

The GBR Hypothesis Revisited

Lewis Nash 

Marrietta, GA, USA

Email: lewis.r.nash@gmail.com

How to cite this paper: Nash, L. (2024)
The GBR Hypothesis Revisited. *Journal of
High Energy Physics, Gravitation and Cos-
mology*, 10, 800-809.
<https://doi.org/10.4236/jhepgc.2024.102048>

Received: December 27, 2023

Accepted: April 20, 2024

Published: April 23, 2024

Copyright © 2024 by author(s) and
Scientific Research Publishing Inc.
This work is licensed under the Creative
Commons Attribution International
License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

The radical hypothesis concerning the physics of gravitational black-body radiation is placed on a more solid statistical mechanics foundation in this study. As the concepts and formalism in the former presentation are only partially developed and furthermore, suffer from an unfortunate misstep regarding Hawking radiation and the hypothetical gravitational black-body temperature of a parcel or distribution of energy; this paper aims to fill in some of the theoretical gaps in the derivation of the Planck radiation formula for gravity (or non-Euclidean space-time), and there by provide a more complete and transparent quantum theory of thermal gravitational radiation.

Keywords

Gravitational Black Body, Gravitons, Thermal Gravitational Radiation, Spiral Galaxy, Rotation Curves

1. Introduction

It is well known since the 1930s, that the observed rotational velocity of matter and energy in the outer arms of flat spiral (disk) galaxies, such as our own Milky Way Galaxy, is greater than that predicted by the inverse square law of Newton's law of universal gravitation (1729). This mysterious phenomenon suggests, in general, that the observable energy¹ content of a flat spiral galaxy alone cannot be the total source of the gravitational force holding the massive system of stars, planets, gas, and dust together.

Generally, a parcel of energy produces a gravitational field. Hypothetical mass less quanta of a gravitational field are called gravitons [1], and their energy, similar to photons, which are quanta of an electromagnetic field², generate packets

¹In accordance with relativity theory which has sufficiently established the equivalence of mass and energy, we shall refer to mass as energy in this study.

²The notion of mass less light quanta was first conceived by Einstein in a paper where he proposed an explanation for the photoelectric effect [2] in 1905.

of coherent oscillations within the Planck lattice [3]. However, because gravitons are essentially particles of spacetime itself, then unlike photons, gravitons not only travel at the speed of light but also oscillate along with the internal oscillations of the Planck lattice. As is known from Maxwell's work on electromagnetism, a moving electric charge radiates electromagnetic waves. Similarly, Einstein's work on gravity showed that a moving gravitational charge radiates a gravitational wave. Thus, here we imagine the gravitational field of a parcel or distribution of energy to be composed of gravitons jiggling about, coupled to standing wave modes of non-Euclidean (curved) space-time [4]. Suppose each standing wave mode of curved space-time is coupled to a harmonic oscillator of the Planck lattice which oscillates at the same frequency as corresponding the standing wave mode of curved space-time. Each oscillator has two degrees of freedom, one for kinetic energy and one for potential energy, so it has an average energy of $k_B T$ according to the equipartition theorem. In thermal equilibrium, the average energy of the Planck oscillators and the standing wave mode of curved space-time must be the same for the two to be in thermal equilibrium [5]. Hence, each mode of oscillation of the gravitational field has an energy $k_B T$ and can be considered as having a temperature T , which is the basis of the Rayleigh-Jeans theory (1900) for the spectral radiance of electromagnetic radiation.

2. Rotation Curves of Spiral Galaxies

As in the case of oscillating electrons in a material radiating electromagnetic energy, we have oscillating gravitons radiating gravitational energy in a similar manner. Because oscillating gravitons increase the strength of a gravitational field, which in turn increases the frequency of graviton oscillations, the resulting strength (or energy) of any gravitational field should be infinite. However, this does not agree with observation. Therefore, we propose that the gravitational energy spectrum be discrete. In accordance with Planck's law [6], which has proven to be adequate thus far, we conjecture that for a given increase in the curvature of spacetime, the oscillation frequency of the gravitons may increase if the increase in the gravitational field energy is greater than or equal to the fundamental unit of energy, as defined by the Planck-Einstein relation, for that particular frequency. This assumption is essential to prevent the energy of the gravitational field from becoming infinite.

Now, we proceed under the general assumption that the gravitational field of any parcel or distribution of energy consists of a familiar classical Newtonian component g_N that includes all forms of energy not made of space-time and a hypothetical gravitational radiation component g_{gr} . Because the Planck lattice absorbs and re-emits all gravitational energy surrounding a parcel or distribution of energy with no reflection or scattering, space-time at the Planck scale is a perfect "gravitational" blackbody [7]. Analogous to electromagnetic black-body radiation, gravitational black-body radiation (GBR) [8] is defined as thermal gravitational radiation [9] surrounding a parcel or distribution of energy in thermodynamic equilibrium with the Planck lattice. The radiation component of

the gravitational field is presumed to be vanishingly small, except in cases involving large distributions of energy like galaxies, or galaxy clusters. Therefore, the gravitational field of a parcel or distribution of energy is defined here by the following expression³:

$$\mathbf{g} \equiv \mathbf{g}_N + \mathbf{g}_{gr}. \quad (1)$$

Let us assume, in accordance with the virial theorem, that the thermal gravitational energy E_{gr} of the spiral galaxy is equal to twice the difference in the rotational kinetic energy (for a solid disk) between the observed rotation speed and the theoretical Newtonian rotation speed. Thus,

$$E_{gr} = M_{gr}c^2 = \frac{1}{2}M_N(v^2 - v_N^2), \quad (2)$$

where M_N is the observable (or Newtonian) mass of the system, M_{gr} is the relativistic mass of the thermal gravitational radiation, v is the observed rotation speed of the galaxy, and $v_N = \sqrt{GM_N/r}$ is the rotation speed of the galaxy, as predicted by Newton's law of gravity for a distribution of energy. It follows from Equation (2) that the observed rotational velocity of the system is determined by

$$v = \sqrt{\frac{GM_N}{r} + \frac{4GM_{gr}}{R_S}}, \quad (3)$$

where r is the radial distance from the center of rotation of the system and $R_S = 2GM_N/c^2$ is the Schwarzschild radius of the gravitational source.

From the equation for circular motion, we obtain an expression for the gravitational acceleration, as follows:

$$\mathbf{g} = \frac{v^2}{r} = \frac{GM_N}{r^2} + \frac{4GM_{gr}}{R_S r}.$$

Hence, from Equation (1) the Newtonian and gravitational radiation components of the gravitational field are, respectively:

$$\mathbf{g}_N = \frac{GM_N}{r^2} \quad \text{and} \quad \mathbf{g}_{gr} = \frac{4GM_{gr}}{R_S r}. \quad (4)$$

Let us suppose the energy per graviton is

$$\epsilon_{gr} = h\nu = hc/2r, \quad (5)$$

where $\nu = c/\lambda$ and $\lambda = 2r$ is the fundamental (or zero-point) wavelength of the gravitational field [10] [11]. The graviton energy can thus be written in terms of R as

$$\epsilon_{gr} = \frac{hcR}{2L}, \quad (6)$$

where $R = 2L/\lambda$ defines a radius in n -space [12] and L is the dimension of the gravitational potential well. The n -space associated with standing wave solutions involves only positive values of n , so the volume must be divided by 8. It then must be multiplied by 2 to account for the two polarization states of the graviton

³For the equations presented in this study G is the gravitational constant, h is Planck's constant, and c is the speed of light in vacuum, unless otherwise stated.

[13]. The number of states is then

$$N = (2) \left(\frac{1}{8} \right) \left(\frac{4\pi}{3} R^3 \right) = \frac{8\pi}{3} \left(\frac{L}{r} \right)^3.$$

The number of states per unit volume is

$$n_s = \frac{N}{L^3} = \frac{8\pi}{3r^3}. \quad (7)$$

From the assumption that the Planck oscillators are quantized in energy with the quantum of energy equal to Planck's constant times the frequency, we shall develop a Planck radiation formula for gravity. The average energy per "mode" or "quantum" is the energy of the quantum $\mathcal{E}_{gr,n} = n\epsilon_{gr}$ times the probability that it will be occupied [14]

$$P_n = \frac{e^{n\epsilon_{gr}/k_B T_{gr}}}{\sum_{n=0}^{\infty} e^{n\epsilon_{gr}/k_B T_{gr}}}.$$

In thermal equilibrium with the Planck oscillators, the average energy has the form

$$\langle E_{gr} \rangle = \frac{\epsilon_{gr}}{e^{\lambda_{p,th}/r} - 1}, \quad (8)$$

where $\lambda_{p,th} = hc/2k_B T$ [15] and $T = T_{gr}$ are the thermal wavelength⁴ and temperature of the Planck lattice, respectively. This average graviton (or Planck oscillator) energy per state (or mode) times the number of states per unit volume times the number of gravitons N_{gr} yields the GBR energy density, which is the Planck radiation formula for gravity:

$$u_{gr}(r) = N_{gr} n_s \langle E_{gr} \rangle = \frac{8\pi}{3r^3} \left(\frac{E_{gr}}{e^{\lambda_{p,th}/r} - 1} \right), \quad (9)$$

where $E_{gr} = N_{gr} \epsilon_{gr}$.

The thermal Planck wavelength is determined so that the computed rotation speed of the galaxy due to the graviton energy density, as prescribed by the relation above, equals the observed rotation speed of the galaxy. The GBR energy and thermal Planck wavelength of a few spiral galaxies were determined from existing rotation curve data [16] [17] and are listed in **Table 1**.

The energy is related to the energy density by the following equation:

$$E_{gr} = u_{gr} \mathcal{V}_{gr}, \quad (10)$$

where $\mathcal{V}_{gr} = \mathcal{V}_{tot} (1 - M_N/M_{tot})$ is the gravitational volume⁵, within a region of space \mathcal{V}_{tot} is the bounded by a disk of radius r , and thickness $t = 1$ kpc centered about the galactic plane and $M_{tot} = v^2 r/G$ is the total gravitational energy of the spiral galaxy, as determined from its observed rotation speed. From Equations (2) and (10), the relativistic mass of the thermal gravitational radiation is given by

⁴This quantity is determined from the equipartition theorem for a system in thermal equilibrium with four degrees of freedom.

⁵A region of space surrounding a parcel or distribution of energy that is essentially empty of matter. In general, as the total volume of a distribution of energy increases the energy density decreases, which causes the gravitational volume and the number of gravitons to increase.

Table 1. Galaxy properties.

Galaxy	E_{gr} (J)	$\lambda_{p,th}$ (kpc)
Milky Way	5.52E+51	12
NGC 1560	1.23E+49	10
NGC 3198	5.65E+50	14
NGC 4157	1.54E+51	14

$$M_{gr} = \frac{8\pi\mathcal{V}_{gr}}{3c^2r^3} \left(\frac{E_{gr}}{e^{\lambda_{p,th}/r} - 1} \right). \quad (11)$$

Inserting Equation (11) into Equation (3) yields a relation for the theoretical rotation speed of the spiral galaxy. This relation can be used to generate the theoretical rotation curves plotted in **Figure 1**.

From these theoretical considerations, one recognizes that as the gravitational volume of a spiral galaxy increases, the number of allowed energy states increases,

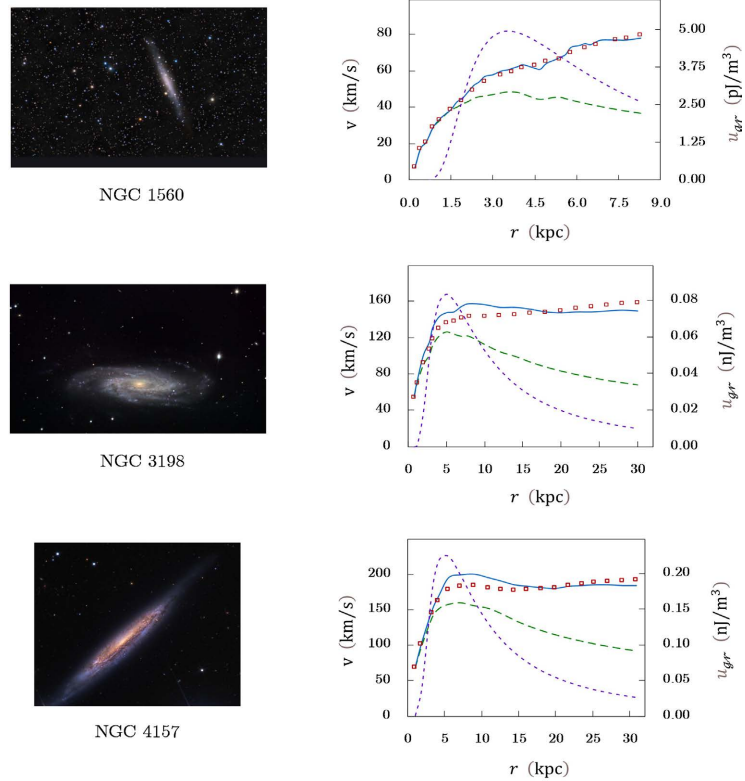


Figure 1⁷. The rotation curves of a few spiral galaxies are calculated by assuming the gravitons of their pervasive gravitational fields oscillate and radiate thermal gravitational energy. The Newtonian rotation curve for light traces mass is shown by the dashed line, and the rotation curve observed in the 21-cm line of neutral hydrogen is shown by the solid line, which extends far beyond the bright galactic center. The small squares lie along the theoretical rotation curve of the galaxy due to GBR, and the dotted (short dashed) curve is the corresponding GBR energy density profile. As one can see, the theoretical rotation speeds due to GBR are in good agreement with observation.

⁷The plotted curves were generated from galaxy rotation curve data provided from Refs. [16] [17] and the computed galaxy properties listed in **Table 1**.

which decreases the spacing between energy levels and subsequently causes the rotation curve of the galaxy to flatten⁸, as shown in **Figure 1**. Although this empirical result manifestly contradicts the “behavior” of the typical rotation curve as predicted by Newton’s well-known classical theory of gravitation, it is in complete agreement with the predictions of our non classical Newtonian (or Planckian) theory of gravity.

By differentiating $u_{gr}(r)$ with respect to r , to find the location r_m of the maximum energy density, one obtains the following equation:

$$\frac{\lambda_{p,th}}{r_m} = 3\left(1 - e^{-\lambda_{p,th}/r_m}\right),$$

which can be iteratively solved. After a few iterations, one finds

$$\frac{\lambda_{p,th}}{r_m} = 2.821. \quad (12)$$

Once the thermal Planck wavelength is determined from observational rotation curve data, it is a straightforward matter to compute the location of the peak in the energy density distribution. We found that the peak GBR energy density of the Milky Way is approximately 1.29 nJ/m^3 and is located 4.25 kpc from the center of the galaxy.

3. Gravitational Black-Body Temperature⁹

The GBR of a spiral galaxy consists of gravitational radiation in thermal equilibrium with the harmonic oscillators of the Planck lattice. When they are in thermal equilibrium, the average rate of emission of gravitational radiation by the Planck oscillators equals the average rate of absorption of gravitational radiation. At thermal equilibrium, the temperature T of the Planck lattice is equal to the temperature of the gravitational radiation T_{gr} .

The black-body temperature of the gravitational field of a parcel or distribution of energy can be found from the wavelength at which the GBR curve peaks. From Equation (12) the relation for the gravitational black-body temperature (GBT) of a parcel or distribution of energy can be written in a form similar to that for Hawking radiation [21]

$$T = \frac{\eta \hbar c}{4\pi k_B r_m} \quad (13)$$

where $\eta = k_{p,th} L_m$ with $k_{p,th} = 2\pi/\lambda_{p,th}$ and $L_m = 2\pi r_m$. Hence, the GBT of the Milky Way Galaxy is 0.0194 zK . In accordance with Wien’s displacement law the product of the peak wavelength and the black-body temperature of curved space-time is expected to be a constant, $\lambda_m T = 0.005103 \text{ m}\cdot\text{K}$. In a similar manner, one may also recognize that the product of the thermal Planck wavelength and the corresponding GBT is also a constant, $\lambda_{p,th} T = 0.007199 \text{ m}\cdot\text{K}$.

⁸This mysterious phenomenon is part of the more general dark matter problem [18] which in principle can be resolved in the framework of extended theories of gravity [19].

⁹The hypothetical temperature of (the Planck lattice (or curved) space-time) outside a parcel or distribution of energy.

The Planck radiation formula for gravity can be expressed in terms of the wavelength λ , as

$$u_{gr}(\lambda) = N_{gr} n_s(\lambda) \langle E_{gr} \rangle = \frac{64\pi}{3\lambda^3} \left(\frac{E_{gr}}{e^{\beta hc/\lambda} - 1} \right), \quad (14)$$

where $\beta = 1/k_B T$ and $n_s(\lambda) = 64\pi/3\lambda^3$ is the number of states per unit volume.

In general, the GBR spectral density, which is plotted in **Figure 2** for the Milky Way Galaxy, has the form

$$S_{gr}(\lambda) = N_{gr} \rho_s(\lambda) \langle E_{gr} \rangle = \frac{64\pi}{\lambda^4} \left(\frac{E_{gr}}{e^{\beta hc/\lambda} - 1} \right), \quad (15)$$

where $\rho_s(\lambda) = |dn_s/d\lambda| = 64\pi/\lambda^4$ denotes the density of states per unit wavelength. Similar to electromagnetic black-body radiation, the total energy density

$$u_{gr} = \int S_{gr}(\lambda) d\lambda$$

is finite and for a constant number of gravitons $N_{gr} = \beta E_{gr}/2$, varies as T^4 :

$$u_{gr} \approx \frac{\pi^7}{50} \left(\frac{E_{gr}}{\lambda_{p,th}^3} \right) = \frac{8\pi^7}{25} \left(\frac{k_B^4 T^4}{h^3 c^3} \right) N_{gr} \quad (16)$$

where $\lambda_{p,th} = \beta hc/2$. Hence, according to the galactic properties listed in **Table 1**, the total GBR energy density of the Milky Way is less than 10 nJ/m^3 .

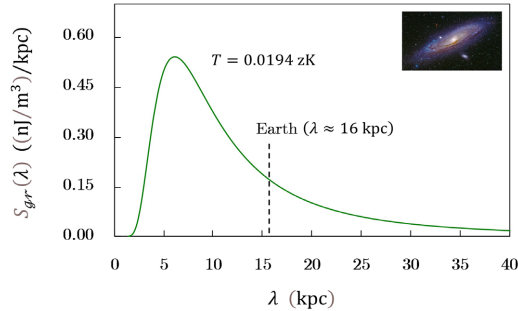


Figure 2¹¹. The background GBR spectral density [20] of the Milky Way Galaxy. The frequency of the thermal gravitational radiation of the Milky Way near Earth is 10^{23} times smaller than the lowest frequency ($\sim 300 \text{ GHz}$) of solar electromagnetic radiation, which is about 10^{13} times smaller than the lowest frequency ($\sim 10 \text{ Hz}$) that can be detected by LIGO [22].

4. Interesting Theoretical Results

The distance R from the Earth to the Moon is approximately $3.84 \times 10^8 \text{ m}$. The computed energy density of gravitational radiation that permeates the Solar System is approximately¹⁰ 0.88 nJ/m^3 . The strength of the gravitational field produced by the background GBR [23] [24] of the Milky Way that slowly pulls the Moon toward the Earth is approximately:

¹⁰The energy density is computed with rotation curve data for the Milky Way provided in Ref. [17].

¹¹The plotted curve and gravitational black body temperature T of the Milky Way were derived from galaxy rotation curve data obtained from Ref. [17] and the computed galaxy properties listed in **Table 1**.

$$g_{gr} = \frac{4GM_{gr}}{R_{s,E}R} = \frac{2M_{gr}c^2}{M_ER} \approx 1.82 \times 10^{-16} \text{ m/s}^2,$$

$$M_{gr} = \frac{u_{gr}\mathcal{V}_{gr}}{c^2} \approx 2.32 \text{ kg},$$

$$\mathcal{V}_{gr} = \frac{4\pi}{3}R^3 \approx 2.38 \times 10^{26} \text{ m}^3,$$

$$M_E = 5.972 \times 10^{24} \text{ kg},$$

which is 13 orders of magnitude smaller than the gravitational field of Earth pulling on the Moon $\sim 2.70 \text{ mm/s}^2$. Here, M_{gr} is the relativistic mass of the GBR contained within a spherical volume centered on Earth with a radius equal to the distance between Earth and Moon. This outcome confirms our earlier claim that the gravitational field generated by GBR is indeed extremely small, except in the case of very large astrophysical sources or events, *i.e.*, galaxies, supermassive black hole collisions, etc.

At the edge of the Solar System, a distance of 143.7 billion kilometers¹² from the Sun, where the gravitational force of our Sun fades, the additional increase in the Sun's gravitational pull due to the background GBR of the Milky Way is approximately:

$$g_{gr} = \frac{2M_{gr}c^2}{M_{\odot}R} \approx 0.76 \times 10^{-10} \text{ m/s}^2,$$

$$M_{gr} = \frac{u_{gr}\mathcal{V}_{gr}}{c^2} \approx 1.21 \times 10^{17} \text{ kg},$$

$$\mathcal{V}_{gr} = \frac{4\pi}{3}R^3 \approx 1.24 \times 10^{43} \text{ m}^3,$$

$$M_{\odot} = 1.988 \times 10^{30} \text{ kg},^{13}$$

which is almost one-tenth the magnitude of the anomalous acceleration $8.74 \times 10^{-10} \text{ m/s}^2$ towards the Sun experienced by the Pioneer spacecrafts [25] (launched in 1972 and 1973) as they moved beyond the orbit of Uranus (approximately 2.8 billion km from the Sun) on their way out of the Solar System.

Conflicts of Interest

The author declares that they have no known competing financial interests or personal relationships that could have influenced the work reported in this study.

References

- [1] Schwinger, J. (1968) Sources and Gravitons. *Physical Review Journals Archive*, **173**, 1264-1272. <https://doi.org/10.1103/PhysRev.173.1264>
- [2] Einstein, A. (1905) On a Heuristic Point of View about the Creation and Conversion of Light. *Annals of Physics*, **17**, 132-148.

¹²This value is based upon Sedna the most distant observable object known in our Solar System.

¹³The mass of the Sun.

- [3] Nash, L. (2022) On the Dynamics of Euclidean Space-Time at the Planck Scale. *Reports in Advances of Physical Sciences*, **6**, Article ID: 2250002. <https://doi.org/10.1142/S2424942422500025>
- [4] Stephani, H. (2003) Some Remarks on Standing Gravitational Waves. *General Relativity and Gravitation*, **35**, 467-474. <https://doi.org/10.1023/A:1022330218708>
- [5] Bowley, R. and Sánchez, M. (1999) Introductory Statistical Mechanics. 2nd Edition, Oxford University Press, Oxford, 166-169. <https://doi.org/10.1093/hesc/9780198505761.001.0001>
- [6] Planck, M. (1901) On the Law of Energy Distribution in the Normal Spectrum. *Annals of Physics*, **4**, 553-563.
- [7] Kirchhoff, G. (1860) On the Relation between the Radiating and Absorbing Powers of Different Bodies for Light and Heat. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **20**, 1-21. <https://doi.org/10.1080/14786446008642901>
- [8] Nash, L. (2022) Gravitational Black-Body Radiation. *Journal of High Energy Physics, Gravitation and Cosmology*, **8**, 527-535. <https://doi.org/10.4236/jhepgc.2022.83038>
- [9] Sivaram, C. and Arun, K. (2007) Thermal Gravitational Waves. *The Open Astronomy Journal*, **4**, 65-71. <https://doi.org/10.2174/1874381101004010065>
- [10] Cacciatori, S., Preparata, G., Rovelli, S., Spagnolatti, I. and Xue, S.S. (1998) On the Ground State of Quantum Gravity. *Physics Letters B*, **427**, 254-260. [https://doi.org/10.1016/S0370-2693\(98\)00349-9](https://doi.org/10.1016/S0370-2693(98)00349-9)
- [11] Parikh, M., Wilczek, F. and Zahariade, G. (2021) Signatures of the Quantization of Gravity at Gravitational Wave Detectors. *Physical Review D*, **104**, Article ID: 046021. <https://doi.org/10.1103/PhysRevD.104.046021>
- [12] Jeans, J. (1905) The Dynamical Theory of Gases and of Radiation. *Nature*, **72**, 101-102. <https://doi.org/10.1038/072101d0>
- [13] Manoukian, E.B. (2016) 45° Relative Orientations of Planes of Polarizations States of Gravitational Waves and the Graviton. *International Journal of Theoretical Physics*, **55**, 4878-4884. <https://doi.org/10.1007/s10773-016-3111-y>
- [14] Boltzmann, L. (1868) Studies on the Balance of Living Force between Moving Material Points. *Wiener Berichte*, **58**, 517-560.
- [15] Boltzmann, L. (1871) Some General Statements on Thermal Equilibrium. *Wiener Berichte*, **63**, 679-711.
- [16] Brownstein, J.R. and Moffat, J.W. (2006) Galaxy Rotation Curves without Non-Baryonic Dark Matter. *The Astrophysical Journal*, **636**, 721-741. <https://doi.org/10.1086/498208>
- [17] Sofue, Y. (2015) Dark Halos of M31 and the Milky Way. *Publications of the Astronomical Society of Japan*, **67**, 75. <https://doi.org/10.1093/pasj/psv042>
- [18] Sanders, R. (2010) The Dark Matter Problem: A Historical Perspective. Cambridge University Press, Cambridge. <https://doi.org/10.1017/CBO9781139192309>
- [19] Capozziello, S. and De Laurentis, M. (2011) Extended Theories of Gravity. *Physics Reports*, **509**, 167-321. <https://doi.org/10.1016/j.physrep.2011.09.003>
- [20] Riles, K. (2012) Gravitational Waves: Sources, Detectors and Searches. *Progress in Particle and Nuclear Physics*, **68**, 1-54. <https://doi.org/10.1016/j.pnpnp.2012.08.001>
- [21] Hawking, S.W. (1975) Particle Creation by Black Holes. *Communications in Mathematical Physics*, **43**, 199-220. <https://doi.org/10.1007/BF02345020>

- [22] Barish, B. and Weiss, R. (1999) LIGO and the Detection of Gravitational Waves. *Physics Today*, **52**, 44-50. <https://doi.org/10.1063/1.882861>
- [23] Michelson, P. (1987) On Detecting Stochastic Background Gravitational Radiation with Terrestrial Detectors. *Monthly Notices of the Royal Astronomical Society*, **227**, 933-941. <https://doi.org/10.1093/mnras/227.4.933>
- [24] Agazie, G., *et al.* (2023) The NANOGrav 15 yr Data Set: Evidence for a Gravitational-Wave Background. *The Astrophysical Journal Letters*, **951**, L8.
- [25] Anderson, J., Laing, P., Lau, E., Liu, A., Nieto, M., *et al.* (2002) Study of the Anomalous Acceleration of Pioneer 10 and 11. *Physical Review D*, **65**, Article ID: 082004. <https://doi.org/10.1103/PhysRevD.65.082004>