark physics.

Keywords

Quarks, CKM Matrix, PMNS Matrix, Wolfenstein Parameterization, CP Violation Phase, Jarlskog Invariant

1. Introduction

In order to understand the key idea on which the structure of this work is based, we will start from the definition for neutrinos via the PMNS mixing matrix:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{PMNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

where ν_e, ν_μ ,and ν_τ are three generations or flavors of neutrinos with subscript markings that show with which charged lepton it is a partner in the charged-current weak interaction. These three weakly interacting eigenstates form the orthonormal basis for the Standard Model of neutrinos. On the other hand, a mass eigenbasis is constructed which is defined with mass eigenstes ν_1, ν_2 and ν_3 which diagonalize the neutrino's free-particle Hamiltonian. Here, the difference between flavor eigenstates and mass eigenstates is highlighted, which in physics represents the key condition for neutrino oscillation, which has been experimentally confirmed.

The following relations are also written for down and up quarks, respectively:

$$\begin{pmatrix} d^I \\ s^I \\ b^I \end{pmatrix} = V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

and

$$\begin{pmatrix} u^I \\ c^I \\ t^I \end{pmatrix} = V_{CKM} \begin{pmatrix} u \\ c \\ t \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix}$$

Based on these records, it can be seen that there is a difference between interaction eigenstates and mass eigenstates, which confirms the possibility of quark oscillation.

Depending on the nature of the interaction of outgoing particles with their environment, it can be determined whether oscillations can be detected experimentally or not. In the case of neutrinos, neutrino oscillations are observed through the detector, which proves that neutrinos have a mass of the order 10^{-6} of the mass of electrons. With a note that it is not possible to measure their individual masses, but their corresponding square masses.

Unlike neutrinos, quarks interact significantly with their environment, which is why their oscillations cannot be observed experimentally.

The quantum mechanical record of outgoing quark states, for example for down quarks, can be written like this:

$$d^I = V_{ud}\left[d\right) + V_{us}\left[s\right) + V_{ub}\left[b\right).$$

These states are mixed with other interacting eigenstates via the time evolution operator when the departing particle begins to oscillate. Unlike neutrinos, quarks interact significantly with their environment, which is why their oscillations cannot be observed experimentally. Under the influence of the surrounding environment, additional particle creation and hadronization occur on a time scale from the femtometer. At the moment of hadronization, decoherence and destruction of the linear combination of quantum states occur, thereby crossing over the initial oscillations.

However, by applying the rule on the sum of quark oscillation probabilities in the space of the order of the femtometer in the developed mathematical model, we will show that in essence these invisible quark oscillations are produced by the CP violation phase, which is a measurable physical quantity in experiments.

However, we believe that such quark oscillations, which are considered invisible, they must have an influence on the quark decay processes. That is why we devoted this work to the quark oscillation processes and their influence on the formation of the final value for the CP violation phase.

The well-known β^- decay occurs under the influence of the weak interaction, which turns the atomic nucleus into a nucleus whose atomic number is increased by one, simultaneously emitting one electron and one electron antineutrino. This decay generally occurs in neutron-rich nuclei and it is described by the following equation:

$$ {}_{Z}^{A}X \rightarrow {}_{Z+1}^{A}X' + e^- + \bar{\nu}_e $$

Where the above labels have the following meanings:

A —atomic number,

Z —mass number,

X —initial nucleus,

X' —decayed nucleus.

This example can be joined by the decay of a free neutron $\left({}_{0}^{1}n\right)$ according to the Feynman diagram:

$$ n \rightarrow p' + e^- + \bar{\nu}_e $$

In this process, as shown by the Feynman diagram at the fundamental level, the transformation of the negatively charged $(-1/3)e$ down quark into a positively charged $(+2/3e)$ up quark occurs through the emission of the W^- boson, while the W^- boson decays into e^- and $\bar{\nu}_e$.

Therefore, within the atomic nucleus, which is rich in neutrons, the decay of neutrons into protons occurs, in the areas of neutrons that are of the order of magnitude on the femtometer scale, according to the following equation:

$$ d \rightarrow u + e^- + \bar{\nu}_e $$

Positron emission, known as β^+ decay, occurs under the influence of the weak interaction, which converts an atomic nucleus into a nucleus with an atomic number decreased by one, simultaneously emitting one positron e^+ and the electron neutrino ν_e.This decay generally occurs in proton-rich nuclei and it is described by the following equation:

$$ {}_{Z}^{A}X \rightarrow {}_{Z-1}^{A}X' + e^+ + \nu_e . $$

This equation can be shown in a simplified form via the Feynman diagram:

$$ p \rightarrow n + e^+ + \nu_e \rightarrow \left(u \rightarrow d + e^+ + \nu_e\right). $$

The weak interaction converts a proton into a neutron, during which the up quark transforms into a down quark with the emission W^+ or absorption of the W^- bosons. After emission W^+ or absorption W^-, they decay into one positron and one electron neutrino.

This process is the opposite of the β^- decay process. Therefore, the decay of a proton into a neutron is possible only within the atomic nucleus, which is rich in protons, but the decay of an isolated proton into a neutron is impossible due to the fact that the mass of the neutron is greater than the mass of the proton. These decays are followed by the transformation of down quarks into up quarks and vice versa with the participation of the corresponding W bosons, and we will observe these transformations as oscillations of quarks in those spaces These transformations are expressed by measuring the absolute value of the matrix element equal to $|V_{ud}|$.

From the point of view of experimental measurements, these quark oscillations are un-measurable. However, according to the mathematical model presented in this paper, the presence of these quark oscillations results in a CP violating phase, which is measured in experiments. So, based on what has been said, the main point of the entire work is to determine the numerical value of the CP violation phase for quarks.

In order to accomplish this task, we will apply the rule for the sum of quark oscillation probabilities on the femtometer scale for both the original CKM matrix and the Wolfenstein parameterization and establish a connection between them.

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ n \to p + \bar{\nu} + e^- & K \to \pi + \bar{\nu} + l^- & B \to \pi + \bar{\nu} + l^- \\ V_{cd} & V_{cs} & V_{cb} \\ D \to \pi + \bar{\nu} + l^- & D \to K + \bar{\nu} + l^- & B \to D + \bar{\nu} + l^- \\ V_{td} & V_{ts} & V_{tb} \\ B^0 \to \bar{B}^0 & B_S \to \bar{B}_S & t \to b + W \end{pmatrix} = \begin{pmatrix} 1 - s_{12}^2/2 & s_{12} & s_{13}e^{-i\delta} \\ -s_{12} & 1 - s_{12}^2/2 & s_{23} \\ s_{12}s_{23} - s_{13}e^{i\delta} & -s_{23} & 1 \end{pmatrix}$$

Below each element of the CKM matrix is shown the decay scheme on the basis of which they are measured. In order to derive the equation in which the CP violation phase appears as an unknown physical quantity, the rule on the sum of the quark oscillation probabilities in the femtometer space was applied, which formally mathematically achieved integration of these elements.

We have determined the spatial dimensions in which these transformations take place. The frames of those spaces are defined with the largest wavelength of quark oscillations, and how it looks can be seen in **Appendix A**.

2. The Rule for the Sum of the Oscillation Probabilities as a Link between PMNS (Pontecorvo-Maki-Nakagawa-Sakata) and CKM (Cabibbo-Kobayasi-Maskawa) Matrices

We established a link between the PMNS matrix and the CKM matrix observing the results of external manifestations that can be measured.

Namely, when an electron appears in the neutrino detector, for example, then it is clear that the electron neutrino has interacted with the detector. And these processes represent neutrino oscillations that are under the control of the PMNS matrix.

In the physics of quarks, quark decay processes occur in which down quarks change into up quarks and vice versa. They are also under the control of the CKM matrix.

The goal of this chapter is to become more familiar with the nature of the CP violation phase for quarks. As we will see in the following sections, we will apply the rule about the sum of quark oscillation probabilities in all transitions in a similar way as it was done for neutrinos [1] [2]. So that there are no doubts about quarks, we have introduced two possible conventions with the application of the same CKM mixing matrix.

In today's quark physics, a convention has been adopted which is expressed in the following paragraphs.

1) The interaction eigenstates and mass eigenstates are chosen to be equal for down type quarks and this is expressed as follows [3] [4]:

$$d_i^{Interaction} = d_j \tag{1}$$

2) Down type quarks are taken as rotating from the interacting basis towards the mass basis which is described by the following relation:

$$\begin{aligned} d_i^{interaction} &= V_{CKM} d_j \to \begin{pmatrix} d^I \\ s^I \\ b^I \end{pmatrix} = V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = V_{CKM} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -A - Be^{i\delta} & C - De^{i\delta} & V_{cb} \\ E - Fe^{i\delta} & -G - He^{i\delta} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \end{aligned} \tag{2}$$

In addition to the usual convention that we mentioned above and that is present in today's quark physics, in this paper we introduce another possible convention.

3) The interaction eigenstates and mass eigenstates are chosen to be equal for up type quarks and this is expressed as follows:

$$u_i^{Interaction} = u_j \tag{3}$$

4) Up type quarks are taken as rotating from the interacting basis towards the mass basis which is described by the following relation:

$$\begin{aligned} u_i^{interaction} &= V_{CKM} u_j \to \begin{pmatrix} u^I \\ c^I \\ t^I \end{pmatrix} = V_{CKM} \begin{pmatrix} u \\ c \\ t \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -A - Be^{i\delta} & C - De^{i\delta} & V_{cb} \\ E - Fe^{i\delta} & -G - He^{i\delta} & V_{tb} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix}. \end{aligned} \tag{4}$$

2.1. Sum Rule for Quark Oscillation Probabilities

Here we first show the procedure of how we used the rule for the sum of the oscillation probabilities of three neutrinos [5] [6] because we believe that we could also apply such a formal mathematical approach to quarks in order to find a numerical value for their CP violation phase.

2.1.1. Example for Three Neutrinos

For neutrinos, the well-known PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix is used [7]-[11]:

$$U_{PMNS}=\begin{pmatrix} U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}$$
$$=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta}\\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23}\\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23}\end{pmatrix} \quad (5)$$

the following relation applies to the probability of neutrino oscillation for all three possible transitions $(\nu_e\to\nu_\mu),(\nu_e\to\nu_\tau),(\nu_e\to\nu_e)$; $(\nu_\mu\to\nu_e),(\nu_\mu\to\nu_\tau),(\nu_\mu\to\nu_\mu)$; $(\nu_\tau\to\nu_e),(\nu_\tau\to\nu_\mu),(\nu_\tau\to\nu_\tau)$:

$$P(\nu_\alpha\to\nu_\beta)$$
$$=\delta_{\alpha\beta}-4\sum_{i<j}R_e\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin^2\left(\frac{\Delta m^2_{ji}Lc^3}{4E\hbar}\right) \quad (6)$$
$$+2\sum_{i<j}\mathrm{Im}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin\left(\frac{\Delta m^2_{ji}Lc^3}{2E\hbar}\right);i,j=1,2,3;\alpha,\beta=e,\mu,\tau.$$

In general, we can write sum rules for oscillation probabilities for three neutrinos in the form of the following relations:

$$P(\nu_e\to\nu_\mu)+P(\nu_e\to\nu_\tau)+P(\nu_e\to\nu_e)=1 \quad (7)$$

$$P(\nu_\mu\to\nu_e)+P(\nu_\mu\to\nu_\tau)+P(\nu_\mu\to\nu_\mu)=1 \quad (8)$$

$$P(\nu_\tau\to\nu_e)+P(\nu_\tau\to\nu_\mu)+P(\nu_\tau\to\nu_\tau)=1 \quad (9)$$

In further considerations in this paper, we set the CP violation phase for quarks as the main problem that should be solved to the end, and the following chapters are devoted to it.

2.1.2. An Example of Down-Quarks

There are two conventions in quark physics: the first convention refers to down quarks and the second convention refers to up quarks, and both will be included in research.

The first convention, which is generally accepted, is described by the following relations:

$$\begin{pmatrix} d^I\\ s^I\\ b^I\end{pmatrix}=V_{CKM}\begin{pmatrix} u\\ c\\ t\end{pmatrix}=\begin{pmatrix} V_{ud} & V_{us} & V_{ub}\\ V_{cd} & V_{cs} & V_{cb}\\ V_{td} & V_{ts} & V_{tb}\end{pmatrix}\begin{pmatrix} d\\ s\\ b\end{pmatrix}$$
$$=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta}\\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23}\\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23}\end{pmatrix}\begin{pmatrix} d\\ s\\ b\end{pmatrix} \quad (10)$$
$$=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta}\\ -A-Be^{i\delta} & C-De^{i\delta} & c_{13}s_{23}\\ E-Fe^{i\delta} & -G-He^{i\delta} & {}_{13}c_{23}\end{pmatrix}\begin{pmatrix} d\\ s\\ b\end{pmatrix}.$$

where the interaction eigenstates d^I,s^I,b^I are connected via the CKM unitary matrix with mass eigenstates d, s, b for down-quarks.

Here are all three possible transitions $d^I \to s^I, d^I \to b^I, d^I \to d^I; s^i \to d^I$, $s^i \to b^I, s^i \to s^i; b^I \to d^I, b^I \to s^i, b^I \to b^I$ are described with the rule for the sum of quark oscillation probabilities in a similar way as already defined in neutrino physics (7, 8, 9):

$$\begin{aligned} &P\left(d^I \to s^I\right) + P\left(d^I \to b^I\right) + P\left(d^I \to d^I\right) = 1. \\ &P\left(s^i \to d^I\right) + P\left(s^i \to b^I\right) + P\left(s^i \to s^i\right) = 1. \\ &P\left(b^I \to d\right)^I + P\left(b^I \to s^i\right) + P\left(b^I \to b^I\right) = 1. \end{aligned} \tag{11}$$

as well as the general relation which reads:

$$\begin{aligned} &P\left(V_\alpha \to V_\beta\right) \\ &= \delta_{\alpha\beta} - 4\sum_{i<j} R_e\left(V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* V_{\beta j}\right) \sin^2\left(\frac{\Delta m_{ji}^2 L c^3}{4E\hbar}\right) \\ &\quad + 2\sum_{i<j} \operatorname{Im}\left(V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* V_{\beta j}\right) \sin\left(\frac{\Delta m_{ji}^2 L c^3}{2E\hbar}\right); i, j = d, s, b; \alpha, \beta = u, c, t. \end{aligned} \tag{12}$$

Here we especially emphasize that we copied the procedure we applied in the research of neutrinos [5] [6]. Namely, in the case of down-quarks, as we can see here the role of mass eigenstates is taken over by the measured values for down-quarks. And we will see their application in the following chapters.

3. Derivation of the Equation of Motion for Three Quarks for Transitions: $d^I \to s^I, d^I \to b^I, d^I \to d^I$; $s^I \to d^I, s^I \to b^I, s^I \to s^I$; and $b^I \to d^I, b^I \to s^I, b^I \to b^I$

The goal of this chapter is to derive the quark motion equation in which the CP violation phase for quarks will appear as an unknown quantity. We analyze all three possible transitions: $d^I \to s^I, d^I \to b^I, d^I \to d^I$, $s^I \to d^I, s^I \to b^I, s^I \to s^I$, and $b^I \to d^I, b^I \to s^I, b^I \to b^I$.

In this case, the role of mass eigenstates is taken over by the measured values for down-quarks.

In quark physics, the squared differences of the corresponding quark masses for both down and up quarks are neither mentioned nor measured. Also, quark oscillations that occur in spaces of the size of a femtometer have not been the subject of research in today's quark physics due to the impossibility of measurements in such small spaces.

However, despite such circumstances, we have developed a mathematical formalism about possible quark oscillations using the rule about the sum of quark oscillation probabilities in those spaces.

We applied an algebraic presentation to quarks that is similar to that in Refs. [5] [6] and as the main result we highlight the derived formula in **Appendix** (A11) which figures in the derivation of the equation for the CP violating phase for quarks.

We especially note that the value for the CP violating phase for quarks obtained in the CKM parameterization δ_{CKM} does not represent the measured value in quark physics

However, it will be seen that it will have a decisive influence when defining the equation for the CP violating phase in the Wolfenstein parameterization in which its measured value δ_w is found.

In particular, we emphasize the key difference between neutrinos and quarks. Namely, in physics for neutrinos, only the square of the mass difference is measured, while in quark physics, quark masses are measured for each quark separately, whether it is down or up quarks.

Therefore, according to what was said for the application of the rule on the sum of probabilities of quark oscillations, the following relation can be written.

3.1. Transition: $d^I \to s^I, d^I \to b^I, d^I \to d^I$

$$
\begin{aligned}
& P\left(d^I \to S^I\right) + P\left(d^I \to b^I\right) + P\left(d^I \to d^I\right) \\
& = 1 - 4R\left\{V_{ud}V_{cd}^*V_{us}^*V_{cs}\sin^2\left(\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\} + 2\,\mathrm{Im}\left\{V_{ud}V_{cd}^*V_{us}^*V_{cs}\sin\left(2\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\} \\
& \quad - 4R\left\{V_{ud}V_{cd}^*V_{ub}^*V_{cb}\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\} + 2\,\mathrm{Im}\left\{V_{ud}V_{cd}^*V_{ub}^*V_{cb}\sin\left(2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\} \\
& \quad - 4R\left\{V_{us}V_{cs}^*V_{ub}^*V_{cb}\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\} + 2\,\mathrm{Im}\left\{V_{us}V_{cs}^*V_{ub}^*V_{cb}\sin\left(2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\} \\
& \quad - 4R\left\{V_{ud}V_{td}^*V_{us}^*V_{ts}\sin^2\left(\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\} + 2\,\mathrm{Im}\left\{V_{ud}V_{td}^*V_{us}^*V_{ts}\sin\left(2\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\} \\
& \quad - 4R\left\{V_{ud}V_{td}^*V_{ub}^*V_{tb}\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\} + 2\,\mathrm{Im}\left\{V_{ud}V_{td}^*V_{ub}^*V_{tb}\sin\left(2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\} \\
& \quad - 4R\left\{V_{us}V_{ts}^*V_{ub}^*V_{tb}\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\} + 2\,\mathrm{Im}\left\{V_{us}V_{ts}^*V_{ub}^*V_{tb}\sin\left(2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\} \\
& \quad - 4\left|V_{ud}\right|^2\left|V_{us}\right|^2\sin^2\left(\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right) - 4\left|V_{ud}\right|^2\left|V_{ub}\right|^2\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right) \\
& \quad - 4\left|V_{us}\right|^2\left|V_{ub}\right|^2\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right) \\
& = 1.
\end{aligned} \tag{13}
$$

By making appropriate simplifications, we get an equation in which the CP violating phase is an unknown quantity:

$$
\begin{aligned}
& -4W_d\,\mathrm{Re}\left\{V_{ud}V_{cd}^*V_{ub}^*V_{cb}\right\} + 2V_d\,\mathrm{Im}\left\{V_{ud}V_{cd}^*V_{ub}^*V_{cb}\right\} - 4W_d\,\mathrm{Re}\left\{V_{us}V_{cs}^*V_{ub}^*V_{cb}\right\} \\
& + 2V_d\,\mathrm{Im}\left\{V_{us}V_{cs}^*V_{ub}^*V_{cb}\right\} - 4W_d\,\mathrm{Re}\left\{V_{ud}V_{td}^*V_{ub}^*V_{tb}\right\} + 2V_d\,\mathrm{Im}\left\{V_{ud}V_{td}^*V_{ub}^*V_{tb}\right\} \\
& - 4W_d\,\mathrm{Re}\left\{V_{us}V_{ts}^*V_{ub}^*V_{tb}\right\} + 2V_d\,\mathrm{Im}\left\{V_{us}V_{ts}^*V_{ub}^*V_{tb}\right\} - 4W_d\left|V_{ud}\right|^2\left|V_{ub}\right|^2 \\
& - 4W_d\left|V_{us}\right|^2\left|V_{ub}\right|^2 = 0.
\end{aligned} \tag{14}
$$

Where the corresponding factors are equal to:

$$\begin{aligned} V_d &= \sin\left(2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right) = \sin\left(2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right); \\ W_d &= \sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right) = \sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right); \\ \Delta m_{bd}^2 &= m_b^2 - m_d^2, \Delta m_{bs}^2 = m_b^2 - m_s^2, \Delta m_{sd}^2 = m_s^2 - m_d^2. \end{aligned} \tag{15}$$

Taking the elements of the CKM matrix (13) into the calculation, we see that in (14) along with $\cos\delta$ and $\sin\delta$ there are the same algebraic expressions that we extract as a common factor, while all the free terms cancel each other out, so we arrive at the equation:

$$K \times \left(2W_d \cos\delta - V_d \sin\delta\right) = 0 \tag{16}$$

where δ is the CP violation phase as an unknown quantity of this equation.

We are looking for a solution to Equation (16), which we write in the form:

$$\left(2W_d \cos\delta - V_d \sin\delta\right) = 0 \tag{17}$$

In Equation (16), the factor K multiplies the equation and it is obvious that it has no influence on the final solution of the equation, which, as we can see, makes physical sense. However, with a subsequent check, we find that the factor K is equal to zero, so the final form of Equation (16) reads:

$$0 \times \left(2W_d \cos\delta - V_d \sin\delta\right) = 0 \tag{18}$$

By inserting explicit values for parameters (15) into Equation (17), it is reduced to the following two identical forms. The first form looks like this:

$$\begin{aligned} &\left(2\sin^2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\cos\delta - \sin 2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\sin\delta\right) = 0 \\ &\rightarrow 2\sin\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\left(\sin\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\cos\delta - \cos\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\sin\delta\right) = 0 \\ &\rightarrow 2\sin\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\left[\sin\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} - \delta\right)\right] = 0; \\ &\left(2\sin^2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\cos\delta - \sin 2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\sin\delta\right) = 0 \\ &\rightarrow 2\sin\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\left(\sin\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\cos\delta - \cos\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\sin\delta\right) = 0 \\ &\rightarrow 2\sin\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\left[\sin\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} - \delta\right)\right] = 0. \end{aligned} \tag{19}$$

Mathematically speaking, we see that the solution of Equation (18) consists of a general solution and a particular solution. The general solution satisfies all the values from the set $\delta \in [0.2\pi)$, which are countless and such solutions have no physical meaning.

But the particular Equation (19) have roots and they can be expressed in the following way:

$$2\sin\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\left[\sin\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\delta\right)\right]=0\to\sin\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\delta\right)=0\to$$
$$\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\delta\right)=0,\pm\pi\to$$
$$A:\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\delta=0\to\delta=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2};$$
$$B:\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\delta=+\pi\to\delta=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\pi=\pi\left(\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}+1\right)-\pi=\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2};$$
$$C:\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\delta=-\pi\to\delta=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}+\pi=\pi\left(\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}+1\right)+\pi=\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}+2\pi=\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}.\quad(20)$$
$$2\sin\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\left[\sin\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\delta\right)\right]=0\to\sin\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\delta\right)=0\to$$
$$\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\delta\right)=0,\pm\pi\to$$
$$A:\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\delta=0\to\delta=\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2};$$
$$B:\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\delta=+\pi\to\delta=\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\pi=\pi\left(\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-1\right)-\pi=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-2\pi=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2};$$
$$C:\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}-\delta=-\pi\to\delta=\pi\left(\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-1\right)+\pi=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}-\pi+\pi=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}.$$

We single out two of these solutions and they make physical sense:

$$\delta_{CKM}=\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2},\delta_{CKM}=\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}.\quad(21)$$

Note. We especially emphasize that if we do not have the same algebraic expressions for the cosine of the angle and the sine of the angle, then the Equation (16) would look like this:

$$K_1\times 2W_d\cos\delta-K_2\times V_d\sin\delta=0,K_1\neq K_2.\quad(22)$$

Assuming that the coefficients K1 and K2 are different from each other, but are each equal to zero, then Equation (22) takes the form:

$$0\times 2W_d\cos\delta-0\times V_d\sin\delta=0\quad(23)$$

In this form of the equation, we would have countless solutions from the set $\delta\in[0.2\pi)$ that satisfy this equation, and such solutions have no physical meaning. Here, too, it is important to emphasize that if we had factors along with $\cos\delta$ and $\sin\delta$, which according to the algebraic structure of mathematical expressions are mutually different, but are each equal to zero, then in that case we would not have solutions that would make physical sense.

As you can see, the decisive role was played by the equality of factors along with cosine and sine, and they could be extracted as a common factor with which we arrived at a particular solution that makes physical sense.

Applying an analogous theoretical procedure for the other two transitions

$s^I \to d^I, s^I \to b^I, s^I \to s^I$ and $b^I \to d^I, b^I \to s^I, b^I \to b^I$

We will get equations in which the equations for the CP violation phase look like this:

$$2W_d \cos\delta_{CKM} + V_d \sin\delta_{CKM} = 0 \tag{24}$$

Solutions of this type of equation for the CP violation phases are roots with opposite signs compared to formulas (21):

$$\delta_{CKM} = -\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}, \delta_{CKM} = -\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}. \tag{25}$$

We especially note that formulas (21) and (25) do not essentially represent CP violating phases, but they will be used to determine the coefficients V_d and W_d which will later be used in the equation obtained in Wolfenstein's parameterization, in which the actual measurements are values for the CP violation phase.

3.2. An Example of up Quarks

Here, transitions between weak interactions are described in the following form: $u^I \to c^I, u^I \to t^I, u^I \to u^I; c^I \to u^I, c^i \to t^I, c^I \to c^I; t^I \to u^I, t^I \to c^I, t^I \to t^I$, and the sum of oscillation probabilities between them could be described by the rule for the sum of oscillation probabilities in a similar way as already defined in neutrino physics (7, 8, 9):

$$\begin{aligned} &P\left(u^I \to c^I\right) + P\left(u^I \to t^I\right) + P\left(u^I \to u^I\right) = 1 \\ &P\left(c^I \to u^I\right) + P\left(c^I \to t^I\right) + P\left(c^I \to c^I\right) = 1. \\ &P\left(t^I \to u^I\right) + P\left(t^I \to c^I\right) + P\left(t^I \to t^I\right) = 1. \end{aligned} \tag{26}$$

as well as the general relation which reads:

$$\begin{aligned} &P\left(V_\alpha \to V_\beta\right) \\ &= \delta_{\alpha\beta} - 4\sum_{i<j} R_e\left(V_{\alpha i}V_{\beta i}^* V_{\alpha j}^* V_{\beta j}\right)\sin^2\left(\frac{\Delta m_{ji}^2 L c^3}{4E\hbar}\right) \\ &\quad + 2\sum_{i<j} \operatorname{Im}\left(V_{\alpha i}V_{\beta i}^* V_{\alpha j}^* V_{\beta j}\right)\sin\left(\frac{\Delta m_{ji}^2 L c^3}{2E\hbar}\right); i, j = u, c, t; \alpha, \beta = d, s, b. \end{aligned} \tag{27}$$

In this case, the role of mass eigenstates is taken over by the measured values for up quarks and then we apply the same methodology as for down quarks using the rule for the sum of quark oscillation probabilities for all three transitions where the corresponding coefficients are determined with the following relations:

$$\begin{aligned} &V_u = \sin\left(2\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}\right) = \sin\left(2\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}\right); W_u = \sin^2\left(\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}\right) = \sin^2\left(\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}\right); \\ &\Delta m_{tu}^2 = m_t^2 - m_u^2, \Delta m_{tc}^2 = m_t^2 - m_c^2, \Delta m_{cu}^2 = m_c^2 - m_u^2. \end{aligned} \tag{28}$$

In this case too, we get equations that are similar in structure to Equations (21), (25) with finite solutions:

$$\delta_{CKM} = \pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}, \delta_{CKM} = \pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}. \tag{29}$$

$$\delta_{CKM} = -\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}, \delta_{CKM} = -\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}. \quad (30)$$

3.3. Unified Equation of Motion for Quarks

Applying the mathematical formalism in applying the rule for the sum of oscillation probabilities for both down and up quarks, we obtained two types of equations:

Down quarks

$$\begin{aligned} 2W_d \cos\delta_{CKM} - V_d \sin\delta_{CKM} &= 0, \\ 2W_d \cos\delta_{CKM} + V_d \sin\delta_{CKM} &= 0. \end{aligned} \quad (31)$$

Adding these two equations gives the unification equation:

$$4W_d \cos\delta_{CKM} = 0 \to \delta_{CKM} = \pm\frac{\pi}{2}. \quad (32)$$

Up quarks

$$\begin{aligned} 2W_u \cos\delta_{CKM} - V_u \sin\delta_{CKM} &= 0, \\ 2W_u \cos\delta_{CKM} + V_u \sin\delta_{CKM} &= 0. \end{aligned} \quad (33)$$

Adding these two equations gives the unification equation:

$$4W_u \cos\delta_{CKM} = 0 \to \delta_{CKM} = \pm\frac{\pi}{2}. \quad (34)$$

In the following sections, it will be seen what role the coefficients play in Equations (31, 33). The same coefficients figure in the equation that will be derived in the following chapters based on the Wolfenstein parameterization. Therefore, all the research conducted above in the basis of the CKM matrix is intended exclusive for determining these coefficients, while the equation that will be derived in Wolfenstein's parameterization serves to determine the numerical value for the CP violation phase.

The dilemma about the sign $\pm\pi/2$ will be resolved when, in the next chapter, the equation in Wolfenstein's parameterization will be derived, in which the mapping in that equation will be seen.

4. Determining the Numerical Value of the Coefficients (V_d, W_d) and (V_u, W_u): The Process of Correcting the Measured Values of Quarks

The determination of coefficients (V_d, W_d) and (V_u, W_u) is based on the correction of the corresponding selected parameters for the measured mass of quarks and it is obtained by equating the corresponding formulas for CP violating phases for quarks, which is shown in the following text.

The following measured values for quark masses in the best fit are taken into account [1] [2] [12]-[17]:

Down quarks

$$\begin{aligned} m_d &= 4.67^{+0.18}_{-0.17}\ \text{MeV}, \\ m_s &= 93.4^{+8.6}_{-3.4}\ \text{MeV}, \\ m_b &= 4.18^{+0.03}_{-0.02}\ \text{GeV}. \end{aligned} \tag{35}$$

Up quarks

$$\begin{aligned} m_u &= 2.16^{+0.49}_{-0.26}\ \text{MeV}, \\ m_c &= 1.27 \pm 0.02\ \text{GeV}, \\ m_t &= 172.69 \pm 0.30\ \text{GeV}. \end{aligned} \tag{36}$$

4.1. Calculation Flow for down Quarks

We include the formulas (21) and (25) in the calculations, so we will have:

$$\begin{aligned} \delta_{BFbs+} &= 180^\circ \times \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} \\ &= \left\{180^\circ \times \left[\left(4180^2 - 93.4^2\right)/\left(93.4^2 - 4.67^2\right)\right]/360^\circ - 1003\right\} \times 360^\circ \approx 164.734^\circ. \\ \delta_{BFbs-} &= -180^\circ \times \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} \\ &= \left\{-180^\circ \times \left[\left(4180^2 - 93.4^2\right)/\left(93.4^2 - 4.67^2\right)\right]/360^\circ + 1004\right\} \times 360^\circ \approx 195.266^\circ. \\ \delta_{BFbd+} &= 180^\circ \times \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} \\ &= \left\{\left[180^\circ \times \left[\left(4180^2 - 4.67^2\right)/\left(93.4^2 - 4.67^2\right)\right]/360^\circ - 1003\right]\right\} \times 360^\circ \cong 344.734^\circ. \\ \delta_{BFbd-} &= -180^\circ \times \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} \\ &= \left\{\left[-180^\circ \times \left[\left(4180^2 - 4.67^2\right)/\left(93.4^2 - 4.67^2\right)\right]/360^\circ + 1004\right]\right\} \times 360^\circ \approx 15.266^\circ. \end{aligned} \tag{37}$$

From the point of view of physical processes between quarks, there is only one unique value for the CP violation phase. In the theoretical consideration of the oscillation process between quarks, we have derived formulas for the CP violation phases which differ in structure and sign (21, 25). It is obvious that the calculated values are all mutually different (26). The reason for this is due to insufficient precision in measurements of quark masses. Therefore, in order to obtain a unique value for the CP violation phase, two procedures are applied:

The first is in finding the mean values between two symmetrically arranged CP violating phases in relation to the +y-axis.

$$\langle \delta_{CKM} \rangle = \frac{\delta_{BFbs+} + \delta_{BFbd-}}{2} \approx \frac{164.734^\circ + 15.266^\circ}{2} = 90^\circ. \tag{38}$$

It is important to understand that this value for the CP violating phase is not measured in experiments. The role of this value will be seen in the section dealing with the derivation of the equation in Wolfenstein's parameterization for the CP violating phase.

Another way is to correct the measured quark masses. This will be applied in a separate section for the procedure for correcting the measured quark mass value.

4.2. Calculation Flow for up Quarks

We include the formulas (29) and (30) in the calculations, so we will have:

$$\begin{aligned}
\delta_{BFtu+} &= 180^\circ \times \frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} \\
&= \left\{\left[180^\circ \times \left(172690^2 - 2.16^2\right) / \left(1270^2 - 2.16^2\right)\right] / 360^\circ - 9244\right\} \times 360^\circ \approx 293.192^\circ. \\
\delta_{BFtu-} &= -180^\circ \times \frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} \\
&= \left\{\left[-180^\circ \times \left(172690^2 - 2.16^2\right) / \left(1270^2 - 2.16^2\right)\right] / 360^\circ + 9245\right\} \times 360^\circ \approx 66.808^\circ. \\
\delta_{BFtc+} &= 180^\circ \times \frac{\Delta m_{tc}^2}{\Delta m_{cu}^2} \\
&= \left\{\left[180^\circ \times \left(172690^2 - 1270^2\right) / \left(1270^2 - 2.16^2\right)\right] / 360^\circ - 9244\right\} \times 360^\circ \approx 113.192^\circ. \\
\delta_{BFtc-} &= -180^\circ \times \frac{\Delta m_{tc}^2}{\Delta m_{cu}^2} \\
&= \left\{\left[-180^\circ \times \left(172690^2 - 1270^2\right) / \left(1270^2 - 2.16^2\right)\right] / 360^\circ + 9245\right\} \times 360^\circ \approx 246.808^\circ.
\end{aligned} \tag{39}$$

Based on these results (39), we calculate the mean arithmetic value for the CP violating phase:

$$\langle \delta \rangle_{CKM} = \frac{\delta_{BFtu-} + \delta_{BFtc+}}{2} \approx \frac{66.808 + 113.192}{2} = 90^\circ. \tag{40}$$

4.3. The Process of Correcting the Measured Values of down Quarks

The application of the mean value means that the parameters of the quarks, namely the corresponding masses of the quarks, are not determined precisely enough during the measurements. Since a unique value for the CP violation phase must work in all transitions, then in that case we equate the formulas δ_{BFbs+} and δ_{BFbd-}. In that case, we single out the parameters that we consider to be measured accurately enough, as well as the parameter that needs to be corrected.

Therefore, the process of correcting one of the selected physical quantities is approached by equating both formulas with opposite signs. In this connection, we can formulate the following task:

If the measured values in the best fit $m_d = 4.67$ MeV, $m_s = 93.4$ MeV calculate the value for m_b when $\delta_{BFbs+} = \delta_{BFbd-}$.

Calculation flow

$$\begin{aligned}
&\delta_{BFbs+} = 180^\circ \times \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} = \delta_{BFbd-} = -180^\circ \times \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} \to \\
&\frac{1}{2}\left[\Delta m_{bd}^2 / \left(93.4^2 - 4.67^2\right) - 1\right] - 1003 = -\frac{1}{2}\Delta m_{bd}^2 / \left(93.4^2 - 4.67^2\right) + 1004 \to \\
&\Delta m_{bd}^2 = m_b^2 - m_d^2 = 2007.5 \times \left(93.4^2 - 4.67^2\right) \to \\
&m_b = \sqrt{2007.5 \times \left(93.4^2 - 4.67^2\right) + 4.67^2} = 4179.567817672 \text{ MeV} \\
&\quad \approx 4179.568 \text{ MeV}.
\end{aligned} \tag{41}$$

Using the corrected value for m_b (41) we find:

$$
\begin{aligned}
V_d &= \sin 2\pi \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} = \sin 2\pi \frac{4179.567817672^2 - 93.4^2}{93.4^2 - 4.67^2} \approx -2.668\times 10^{-7}, \\
W_d &= \sin^2 \pi \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}. = \sin^2 \pi \frac{4179.567817672^2 - 93.4^2}{93.4^2 - 4.67^2} = 1. \\
V_d &= \sin 2\pi \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} = \sin 2\pi \frac{4179.567817672^2 - 4..67^2}{93.4^2 - 4.67^2} \approx -2.668\times 10^{-7}, \\
W_d &= \sin^2 \pi \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} = \sin^2 \pi \frac{4179.567817672^2 - 4.67^2}{93.4^2 - 4.67^2} = 1.
\end{aligned} \tag{42}
$$

These are theoretical results that show on which value scale the coefficients (42) are located.

The corrected value compared to the measured value can be displayed as follows:

$$
\begin{aligned}
& m_{b\exp} - m_{bth} = (4180 - 4179.5678)\,\text{MeV} \approx 0.40\ \text{MeV} \to \\
& (m_{bth}) = (4180)_{-0.40}\ \text{MeV} = (4.18)_{-0.00040}\ \text{GeV}.
\end{aligned} \tag{43}
$$

Based on the measured values (35), we see that the corrected value (41) is in the domain of -1σ .

Note. The values (42) will be used to define the equation in Wolfenstein's parameterization.

4.4. The Process of Correcting the Measured Values of up Quarks

As in the case of down quarks, here we also select physical quantities that we assume are sufficiently accurately measured, and we single out the physical quantity that we need to correct. and that theoretical procedure looks like this:

$$
\begin{aligned}
& \delta_{BFtc+} = 180^\circ \times \frac{\Delta m_{tc}^2}{\Delta m_{cu}^2} = \delta_{BFtu-} = -180^\circ \times \frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} \to \\
& \frac{1}{2}\left[\frac{\Delta m_{tu}^2}{1270^2 - 2.16^2} - 1\right] - 9244 = -\frac{1}{2}\frac{\Delta m_{tu}^2}{1270^2 - 2.16^2} + 9245 \to \\
& m_t = \sqrt{18489.5\times(1270^2 - 2.16^2) + 2.16^2} = 172689.398314\ \text{MeV} \\
& \quad \approx 172689.398\ \text{MeV}.
\end{aligned} \tag{44}
$$

Using the corrected value for m_t (44) we find:

$$
\begin{aligned}
V_u &= \sin\left(2\pi\times\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}\right) = \sin\left[2\pi\times\frac{172689.398314^2 - 1270^2}{1270^2 - 2.16^2}\right] = 1.192\times 10^{-8}. \\
V_u &= \sin\left(2\pi\times\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}\right) = \sin\left[2\pi\times\frac{172689.398314^2 - 2.16^2}{1270^2 - 2.16^2}\right] = 1.192\times 10^{-8}. \\
W_u &= \sin^2\left(\pi\times\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}\right) = \sin^2\left[\frac{172689.398314^2 - 2.16^2}{1270^2 - 2.16^2}\right] = 1. \\
W_u &= \sin^2\left(\pi\times\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}\right) = \sin^2\left[\frac{172689.398314^2 - 1270^2}{1270^2 - 2.16^2}\right] = 1.
\end{aligned} \tag{45}
$$

These are theoretical results that show on which value scale the coefficients (45) are located.

The corrected value compared to the measured value can be displayed as follows:

$$\begin{aligned} & m_{texp} - m_{tth} \approx (172690 - 172689.3983)\ \text{MeV} \approx 0.60\ \text{MeV} \rightarrow \\ & m_{bth} = 172690_{-0.60}\ \text{MeV} = 172.69_{-0.00060}\ \text{GeV}. \end{aligned} \tag{46}$$

Based on the measured values (36), we see that the corrected value (46) is in the domain of -1σ.

Note. The values (45) will be used to define the equation in Wolfenstein's parameterization.

5. Wolfenstein Parameterization: Sum Rule for Quark Oscillation Probabilities

In this chapter, we will apply the Wolfenstein parameterization for both down quarks and up quarks via the following matrix transformations: [18] [19]:

Down type quarks

$$\begin{aligned} \begin{pmatrix} d^I \\ s^I \\ b^I \end{pmatrix} &= V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \\ &= \begin{pmatrix} 1 - s_{12}^2/2 & s_{12} & s_{13}e^{-i\delta} \\ -s_{12} & 1 - s_{12}^2/2 & s_{23} \\ s_{12}s_{23} - s_{13}e^{i\delta} & -s_{23} & 1 \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \end{aligned} \tag{47}$$

Up type quarks

$$\begin{aligned} \begin{pmatrix} u^I \\ c^I \\ t^I \end{pmatrix} &= V_{CKM} \begin{pmatrix} u \\ c \\ t \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \\ &= \begin{pmatrix} 1 - s_{12}^2/2 & s_{12} & s_{13}e^{-i\delta} \\ -s_{12} & 1 - s_{12}^2/2 & s_{23} \\ s_{12}s_{23} - s_{13}e^{i\delta} & -s_{23} & 1 \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \end{aligned} \tag{48}$$

5.1. Derivation of the Equation for CP Violating Phase: Transition $d^I \rightarrow s^I, d^I \rightarrow b^I, d^I \rightarrow d^I$

The goal of this chapter is to derive the quark motion equation in which the CP violation phase for quarks will appear as an unknown quantity. We analyze possible transition: $d^I \rightarrow s^I, d^I \rightarrow b^I, d^I \rightarrow d^I$. In the theory of quarks, the differences of the squares of the quark masses are not mentioned nor are they measured.

The key difference between neutrinos and quarks is in terms of mass. Namely, with neutrinos, only the square of the mass difference is measured, while with quarks, they are measured and obtained explicitly numerical values for both down-type quarks and up-type quarks.

The derivation of the equation for the CP violation phase for quarks is based on the formulas (10, 11) and (12), which we now apply in full in the further procedure.

Transition $d^I \to s^I, d^I \to b^I, d^I \to d^I$

$$
\begin{aligned}
&P\left(d^I \to S^I\right)+P\left(d^I \to b^I\right)+P\left(d^I \to d^I\right)\\
&=1-4R\left\{V_{ud}V_{cd}^{*}V_{us}^{*}V_{cs}\sin^2\left(\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\}+2\,\mathrm{Im}\left\{V_{ud}V_{cd}^{*}V_{us}^{*}V_{cs}\sin\left(2\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\}\\
&\quad-4R\left\{V_{ud}V_{cd}^{*}V_{ub}^{*}V_{cb}\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\}+2\,\mathrm{Im}\left\{V_{ud}V_{cd}^{*}V_{ub}^{*}V_{cb}\sin\left(2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\}\\
&\quad-4R\left\{V_{us}V_{cs}^{*}V_{ub}^{*}V_{cb}\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\}+2\,\mathrm{Im}\left\{V_{us}V_{cs}^{*}V_{ub}^{*}V_{cb}\sin\left(2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\}\\
&\quad-4R\left\{V_{ud}V_{td}^{*}V_{us}^{*}V_{ts}\sin^2\left(\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\}+2\,\mathrm{Im}\left\{V_{ud}V_{td}^{*}V_{us}^{*}V_{ts}\sin\left(2\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)\right\}\\
&\quad-4R\left\{V_{ud}V_{td}^{*}V_{ub}^{*}V_{tb}\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\}+2\,\mathrm{Im}\left\{V_{ud}V_{td}^{*}V_{ub}^{*}V_{tb}\sin\left(2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\right\}\\
&\quad-4R\left\{V_{us}V_{ts}^{*}V_{ub}^{*}V_{tb}\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\}+2\,\mathrm{Im}\left\{V_{us}V_{ts}^{*}V_{ub}^{*}V_{tb}\sin\left(2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\right\}\\
&\quad-4\left|V_{ud}\right|^2\left|V_{us}\right|^2\sin^2\left(\pi\frac{\Delta m_{sd}^2}{\Delta m_{sd}^2}\right)-4\left|V_{ud}\right|^2\left|V_{ub}\right|^2\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)\\
&\quad-4\left|V_{us}\right|^2\left|V_{ub}\right|^2\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right)\\
&=1.
\end{aligned}
\tag{49}
$$

By making appropriate simplifications, we get an equation in which the CP violating phase is an unknown quantity:

$$
\begin{aligned}
&-4W_d\,\mathrm{Re}\left\{V_{ud}V_{cd}^{*}V_{ub}^{*}V_{cb}\right\}+2V_d\,\mathrm{Im}\left\{V_{ud}V_{cd}^{*}V_{ub}^{*}V_{cb}\right\}\\
&-4W_d\,\mathrm{Re}\left\{V_{us}V_{cs}^{*}V_{ub}^{*}V_{cb}\right\}+2V_d\,\mathrm{Im}\left\{V_{us}V_{cs}^{*}V_{ub}^{*}V_{cb}\right\}\\
&-4W_d\,\mathrm{Re}\left\{V_{ud}V_{td}^{*}V_{ub}^{*}V_{tb}\right\}+2V_d\,\mathrm{Im}\left\{V_{ud}V_{td}^{*}V_{ub}^{*}V_{tb}\right\}\\
&-4W_d\,\mathrm{Re}\left\{V_{us}V_{ts}^{*}V_{ub}^{*}V_{tb}\right\}+2V_d\,\mathrm{Im}\left\{V_{us}V_{ts}^{*}V_{ub}^{*}V_{tb}\right\}\\
&-4W_d\left|V_{ud}\right|^2\left|V_{ub}\right|^2-4W_d\left|V_{us}\right|^2\left|V_{ub}\right|^2=0.
\end{aligned}
\tag{50}
$$

Where the corresponding factors are equal to:

$$
\begin{aligned}
&V_d=\sin\left(2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)=\sin\left(2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right);\\
&W_d=\sin^2\left(\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\right)=\sin^2\left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\right);\\
&\Delta m_{bd}^2=m_b^2-m_d^2,\Delta m_{bs}^2=m_b^2-m_s^2,\Delta m_{sd}^2=m_s^2-m_d^2.
\end{aligned}
\tag{51}
$$

If the corresponding elements of the Wolfenstein matrix are inserted into Equation (50), an equation for down quarks of the following form is obtained:

$$\begin{aligned}
&+4W_d S_{12}S_{23}S_{13}\left(1-S_{12}^2/2\right)\cos\delta_w-2V_d S_{12}S_{23}S_{13}\left(1-S_{12}^2/2\right)\sin\delta_w\\
&-4W_d S_{12}S_{23}S_{13}\left(1-S_{12}^2/2\right)\cos\delta_w+2V_d S_{12}S_{23}S_{13}\left(1-S_{12}^2/2\right)\sin\delta_w\\
&-4W_d S_{12}S_{23}S_{13}\left(1-S_{12}^2/2\right)\cos\delta_w+4W_d\left(1-S_{12}^2/2\right)S_{13}^2\\
&+2V_d S_{12}S_{23}S_{13}\left(1-S_{12}^2/2\right)\sin\delta_w+4W_d S_{12}S_{23}S_{13}\cos\delta_w\\
&-2V_d S_{12}S_{23}S_{13}\sin\delta_w-4W_d\left(1-S_{12}^2/2\right)^2 S_{13}^2-4W_d S_{12}^2S_{13}^2=0\rightarrow\\
&\frac{S_{12}^2}{2}\cos\delta_w-\frac{V_d}{2W_d}\frac{S_{12}^2}{2}\sin\delta_w+\frac{S_{13}}{S_{12}S_{23}}\left[1-\frac{S_{12}^2}{2}-\left(1-\frac{S_{12}^2}{2}\right)^2-S_{12}^2\right]=0.
\end{aligned}\tag{52}$$

From here, the final form of the equation for the CP violating phase is obtained, in which it appears as an unknown physical quantity:

$$\begin{aligned}
&\cos\delta_W-\frac{V_d}{2W_d}\sin\delta_w-\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}=0\rightarrow\\
&\frac{V_d}{2W_d}=\frac{\sin 2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}}{2\sin^2\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}}=\frac{\sin 2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}}{2\sin^2\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}}=\cot\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}=\cot\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\rightarrow\\
&\cos\delta_W-\cot\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\sin\delta_w-\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}=0\rightarrow\\
&\cos\delta_W-\cot\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\sin\delta_w-\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}=0.
\end{aligned}\tag{53}$$

If we take into account that the factors along $\sin\delta_w$, both in the CKM matrix and in Wolfenstein's parameterization, are equal to each other, then a connection can be established between the corresponding CP violating phases and it can be written like this:

$$\begin{aligned}
&\cos\delta_W-\frac{V_d}{2W_d}\sin\delta_w-\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}=0\rightarrow\\
&\cos\delta_W-\frac{\cos\delta_{CKM}}{\sin\delta_{CKM}}\sin\delta_w=\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}\rightarrow\\
&\cos\delta_W\sin\delta_{CKM}-\cos\delta_{CKM}\sin\delta_w=\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}\sin\delta_{CKM}\rightarrow\\
&\cos\delta_W-\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}\sin\delta_{CKM}=0\leftarrow\delta_{CKM}=+\frac{\pi}{2}.
\end{aligned}\tag{54}$$

If the question of the sign of $\delta_{CKM}=\pm\pi/2$ in the union Equations (32) and (34) was undetermined, now it can be seen that δ_{CKM} is equal to $+\pi/2$ and Equation (54) takes its final form:

$$\begin{aligned}
&\cos\delta_W=\left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}\rightarrow\\
&\delta_w=\arccos\left[1+0.2265^2/2\right]\times\frac{0.00361}{0.2265\times0.04053}\approx 66.21^{\circ}.
\end{aligned}\tag{55}$$

We derive a similar equation for Up quarks:

$$\cos\delta_W - \frac{V_u}{2W_u}\sin\delta_W - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \to$$

$$\frac{V_u}{2W_u} = \frac{\sin 2\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}}{2\sin^2\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}} = \frac{\sin 2\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}}{2\sin^2\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}} = \cot\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} = \cot\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2} \to \quad (56)$$

$$\cos\delta_W - \cot\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}\sin\delta_W - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \to$$

$$\cos\delta_W - \cot\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}\sin\delta_W - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0.$$

If we take into account that the factors along $\sin\delta_w$, both in the CKM matrix and in Wolfenstein's parameterization, are equal to each other, then a connection can be established between the corresponding CP violating phases and it can be written like this:

$$\cos\delta_W - \frac{V_u}{2W_u}\sin\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \to$$

$$\cos\delta_W - \frac{\cos\delta_{CKM}}{\sin\delta_{CKM}}\sin\delta_w = \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} \to \quad (57)$$

$$\cos\delta_W\sin\delta_{CKM} - \cos\delta_{CKM}\sin\delta_w = \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}\sin\delta_{CKM} \leftarrow \delta_{CKM} = +\frac{\pi}{2}.$$

If the value for $\delta_{CKM} = +\pi/2$ in the CKM parameterization is taken into account, the Equation (57) takes the final form:

$$\cos\delta_W = \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} \to$$

$$\delta_w = \arccos\left[1+0.2265^2/2\right]\times\frac{0.00361}{0.2265\times 0.04053} \approx 66.21^\circ. \quad (58)$$

Remark: The same value (58) is also obtained for $\delta_{CKM} = -\pi/2$

6. Calculation Examples for Applying Equations (53) and (56)

The following data are given for the measured values of down quarks in the best fit:

Down quarks

$$m_d = 4.67^{+0.18}_{-0.17}\ \text{MeV} \to (m_d)_{BF} = 4.67\ \text{MeV}.$$

$$m_s = 93.4^{+8.6}_{-3.4}\ \text{MeV} \to (m_s)_{BF} = 93.4\ \text{MeV}. \quad (59)$$

$$(m_b)_{BF} = ?$$

1) Using the Equation (53) predict the value for $(m_b)_{BF}$ that should be measured in the experiments so that the measured value for the CP violating phase would be $(\delta_w)_{BF} = 66.21^\circ$.

Answer

a) Calculation Flow

The calculation flow that leads us to the value for $(m_b)_{BF}$ is as follows:

$$\cos\delta_W - \cot\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}\sin\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \rightarrow$$

$$\tan\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} = \frac{\sin\delta_w}{\cos\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}} \rightarrow$$

$$\left(-180^\circ\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}/360^\circ + 1004\right)\times 360^\circ = \arctan\frac{\sin\delta_w}{\cos\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}} \rightarrow$$

$$m_b = \left[\left|4.67^2 + 2\times\left|1004 - \frac{1}{360^\circ}\arctan\frac{\sin 66.21^\circ}{\cos 66.21^\circ - \left(1+0.2265^2/2\right)\times\frac{0.00361}{0.2265\times 0.04053}}\right|\times\left(93.4^2 - 4.67^2\right)\right|\right]^{1/2} \quad (60)$$

$= 4179.567837\ \text{MeV} \approx 4179.568\ \text{MeV}.$

$$\cos\delta_W - \cot\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}\sin\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \rightarrow$$

$$\left(180^\circ\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}/360^\circ - 1003\right)\times 360^\circ = \arctan\frac{\sin\delta_w}{\cos\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}} \rightarrow$$

$$m_b = \left[\left|93.4^2 + 2\times\left|1003 + \frac{1}{360^\circ}\arctan\frac{\sin 66.21^\circ}{\cos 66.21^\circ - \left(1+0.2265^2/2\right)\times\frac{0.00361}{0.2265\times 0.04053}}\right|\times\left(93.4^2 - 4.67^2\right)\right|\right]^{1/2}$$

$= 4179.567798\ \text{MeV} \approx 4179.568\ \text{MeV}.$

Up quarks

The following data are given for the measured values of Up quarks in the best fit:

$$m_u = 2.16^{+0.49}_{-0.26}\ \text{MeV} \rightarrow \left(m_u\right)_{BF} = 2.16\ \text{MeV}.$$
$$m_c = 1.27 \pm 0.02\ \text{GeV} \rightarrow \left(m_c\right)_{BF} = 1270\ \text{MeV}. \quad (61)$$
$$\left(m_t\right)_{BF} = ?$$

2) Using the Equation (56) predict the value for $\left(m_t\right)_{BF}$ that should be measured in the experiments so that the measured value for the CP violating phase would be $\left(\delta_w\right)_{BF} = 66.21^\circ$.

Answer

a) Calculation Flow

The calculation flow that leads us to the value for $\left(m_t\right)_{BF}$ is as follows:

$$\cos\delta_W - \cot\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}\sin\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \rightarrow$$

$$\tan\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} = \frac{\sin\delta_w}{\cos\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}} \rightarrow$$

$$\left(-180^\circ\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}/360^\circ + 9245\right)\times 360^\circ = \arctan\frac{\sin\delta_w}{\cos\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}} \rightarrow$$

$$m_t = \left[2.16^2 + 2 \times \left(9245 - \frac{1}{360^\circ} \arctan \frac{\sin 66.21^\circ}{\cos 66.21^\circ - \left(1 + 0.2265^2/2\right) \times \dfrac{0.00361}{0.2265 \times 0.04053}} \right) \times \left(1270^2 - 2.16^2\right) \right]^{1/2}$$

$$= 172689.398405 \text{ MeV} \approx 172689.398 \text{ MeV}.$$

$$\cos\delta_W - \cot\pi \frac{\Delta m_{tc}^2}{\Delta m_{cu}^2} \sin\delta_w - \left(1 + s_{12}^2/2\right) \frac{s_{13}}{s_{12}s_{23}} = 0 \rightarrow$$

$$\left(180^\circ \frac{\Delta m_{tc}^2}{\Delta m_{cu}^2} / 360^\circ - 9244\right) \times 360^\circ = \arctan \frac{\sin\delta_w}{\cos\delta_w - \left(1 + s_{12}^2/2\right) \dfrac{s_{13}}{s_{12}s_{23}}} \rightarrow \quad (62)$$

$$m_t = \left[1270^2 + 2 \times \left(9244 + \frac{1}{360^\circ} \arctan \frac{\sin 66.21^\circ}{\cos 66.21^\circ - \left(1 + 0.2265^2/2\right) \times \dfrac{0.00361}{0.2265 \times 0.04053}} \right) \times \left(1270^2 - 2.16^2\right) \right]^{1/2}$$

$$= 172689.398223 \text{ MeV} \approx 172689.398 \text{ MeV}.$$

7. Summary and Conclusions

The main goal of this work is to form a mathematical structure through which the numerical value for the CP violation phase for quarks could be calculated using the measured parameters obtained specifically for the Wolfenstein parameterization.

In essence, this mathematical structure consists of the following parts:

1) Original CKM matrix,

2) Wolfenstein parameterization,

3) Application of the rule for the sum of quark oscillation probabilities,

4) A procedure for determining the largest wavelength within which all quark oscillations are included based on the rule for the sum of quark oscillation probabilities, both for down quarks and up quarks, on the femtometer space scale (given separately in **Appendix A**).

5) In **Appendix B**, the unitary triangles in the first and second quadrants for both parameterizations are specifically determined in order to establish the possible occurrence of degeneration or mutual dependence of those triangles.

Based on the thus established mathematical structure, it was shown that δ_{CKM} found in the CKM parameterization is mirrored in Wolfenstein's parameterization, which shows the necessary presence of both parameterizations:

Down quarks

$$\cos\delta_W - \frac{V_d}{2W_d} \sin\delta_w - \left(1 + s_{12}^2/2\right) \frac{s_{13}}{s_{12}s_{23}} = 0 \rightarrow$$

$$\cos\delta_W - \frac{\cos\delta_{CKM}}{\sin\delta_{CKM}} \sin\delta_w = \left(1 + s_{12}^2/2\right) \frac{s_{13}}{s_{12}s_{23}} \rightarrow$$

$$\cos\delta_W \sin\delta_{CKM} - \cos\delta_{CKM} \sin\delta_w = \left(1 + s_{12}^2/2\right) \frac{s_{13}}{s_{12}s_{23}} \sin\delta_{CKM} \leftarrow \delta_{CKM} = \frac{\pi}{2} \rightarrow$$

$$\cos\delta_W - \left(1 + s_{12}^2/2\right) \frac{s_{13}}{s_{12}s_{23}} = 0.$$

Up quarks

$$\cos\delta_W - \frac{V_u}{2W_u}\sin\delta_w - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0 \to$$

$$\cos\delta_W - \frac{\cos\delta_{CKM}}{\sin\delta_{CKM}}\sin\delta_w = \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} \to$$

$$\cos\delta_W \sin\delta_{CKM} - \cos\delta_{CKM}\sin\delta_w = \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}}\sin\delta_{CKM} \leftarrow \delta_{CKM} = \frac{\pi}{2} \to$$

$$\cos\delta_W - \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} = 0.$$

Numerical value for the CP violating phase for quarks

$$\cos\delta_W = \left(1+s_{12}^2/2\right)\frac{s_{13}}{s_{12}s_{23}} \to$$

$$\delta_w = \arccos\left[1+0.2265^2/2\right]\times\frac{0.00361}{0.2265\times0.04053} \approx 66.21^\circ.$$

From a methodological point of view, the common thread that unites CKM and Wolfenstein's parameterization is in the application of the rule for the sum of quark oscillation probabilities, which contains the same factors in the form: $\frac{V_d}{2W_d} = \cot\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} = \cot\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}$ and $\frac{V_u}{2W_u} = \cot\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} = \cot\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}$. One could get the impression that it is enough to apply only Wolfenstein's parameterization, but in that case we would have the following numerical values:

$$\frac{V_d}{2W_d} = \cot\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} = \cot\pi\frac{4180^2-4.67^2}{93.4^2-4.67^2}$$

$$= \cot\pi\frac{4180^2-93.4^2}{93.4^2-4.67^2} = -3.66386 \to$$

$$\frac{V_u}{2W_u} = \cot\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2} = \cot\pi\frac{172690^2-2.16^2}{1270^2-2.16^2}$$

$$= \cot\pi\frac{172690^2-1270^2}{1270^2-2.16^2} = -0.428429.$$

These numerical values show that the equation for the CP violating phase for down quarks would differ from the equation for up quarks, which does not make any physical sense. Due to these facts, the importance and necessity of including the CKM matrix can be seen, which is clearly indicated by the results obtained in (42, 45) and especially in (C1, C2, C3).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A: Determining the Space in Which Quark Oscillations Take Place

The goal of this appendix is to determine the region where oscillations occur for down and up quarks. These are spaces that are invisible to experiments and they are defined with the largest wavelength that determines the space in which these oscillations occur.

In the following text, the flow of calculation of various mathematical relations is presented, assuming that all types of quarks are free relativistic particles whose velocities in the process of oscillation are close to the speed of light.

$$\begin{aligned} E_1 &= \sqrt{p_d^2c^2 + m_d^2c^4} \rightarrow p_d^2 = \frac{E_1^2}{c^2} - m_d^2c^2 \rightarrow p_d = \frac{E_1}{c}\sqrt{1-\frac{m_d^2c^4}{E_1^2}}. \\ E_2 &= \sqrt{p_s^2c^2 + m_s^2c^4} \rightarrow p_s^2 = \frac{E_2^2}{c^2} - m_s^2c^2 \rightarrow p_s = \frac{E_2}{c}\sqrt{1-\frac{m_s^2c^4}{E_2^2}}. \\ E_3 &= \sqrt{p_b^2c^2 + m_b^2c^4} \rightarrow p_b^2 = \frac{E_3^2}{c^2} - m_b^2c^2 \rightarrow p_b = \frac{E_3}{c}\sqrt{1-\frac{m_b^2c^4}{E_3^2}}. \end{aligned} \tag{A1}$$

We introduce the assumption that quark energies are mutually equal:

$$E_1 = E_2 = E_3 = E \tag{A2}$$

then we can write equations for momentums for all three quarks:

$$\begin{aligned} p_d &= \frac{E}{c}\sqrt{1-\frac{m_d^2c^4}{E^2}} \approx \frac{E}{c}\left(1-\frac{m_d^2c^4}{2E^2}\right) = \frac{E}{c} - \frac{m_d^2c^3}{2E} = \frac{h}{\lambda_d} \cdot \frac{m_d^2c^4}{E^2} \ll 1. \\ p_s &= \frac{E}{c}\sqrt{1-\frac{m_s^2c^4}{E^2}} \approx \frac{E}{c}\left(1-\frac{m_s^2c^4}{2E^2}\right) = \frac{E}{c} - \frac{m_s^2c^3}{2E} = \frac{h}{\lambda_s} \cdot \frac{m_s^2c^4}{E^2} \ll 1. \\ p_b &= \frac{E}{c}\sqrt{1-\frac{m_b^2c^4}{E^2}} \approx \frac{E}{c}\left(1-\frac{m_b^2c^4}{2E^2}\right) = \frac{E}{c} - \frac{m_b^2c^3}{2E} = \frac{h}{\lambda_b} \cdot \frac{m_b^2c^4}{E^2} \ll 1. \end{aligned} \tag{A3}$$

In the next step, we write relations that connect the differences between the corresponding momentums:

$$\begin{aligned} & p_d - p_s = \frac{E}{c} - \frac{m_d^2c^3}{2E} - \frac{E}{c} + \frac{m_s^2c^3}{2E} = \frac{h}{\lambda_d} - \frac{h}{\lambda_s} \rightarrow \\ & p_d = \hbar k_d = \hbar\frac{2\pi}{\lambda_d} = \frac{h}{\lambda_d} \\ & p_s = \hbar k_s = \hbar\frac{2\pi}{\lambda_s} = \frac{h}{\lambda_s} \rightarrow \lambda_d < \lambda_s \\ & p_d - p_s = \frac{c^3}{2E}\left(m_s^2 - m_d^2\right) = \frac{c^3}{2E}\Delta m_{sd}^2 \rightarrow \\ & p_d - p_s = h\left(\frac{\lambda_s - \lambda_d}{\lambda_s\lambda_d}\right) \rightarrow \left(p_d - p_s\right)\frac{\lambda_s\lambda_d}{\lambda_s - \lambda_d} = h. \\ & \frac{c^3}{2E}\Delta m_{sd}^2\frac{\lambda_s\lambda_d}{\lambda_s - \lambda_d} = h \rightarrow \frac{c^3}{2E\hbar}\Delta m_{sd}^2\frac{\lambda_s\lambda_d}{\lambda_s - \lambda_d} = 2\pi. \\ & L_{sd} = \frac{\lambda_s\lambda_d}{\lambda_s - \lambda_d} = \frac{2Eh}{c^3\Delta m_{sd}^2} \end{aligned} \tag{A4}$$

$$\begin{aligned}
&p_d - p_b = \frac{E}{c} - \frac{m_d^2 c^3}{2E} - \frac{E}{c} + \frac{m_b^2 c^3}{2E} = \frac{h}{\lambda_d} - \frac{h}{\lambda_b} \\
&p_d - p_b = h\frac{\lambda_b - \lambda_d}{\lambda_b \lambda_d} \to \left(p_d - p_b\right)\frac{\lambda_b \lambda_d}{\lambda_b - \lambda_d} = h \to \\
&\frac{c^3}{2E}\Delta m_{bs}^2 \frac{\lambda_b \lambda_d}{\lambda_b - \lambda_d} = h \to \\
&\frac{c^3}{2E\hbar}\Delta m_{bd}^2 \frac{\lambda_b \lambda_d}{\lambda_b - \lambda_d} = 2\pi \to \\
&L_{bd} = \frac{\lambda_b \lambda_d}{\lambda_b - \lambda_d} = \frac{2Eh}{c^3 \Delta m_{bd}^2}.
\end{aligned} \tag{A5}$$

$$\begin{aligned}
&p_s - p_b = \frac{E}{c} - \frac{m_s^2 c^3}{2E} - \frac{E}{c} + \frac{m_b^2 c^3}{2E} = \frac{h}{\lambda_s} - \frac{h}{\lambda_b} \to \\
&p_s - p_b = \frac{c^3}{2E}\left(m_b^2 - m_s^2\right) = \frac{c^3}{2E}\Delta m_{bs}^2 \to \\
&p_s - p_b = h\frac{\lambda_b - \lambda_s}{\lambda_b \lambda_s} \to \left(p_s - p_b\right)\frac{\lambda_b \lambda_s}{\lambda_b - \lambda_s} = h \to \\
&\frac{c^3}{2E}\Delta m_{bs}^2 \frac{\lambda_b \lambda_s}{\lambda_b - \lambda_s} = h \to \frac{c^3}{2E\hbar}\Delta m_{bs}^2 \frac{\lambda_b \lambda_s}{\lambda_b - \lambda_s} = 2\pi \to \\
&L_{bs} = \frac{\lambda_b \lambda_s}{\lambda_b - \lambda_s} = \frac{2Eh}{c^3 \Delta m_{bs}^2}.
\end{aligned} \tag{A6}$$

In the next step, we determine the relationship between the values for the equivalent values of the wavelengths:

$$\begin{aligned}
&L_{bd} = \frac{\lambda_b \lambda_d}{\lambda_b - \lambda_d} = \frac{h}{p_b}\frac{h}{p_d}\frac{1}{\frac{h}{p_b} - \frac{h}{p_d}} = \frac{h}{p_b}\frac{h}{p_d}\frac{p_b p_d}{h\left(p_d - p_b\right)} = \frac{h}{p_d - p_b} = \frac{2Eh}{c^3 \Delta m_{bd}^2}. \\
&L_{sd} = \frac{\lambda_s \lambda_d}{\lambda_s - \lambda_d} = \frac{h}{p_s}\frac{h}{p_d}\frac{1}{\frac{h}{p_s} - \frac{h}{p_d}} = \frac{h}{p_s}\frac{h}{p_d}\frac{p_s p_d}{h\left(p_d - p_s\right)} = \frac{h}{p_d - p_s} = \frac{2Eh}{c^3 \Delta m_{sd}^2}. \\
&L_{bs} = \frac{\lambda_b \lambda_s}{\lambda_b - \lambda_s} = \frac{h}{p_b}\frac{h}{p_s}\frac{1}{\frac{h}{p_b} - \frac{h}{p_s}} = \frac{h}{p_b}\frac{h}{p_s}\frac{p_b p_s}{h\left(p_s - p_b\right)} = \frac{h}{p_s - p_b} = \frac{2Eh}{c^3 \Delta m_{bs}^2}.
\end{aligned} \tag{A7}$$

Based on the relations between the differences of the squares of the masses, we find the relation between the equivalent wavelengths:

$$\begin{aligned}
&L_{sd} = \frac{2Eh}{c^3 \Delta m_{sd}^2}, L_{bs} = \frac{2Eh}{c^3 \Delta m_{bs}^2}, L_{bd} = \frac{2Eh}{c^3 \Delta m_{bd}^2} \to \\
&\frac{2Eh}{c^3 \Delta m_{sd}^2} > \frac{2Eh}{c^3 \Delta m_{bs}^2} > \frac{2Eh}{c^3 \Delta m_{bd}^2} \leftarrow \Delta m_{sd}^2 < \Delta m_{bs}^2 < \Delta m_{bd}^2 \to \\
&L_{sd} > L_{bs} > L_{bd}.
\end{aligned} \tag{A8}$$

In order to determine coefficients W_d and V_d all other equivalent wavelengths are reduced to the largest equivalent wavelength L_{sd}. This procedure is given next.

$$
\begin{aligned}
&\left(p_d - p_s\right)\frac{\lambda_s \lambda_d}{\lambda_s - \lambda_d} = \frac{c^3}{2E}\Delta m_{sd}^2 L_{sd} = h \to \frac{c^3}{2E\hbar}\Delta m_{sd}^2 L_{sd} = 2\pi \to \\
&L_{sd} = 2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2} \to \\
&W_d = \sin^2 \frac{\Delta m_{sd}^2 c^3}{4E\hbar}\left(L = L_{sd}\right) = \sin^2 \frac{\Delta m_{sd}^2 c^3}{4E\hbar}\left(2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2}\right) = \sin^2 \pi = 0. \\
&V_d = \sin \frac{\Delta m_{sd}^2 c^3}{2E\hbar}\left(L = L_{sd}\right) = \sin \frac{\Delta m_{sd}^2 c^3}{2E\hbar}\left(2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2}\right) = \sin 2\pi = 0.
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
&\left(p_s - p_b\right)\frac{\lambda_b \lambda_s}{\lambda_b - \lambda_s} = \left(p_s - p_b\right) L_{bs} = h = 2\pi\hbar \to \frac{c^3}{2E\hbar}\Delta m_{bs}^2 L_{bs} = 2\pi. \\
&L_{sd} = 2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2} \to \\
&V_d = \sin \frac{c^3}{2E\hbar}\Delta m_{bs}^2 \left(L = L_{sd}\right) = \sin \frac{c^3}{2E\hbar}\Delta m_{bs}^2 L_{sd} \\
&\quad = \sin \frac{c^3}{2E\hbar}\Delta m_{bs}^2 \left(2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2}\right) = \sin 2\pi \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}. \\
&W_d = \sin^2 \frac{c^3}{4E\hbar}\Delta m_{bs}^2 \left(L = L_{sd}\right) = \sin^2 \frac{c^3}{4E\hbar}\Delta m_{bs}^2 L_{sd} \\
&\quad = \sin^2 \frac{c^3}{4E\hbar}\Delta m_{bs}^2 \left(2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2}\right) = \sin^2 \pi \frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}.
\end{aligned}
\tag{A10}
$$

We now reduce to that (A10) the other two and write it like this:

$$
\begin{aligned}
&\left(p_d - p_b\right)\frac{\lambda_b \lambda_d}{\lambda_b - \lambda_d} = \left(p_d - p_b\right) L_{bd} = h = 2\pi\hbar \to \frac{c^3}{2E\hbar}\Delta m_{bd}^2 L_{bd} = 2\pi. \\
&L_{sd} = 2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2} \to \\
&V_d = \sin \frac{c^3}{2E\hbar}\Delta m_{bd}^2 \left(L = L_{sd}\right) = \sin \frac{c^3}{2E\hbar}\Delta m_{bd}^2 \left(2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2}\right) = \sin 2\pi \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}. \\
&W_d = \sin^2 \frac{c^3}{4E\hbar}\Delta m_{bd}^2 \left(L = L_{sd}\right) = \sin^2 \frac{c^3}{4E\hbar}\Delta m_{bd}^2 \left(2\pi \frac{2E\hbar}{c^3 \Delta m_{sd}^2}\right) = \sin^2 \pi \frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}.
\end{aligned}
\tag{A11}
$$

Applying the same procedure, mathematical relations for up quarks are found. In order to determine coefficients W_u and V_u all other equivalent wavelengths are reduced to the largest equivalent wavelength L_{cu}. This procedure is given next

$$
\begin{aligned}
&L_{cu} = \frac{2Eh}{c^3 \Delta m_{cu}^2}, L_{tc} = \frac{2Eh}{c^3 \Delta m_{tc}^2}, L_{tu} = \frac{2Eh}{c^3 \Delta m_{tu}^2} \to \\
&\frac{2Eh}{c^3 \Delta m_{cu}^2} > \frac{2Eh}{c^3 \Delta m_{tc}^2} > \frac{2Eh}{c^3 \Delta m_{tu}^2} \to \\
&L_{cu} > L_{tc} > L_{tu}.
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
&\left(p_u - p_c\right)\frac{\lambda_c \lambda_u}{\lambda_c - \lambda_u} = \hbar \frac{c^3}{2E}\Delta m_{cu}^2 L_{cu} = h \to \frac{c^3}{2E\hbar}\Delta m_{cu}^2 L_{cu} = 2\pi \to \\
&L_{cu} = 2\pi \frac{2E\hbar}{c^3 \Delta m_{cu}^2}.
\end{aligned}
\tag{A13}
$$

We now reduce to that (A13) the other two and write it like this:

$$\begin{aligned}
&\left(p_c - p_t\right)\frac{\lambda_t \lambda_c}{\lambda_t - \lambda_c} = \left(p_c - p_t\right)L_{tc} = h = 2\pi\hbar \to \frac{c^3}{2E\hbar}\Delta m_{tc}^2 L_{tc} = 2\pi. \\
&L_{cu} = \left(2\pi\frac{2E\hbar}{c^3\Delta m_{cu}^2}\right) \to \\
&V_u = \sin\frac{c^3}{2E\hbar}\Delta m_{tc}^2\left(L = L_{cu}\right) = \sin\left[\frac{c^3}{2E\hbar}\Delta m_{tc}^2\left(2\pi\frac{2E\hbar}{c^3\Delta m_{cu}^2}\right)\right] = \sin 2\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}. \\
&W_u = \sin^2\left[\frac{c^3}{4E\hbar}\Delta m_{tc}^2\left(L = L_{cu}\right)\right] = \sin^2\left[\frac{c^3}{4E\hbar}\Delta m_{tc}^2\left(2\pi\frac{2E\hbar}{c^3\Delta m_{cu}^2}\right)\right] = \sin^2\pi\frac{\Delta m_{tc}^2}{\Delta m_{cu}^2}.
\end{aligned} \tag{A14}$$

$$\begin{aligned}
&\left(p_u - p_t\right)\frac{\lambda_t \lambda_u}{\lambda_t - \lambda_u} = \left(p_u - p_t\right)L_{tu} = h = 2\pi\hbar \to \frac{c^3}{2E\hbar}\Delta m_{tu}^2 L_{tu} = 2\pi. \\
&L_{cu} = 2\pi\frac{2E\hbar}{c^3\Delta m_{cu}^2} \to \\
&V_u = \sin\frac{c^3}{2E\hbar}\Delta m_{tu}^2\left(L = L_{cu}\right) = \sin\frac{c^3}{2E\hbar}\Delta m_{tu}^2\left(2\pi\frac{2E\hbar}{c^3\Delta m_{cu}^2}\right) = \sin 2\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}. \\
&W_u = \sin^2\frac{c^3}{4E\hbar}\Delta m_{tu}^2\left(L = L_{cu}\right) = \sin^2\frac{c^3}{4E\hbar}\Delta m_{tu}^2\left(2\pi\frac{2E\hbar}{c^3\Delta m_{cu}^2}\right) = \sin^2\pi\frac{\Delta m_{tu}^2}{\Delta m_{cu}^2}.
\end{aligned} \tag{A15}$$

All mathematical relations are also applied to up quarks, so corresponding connections are obtained with the relations between the differences of the squares of the masses of the quarks.

Appendix B: Calculation of Angles

The goal of this section is to first of all show that the solutions for the CP violation phases are symmetric to the +y-axis. In this regard, the following statement is proved:

CP phases $\delta_1 = -\pi\Delta m_{bd}^2/\Delta m_{sd}^2$ and $\delta_2 = \pi\Delta m_{bs}^2/\Delta m_{sd}^2$ are symmetrically distributed in relation to the +y-axis.

The proof of this statement is based on finding the sine of the angle δ_1:

$$\begin{aligned}
\sin\delta_1 &= \sin\left(-\pi\Delta m_{bd}^2/\Delta m_{sd}^2\right) = \sin\left[-\pi\left(\Delta m_{bs}^2/\Delta m_{sd}^2 + 1\right)\right] \\
&= -\sin\left[\pi\Delta m_{bs}^2/\Delta m_{sd}^2 + \pi\right] = \sin\left(\pi\Delta m_{bs}^2/\Delta m_{sd}^2\right) = \sin\delta_2
\end{aligned}$$

which concludes the proof.

The consequence of this statement is reflected in the equality of the Jarlskog invariant for both phases:

$$J_{CP1} = J_{CP}^{\max}\sin\delta_1 = J_{CP}^{\max}\sin\left(-\pi\Delta m_{bd}^2/\Delta m_{sd}^2\right) = J_{CP}^{\max}\sin\left(\pi\Delta m_{bs}^2/\Delta m_{sd}^2\right) = J_{CP}^{\max}\sin\delta_2.$$

With this simple calculation, we have shown that the sines of both phase angles are mutually equal, which means that they are symmetrically distributed in relation to the +y-axis. It should be noted that the form of the formula is not identical, but the sines of these angles are equal to each other, as are the areas of the unitary triangles for these angles. From the point of view of this mathematical model, it follows that the official value for the CP violation phase of $\delta = 1.144$ rad should be associated with a symmetrical component in relation to the +y-axis equal to $\pi - 1.144$ rad.

The existence of these two symmetrical CP phases makes it possible to construct two unitary triangles. For those two triangles, the corresponding angles are calculated, and then the corresponding connections between those angles are established based on the following calculations.

B1. CKM Matrix

First quadrant: Case $\delta = 1.144$ rad [16]

$$\beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right] \to$$

$$\cos\beta = \frac{AE - BF + (BE - AF)\cos\delta}{\sqrt{\left[AE - BF + (BE - AF)\cos\delta\right]^2 + \left[(BE + AF)\sin\delta\right]^2}} \to$$

$$\beta \approx 22.463^\circ.$$

$$\alpha = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right] \to \cos\alpha = \frac{F - E\cos\delta}{\sqrt{\left[F - E\cos\delta\right]^2 + \left[E\sin\delta\right]^2}} \to \alpha \approx 92.025^\circ. \quad \text{(B1)}$$

$$\gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right] \to \cos\gamma = \frac{A\cos\delta + B}{\sqrt{\left[A\cos\delta + B\right]^2 + \left[A\sin\delta\right]^2}} \to \gamma \approx 65.511^\circ.$$

$$A = S_{12}C_{23} = 0.22480316, B = C_{12}S_{23}S_{13} = 0.00015035,$$

$$E = S_{12}S_{23} = 0.00940950, F = \sqrt{1 - S_{12}^2}\sqrt{1 - S_{23}^2}S_{13} = 0.0035922.$$

$$(S_{12})_{BF} = S_{12} = 0.225; (S_{23})_{BF} = S_{23} = 0.04182; (S_{13})_{BF} = S_{13} = 0.00369.$$

Second quadrant: The case $\delta \to \pi - 1.144$ rad

$$\beta_{\pi-\delta} = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right]_{\pi-\delta} \to$$

$$\cos\beta_{\pi-\delta} = \frac{AE - BF - (BE - AF)\cos\delta}{\sqrt{\left[AE - BF - (BE - AF)\cos\delta\right]^2 + \left[(BE + AF)\sin\delta\right]^2}} \to$$

$$\beta_{\pi-\delta} \approx 16.739^\circ.$$

$$\alpha_{\pi-\delta} = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right]_{\pi-\delta} \to \cos\alpha_{\pi-\delta} = \frac{F + E\cos\delta}{\sqrt{\left[F + E\cos\delta\right]^2 + \left[E\sin\delta\right]^2}} \to$$

$$\alpha_{\pi-\delta} \approx 48.842^\circ. \quad \text{(B2)}$$

$$\gamma_{\pi-\delta} = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]_{\pi-\delta} \to \cos\gamma_{\pi-\delta} = \frac{-A\cos\delta + B}{\sqrt{\left[-A\cos\delta + B\right]^2 + \left[A\sin\delta\right]^2}} \to$$

$$\gamma_{\pi-\delta} \approx 114.418^\circ.$$

$$A = S_{12}C_{23} = 0.22480316, B = C_{12}S_{23}S_{13} = 0.00015035,$$

$$E = S_{12}S_{23} = 0.00940950, F = \sqrt{1 - S_{12}^2}\sqrt{1 - S_{23}^2}S_{13} = 0.0035922.$$

By comparing the numerical values for the angles of unitary triangles, we see that by mapping the triangle from the first quadrant to the second quadrant, their numerical values do not match.

And it can be written like this:

$$\gamma_\pi \neq \alpha_{\pi-\delta} + \beta_{\pi-\delta}$$
$$\gamma_{\pi-\delta} \neq \alpha_\pi + \beta_\pi \quad \text{(B3)}$$

These relations between the angles can be interpreted that these two triangles are independent. However, two independent CP violating phases cannot exist in principle, because in nature there can only be one unique CP phase. This means that in the mathematical model, two different formulas are generated structures and signs must be given the same numerical value for the CP violating phase in all transitions. In order for this to happen, both formulas must be equalized. An example of that equalization can look like this:

$$-\pi \Delta m_{bd}^2 / \Delta m_{sd}^2 = \pi \Delta m_{bs}^2 / \Delta m_{sd}^2 \quad \text{(B4)}$$

In the original CKM matrix, it is shown that the CP violating phase could be 90 degrees. However, this value is not measured in the experiments, but this equality (B4) is plotted in order to make a correction by the mass of one of the selected quarks, whether it is for down or for up quarks, which is shown above in the main text in a separate chapter.

We perform this calculation for a symmetrical CP violating phase in the second quadrant to see the nature of the connection with the triangle in the first quadrant. From the nature of their connection, we deduce whether there is degeneration. If it is established that there is degeneration, then a unitary triangle is adopted in the first quadrant. In the application of the CKM matrix, we showed that there are two formulas for calculating the CP violating phase and that they are symmetrically distributed in relation to the +y-axis of the trigonometric circle. Then, using those two formulas, we calculated the angles for two unitary triangles and concluded that these triangles mutually independent.

The consequence of that is that both triangles cannot exist at the same time, but only one unitary triangle that represents their resultant. We presented that result in two ways:

1) By finding the mean value for the CP violating phase (38, 40).

2) In the process of correcting one of the selected measured quark masses by equating the formulas in (41, 44), we find a numerical value for the selected quark mass that should be measured.

B2. Wolfenstein Parameterization

First quadrant: Case $\delta = 1.144$ rad [16]

$$\beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right] \to \cos\beta = \frac{1-k\cos\delta}{\sqrt{1-2k\cos\delta+k^2}} \to \beta \approx 23.082^\circ.$$

$$\alpha = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right] \to \cos\alpha = \frac{k-\cos\delta}{\sqrt{1-2k\cos\delta+k^2}} \to \alpha \cong 91.372^\circ.$$

$$\gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right] \to \cos\gamma = \cos\delta \to \gamma = 65.546^\circ. \quad \text{(B5)}$$

$$(S_{12})_{BF} = S_{12} = 0.225; (S_{23})_{BF} = S_{23} = 0.04182;$$

$$(S_{13})_{BF} = S_{13} = 0.00369, k = \frac{S_{13}}{S_{12}S_{23}}.$$

Second quadrant: The case $\delta \to \pi - 1.144$ rad

$$\beta_{\pi-\delta} = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right]_{\pi-\delta} \to \cos\beta_{\pi-\delta} = \frac{1+k\cos\delta}{\sqrt{1+2k\cos\delta+k^2}} \to \beta_{\pi-\delta} \approx 17.073^\circ.$$
$$\alpha_{\pi-\delta} = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right]_{\pi-\delta} \to \cos\alpha_{\pi-\delta} = \frac{k+\cos\delta}{\sqrt{1+2k\cos\delta+k^2}} \to \alpha_{\pi-\delta} \approx 48.473^\circ.$$
$$\gamma_{\pi-\delta} = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]_{\pi-\delta} \to \cos\gamma_{\pi-\delta} = -\cos\delta \to \gamma_{\pi-\delta} \approx 114.454^\circ. \quad \text{(B6)}$$
$$(S_{12})_{BF} = S_{12} = 0.225; (S_{23})_{BF} = S_{23} = 0.04182;$$
$$(S_{13})_{BF} = S_{13} = 0.00369, k = \frac{S_{13}}{S_{12}S_{23}}.$$

Based on the numerical values for the angles of the unitary triangles in the first and second quadrants, it can be seen that the mirrored triangle from the first quadrant to the second quadrant changes its shape.

However, what is interesting is that, unlike the application of the original CKM matrix where one sees that there is no dependence between the corresponding gamma angles for the triangles in the first and second quadrants, here in the application of the Wolfenstein parameterization, this dependence is obvious and it can be written as

$$\gamma = \alpha_{\pi-\delta} + \beta_{\pi-\delta} = 48.473^\circ + 17.073^\circ = 65.546^\circ.$$
$$\gamma_{\pi-\delta} = \alpha + \beta = 23.082^\circ + 91.372^\circ = 114.454^\circ. \quad \text{(B7)}$$

Relation (B7) means the occurrence of degeneracy of unitary triangles and that it is justified to adopt a unitary triangle in the first quadrant which is related to $\delta = 1.144\ \text{rad}$.

Appendix C: Calculation of the Resulting CP Violating Phase without Numerical Values for Quark Masses

C1. Calculation for down Quarks

Use the following formulas in the following calculation:

$$\delta_1 = \pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} \to \left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}/2\pi - 1003\right)\times 2\pi,$$
$$\delta_2 = \pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} \to \left(-\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}/2\pi + 1004\right)\times 2\pi \quad \text{(C1)}$$

I. Show that the following relation holds: $\frac{\delta_1+\delta_2}{2} = \frac{\pi}{2}$.

Proof:

$$\frac{\delta_1+\delta_2}{2} = \frac{\left(\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}/2 - 1003\right)\times 2\pi + \left(-\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}/2 + 1004\right)\times 2\pi}{2}$$
$$= \frac{\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}/2 - 1003 + \left(-\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} - 1\right)/2 + 1004}{2}\times 2\pi \quad \text{(C2)}$$
$$= \frac{1/2}{2}\times 2\pi = \frac{\pi}{2}.$$

II. Calculate δ_1 **and** δ_2 **provided that:** $\delta_1 = \delta_2$.

Calculation flow:

$$\delta_1 = \pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} = \left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}/2\pi - 1003\right)\times 2\pi,$$
$$\delta_2 = \pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} \to \left(-\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}/2\pi + 1004\right)\times 2\pi \to$$
$$\delta_1 = \delta_2 \to \left(\pi\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2}/2\pi - 1003\right)\times 2\pi = \left[\pi\left(-\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} - 1\right)/2\pi + 1004\right]\times 2\pi \to$$
$$\frac{\Delta m_{bs}^2}{\Delta m_{sd}^2} = 2006.5 \to \delta_1 = \left(\frac{1}{2}2006.5 - 1003\right)\times 2\pi = \frac{\pi}{2} \to \tag{C3}$$
$$\left[\pi\left(\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} - 1\right)/2\pi - 1003\right]\times 2\pi = \left(-\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}/2\pi + 1004\right)\times 2\pi$$
$$\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2} = 2007.5 \to \delta_2 = \left(-\pi\frac{\Delta m_{bd}^2}{\Delta m_{sd}^2}/2\pi + 1004\right)\times 2\pi$$
$$= \left(-\frac{1}{2}2007.5 + 1004\right)\times 2\pi = \frac{\pi}{2}.$$

These calculations show that in the CKM basis the value for the CP violation phase is equal to π/2, and that it is not measured in the experiments, but that it is present in the equation for the CP violating phase δ_w which is measured and derived in Wolfenstein's parameterization.

Note: Applying the same calculation procedure for Up quarks leads to the same results.