Journal of High Energy Physics, Gravitation and Cosmology, 2024, 10, 1367-1379
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

Neutrino Masses Prediction

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How to cite this paper: Majkic, R. (2024) Neutrino Masses Prediction. *Journal of High Energy Physics, Gravitation and Cosmology*, **10**, 1367-1379.
https://doi.org/10.4236/jhepgc.2024.104076

Received: May 6, 2024
Accepted: August 26, 2024
Published: August 29, 2024

Abstract

Despite the vast theoretical and experimental knowledge about neutrino physics: neutrino oscillation, neutrino astrophysics, as today, there is no theory to offer the neutrino rest mass number. This paper is not a theory of the particle masses but a simple model built on the existing experimental and theoretical knowledge to predict neutrino masses. The elastic weak current neutrino interaction $(\nu_i, n) \to (l, n') \sim (\nu_e, n) \to (e, p)$, and a few very fundamental principles predict the direct or the neutrino mass increasing hierarchy vector $\mathbf{m} = \left| m_e, m_\mu, m_\tau \right\rangle = \left| 1.35 \times 10^{-8}, 5.81 \times 10^{-4}, 1.64 \times 10^{-1} \right\rangle \text{eV/c}^2$.

Keywords

Interaction Probability, Cross-Section, Interaction Potential, Minimal Coupling

1. Introduction

A physical particle $\mathcal{T}\mathbf{P}$ is an object characterized by its internal variables, $\{m, e, \sigma, \mu, \cdots\}$ motion state $(E, \vec{p})$ and its state function $\mathbf{\Psi}$. Fundamental particle state is its rest state, all internal particle variables are associated with the particle rest state, and all other particle states are the evolution of the fundamental particle under interaction transformation. Simply, the particle $\mathbf{P}$ is all of this $\{m, e, \sigma, \mu, \cdots, (E, \vec{p}), \mathbf{\Psi}\}$.

Here, we consider the neutrino-nucleon interaction $(\nu_i, n) \to (l, \overline{n}')$. Neutrino acts on a nucleon creates a lepton and transforms nucleon into nucleon of opposite charge in an excited state. The detailed process is the neutrino-quark or the quasi-elastic neutrino-quark scattering. In particular, we look at the $(\nu_e, n) \to (p, e)$ interaction presented by the following mapping diagram

$$\hat{\nu} : n \to (p, e) \to (p, e)^*. \tag{1}$$

Initially, electron neutrino, the free state particle, acts as an operator $\hat{\nu}$ transforms neutron to a proton, creates an electron and brings the pair (p,e) in an excited state. The neutrino undergoes complete decay in the course of the process. During the process, the neutrino and electron are in (1:1) correspondence related by the amount of the neutrino energy to create the electron in an energy state. Thus, there is a functional measure of the level of neutrino decay by the "level" of electron creation. The construction of such a function rests on the following facts.

Free neutrino disappears in a complex interaction potential, and its state decay function defines the probability of the neutrino decay and, consequently, the probability of the electron creation in terms of the interaction potential. The theoretical cross-section of the neutrino-electron interaction depends on the electron energy [1], and the interaction probability. Consequently, the cross-section relates the interaction potential and electron energy. Finally, the interaction potential is minimally coupled to the matter in the Kline-Gordon-Dirac equation, see the **Appendix**, and the neutrino decay endpoint, or equivalently the point of its disappearance, relates the neutrino and electron rest energies.

2. Interaction Probability

The neutrino decay starts with its sinking in the constant complex potential Γ at $t=0$, and continues together with the complementary process of electron creation. The neutrino state decay function

$$\Psi = A'\psi(\vec{r},t)e^{-\frac{\Gamma}{\hbar}t}, \quad A\in\mathbb{C}, \ \psi\in \mathrm{L}^2(\mathrm{R},\mathrm{R}),$$

defines the density function of the neutrino decay probability,

$$\rho(\vec{r},t)=\frac{\mathrm{d}P}{\mathrm{d}\vec{r}^{3}\mathrm{d}t}=|A'|^2|\psi(\vec{r},t,*)|^2 e^{-\varpi t}, \quad \varpi=2\Gamma/\hbar.$$

We may assume that the integration of $|\psi(\vec{r},t,*)|^2$ reduces to one so that

$$\dot{\mathrm{P}}=|A'|^2 e^{-\varpi t}.$$

Consequently, the probability that the neutrino exists at a $t\geq 0$ is

$$\mathrm{P}(t)=\int P(t')\mathrm{d}t'=|A'|^2\int e^{-\varpi t'}\mathrm{d}t'+B=-\frac{|A'|^2}{\varpi}e^{-\varpi t}+B=Ae^{-\varpi t}+B.$$

Boundary conditions are set to be $P(0)=1$ and $P(\infty)=0$, which implies that $B=0$ and $A=1$. Hence the neutrino probability is

$$\mathrm{P}(t)=e^{-\varpi t}.$$

Since electron creation and neutrino destruction are complimentary events, the probability of electron creation is

$$\mathrm{P}^c=1-\mathrm{P}=1-e^{-\varpi t}.$$

At equal probabilities point

$$\mathrm{P}=\mathrm{P}^c \Rightarrow e^{-\varpi\tau}=1/2 \Rightarrow \varpi\tau=\ln 2,$$

the decay variable has the value $\xi_{1/2}=\varpi\tau=\ln 2$, and the $\varpi\tau>\ln 2$ is the electron creation condition, which eventually imposes some conditions on the neutrino decay potential.

3. Minimal Uncertainty

Interaction probability and the electron creation probability explicitly depend on the neutrino decay potential. The energy-time uncertainty principle states that the uncertainty $\mathrm{A} = \mathcal{E}\mathrm{T}$ to create an electron of energy $\mathcal{E}$ in a time interval t cannot be smaller than uncertainty minimum $\hbar/2$. This means that all possible creation realizations of the electron are defined by an uncertainty parameter $\theta \geq 1$, and that

$$\mathrm{A} = \mathcal{E}\mathrm{T} = \frac{\theta\hbar}{2} \geq \frac{\hbar}{2} \Rightarrow \mathrm{T} = \frac{\mathrm{A}}{\mathcal{E}} = \frac{\theta\hbar}{2\mathcal{E}} \geq \frac{\hbar}{2\mathcal{E}} = \mathrm{T}_{\mathrm{O}}.$$

The time T_{O} is the minimal uncertainty creation or minimal creation time. For each uncertainty $\mathrm{A}(\theta)$ or for each $\theta \geq 1$ an electron of an energy $\mathcal{E}$ is created in the time interval T with the probability $\mathrm{P}^{c}(T) = s$. Consequently,

$$\varpi T = \frac{\varpi \mathrm{A}}{\mathcal{E}} = \frac{2\Gamma}{\hbar\mathcal{E}}\frac{\theta\hbar}{2} = \frac{\theta\Gamma}{\mathcal{E}} \geq \frac{\Gamma}{\mathcal{E}}$$

$$\Rightarrow e^{-\varpi T} \leq e^{-\Gamma/\mathcal{E}} \Leftrightarrow \varpi T \geq \Gamma/\mathcal{E}.$$

Here, $\xi_{\mathrm{O}} = \Gamma/\mathcal{E}$ is the minimum uncertainty decay variable. Further, the $\alpha_{\mathrm{O}} = e^{-\Gamma/\mathcal{E}}$ and $\beta_{\mathrm{O}} = 1 - \alpha_{\mathrm{O}}$ are the minimal uncertainty probabilities of the neutrino existence and electron creation. The minimal uncertainty decay variable is either greater or smaller than its value at the half-decay probability point. When it is greater than its value at the half-decay probability point

$$\beta \geq e^{-\Gamma/\mathcal{E}} + \left(1 - 2e^{-\Gamma/\mathcal{E}}\right) \geq e^{-\Gamma/\mathcal{E}}$$

$$\alpha \leq e^{-\Gamma/\mathcal{E}} \leq \beta.$$

Consequently, for $\xi_{\mathrm{O}} \geq \xi_{1/2}$ the decay variable ξ is bounded by the interaction probabilities and for all θ

$$\ln a \geq \frac{\Gamma}{\mathcal{E}} \geq \ln b, \quad a = 1/\alpha, \ b = 1/\beta.$$

The ordering is valid for $\alpha < 1/2 \Leftrightarrow a > 2$ and $\beta > 1/2 \Leftrightarrow b < 2$. Consequently

$$\xi(\theta) \geq \xi(\theta = 1) = \xi_{\mathrm{O}} = \frac{\Gamma_{\mathrm{O}}}{\mathcal{E}_{\mathrm{O}}} \geq \ln b = (\ln 2).$$

Definition: *The potential of the neutrino state decay interaction is measured by the energy of the created electron, and*

$$\Gamma : \frac{\Gamma}{\mathcal{E}} = g(\mathcal{E}) \geq \ln b, \quad \left(= g(\mathcal{E}) \geq \ln 2\right).$$

4. Cross Section

All knowledge about the neutrino decay potential is in the theoretical, experimentally well-known, cross-section [1] of the low energies decay processes $(\nu_e, n) \rightarrow (p, e)$. If $(\mathcal{E}, \vec{p}, m)$ refers to the electron, if $\mathcal{E}_{\mathrm{O}} = m_e c^2$ and $f_{\tau/2}$ is the comparative half-life characteristic of the process, the process theoretical cross section [1] is

$$\sigma = \frac{2\pi^2 c^3 \hbar^3 p\mathcal{E}}{\mathcal{E}_O^5 f_{\tau/2}} = k_O \frac{cp\mathcal{E}}{\mathcal{E}_O^2}, \tag{2}$$

$$k_O = \frac{2\pi^2 c^2 \hbar^3}{\mathcal{E}_O^3 f_{\tau/2}} = 2.3\times 10^{-44}\ \text{cm}^2, \tag{3}$$

or, by the use of the $cp = \sqrt{\mathcal{E}^2 - \mathcal{E}_O^2} = \mathcal{E}\Phi$, $\Phi = \sqrt{1-\left(\mathcal{E}_O/\mathcal{E}\right)^2}$,

$$\sigma = k_o \frac{\mathcal{E}^2}{\mathcal{E}_O^2}\Phi\left(\frac{\mathcal{E}}{\mathcal{E}_O}\right). \tag{4}$$

Interacting neutrinos are in (1:1) correspondence with produced electrons, so that σ is also the electron production cross-section. However, the interaction cross-section $\sigma = \sigma(\mathrm{P}) = \sigma(\mathrm{P}(\Gamma))$ implicitly holds the probability of the neutrino decay, which implies the existence of the inverse relation $\Gamma = \Gamma(\sigma)$. Further, we look for an explicit cross-section probability relation.

Let N be the total number of neutrinos in the beam per unit time of the beam, and let the number of the neutrinos on a single electron target be N_ν, let the number of the neutrino electron interactions is $N_{e\cap\nu}$. The electron conditional probability, when the neutrino event happens, is

$$\mathrm{P}_{e/\nu} = \frac{N_{e\cap\nu}}{N_\nu} = \frac{N_{e\cap\nu}/N}{N_\nu/N} = \frac{\mathrm{P}(e\cap\nu)}{\mathrm{P}_\nu}.$$

The electron conditional probability measures by the neutrino interaction cross-section and the electron geometric cross section $A_e \sim a_e^2$, $a_e = e^2/m_e c^2$, so that $\mathrm{P}_{e/\nu} = \sigma/A_e$. The probability $\mathrm{P}(e\cap\nu)$ is the probability P^c of all electrons created by the neutrinos, and the probability P_ν of the neutrino interaction is proportional to the comparative half-time $t_{1/2}$ of the neutrino on the electron target in the unit time of the beam. Consequently $\mathrm{P}_\nu \sim 1/t_{1/2}$ and

$$\frac{\sigma}{A_e} = \frac{\mathrm{P}^c}{1/t_{1/2}} \Rightarrow \mathrm{P}^c = k_\mathrm{P}\sigma,\quad k_\mathrm{P} = \frac{1}{t_{1/2}A_e}. \tag{5}$$

Finally, the electron creation probability is

$$\mathrm{P}^c = k_\mathrm{P} k_O \frac{\mathcal{E}^2}{\mathcal{E}_O^2}\Phi\left(\frac{\mathcal{E}}{\mathcal{E}_O}\right). \tag{6}$$

Remark: The original cross-section is zero when $\vec{p} = 0$ and the production of the electrons of the rest energy is zero. However, the probability of electron production at $t = T$ is $\beta = 1 - \alpha > 0$ for all electron energies $\mathcal{E} \geq \mathcal{E}_O$, and the cross-section at the rest energy cannot be zero. Hence, the cross-section should be corrected. For this we notice that for all $\eta = \mathcal{E}_O/\mathcal{E} < 1$ the momentum

$$cp = \mathcal{E}\sqrt{1-\eta^2} = \mathcal{E}\left(1 - \frac{\eta^2}{2} - \cdots\right) = \mathcal{E}\Omega \geq 0,$$

is approaching zero from above when the electron energy is approaching the rest energy rom above. Hence, any truncated function Ω_* of Ω will give non-zero production of the electrons at the rest energy. As the first approximation, we may take $\Omega_* = 1$ and

Definition: *The electrons are produced at the rest energy and the electron production cross section at an energy* $\mathcal{E} \geq \mathcal{E}_O$, *is*

$$\sigma = k_{\mathcal{E}} k_O \begin{cases} \frac{\mathcal{E}^2}{\mathcal{E}_O^2}, & \text{if } \mathcal{E} \geq \mathcal{E}_O, \\ 0, & \text{if } \mathcal{E} < 0. \end{cases}$$

The electron energy coefficient $k_{\mathcal{E}}$, in the first approximation is 1.

Corollary 1. *The electron creation probability is*

$$P^c = k_P k_A k_{\mathcal{E}} k_O \frac{\mathcal{E}^2}{\mathcal{E}_O^2} \xrightarrow{\mathcal{E} \downarrow \mathcal{E}_O} k_P k_A k_{\mathcal{E}} k_O.$$

□ This follows directly from the $P^c = k_P (k_A) \sigma$. The electron cross-section area coefficient k_A may be chosen to be $1/\pi$ or a better estimated number. In the first approximation, we will take $k_A = 1 = k_{\mathcal{E}}$. ■

5. Interaction Potential

The electron creation probability $P^c = 1 - e^{-\varpi\Gamma}$, explicitly defined by the decay interaction potential, is directly related to the theoretical electron creation cross-section, Equation (4), a function of the created electron energy, Equations (1), (2) and (3). Hence

$$\varpi\Gamma = -\ln\left(1 - P^c\right) = -\ln\left(1 - k_P \sigma\right) = k_P \sigma + \frac{\left(k_P \sigma\right)^2}{2} + \cdots$$

$$\Rightarrow \varpi\Gamma = k_P k_A k_{\mathcal{E}} k_O \frac{\mathcal{E}^2}{\mathcal{E}_O^2} + \cdots.$$

The variable $\varpi\Gamma = \Gamma/\mathcal{E}$ at the minimum uncertainty electron production reduces to the

$$\frac{\Gamma}{\mathcal{E}} = k_P \sigma + \cdots = k_P k_A k_{\mathcal{E}} k_O \frac{\mathcal{E}^2}{\mathcal{E}_O^2} + \cdots.$$

The last defines the function $g(\mathcal{E})$ and, consequently, the decay interaction potential is

$$\Gamma = k_P \sigma \mathcal{E} = k_P k_A k_{\mathcal{E}} k_O \frac{\mathcal{E}^2}{\mathcal{E}_O^2} \mathcal{E} \xrightarrow{\mathcal{E} \to \mathcal{E}_O} k_P k_A k_{\mathcal{E}} k_O \mathcal{E}_O = k_A k_{\mathcal{E}} k_P \frac{2\pi^2 c^2 \hbar^3}{\mathcal{E}_O^3 f_{\tau/2}} m_e c^2. \quad (7)$$

6. Kline-Gordon-Dirac Equations

This part rests on the concept of minimal coupling presented in the **Appendix**. To complete the system of equations defining the neutrino masses, we will write the energy equation, Kline-Gordon equation for the free massive particle minimal coupling extension by the neutrino interaction potential. Further, the particle is the vector $R = \left|E, \vec{p}, m, z\right\rangle$, $z = (u_o, u)$ in the $\mathbb{R}_1^5 \otimes \mathbb{R}_1^2 = \mathbb{R}_1^4 \otimes \mathbb{R}^1 \otimes \mathbb{R}_1^2$ Minkowski space.

Definition: *The energy function of a free particle in the complex decay potential is the non-negative function, the norm* $\mathcal{Z} = |\mathrm{R}|^2$ *, and the particle energy conservation law is the equation* $\mathcal{Z} = 0$ *. Exactly,*

$$|\mathrm{R}|^2 = E^2 - \vec{p}^2 - z = P^2 - z, \quad z \equiv \mu^2 = m^2 + u_O^2 - u^2.$$

$$\mathcal{Z} = 0 \iff P^2 - z = E^2 - \vec{p}^2 - z = 0.$$

The function $\mu : \mu^2 = z$ is the particle rest energy. If the particle energy E is real number, the particle is real. Else it is an exotic particle. The particle is regular if its interaction rest energy μ is real.

A collection of quantum operators residing in the operator space $\hat{\mathbf{R}} = \hat{\mathbf{R}}_1^4 \otimes \hat{\mathbf{R}}_1^1 \otimes \hat{\mathbf{R}}_1^2$, inheriting all algebraic properties of the vector space $\mathbf{R}$. is associated with each particle. The particle quantum state equation is its quantized energy equation. Further, the speed of the light is one, and $\mu : \mu^2 = m^2 + u_O^2 - u^2$ is the particle interaction rest energy, i-rest energy or the interaction rest mass, i-rest mass.

Definition: *The particle state variables and its state decay function* Ψ *are entangled in the quantum state Kline-Gordon equation*

$$\hat{\mathcal{H}}^2\Psi = \left(\hat{P}^2 - \hat{\mu}^2\right)\Psi = \left(\hat{E}^2 - \hat{\vec{p}}^2 - \hat{\mu}^2\right)\Psi = 0.$$

Corollary 2. The energy function operator
$\hat{\mathcal{H}}^2 = \hat{P}^2 - \hat{\mu}^2 = \left(\hat{P} - \hat{\mu}\right)\left(\hat{P} + \hat{\mu}\right) = \hat{\Pi}\hat{\Pi}^*$ and

$$\hat{\mathcal{H}}^2\mathbf{\Psi} = \left(\hat{P}^2 - \hat{\mu}^2\right)\Psi = \left(\hat{E}^2 - \hat{\vec{p}}^2 - \hat{\mu}^2\right)\Psi.$$

In addition, the linear factorization or the Kline-Gordon equation, or the Kline-Gordon-Dirac equations for the free particle state decay hold and

$$\hat{\Pi}\psi = \left(\hat{P} - \mu\right)\psi = \left(E - \vec{p} - \mu\right)\psi = 0,$$

$$\hat{\Pi}^*\psi = \left(\hat{P} + \mu\right)\psi = \left(E + \vec{p} + \mu\right)\psi = 0.$$

■ To shorten notation we use $\hat{P} = \hat{E} + \hat{\vec{p}}$, $\hat{\mu} = m + u_o - u$. By definition $\hat{P}\Psi = P\Psi$, $\hat{\mu}\Psi = \mu\Psi$, and $\hat{P}^2\Psi = P^2\Psi$ and $\hat{\mu}^2\Psi = \mu^2\Psi$, so that

$$\hat{\mathcal{H}}^2\Psi = \left(\hat{P}^2 - \hat{\mu}^2\right)\Psi = \left(P^2 - \mu^2\right)\Psi = P^2\Psi - \mu^2\Psi$$
$$= P\hat{P}\Psi - \mu\hat{\mu}\Psi = \hat{P}^2\Psi - \hat{\mu}^2\Psi = \left(\hat{P}^2 - \hat{\mu}^2\right)\Psi.$$

However, the operators inherit the scalar product of the space $\mathbb{R}_1^3 \otimes \mathbb{R}^1 \otimes \mathbb{R}_1^2$ and

$$\left(\hat{P}^2 - \hat{\mu}^2\right)\Psi = \left\langle \hat{P} - \hat{\mu} \middle| \hat{P} + \hat{\mu} \right\rangle \Psi = \hat{\Pi}\hat{\Pi}^*\Psi.$$

Suppose that $\hat{\Pi}\Psi \neq 0$ and $*\Psi \neq 0$. Then

$$0 \equiv \hat{\mathcal{H}}^2 = \Pi\Pi * \Psi = \Pi * \Psi = *\Pi\Psi = *\hat{\Pi}\Psi \neq 0.$$

contradiction. Thus, the linear state equations must hold. We notice that the Kline-Gordon-Dirac linear equations differ from the Dirac linear relativistic spinor equation. ■

Definition: *The pair* $(\mathbf{P}, \mathbf{P}^*)$ *of the objects in the states described by the linear Kline-Gordon or Dirac equation are dual particles.*

7. Kline-Gordon Particles

We notice that the particle $\mathbf{P}$ is the evolution of the original particle $\mathbf{P}_O$ in the imaginary decay potential $G = iu$, in which it gains the interaction rest mass, i-mass, or equivalently the interaction i-rest energy $z = \mu^2$. Such are the particles that satisfy the Kline-Gordon equation, their energy solution $E^2 = \vec{p}^2 + z$ involves the i-rest energy function $z = m^2 - u^2$. Let $M = m - u$ and $M_* = m + u$. Then $M_* + M = 2m$, $M_* - M = 2u$ and i-rest energy $z = M_* M$ is

$$z = \begin{cases} +M_* M, & E_O = \pm\sqrt{|M_* M|}, & \text{if } m > u, \\ 0, & E_O = 0, & \text{if } m = u, \\ -M_* M, & E_O = \pm i\sqrt{|M_* M|}, & \text{if } m < u. \end{cases}$$

Positive and negative energy solutions correspond to the dual particle pairs $(\mathbf{P}, \mathbf{P}^*)$. To understand the meaning of the imaginary energy solutions, we set the particle in the interaction potential hole of the increasing sequence of the constant energy depths. All possible cases $m > u$, $m = u$ and $m < u$ of the pairs (m, u), are the $z > 0$, $z = 0$ and $z < 0$ i-rest energy cases, and the following situations appear.

1) The choice of (m, u) sets the zero i-rest energy point and places the particle into the class of the positive or negative i-rest energy function *z*. The particle is real as long as its energy $E^2 = \vec{p}^2 + z$ is non-negative, this is true for all regular particles, $m \geq u$. All free particles start in a depth-increasing potential as the regular particle and hold their regularity until they reach the zero i-rest energy.

Definition: *The zero i-rest energy is the decay particle splitting point.*

2) When the depth of the decay potential reaches the particle rest energy, the particle i-rest energy becomes zero, and the Kline-Gordon's equation splits into two equations

$$P^2 = E^2 - \vec{p}^2 = 0, \quad z = m^2 - u^2 = 0. \tag{8}$$

The real particle $\mathbf{P}$ enters the zero i-rest energy point with either $(E, \vec{p}) \equiv 0$ or $(E, \vec{p}) > 0$.

When $(E, \vec{p}) \equiv 0$ the particle split is $\mathbf{P}_O \sim \mathbf{P} \wedge \mathbf{P}_\varnothing$, where $\mathbf{P}_\varnothing = (0,0,0,*)$ is the zero Kline-Gordon photon, and the $\mathbf{P}$ the particle of the mass $m = m$, sitting in the rest mass field. Such a particle decouples from the motion, and the splitting point is the particle vanishing point.

When $(E, \vec{p}) \neq 0$ the particle split is $\mathbf{P}_O \sim \mathbf{P} \wedge \mathbf{P}_\gamma$ where $\mathbf{P}_\gamma$, is the Kline-Gordon photon, the original particle decouples into a Kline-Gordon photon and the massive particle $\mathbf{P}$ of the mass $m = m$, sitting in the rest mass field.

3) The decay potential may exceed the particle rest energy and the i-rest particle energy function becomes negative. In that case

$$E^2 = \vec{p}^2 - \mu^2 \geq 0, \quad \forall \vec{p} \geq \vec{p}_m = \mu^2.$$

and the particle is real as long as $E^2 \geq 0$. For such particle $\vec{p} \geq \vec{p}_m = \mu^2 \geq 0$, and such particles never achieve the rest state, an example is the photon. The energy equation for such particles is

$$\vec{p}^2 = E^2 + \mu^2 \geq 0, \quad \mu^2 \geq 0.$$

Remark: *We recall that the upper and lower extremal differ in the sign of the four-momentum vector, and that all extensions above minimal coupling done to the lower extremal* $-x_O^2 + x^2$ *would produce the energy function*

$$\mathcal{Z} = -E^2 + \vec{p}^2 - z, \quad z = \mu^2 = \left|m^2 - u^2\right|.$$

$$\Rightarrow \mathcal{Z} = 0 \Rightarrow \vec{p}^2 = E^2 + \mu^2.$$

When $z > 0$ the particle energy is the same as the energy on the upper extremal solution for the negative i-rest energy. Hence

$$E^2 = \begin{cases} \vec{p}^{\,2} + \mu^2 \geq 0, & if \;\; z > 0, \\ 0, & if \;\; z = 0, \\ \vec{p}^{\,2} - \mu^2 \geq 0, & if \;\; z < 0. \end{cases}$$

8. Dirac Particles

Dirac particles are the particle pair $\left(\mathbf{P}, \mathbf{P}^*\right)$ solution to the equations linear factors of the equations of the Kline-Gordon equation. The particle rest energies are $M = m - u$ and $M_* = m + u$, and if the particle i-rest energy is M the i-rest energy of its dual particle is $M_* = 2m - M$. At the splitting point, the i-rest mass is zero, and the pair $\left(\mathbf{P}, \mathbf{P}^*\right)$ is a massless particle $\mathbf{P} \sim M = 0$ and massive dual particle $\mathbf{P}^* \sim M_* = 2m$.

9. Neutrino Masses

The minimal creation of an electron in the neutrino-nucleon process is the creation of an electron at rest state. Consequently, the minimum requirement is that the neutrino state decay ends at the splitting point. Equivalently, the splitting point is the neutrino disappearance point. There, $\mu = 0$, an electron at the rest state followed by the $\mathbf{P}_\varnothing$ zero Kline-Gordon photon is created. Thus

$$\mu^2 c^4 = 0 \Leftrightarrow m^2 c^4 - \Gamma^2 = 0 \Rightarrow m = \Gamma = k_P \sigma m_e$$

$$\therefore m = k_A k_\mathcal{E} k_P \frac{2\pi^2 c^2 \hbar^3}{\mathcal{E}_O^3 f_{\tau/2}} m_e = k_A k_\mathcal{E} k_P k_O m_e. \tag{9}$$

For the $\left(\nu_e, n\right) \rightarrow \left(p, e\right)$ process $t_{1/2} = 1.1 \times 10^3$, and we take $k_A = 1 = k_\mathcal{E}$ in the first approximation. According to Equation (2), the coefficient $k_O = 2.3 \times 10^{-44}$ s, for an electron $a_e^2 = 7.90 \times 10^{-34}$ cm^2 and by the Equation (4), $k_P = 1/a_e^2 t_{1/2} = 1.15 \times 10^{30}$. Consequently, the electron neutrino mass is

$$m_{\nu_e} = k_P k_O m_e = 1.15 \times 10^{30} \times 2.3 \times 10^{-44} \times 0.511$$
$$= 1.35 \times 10^{-14}\ \mathrm{MeV/c^2} = 1.35 \times 10^{-8}\ \mathrm{eV/c^2}.$$

Since muon and the tau neutrinos follow the $(\nu_i, N) \to (\bar{N}, l)$ charged weak currents interactions, the electron neutrino elastic scattering is the prototype for all neutrino generation interactions with the nucleons. Based on this similarity, we may assume that the corresponding cross-sections have the same analytic structure and that they differ among themselves by the charged lepton mass entries, and, it may be, the characteristic half-time $t_{1/2}$. If m_i is a neutrino and m_i' its lepton mass and e refers to an electron, then

$$\frac{\mathrm{m}_i}{m_{\nu_e}} = \frac{\sigma^i}{\sigma^e}\frac{m_i'}{m_e} = \frac{k_O^i}{k_O^e}\frac{m_i'}{m_e} = \frac{m_e^3}{m_i'^3}\frac{t_{1/2}^e}{t_{1/2}^i}\frac{m_i'}{m_e} = F\left(m_i', m_e\right)\frac{m_i'}{m_e}.$$

Table 1 shows the calculation, and the masses are in the eV/c^2.

We assume that the half-life time does not vary over the leptons generations and further depending on the cross-section lepton masses dependence and consider the following cases

Case I: $F = 1$, the cross-section does not vary by the lepton generation/masses,

Case II: $F \neq 1$, the cross-section does vary by the lepton generation/masses.

Table 1. Calculation of the neutrino masses.

	e	μ	τ	Σ
m_i	5.11×10^{-1}	1.06×10^{-2}	1.78×10^{-3}	
$\frac{m_i'}{m_e}$	1	2.07×10^{2}	3.48×10^{3}	
$\frac{m_e}{m_i'}$	1	4.83×10^{-3}	2.87×10^{-4}	
m_i^{I}	1.35×10^{-8}	5.81×10^{-4}	1.64×10^{-1}	0.165
m_i^{II}	1.35×10^{-8}	3.15×10^{-13}	1.11×10^{-15}	1.35×10^{-8}
	$\Delta m^2_{\lvert 12 \rvert}$	$\Delta m^2_{\lvert 23 \rvert}$	$\Delta m^2_{\lvert 31 \rvert}$	
I	3.37×10^{-7}	2.70×10^{-2}	2.7×10^{-2}	eV^2/c^4
II	1.83×10^{-16}	9.94×10^{-26}	1.83×10^{-16}	eV^2/c^4

10. Conclusion

The cases I and II generate two chains of the neutrino masses, each starting with the electron neutrino mass. In the case I, the interaction cross-sections do not vary over the lepton generations and the neutrinos constitute an increasing order mass chine $\nu_e \prec \nu_\mu \prec \nu_\tau$. Else, the decreasing mass chain is obtained when the cross-section variation by the lepton masses is allowed. In addition, both chains satisfy the mass sum limit [2] $\Sigma m_i < 0.17$ eV with 95% confidence level, set by the Planck satellite [2] experimental data. However, only the mass-increasing neutrino hierarchy is consistent with the neutrino oscillation [2] experimental square mass differences data. Under the limitations of the assumptions above, perhaps the leptons geometry, and the simplicity of the model, the increasing neutrino mass-ordering chain $\left|\mathrm{m}_e, \mathrm{m}_\mu, \mathrm{m}_\tau\right\rangle = \left|1.35\times10^{-8}, 5.81\times10^{-4}, 1.64\times10^{-1}\right\rangle \mathrm{eV/c^2}$ is predicted [3].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Frauenfelder, H. and Hennley, E.M. (1995) Subatomic Physics. Prentice Hall.

[2] de Salas, P.F., *et al*. (2018) Neutrino Mass Ordering from Oscillations and beyond: 2018 Status and Future Prospects. *Frontiers in Astronomy and Space Sciences*, **5**. https://doi.org/10.3389/fspas.2018.00036

[3] Majkic, R. (2021) Neutrino Masses. *Energy Physics*, viXra:2104.0014.

Appendix

Minimal Coupling

In this part $\mathbb{R}^{m+n}$ is Euclidean space spanned by the base $(\hat{x}_o, \hat{\mathrm{x}})$ and equipped with the scalar product. The square of the general vector $\mathrm{X} = \vec{x} + \vec{\mathrm{x}} = x_{\mathrm{O}}\hat{x} + \mathrm{x}\hat{\mathrm{x}}$

$$\mathrm{X}^2 = x_{\mathrm{O}}^2 + 2\vec{x}_{\mathrm{O}}\vec{\mathrm{x}} + \mathrm{x}^2 = x_{\mathrm{O}}^2 + 2x_{\mathrm{O}}\mathrm{x}\cos\theta + \mathrm{x}^2 = (x_{\mathrm{O}} + \mathrm{x}\cos\theta)^2 + \mathrm{x}^2\sin^2\theta > 0,$$

is the paraboloid function $y(x_{\mathrm{O}}, \mathrm{x})$ of the magnitudes of variables $(x_{\mathrm{O}}, \mathrm{x})$ in the $\mathbb{R}^{m+n}$ space. Its upper branch is the Euclidean norm $|\mathrm{X}| : |\mathrm{X}|^2 = y(x_{\mathrm{O}}, \mathrm{x})$. The norm has minimum $y = 0$, at the point $(0,0)$, and does not have other extremes. Further, we are looking to find the extremes of the functions restrictions $\mathrm{U}(x_{\mathrm{O}}) = y(x_{\mathrm{O}}, const)$ and $\mathrm{V}(\mathrm{x}) = y(const, \mathrm{x})$, the cuts $x = const$ and $x_{\mathrm{O}} = const$ of the vector square function. It follows that

$$\partial_{\mathrm{x}}\mathrm{V} = (2\vec{x}_{\mathrm{O}} + 2\vec{\mathrm{x}})\hat{\mathrm{x}} = 0 \Rightarrow (2\vec{x}_{\mathrm{O}}\vec{x} + 2\vec{\mathrm{x}}^2) = 0,\ \partial_{\mathrm{xx}}y = 2\hat{\vec{\mathrm{x}}}^2 = 2 > 0$$

$$\partial_{x_{\mathrm{O}}}\mathrm{U} = (2\vec{x}_{\mathrm{O}} + 2\vec{\mathrm{x}})\hat{x}_{\mathrm{O}} = 0 \Rightarrow (2\vec{x}_{\mathrm{O}}^2 + 2\vec{x}_{\mathrm{O}}\vec{\mathrm{x}}) = 0,\ \partial_{x_{\mathrm{O}}x_{\mathrm{O}}}y = 2\hat{x}_{\mathrm{O}}^2 = 2 > 0,$$

and the extremals are the upper and lower hyperbolic saddles at the point $(\vec{0}, \vec{0})$ over $\mathrm{N} = n + m$ dimensional space. Thus

$$y = \begin{cases} Z = +x_{\mathrm{O}}^2 - x^2 \subset \mathbb{R}_n^{m+n}, & \text{upper saddle at } x_{\mathrm{O}} \text{ fixed cut,} \\ Y = -x_{\mathrm{O}}^2 + x^2 \subset \mathbb{R}_m^{m+n}, & \text{lover saddle at x fixed cut.} \end{cases}$$

Definition: *The space* $\mathbb{R}^n$ *is minimally coupled to the space* $\mathbb{R}^m$ *by the hyperbolic norm function*

$$\mathcal{Z} = |\mathrm{X}|^2 = \vec{x}^2 - \vec{\mathrm{x}}^2.$$

into Minkowski space $\mathbb{R}_n^{m+n}$. The norm induces the hyperbolic scalar product

$$\langle \mathrm{X} | \mathrm{X} \rangle = \langle \vec{x} \quad -\vec{\mathrm{x}} | \vec{x} \quad \vec{\mathrm{x}} \rangle,$$

on $\mathbb{R}_n^{m+n}$, where $\langle \mathrm{X} | = \langle \vec{x} \quad -\vec{\mathrm{x}} |$ is the conjugate vector of the vector $| \mathrm{X} \rangle = | \vec{x} \quad \vec{\mathrm{x}} \rangle$. The norm and hyperbolic scalar product are mutually consistent.

The hyperbolic norm in the physical application is the particle energy square function $\mathcal{Z} = \mathcal{H}^2$.

Example An example of the minimal coupling is the pairing of the space and time into a space-time four-vector $\mathrm{X} = (x_{\mathrm{O}}, \vec{\mathrm{x}})$ in the Minkowski space $\mathbb{R}_3^4$. The speed of the light is one. The norm there is $|\mathrm{X}|^2 = x_{\mathrm{O}}^2 - \vec{\mathrm{x}}^2$, and the scalar product consistent with the norm $\langle \mathrm{X} | \mathrm{X} \rangle = \langle x_{\mathrm{O}} \quad -\vec{\mathrm{x}} | x_{\mathrm{O}} \quad \vec{\mathrm{x}} \rangle$.

Corollary P1. The energy and momentum of a massless particle couple in a four-vector in the Minkowski space $\mathbb{R}_3^4$, the particle identifies with the vector $\mathrm{P} = (E \quad \vec{p})$ in the $\mathbb{R}_3^4$, and its energy function

$$\mathcal{Z} = E^2 - \vec{p}^2.$$

Scalar product consistent with the norm is $\langle P | P \rangle = \langle E \quad -\vec{p} | E \quad \vec{p} \rangle$. The free non-interacting particle preserves energy and $\mathcal{Z} = |P|^2 = E^2 - \vec{p}^2 = 0$.

■ Minimal momentum coupling condition applied to the squire of the vector

$P = (E, \vec{p})$ implies

$$P^2 = E^2 + 2E\vec{p} + \vec{p}^2 \Rightarrow \partial_p P^2 = 2E\hat{1} + 2p\hat{p},\ \partial_{pp} P^2 = 2 > 0$$

$$\partial_p P^2 = 0 \Rightarrow 2(E\vec{p} + \vec{p}^2) = 0 \Rightarrow \mathcal{Z} = E^2 - \vec{p}^2.$$

The function $\mathcal{Z}$ is the norm of the vector *P*, and the corresponding scalar product is $\langle P | P \rangle = \langle E \ \ -\vec{p} | E \ \ \vec{p} \rangle$. The energy function $\mathcal{Z}$ is the energy-momentum four-vector equation for non-interacting physical particles. In the case of a free massless particle, the energy momentum preserves, and $\mathcal{Z} = E^2 - \vec{p}^2 = 0$. ■

Remark: *The evaluation* $\mathcal{Z} = 0$ *is the energy-momentum conservation law for the free massless particle, and* $\mathcal{Z} = \text{const} > 0$ *is the well-known statement of the energy-momentum conservation law for massive particles. It is important to notice that massive particle is the minimal coupling of matter to space-time.*

Corollary P2. *Hyperbolic norm* $|\mathcal{P}|^2 = P^2 - m^2$ *couples minimally matter and the space-time into five-vector* $\mathcal{P} = (E \ \ \vec{p} \ \ m)$ *in the Minkowski space* $\mathbb{R}^5_{3+1}$. *The scalar product* $|\mathcal{P}|^2 = \langle E \ -\vec{p} \ -m | E \ \ \vec{p} \ \ m \rangle = \langle \mathcal{P} | \mathcal{P} \rangle$ *is induced by the norm.*

The zero norm condition $|\mathcal{P}|^2 = 0$ is the free particle energy-momentum conservation law for both the massive and massless particles.

■ Independently on that whether $P = (E, \vec{p})$ is minimally coupled into energy-momentum four-vector, the vector $\mathcal{P} = (P, mc^2) \sim (P, m)$ is in the five-dimensional space and

$$\mathcal{P}^2 = P^2 + 2Pm + m^2 \Rightarrow \partial_m \mathcal{P}^2 = 2P + 2m,\ \partial_{mm} \mathcal{P}^2 = 2 > 0$$

$$\partial_m \mathcal{P}^2 = 0 \Rightarrow 2(Pm + m^2) = 0 \Rightarrow \mathcal{Z} = P^2 - m^2.$$

$$\therefore \mathcal{Z} = P^2 - m^2 = E^2 - \vec{p}^2 - m^2 \geq 0$$

The non-negative energy function, hyperbolic norm $\mathcal{Z}$, represents the energy-momentum equation for a massive particle. The scalar product
$\langle P \ -m | P \ \ m \rangle = \langle E \ -\vec{p} \ -m | E \ \ \vec{p} \ \ m \rangle = \langle \mathcal{P} | \mathcal{P} \rangle$ is consistent with the norm. Further, the zero norm is a free massive particle energy-momentum conservation law.
■

Remark: Further, a free particle $\mathrm{P_O}$ is a physical object characterized by the five-vector $\mathcal{P} = (P, m) = (E, \vec{p}, m)$ in the five-dimensional space
$\mathbb{R}^4_3 \otimes \mathbb{R}_1 = \mathbb{R}^5_{3+1}$. However, an interacting particle requires the presence of an interaction potential or, equivalently, an additional extension of the space associated with the particle.

Definition: *An interaction potential of a physical particle* $\mathcal{P}$ *is a complex function* $G = u_O + iu$ *minimally coupled to the particle. The interacting particle is the vector* $\mathrm{R} = (\mathcal{P}, G)$ *in the seven dimensional space.*

Corollary P3. The energy function of a particle in the complex interaction potential $G = u_O + u$ is

$$\mathcal{Z} = |\mathcal{P}|^2 - G^2.$$

If interaction $G = u_O + u$ measures the particle interaction the complex interaction potential with minimally coupled components also measures the particle interaction, the complex potential $G \Rightarrow u_O^2 - u^2$ and the interacting particle energy function is

$$\mathcal{Z} = |P|^2 - z = E^2 - \vec{p}^2 - z.$$

The particle i-rest mass or its i-rest energy $z = m^2 + u_O^2 - u^2$ consistently reproduces the rest muss of the free particle. The interacting particle is the vector R in the $\mathbb{R}_3^6 \otimes \mathbb{C} \sim \mathbb{R}_3^4 \otimes \mathbb{R}_1 \otimes \mathbb{R}_1^2 \sim \mathbb{R}_{3+1}^5 \otimes \mathbb{R}_1^2$ vector space, and the scalar product consistent with the particle energy function is

$$\langle \mathrm{R} | \mathrm{R} \rangle = \langle E \quad -\vec{p} \quad -m \quad -u_O \quad u | E \quad \vec{p} \quad m \quad u_O \quad u \rangle = |\mathrm{R}|^2.$$

■ The interacting particle is the vector $\mathrm{R} = (\mathcal{P}, G)$ in the $\mathbb{R}_3^5 \otimes \mathbb{C} \sim \mathbb{R}^7$ space. The interaction potential is minimally coupled to the particle $\mathcal{P}$ and the particle square

$$\mathrm{R}^2 = \mathcal{P}^2 + 2\mathcal{P}G + G^2 \quad \therefore \; \partial_G \mathcal{P}^2 = 2(\mathcal{P} + G) = 0$$

$$\Rightarrow 2\mathcal{P}G + 2G^2 = 0 \; \Rightarrow \; \mathcal{Z} = \mathcal{R}^2 = |\mathcal{P}|^2 - G^2.$$

The function $G^2 : G = u_O + iu$ measures the particle interaction energy. The physical particle variables are real so that

$$\mathrm{R} = (\mathcal{P}, G) \sim (\mathcal{P}; u_O + iu) \sim (\mathcal{P}, u_O; iu).$$

If the function G measures the particle interaction then the complex interaction potential G with the minimally coupled components measures also the particle interaction. The real number nature of the particle variables requires that the real part u_O of all interaction potential G couples to the four-momentum as the mass does to constitute the rest interaction mass or the rest interaction energy. Thus, the imaginary component of the interaction potential is minimally coupled to its real part and

$$G^2 = u_O^2 + 2iu_O \cdot u + u^2 \quad \therefore \; \partial_u G^2 = 2iu_O \hat{u} = 0$$

$$\Leftrightarrow 2iu_O u + 2u^2 = 0 \; \Rightarrow \; G^2 = u_O^2 - u^2.$$

Since $|\mathcal{P}^2| = P^2 - m^2 = E^2 - \vec{p}^2 - m^2$ the energy function

$$\mathcal{Z} = P^2 - G^2 = E^2 - \vec{p}^2 - G^2 - u_O^2 + u^2 = E^2 - \vec{p}^2 - z.$$

The particle interaction rest energy function $z = m^2 + u_O^2 - u^2$ may be positive, negative and zero. The particle is the vector $\mathrm{R} = |E \quad \vec{p} \quad m \quad u_O \quad u\rangle$ in the $\mathbb{R}_{3+1}^5 \otimes \mathbb{C} \sim \mathbb{R}_3^4 \otimes \mathbb{R}^1 \otimes \mathbb{R}_1^2$. Non-negative energy function is the norm $|\mathrm{R}|^2$, and the scalar product induced by the norm is

$$\langle \mathrm{R} | \mathrm{R} \rangle = \langle E \quad -\vec{p} \quad -m \quad -u_O \quad u | E \quad \vec{p} \quad m \quad u_O \quad u \rangle = |\mathrm{R}|^2.$$

■

The interaction mass or the interaction rest energy function for the particle in the complex interaction potential $G = u_O + iu$ is

$$z = \begin{cases} m^2 + u_O^2 - u^2, & \text{if } m^2 + u_O^2 - u^2 > 0, \\ 0, & \text{if } m^2 + u_O^2 - u^2 = 0, \\ u^2 - m^2 - u_O^2, & \text{if } m^2 + u_O^2 - u^2 < 0. \end{cases}$$