Journal of High Energy Physics, Gravitation and Cosmology, 2024, 10, 1017-1028
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

A Novel Derivation of Black Hole Entropy in all Dimensions from Truly Point Mass Sources

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How to cite this paper: Castro Perelman, C. (2024) A Novel Derivation of Black Hole Entropy in all Dimensions from Truly Point Mass Sources. *Journal of High Energy Physics, Gravitation and Cosmology*, **10**, 1017-1028.
https://doi.org/10.4236/jhepgc.2024.103062

Received: March 18, 2024
Accepted: July 9, 2024
Published: July 12, 2024

Abstract

It is explicitly shown how the Schwarzschild Black Hole Entropy (in all dimensions) emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature involving the Dirac delta distribution. In order to achieve this, one is required to *extend* the domain of r to *negative* values $-\infty \le r \le +\infty$. It is the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ which furnish the Schwarzschild black hole entropy in all dimensions $D \ge 4$ after evaluating the Euclidean Einstein-Hilbert action. Two of the most salient results are i) that the observed spacetime dimension $D = 4$ is precisely singled out from all the other dimensions when the strong and weak energy conditions are met, and ii) the point mass source described in this work is *not* the result of a spherically symmetric gravitational collapse of a star as described by the Oppenheimer-Snyder model because we are *not* neglecting the pressure. As usual, it is required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. This approach can be generalized to the Reissner-Nordstrom and Kerr-Newman metrics. The physical implications of this finding warrant further investigation since it suggests a profound connection between the notion of gravitational entropy and spacetime singularities.

Keywords

General Relativity, Black Holes, Entropy, Strings

1. Introduction: Modification of the Schwarzschild Solution

The static spherically symmetric (SSS) *vacuum* solution of Einstein's field equations

[1] (in Lorentzian signature) was originally found by Schwarzschild [2], but is historically more widely known in terms of the solution provided by Hilbert [3] as

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}\right)(\mathrm{d}t)^2 + \left(1-\frac{2GM}{r}\right)^{-1}(\mathrm{d}r)^2 + r^2(\mathrm{d}\Omega)^2 \tag{1}$$

where the solid angle infinitesimal element is $(\mathrm{d}\Omega)^2 = (\mathrm{d}\theta)^2 + \sin^2(\theta)(\mathrm{d}\phi)^2$. We shall use throughout this work the units of $\hbar = c = k_B = 1$.

The higher-dimensional extension of the metric (1) was found by Tangherlini [4] and can be obtained by simply replacing $(\mathrm{d}\Omega)^2 \to (\mathrm{d}\Omega_{D-2})^2$ (the $D-2$ -dim solid angle) and $1-\frac{2GM}{r} \to 1-\left(\frac{r_h}{r}\right)^{D-3}$ where r_h is the horizon radius expressed in terms of M and the gravitational coupling G_D in D dimensions whose units are $(length)^{D-2}$. The higher dimensional metric is given by

$$\mathrm{d}s^2 = -f(r)(\mathrm{d}t)^2 + \frac{(\mathrm{d}r)^2}{f(r)} + r^2(\mathrm{d}\Omega_{D-2})^2, \quad f(r) = 1-\frac{16\pi G_D M}{(D-2)\Omega_{D-2} r^{D-3}} \tag{2a}$$

where G_D is the D-dim Newton's constant, M the black hole mass. The solid angle of a $D-2$ -dim hypersphere is $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}} \Big/ \Gamma\left(\frac{D-1}{2}\right)$. The horizon radius is determined from the condition $f(r_h) = 0$ giving

$$r_h = \left(\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}\right)^{\frac{1}{D-3}} \tag{2b}$$

such that the metric (2a) can be rewritten as

$$\mathrm{d}s^2 = -\left[1-\left(\frac{r_h}{r}\right)^{D-3}\right](\mathrm{d}t)^2 + \left[1-\left(\frac{r_h}{r}\right)^{D-3}\right]^{-1}(\mathrm{d}r)^2 + r^2(\mathrm{d}\Omega_{D-2})^2 \tag{3}$$

The Schwarzschild metric leads to a vanishing Ricci tensor and scalar curvature $R = 0$, hence in order to arrive at a key delta function singularity at the origin one has to *extend* the domain of r to *negative* values $-\infty \le r \le +\infty$, and replace r for $|r|$ in the metric (1). More precisely, one needs to make the replacement

$$1-\frac{2GM}{r} \to 1-\frac{2GM}{|r|} = 1-\frac{2GM}{r}\frac{r}{|r|} = 1-\frac{2GM\,\mathrm{sgn}(r)}{r}, \quad r = |r|\,\mathrm{sgn}(r) \tag{4a}$$

so the metric is actually of the form

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{|r|}\right)(\mathrm{d}t)^2 + \left(1-\frac{2GM}{|r|}\right)^{-1}(\mathrm{d}r)^2 + |r|^2(\mathrm{d}\Omega)^2 \tag{4b}$$

where $\mathrm{sgn}(r)$ is the sign function. The sign function is defined by $\mathrm{sgn}(r) = 1$, for $r > 0$; $\mathrm{sgn}(r) = -1$, for $r < 0$; and $\mathrm{sgn}(r = 0) = 0$, the arithmetic mean of 1, −1, and it will be instrumental in deriving the non-zero scalar curvature. The derivative of the sign function is $\frac{d}{dr}\mathrm{sgn}(r) = 2\delta(r)$ [1]. It is the derivatives of the sign function appearing in eq-(4) which will generate the key $\delta(r)$ terms in the scalar curvature. If one wishes to be mathematically rigorous in using distributions in nonlinear theories like general relativity one needs to recur to the Colombeau's theory of distributions [5] instead of the Dirac delta distributions.

Recently, the authors [6] have argued that unphysical equations of state result from the unrestricted use of the Synge *G*-trick of running the Einstein field equations backwards, which is what we are precisely doing in this work. Often this results in $\rho + p < 0$ which implies negative inertial mass density, which does not occur in reality. This is the basis of some unphysical spacetime models including phantom energy in cosmology and traversable wormholes [6]. It shall be shown below that the observed spacetime dimension $D = 4$ is precisely *singled* out from all the other dimensions when both the strong and weak energy conditions associated with the stress energy tensor are satisfied. The stress energy tensor originates from a variation of the matter terms with respect to the metric in the combined Einstein-Hilbert action of general relativity coupled to a point particle. It is the on-shell value of this combined action that leads precisely to the Einstein field equations with the stress energy tensor appearing in the right-hand side.

Thus, by replacing r for $|r|$ in eqs-(4) one finds that the scalar curvature is no longer zero $\mathcal{R} \neq 0$ but has a delta function singularity at $r = 0$ [2]

$$\mathcal{R} = 4GM\left(\frac{\delta'(r)}{r} + 2\frac{\delta(r)}{r^2}\right) = 4GM\frac{\delta(r)}{r^2}, \tag{5}$$

where the identities involving the derivatives of the delta functions have been used

$$\delta'(r) = -\frac{\delta(r)}{r}, \quad \delta^{(n)} = (-1)^n n!\frac{\delta(r)}{r^n} \tag{6}$$

2. The Euclidean Einstein-Hilbert Action and Black Hole Entropy

Because now one has that $\mathcal{R} \neq 0$, the Euclidean Einstein-Hilbert action is no longer zero. The inverse Hawking temperature $\beta_H = 8\pi GM$ is the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory and resulting after a Wick rotation, $it = \tau$, to imaginary time. The non-trivial Euclidean Einstein-Hilbert action is given by the integral

[1]The factor of 2 is due to the jump of 2 from −1 to +1.

[2]The Kretschmann invariant $\mathcal{R}_{\mu\nu\rho\tau}\mathcal{R}^{\mu\nu\rho\tau} \sim \left(\frac{2GM}{r^3}\right)^2$ is singular at $r = 0$ for the Schwarzschild metric.

$$
\begin{aligned}
I &= -\frac{i}{16\pi G}\int_0^{\beta_H} \mathrm{d}\tau \int_0^{\infty} \mathcal{R} 4\pi r^2 \mathrm{d}r \\
&= -\frac{i}{16\pi G}\int_0^{\beta_H} \mathrm{d}\tau \int_0^{\infty} \left(4GM\delta(r)/r^2\right) 4\pi r^2 \mathrm{d}r \\
&= -iM\beta_H \int_0^{\infty} \delta(r)\mathrm{d}r
\end{aligned}
\tag{7}
$$

Note the presence of an *-i* factor in the Euclidean action *I* which results from the measure $\sqrt{-g}$ piece since the determinant $g = \det(g_{\mu\nu}) > 0$ is now positive due to the Euclidean signature. The minus sign *-i* is chosen so that $\exp(iS_g) = \exp(-I)$ in the gravitational path integral ($I = -iS_g$).[3] Furthermore, because the end result of the radial integral (7) is symmetric in *r* due to $\delta(-r) = \delta(r)$, one may extend the radial domain of integration as follows

$$
\int_0^{\infty} \delta(r)\mathrm{d}r = \frac{1}{2}\int_{-\infty}^{\infty} \delta(r)\mathrm{d}r = \frac{1}{2} \tag{8}
$$

in order to fully integrate the delta function.

Some important remarks are in order before proceeding. In 3 spatial dimensions the radius is defined as $r = \pm\sqrt{x^2 + y^2 + z^2}$. In general, one must include both ± signs so an analytical extension from $r \to -r$ is possible by using $|r|$ in the metric solution (4b) and *without* having to switch the signs $M \to -M$, as it is required in the Schwarzschild metric (1) when one replaces $r \to -r$ (in order to avoid naked singularities and to maintain invariance of the metric). Hence, it is the *key* presence of $|r|$ in the metric (4b) which permits the analytical extension $r \to -r$ and allows us to perform the integral as shown in (8) by extending the domain of integration to negative values of *r*. Rigorously speaking, one has a branch-cut at $r = 0$, and a proper treatment would require working in the *complex r*-plane[4]. Physically speaking, one could interpret the point mass at $r = 0$ as a "point" wormhole where one goes from a positive *r* to a negative *r* region[5].

To illustrate the physical relevance of *extending* the domain of *r* to *negative* values $-\infty \le r \le +\infty$, and in using the modulus $|r|$, let us evaluate the 3-dim Laplacian (in a flat Euclidean space) of $1/r$ versus $1/|r|$. One finds in spherical coordinates that $\nabla^2(1/r) = \frac{1}{r^2}\partial_r\left(r^2\partial_r\left(\frac{1}{r}\right)\right) = 0$, is trivially zero[6] but $\nabla^2(1/|r|) = -2\delta(r)/r^2 \neq 0$. Hence, one finds that the delta function point-mass density source at $r = 0$ is a solution of the Poisson equation $\nabla^2\Phi = 4\pi G\rho$ when the classical gravitational potential is given by $\Phi = -GM/|r|$, instead of $-GM/r$.

[3]The scalar curvature R remains unaffected due to the fact that the change of sign in g_{tt} and R_{tt} cancels out when one evaluates the trace of the Ricci tensor component $g^{tt}R_{tt}$.

[4]$r = 0$ is also a spacelike singularity.

[5]We thank Eduardo Guendelman and Thomas Curtright for suggesting this.

[6]$\frac{0}{r^2}$ is also zero at $r = 0$. Since the numerator is already 0 it goes faster to 0 than r^2.

The error that one finds in many textbooks is due to the fact that when $r > 0$, the function $f(r) = r$, and its derivatives, coincide with the function $h(r) = |r|$, and its derivatives. So, when $r > 0$, some authors go ahead and replace the Laplacian of ($1/|r|$), with the Laplacian of $1/r$, and claim to generate a delta function. However, the Laplacian (in flat 3-dim) of $1/r$ is trivially zero. Mathematically speaking, the functions r and $|r|$ are *not* the same. At $r = 0$, the derivative of the function $|r|$ has a key discontinuity from 1 to −1, while the function r does not. The second derivative of $|r|$ is what furnishes the delta function.

As mentioned above, to rigorously treat distributions in nonlinear theories like gravity one must recur to the nonlinear distributional calculus developed by Colombeau [5]. The authors [7] devoted a mathematical analysis to the distributional Schwarzschild geometry. The Schwarzschild solution is extended to include the singularity; the energy momentum tensor becomes a δ-distribution supported at $r = 0$. Using generalized distributional geometry in the sense of Colombeau's (special) construction the nonlinearities were treated in a mathematically rigorous way. They also arrived at a scalar curvature given by a δ-distribution. In this work, we bypass these very rigorous mathematical details by simply extending the domain of r to negative values and by replacing r for $|r|$ in the Schwarzschild metric solution.

Another subtle point by replacing $1/|r|$ with $1/r$ in the region $r > 0$ occurs when one replaces the gravitational field $\vec{g} = \nabla\left(GM/|r|\right) = -\mathrm{sgn}(r)\frac{GM}{|r|^2}\hat{u}_r$ ($r^2 = |r|^2$) with $\vec{g} = -\frac{GM}{r^2}\hat{u}_r = -\nabla\Phi(r) = \nabla(GM/r)$ and proceeds to use the divergence theorem (Gauss law) where the boundary of the volume bulk region is a sphere of radius R, and the unit vector $\hat{u}_r$ points outward (so the non-vanishing gravitational flux flows inward) and claim once more, incorrectly, that the Laplacian of ($1/r$) generates a delta function. The reason for this inconsistency is similar as before, the sign function is not equal to 1 for all values of r and $\mathrm{sgn}(r)$ has a discontinuity at $r = 0$ (the location of the gravitational source), so mathematically speaking the expression $\vec{g} = -\mathrm{sgn}(r)\frac{GM}{r^2}\hat{u}_r$ [7] is *not* the same as $\vec{g} = -\frac{GM}{r^2}\hat{u}_r$, despite that they coincide in the region $r > 0$.

Finally, after this detour, given $\beta_H = 8\pi GM$, $\mathcal{R} = 4GM\delta(r)/r^2$, and eq-(8), the magnitude of the integral (7) becomes

$$|I| = \frac{1}{2}M\beta_H = 4\pi GM^2 = \frac{4\pi(2GM)^2}{4G} = \frac{4\pi r_h^2}{4G} = \frac{Area}{4L_P^2} \tag{9}$$

To conclude, the (magnitude of the) Euclidean Einstein-Hilbert action S_E

[7]For $r > 0$ the unit radial vector $\hat{u}_r$ points outwards (increasing r). For $r < 0$ the unit radial vector points inward (increasing r). Hence $\vec{g}$ always points towards the point mass attractive gravitational source. At $r = 0$ the spherical coordinate system is not well defined and must be replaced by Cartesian coordinates.

associated with the delta function point mass source yields precisely the Schwarzschild black hole entropy and given by one quarter of the horizon's area in Planck units. One should note that if one were to include the contribution of the point-mass matter term in the evaluation of the Euclidean action this would amount to introducing an additive constant to the entropy. The issue of an additive constant in the evaluation of entropy has been addressed by [8] in numerous occasions.

It was *not* necessary to introduce the Gibbons-Hawking-York *boundary* term [9], [10] in order to evaluate the entropy and involving the trace of the extrinsic curvature K

$$S = \frac{1}{16\pi G}\int_M \sqrt{|g|}\mathcal{R}\mathrm{d}^4x + \frac{1}{8\pi G}\int_{\partial M}\sqrt{|h|}K\mathrm{d}^3x$$

h is the determinant of the induced metric on the boundary ∂M. The bulk Einstein-Hilbert action for the Schwarzschild metric (1) vanishes (due to the vanishing of $\mathcal{R} = 0$), consequently, the contribution to the entropy stems entirely from the extrinsic curvature K of the boundary term. Gibbons and Hawking argued that in order to obtain an action which depends on the first derivatives of the metric, as is required by the composition property of the path-integral approach, the second derivatives appearing in the curvature scalar $\mathcal{R}$ had to be removed by an integration by parts resulting in the need to introduce the boundary term. Since now the variations of the first derivatives of the metric are no longer zero on the boundary, the Gibbons-Hawking-York boundary term is required in order to reproduce the Einstein field equations. In the case of asymptotically flat metrics the boundary region can be chosen to be the product of the Euclidean time axis (a circle of size β_H) with a sphere S^2 of large radius. Gibbons and Hawking evaluated the action for the gravitational field on a section of the *complexified* spacetime which avoids the singularity. The boundary integral in the limit that the sphere's radius goes to infinity yielded an action I given by $i4\pi GM^2$, and which agrees with the black hole entropy (up to an i factor).

However, in this work we are *not* removing the second derivatives of the metric so the boundary term is *no* longer required in order to reproduce the Einstein's field equations. And due to the non-vanishing scalar curvature given in terms of the Dirac delta distribution the Einstein-Hilbert bulk action is no longer vanishing. Consequently we have found that there are two ways to obtain the black hole entropy. One way is provided by the *boundary* term of eq-(10) when $\mathcal{R} = 0$ for the Schwarzschild metric, and another way is provided by the *bulk* Einstein-Hilbert action for the *modified* metric (4a, 4b) furnishing $\mathcal{R} = 4GM\frac{\delta(r)}{r^2} \neq 0$. One may speculate that some sort of bulk/boundary "duality" is taken place.

Let's proceed with the evaluation of the higher dimensional Schwarzschild black hole entropy. Once more, by replacing $r \to |r|$ in the metric (2a, 3) it gives

$$\mathrm{d}s^2 = -f(|r|)(\mathrm{d}t)^2 + \frac{(\mathrm{d}r)^2}{f(|r|)} + |r|^2(\mathrm{d}\Omega_{D-2})^2$$
$$= -\left(1-\left(\frac{r_h}{|r|}\right)^{D-3}\right)(\mathrm{d}t)^2 + \left(1-\left(\frac{r_h}{|r|}\right)^{D-3}\right)^{-1}(\mathrm{d}r)^2 + |r|^2(\mathrm{d}\Omega_{D-2})^2 \tag{11}$$

After a very lengthy and laborious calculation one learns that the scalar curvature associated with the metric (11) is

$$\mathcal{R} = \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{2(D-2)}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{(D-2)(D-3)}{r^2}(1-f) \tag{12}$$

Taking into account that $\frac{\mathrm{d}|r|}{\mathrm{d}r} = \mathrm{sgn}(r)$ [8] where $\mathrm{sgn}(r)$ is the sign function it leads to the following results

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{sgn}(r) = 2\delta(r),\ \frac{\mathrm{d}f}{\mathrm{d}r} = (D-3)r_h^{D-3}\frac{\mathrm{sgn}(r)}{|r|^{D-2}},$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} = -(D-2)(D-3)r_h^{D-3}\frac{1}{|r|^{D-1}} + 2(D-3)r_h^{D-3}\frac{\delta(r)}{|r|^{D-2}} \tag{13}$$

Inserting the results of eq-(13) into eq-(12) and taking into account the *identity* $r = |r|\mathrm{sgn}(r)$ which leads to key exact *cancellations*, the scalar curvature in eq-(12) turns out to be

$$\mathcal{R}_D = 2\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}(D-3)\frac{\delta(r)}{|r|^{D-2}} = 2r_h^{D-3}(D-3)\delta(r)\frac{\delta(r)}{|r|^{D-2}} \tag{14}$$

The use of $|r|$ in $f(|r|)$ in eq-(11) was instrumental in generating the delta function in (14). Had one used $f(r)$ one would have obtained $\mathcal{R} = 0$. In the case when $D = 4$ one recovers the same result as in eq-(5) for $\mathcal{R}$.

The Hawking temperature of the *D*-dim Schwarzschild black hole is $T_D = (D-3)/4\pi r_h \Rightarrow \beta_D = 4\pi r_h/(D-3)$. The non-trivial Euclidean Einstein-Hilbert action in *D*-dim is given by the integral

$$I = -\frac{i}{16\pi G_D}\int_0^{\beta_D}\mathrm{d}\tau\int_0^{\infty}\mathcal{R}_D\Omega_{D-2}r^{D-2}\mathrm{d}r \tag{15a}$$

Since one is integrating over the region $r \geq 0$, and *no* more derivatives are involved, it is valid now to equate $|r|^{D-2}$ with r^{D-2} without leading to inconsistencies. After setting $\beta_D = 4\pi r_h/(D-3)$, and inserting the expression (14) for $\mathcal{R}_D$ into (15a), it becomes

$$I = -i\frac{\Omega_{D-2}r_h^{D-2}}{2G_D}\int_0^{\infty}\delta(r)\mathrm{d}r \tag{15b}$$

After taking into account eq-(8), the integral involving the $\delta(r)$ function

[8]The derivative of $|r|$ is discontinuous at $r = 0$, but because it jumps from −1 to +1, one may take their arithmetic mean which is 0 and which agrees with the value of $\mathrm{sgn}(r=0) = 0$.

yields a key factor of $\frac{1}{2}$, and the magnitude of the integral eq-(15b) yields finally

$$|I| = \frac{\Omega_{D-2} r_h^{D-2}}{4G_D} = \frac{\Omega_{D-2}}{4G_D} \left(\frac{16\pi G_D M}{(D-2)\Omega_{D-2}} \right)^{\frac{D-2}{D-3}} \tag{16}$$

which is the Schwarzschild black hole entropy in *D*-dimensions given by one-quarter of the horizon area in Planck units.

Next we shall find the expressions for the density and pressure of the point-matter source leading to a non-vanishing scalar curvature and which furnishes the higher dimensional black hole entropy. Given the trace of the stress energy tensor $\mathcal{T}_D = T_\mu^\mu$, the *trace* of the Einstein tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$ obeys the following relation stemming from the field equations

$$-\mathcal{R}_D \frac{D-2}{2} = 8\pi G_D \mathcal{T}_D = -(8\pi G_D) \left(2(D-3) \frac{M}{\Omega_{D-2}} \frac{\delta(r)}{|r|^{D-2}} \right) \tag{17}$$

Since the spherically symmetric energy-mass density ρ in *D*-dim for a point mass source is given by[9]

$$\rho = \frac{2M}{\Omega_{D-2} |r|^{D-2}} \delta(r) \Rightarrow \int_0^\infty \rho\, \Omega_{D-2} r^{D-2} \mathrm{d}r = 2M \int_0^\infty \delta(r) \mathrm{d}r = M \tag{18}$$

one finds that the trace of the stress energy tensor is

$$\mathcal{T}_D = -(D-3) \left[\frac{2M}{\Omega_{D-2} |r|^{D-2}} \delta(r) \right] = -(D-3)\rho$$

Due to the (hyper) spherical symmetry, the $D-2$ transverse pressure components $p_\perp$ to the radial direction are all equal, then the expression in (19) leads to

$$\mathcal{T}_D = -\rho + p_r + (D-2) p_\perp = -(D-3)\rho \tag{20}$$

One must supplement eq-(20) with the Einstein field equations in order to determine ρ, p_r and the $D-2$ transverse pressure components $p_\perp = p_{\theta_i}, i = 1, 2, \cdots, D-2$,

$$\mathcal{R}_t^t - \frac{1}{2} \delta_t^t \mathcal{R} = 8\pi G_D T_t^t = -8\pi G_D \rho,\ \mathcal{R}_r^r - \frac{1}{2} \delta_r^r \mathcal{R} = 8\pi G_D T_r^r = 8\pi G_D p_r \tag{21}$$

$$\mathcal{R}_\perp^\perp - \frac{1}{2} \delta_\perp^\perp \mathcal{R} = 8\pi G_D T_\perp^\perp = 8\pi G_D p_\perp \tag{22}$$

After a lengthy but straightforward algebra one finds that the density and the *anisotropic* pressure components are given by

$$\rho = \frac{2M}{\Omega_{D-2} |r|^{D-2}} \delta(r),\ p_r = -\frac{2(D-3)}{D-2} \rho,$$

[9]Note the key extra factor of 2 in eq-(18) that is required to evaluate the integral of $\delta(r)$.

$$p_{\perp} = \frac{(4-D)(D-2)+2(D-3)}{(D-2)^2}\rho \Rightarrow -\rho + p_r + (D-2)p_{\perp} = -(D-3)\rho \quad (23)$$

The solutions (23) satisfy the *strong* energy conditions $\rho + \sum p_i \ge 0$ when $D = 4,5$, and the weak energy conditions $\rho + p_i \ge 0$ for all $i = 1,2,\cdots,D-1$ when $D = 4$. Thus, interestingly enough, the observed spacetime dimension $D = 4$ is *singled* out from all the others when both the strong and weak energy conditions are satisfied.

One may object to the above expressions (23) because the angular coordinates are not well defined at $r = 0$. This is not a problem because one can simply perform a coordinate change of the stress energy tensor $T_{\mu\nu}$ to Cartesian coordinates which are well defined at $r = 0$ [10]. The solutions (23) are consistent with the conservation equation of the stress energy tensor $\nabla_{\mu} T^{\mu\nu} = 0$. It can be more easily verified in $D = 4$ where one arrives at

$$\rho = -p_r = \frac{2M}{4\pi r^2}\delta(r), \quad p_{\perp} = \frac{1}{2}\rho = \frac{M}{4\pi r^2}\delta(r) \quad (24)$$

satisfying the strong and weak energy conditions. One can check that the expressions (24) are consistent with the conservation equation

$$\nabla_{\mu} T^{\mu\nu} = 0 \Rightarrow p_{\perp} + \rho + \frac{r}{2}\frac{d\rho}{dr} = 0 \quad (25)$$

and which can be verified explicitly after using the identities $r\frac{d}{dr}(\delta(r)) = -\delta(r)$; $r^n \frac{d^n}{dr^n}(\delta(r)) = (-1)^n n!\,\delta(r)$. Similar results as those found in eq-(24) were obtained in [11] by choosing a mass density given by a Gaussian $M(\sigma)^{-3/2}\exp(-r^2/\sigma)$ where the Gaussian width $\sqrt{\sigma}$ was related to the non-commutativity parameter associated with the noncommutative spacetime co-ordinates $[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\Theta^{\mu\nu}\mathbf{1}$ after equating the norm to σ: $\sqrt{\Theta_{\mu\nu}\Theta^{\mu\nu}} = \sigma$. As the width of the Gaussian goes to zero one recovers the product of three delta functions

$$\lim_{\sigma\to 0}\left(M(\sigma)^{-3/2}\exp(-r^2/\sigma)\right) \to \rho = M\delta(x)\delta(y)\delta(z) \quad (26)$$

Our mass density does *not* involve the product of three delta functions (a 3-dim delta function) but involves the term $\frac{\delta(r)}{r^2}$ instead. The one-dim delta function $\delta(r)$ originated directly from the *second* derivatives of the metric (4b), and which in turn, results into an effective "dimensional reduction" of the 3-dim delta function $\delta(x)\delta(y)\delta(z)$ to a one-dimensional one $\delta(r)$.

Because the authors [11] used a Gaussian mass density to *smear* the point mass source and introduce "fuzziness" of the spacetime points into the picture, their value of $\mathcal{R}$ was finite at $r = 0$. Their physical model could be viewed as a self-gravitating *anisotropic* fluid droplet. Our effective mass function in eqs-(4) is $\mathcal{M}(r) = M\,\mathrm{sgn}(r)$, and represents that mass enclosed[11] within a radius r,

[10]In Cartesian coordinates the stress energy tensor will have off-diagonal components.

whereas the mass function $\mathcal{M}(r)$ in [11] was given by an incomplete gamma function as a result of integrating the Gaussian mass density across a spherical region of radius *r*.

3. Concluding Remarks

It is very important to *emphasize* that the point mass source described in this work is *not* the result of a spherically symmetric gravitational *collapse* of a star as described by the Oppenheimer-Snyder model because they neglected the pressure [12]. The Tolman-Oppenheimer-Volkoff equation [13] constrains the structure of a spherically symmetric body of *isotropic* material (fluid) which is in static gravitational equilibrium, as prescribed by general relativity. However the pressure in our case is *not* isotropic. Thus, the point mass source described here cannot be interpreted as a round ball of a fluid of isotropic pressure shrinking to zero size. More likely, it could be the result of gravitational collapse of an *anisotropic* star. Primordial black holes are postulated to result from the gravitational collapse of regions in the very early universe which experience very high density perturbations. It is warranted to explore the consequences that the point mass sources described in this work might have in the study of *primordial* black holes [14].

After this discussion one concludes that the expressions (23) are the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r=0$ and which furnish the Schwarzschild black hole entropy (up to a factor of $-i$) in all dimensions $D \geq 4$ by a direct evaluation of the Euclidean Einstein-Hilbert action. As usual, it was required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. The appealing result is that there was *no* need to introduce the Gibbons-Hawking-York boundary term [15] in order to arrive at the black hole entropy because in our case one has that $\mathcal{R} \neq 0$, and we are working with a *second* derivative theory. And, furthermore, there was no need to introduce a complex integration contour to *avoid* the singularity as done in [10].

On the contrary, we found that the source of the black hole entropy stems entirely from the scalar curvature *singularity* at the origin $r=0$. The physical implications of this finding warrant further investigation since it suggests a profound connection between the notion of gravitational entropy and spacetime singularities. The procedure proposed in this work also works for the Reissner-Nordstrom and the more general Kerr-Newman metric solutions as shown more recently [16].

To finalize, one should mention that a considerable progress in recent years has been made in understanding the quantum aspects of black holes and the

[11]Since there is no mass *enclosed* by $r=0$ then $\mathcal{M}(r=0) = M\,\mathrm{sgn}(r=0) = 0$ despite that the point mass is sitting at $r=0$.

Hawking evaporation process [17]. Most recently, a plethora of activity has been centered concerning the relation between generalized entropy $S_{gen} = \frac{A}{4G} + S_{ext}$ and von Neumann entropy. After reinstating the numerical constants that were set to unity one has $S_{gen} = \frac{k_B c^3 A}{4G\hbar} + S_{ext}$. While the individual terms are ill-defined in the semi-classical limit, their sum is well-defined if one takes into account perturbative quantum gravitational effects. For a detailed discussion of von Neumann algebras, generalized entropy see [18], and the excellent 22 lectures by Witten [8]. Consequently, much more work remains ahead in finding a bridge between our results and the most recent findings in operator algebras and Algebraic Quantum Field Theory (AQFT).

Acknowledgments

We are indebted to M. Bowers for assistance, and to Eduardo Guendelman and Emil Mottola for many illuminating discussions at the mini-conference in the Bahamas that took place in March 3-10, 2024.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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