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Equivalence of the Aharonov-Bohm and Dirac Monopole Potentials

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Abstract

The Aharonov-Bohm effect (experimentally verified) constitutes an undubitable proof of the non local nature of quantum mechanics and of the gauge character of the electromagnetic interaction. On the other hand, the existence of a Dirac monopole (not yet experimentally confirmed) leads to the quantization of the electric charge. Both phenomena can be mathematically described in the context of fiber bundle theory. Using this approach, we briefly review the mutual determination of the corresponding connections ω_{A-B}, ω_D and potentials $A_{A-B\pm}$, $A_{D\pm}$. This mathematical result gives an additional theoretical support to present day active search of the magnetic charge.

Keywords

Aharonov-Bohm Effect, Dirac Monopole, Principal Bundles

1. Introduction

As is well known, both the Aharonov-Bohm (A-B) effect [1] and the Dirac (D) magnetic monopole theory [2] [3] admit a natural description in geometric and topological terms in the context of fiber bundle theory and connections [4] [5]. However, while the A-B effect is experimentally observed [6], the abelian Dirac monopole is not confirmed to exist in the physical world [7]. Initial claims of its existence [8] [9] could not be confirmed even in recent experiments [10] [11].

In this note we exhibit, in a concise manner, the deep relation between the geometrical descriptions of both phenomena: the existence of the D connection and the corresponding local potentials implies the existence of the A-B connection and its potentials, and *vice versa*. It is this "*vice versa*" that supports the expectation of the experimental finding of Dirac monopoles in future experiments. As Polchinski [12] wrote: "...the existence of magnetic monopoles seems like one of the safest bets that one can make about physics not yet seen."

The present paper only describes the main facts of the above relation; more detailed calculations can be found in [13] [14].

2. A-B Bundle ξ_{A-B}, Connection ω_{A-B}, and Local Potentials $A_{(A-B)\pm}$

The $U(1)\cong S^1$ bundle associated with the A-B effect with an infinitesimally thin and infinitely long solenoid is the product (and therefore trivial) bundle [5]

$$\xi_{A-B}: S^1 \to \left(T_0^2\right)^* \xrightarrow{pr_1} \left(D_0^2\right)^* \tag{1}$$

where: $\left(D_0^2\right)^* = \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is the punctured open disk in two dimensions, $\left(T_0^2\right)^* = \left(D_0^2\right)^* \times S^1$ is the open solid 2-torus minus a circle, and pr_1 is the projection in the first entry $pr_1\left(z, \mathrm{e}^{i\varphi}\right) = z$. The action of S^1 on $\left(T_0^2\right)^*$, $\psi_{A-B}: \left(T_0^2\right)^* \times S^1 \to \left(T_0^2\right)^*$, is given by $\psi_{A-B}\left(\left(z, \mathrm{e}^{i\varphi}\right), \mathrm{e}^{i\varphi'}\right) = \left(z, \mathrm{e}^{i(\varphi+\varphi')}\right)$. The reason for (1) is that, because of the symmetry along the solenoid, the available space for the electrically charged particles moving around it is $\mathbb{R}^2\setminus\{0\} \equiv \left(\mathbb{R}^2\right)^* \cong \mathbb{C}^*$. Since $\mathbb{C}^*$ is of the same homotopy type as S^1, ξ_{A-B} is, up to isomorphism, the *unique* $U(1)$-bundle over $\mathbb{C}^*$. This uniqueness is a remarkable fact in relation to the description of the A-B effect.

The A-B potentials (and global connection ω_{A-B} on $\mathbb{C}^*\times S^1$ since ξ_{A-B} is trivial) are the flat but non-exact 1-forms with values in $u(1) = i\mathbb{R}$, the Lie algebra of $U(1)$,

$$A_{(A-B)\pm} = \mp\frac{i}{2}d\varphi = \mp\frac{i}{2}\left(\frac{XdY - YdX}{X^2+Y^2}\right) \tag{2}$$

with $\varphi\in(0,2\pi)$, $X+iY=z$, and X,Y Cartesian coordinates on $\left(\mathbb{R}^2\right)^*$.

3. D Bundle ξ_D, Connection ω_D, and Local Potentials $A_{(D)\pm}$

The $U(1)$ principal bundles associated with Dirac monopoles of magnetic charge $g=\lambda k$ with integer k and λ a number depending on units, are the Hopf bundles [4]

$$\xi_D^k: S^1 \to P_k^3 \xrightarrow{\pi_k} S^2 \tag{3}$$

with $P_0^3 = S^2\times S^1$ (the unique trivial bundle, no magnetic charge);

$\mathbb{R}^3 \supset S^2 = \left\{(x_1,x_2,x_3) \middle| \sum_{j=1}^3 x_j^2 = 1\right\} \overset{\Phi}{\cong} \mathbb{C}\cup\{\infty\}$, the unit 2-sphere, with $\Phi(x_1,x_2,x_3) = (x_1+ix_2)/(1+x_3)$ for $(x_1,x_2,x_3)\neq(0,0,-1)$ and ∞ for $(x_1,x_2,x_3)=(0,0,-1)$. In particular, for the case $k=1$, to which we shall restrict,

$$\xi_D^1: S^1 \to S^3 \xrightarrow{\pi} S^2 \tag{4}$$

with $P_1^3 = S^3 = \left\{(z_1,z_2)\in\mathbb{C}^2 \middle| |z_1|^2+|z_2|^2=1\right\}$ the 3-sphere, and $\pi_1\equiv\pi$ the Hopf map $\pi(z_1,z_2)=z_1/z_2$ for $z_2\neq 0$ and ∞ for $z_2=0$. The action of S^1 on S^3, $\psi_D: S^3\times S^1\to S^3$ is given by $\psi_D\left((z_1,z_2),\mathrm{e}^{i\varphi}\right)=\left(z_1\mathrm{e}^{i\varphi},z_2\mathrm{e}^{i\varphi}\right)$.

If $\chi,\varphi\in(0,2\pi)$ and $\theta\in[0,\pi]$ are the Euler angles in $\mathbb{R}^3$, the $u(1)$-valued non flat D connection on S^3 is given by [15]

$$\omega_D=\frac{i}{2}(d\chi+\cos\theta d\varphi) \tag{5}$$

with D potentials on S^2

$$A_{D\pm}=\mp\frac{i}{2}(1\mp\cos\theta)d\varphi. \tag{6}$$

The curvature of ω_D is $(-i)\times$ the magnetic field of the monopole: $F_D=d\omega_D=\frac{i}{2}\sin\theta d\theta\wedge d\varphi$.

4. Relation between the A-B and D Bundles

The inclusion

$$\iota:\mathbb{C}^*\to\mathbb{C}\cup\{\infty\},\ \iota(z)=z \tag{7}$$

induces:

1) The bundle map (but not isomorphism) $\xi_{A-B}\to\xi_D^1\equiv\xi_D$ (**Figure 1**), with

$$\pi\circ\overline{\iota}=\iota\circ pr_1,\ \psi_D\circ\left(\overline{\iota}\times Id_{S^1}\right)=\overline{\iota}\circ\psi_{A-B} \tag{8}$$

where

$$\overline{\iota}:\mathbb{C}^*\times S^1\to S^3,\ \overline{\iota}\left(z,\mathrm{e}^{i\varphi}\right)=\frac{(z,1)}{\|(z,1)\|}\mathrm{e}^{i\varphi}. \tag{9}$$

$$\begin{array}{ccc}
(\mathbb{C}^*\times S^1)\times S^1 & \xrightarrow{\overline{\iota}\times Id_{S^1}} & S^3\times S^1\\
\psi_{A-B}\downarrow & & \psi_D\downarrow\\
\mathbb{C}^*\times S^1 & \xrightarrow{\overline{\iota}} & S^3\\
pr_1\downarrow & & \pi\downarrow\\
\mathbb{C}^* & \xrightarrow{\iota} & \mathbb{C}\cup\{\infty\}
\end{array}$$

Figure 1. Bundle map $\xi_{A-B}\to\xi_D$.

2) The pull-back of the D bundle (**Figure 2**)

$$\iota^*(\xi_D): S^1\to P_{\iota^*(\xi_D)}\xrightarrow{pr_1}\mathbb{C}^* \tag{10}$$

with total space

$$P_{\iota^*(\xi_D)}=\left\{\left(z,(z_1,z_2)\right)\in\mathbb{C}^*\times S^3\,|\,\iota(z)=\pi(z_1,z_2)\right\}, \tag{11}$$

and action

$$\psi_{\iota^*(\xi_D)}:P_{\iota^*(\xi_D)}\times S^1\to P_{\iota^*(\xi_D)},\ \psi_{\iota^*(\xi_D)}\left(\left(z,(z_1,z_2)\right),\mathrm{e}^{i\lambda}\right)=\left(z,(z_1,z_2)\mathrm{e}^{i\lambda}\right). \tag{12}$$

One then has the bundle map $(pr_2,\iota):\xi_{\iota^*(\xi_D)}\to\xi_D$ given by

$$\pi \circ pr_2 = \iota \circ pr_1,\ \psi_D \circ \left(pr_2 \times Id_{S^1} \right) = pr_2 \circ \psi_{\iota^*(\xi_D)} \tag{13}$$

where $pr_2 : P_{\iota^*(\xi_D)} \to S^3$, $pr_2\left(\left(z,\left(z_1,z_2\right)\right)\right) = \left(z_1,z_2\right)$ is the projection in the second entry.

$$\begin{array}{ccc} P_{\iota^*\xi_D} \times S^1 & \overset{pr_2 \times Id_{S^1}}{\longrightarrow} & S^3 \times S^1 \\ \psi_{\iota^*(\xi_D)} \downarrow & & \psi_D \downarrow \\ P_{\iota^*(\xi_D)} & \overset{pr_2}{\longrightarrow} & S^3 \\ pr_1 \downarrow & & \pi \downarrow \\ \mathbb{C}^* & \overset{\iota}{\longrightarrow} & \mathbb{C} \cup \{\infty\} \end{array}$$

Figure 2. Pull back $\iota^*\left(\xi_D\right)$ of ξ_D.

3) The bundle isomorphism $\iota^*\left(\xi_D\right) \to \xi_{A-B}$ (**Figure 3**): the map

$$\Psi : P_{\iota^*(\xi_D)} \to \mathbb{C}^* \times S^1,\ \Psi\left(z, \frac{(z,1)}{\left\|(z,1)\right\|} \mathrm{e}^{i\varphi} \right) = \left(z, \mathrm{e}^{i\varphi}\right) \tag{14}$$

is continuous, one-to-one and onto, with continuous inverse Ψ^{-1}. Together with ι, Ψ establishes the topological equivalence between the bundles $\iota^*\left(\xi_D\right)$ and ξ_{A-B}. It is easy to verify the equalities

$$pr_1 \circ \Psi = Id_{\mathbb{C}^*} \circ pr_1,\ \psi_{A-B} \circ \left(\Psi \times Id_{S^1}\right) = \Psi \circ \psi_{\iota^*(\xi_D)}. \tag{15}$$

$$\begin{array}{ccc} P_{\iota^*\xi_D} \times S^1 & \overset{\Psi \times Id_{S^1}}{\longrightarrow} & (\mathbb{C}^* \times S^1) \times S^1 \\ \psi_{\iota^*(\xi_D)} \downarrow & & \psi_{A-B} \downarrow \\ P_{\iota^*\xi_D} & \overset{\Psi}{\longrightarrow} & \mathbb{C}^* \times S^1 \\ pr_1 \downarrow & & pr_1 \downarrow \\ \mathbb{C}^* & \overset{Id_{\mathbb{C}^*}}{\longrightarrow} & \mathbb{C}^* \end{array}$$

Figure 3. Bundle isomorphism $\iota^*\left(\xi_D\right) \to \xi_{A-B}$.

5. The Existence of the D Potentials Implies the Existence of the A-B Potentials

In terms of the cartesian coordinates $\left(X,Y,Z\right) = r\left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right)$ with $\theta \in \left(0,\pi\right)$ and $\varphi \in \left[0,2\pi\right)$ (which excludes the Z axis), the monopole potentials $A_{D\pm}$ of Equation (6) are given by

$$A_{D\pm}\left(X,Y,Z\right) = \left(A_{D\pm}\right)_X dX + \left(A_{D\pm}\right)_Y dY \tag{16}$$

with

$$\begin{aligned} \left(A_{D\pm}\right)_X &= \pm \frac{i}{2} \frac{Y}{X^2+Y^2} \left(1 \mp \frac{Z}{\sqrt{X^2+Y^2+Z^2}} \right), \\ \left(A_{D\pm}\right)_Y &= \mp \frac{i}{2} \frac{X}{X^2+Y^2} \left(1 \mp \frac{Z}{\sqrt{X^2+Y^2+Z^2}} \right) \end{aligned} \tag{17}$$

Restricting this 2-form to $Z = 0$, its pull-back operation by $\iota : \left(D_0^2\right)^* \to S^2$

reduces to the identity and gives on $\left(D_0^2\right)^*$ the A-B potentials of Equation (2). It is then clear that the same occurs at the level of connections, *i.e.* the pull-back by $\overline{\iota}$ of ω_D on S^3 gives ω_{A-B} on $\mathbb{C}^* \times S^1$.

6. The Existence of the A-B Connection Implies the Existence of the D Connection

The 2-torus resulting from taking the pre-image by π of the north and south poles of S^2, respectively $N = (0,0,1)$ and $S = (0,0,-1)$,

$$\pi^{-1}\left(\{N,S\}\right) = \left\{(z_1,0) \middle| |z_1| = 1\right\} \cup \left\{(0,z_2) \middle| |z_2| = 1\right\} \cong S^1 \times S^1 = T^2, \tag{18}$$

allows to define the truncated bundle $\hat{\xi}_D$ of ξ_D:

$$\hat{\xi}_D : S^1 \to \left(S^3 \setminus T^2\right) \xrightarrow{\hat{\pi}} \left(S^2 \setminus \{N,S\}\right) \cong \mathbb{C}^*, \tag{19}$$

isomorphic to ξ_{A-B} through the pair of maps $\left(Id_{\mathbb{C}^*}, \overline{\iota}\right)$, with action of S^1 on $S^3 \setminus T^2$ given by the restriction of ψ_D, $\psi_D|$ (**Figure 4**). One can then apply the Proposition 6.1 in ref. [16]: given a connection η (in our case A_{A-B}) in ξ_{A-B} there *exist* and is *unique* a connection ω in $\hat{\xi}_D$ such that the horizontal subspaces of η (kernels of η) in $\mathbb{C}^* \times S^1$ are mapped into the horizontal subspaces of ω (kernels of ω) in $S^3 \setminus T^2$. In our case, this is done through the push-forward linear transformation $d\overline{\iota} \equiv \overline{\iota}_*$. The result is

$$\overline{\iota}_*\left(ker\left(A_{A-B}\right)\right) = \frac{i}{2} d\chi = \omega_D \,|\left(\theta = \pi/2\right) \tag{20}$$

where $\omega_D|$ is the D connection on ξ_D (*i.e.* on S^3) restricted to $S^3 \setminus T^2$ *i.e.* with $\theta \in (0,\pi)$ (ref. [14]). The extension of the domain of θ from $(0,\pi)$ to $[0,\pi]$ recovers the bundle ξ_D and the D connection on it. This completes the proof that the A-B connection on ξ_{A-B} *uniquely* determines the D connection on ξ_D.

$$\begin{array}{ccc}
(\mathbb{C}^* \times S^1) \times S^1 & \stackrel{\overline{\iota} \times Id_{S^1}}{\longrightarrow} & (S^3 \setminus T^2) \times S^1 \\
\psi_{A-B} \downarrow & & \psi_D| \downarrow \\
\mathbb{C}^* \times S^1 & \stackrel{\overline{\iota}}{\longrightarrow} & S^3 \setminus T^2 \\
pr_1 \downarrow & & \hat{\pi} \downarrow \\
\mathbb{C}^* & \stackrel{Id_{\mathbb{C}^*}}{\longrightarrow} & \mathbb{C}^*
\end{array}$$

Figure 4. Bundle isomorphism $\xi_{A-B} \to \hat{\xi}_D$.

7. Conclusion

From its proposal in 1931 by Dirac [2], the abelian magnetic pole has been subject to intense search, however yet unsuccessful. In contradistinction, the Aharonov-Bohm effect proposed in 1959 by Aharonov and Bohm [1] was immediately experimentally verified [6]. Both "phenomena" admit a natural description in terms of the theory of fiber bundles and connections; moreover, the intimate relation between the corresponding $U(1)$-bundles ξ_D and ξ_{A-B}, the "residence" of the phenomena (Section 4), leads to the proof of the equivalence of the A-B and D potentials and connections, that is, the existence of $A_{D\pm}$ implies the existence of

$A_{A-B\pm}$ (the same for ω_D and ω_{A-B}) and *vice versa*. This pure mathematical result can be considered, given the physical existence of the A-B effect, a theoretical support to the present search for finding the magnetic charge.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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