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Quantification of GARCH (1, 1) Model Misspecification with Three Known Assumed Error Term Distributions

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Abstract

Generalized autoregressive conditional heteroscedastic (GARCH) models have become significant tools in the assessment of time series data, largely the traditional normal distribution of GARCH models because of their ease of use in practice. However, it is proven that high frequency financial data have heavy tails leading to the resulting estimates being inefficient. The Student-t and General error (GED) distributions are more capable of representing these financial series. In this paper, we conduct a series of simulations for the GARCH (1, 1) model assuming the error terms follow a Normal distribution. We fit the simulated returns to the GARCH (1, 1) with Normal, Student-t and GED innovations and varying sample sizes. The return series of Samsung electronics daily stock prices, Bitcoin-USD daily cryptocurrency and Moody's seasoned AAA corporate bond yield (BAAA) are fitted to the GARCH (1, 1) with Normal, Student-t and GED innovations. We investigate if these models are subject to model misspecification if the error terms do not assume similar distributions as the simulated data and real data innovations. Model misspecification was identified in the GARCH model building process of the simulated and real datasets.

Keywords

Bond Market, Heavy Tailed Distribution, GARCH Model, Model Misspecification

1. Introduction

The theories and practices regarding the process of determining the fair market value of an asset over time are the focus of financial time series analysis. For decades, several financial time series have been documented and researched as financial time series analysis is of great interest for making inferences and forecasts. According to Tsay (2005), the features of financial time series include uncertainty, volatility, excess kurtosis, high standard deviation, high skewness and sometimes non-normality. For instance, asset volatility can be defined in several ways, and the volatility of a stock return series is not clearly observable hence statistical theory and techniques play a fundamental role in financial time series analysis. Models such as autoregressive conditional heteroscedastic (ARCH), generalized autoregressive conditional heteroscedastic (GARCH) and several variants of these models have been proposed in order to model and capture volatility features of financial time series (Lawrance, 2013).

The ARCH model has been a typical method for analysing the volatility of financial time series over the last few decades. Engle (1982) proposed this framework, which Bollerslev (1986) later expanded to the GARCH model. The GARCH model is a statistical model that is used to assess a range of financial data. The GARCH model is well-known for predicting time-varying and volatility. According to Hansen et al. (2012) the standard GARCH models utilises daily squared returns to obtain information about the present level of volatility, which is then used to construct expectations about the volatility of the next period. The GARCH process exists with the assumption that return innovations follow the Normal, Student-t and Generalized Error Distributions (GED). Originally, GARCH models are built on the assumption that financial time series have a Normal (Gaussian) distribution. However, there is significant evidence that supports the fact that financial time series are rarely Gaussian, but rather leptokurtic and heavy-tailed (Bollerslev, 1987). Hence the Student-t distribution and the GED distribution were proposed by Bollerslev (1987) and Nelson (1991) respectively, to address this issue by capturing the fat-tail behaviour of financial time series. According to Yaya et al. (2014), the effect of misspecification with appropriate sampling probability distribution of GARCH processes was performed and the authors concluded that from the three distributions (i.e., Normal, Student-t and GED), the GED performed better against the Normal and Student-t assumed distributions in terms of parameter estimates and forecasting.

Aside from real-world applications of GARCH models to financial time series data, there is a need to assess and investigate the impact of parameters during estimation that is caused by misspecified distributions of a GARCH model return innovation. The specification of a model is of great importance during the modelling process, hence model risk have been receiving a considerable amount of attention lately. According to Barrieu and Scandolo (2015), model risk is the hazard of using a potentially not well-suited model to make predictions. Model misspecification is a component of model risk among other type of components, such as errors in the estimation of parameters. Any model is just an estimate of reality, meaning that we will inevitably encounter misspecified models. Misspecification refers to the violation of key model assumptions which can lead to biasness in parameter estimation (White, 1982). Thus, a model is said to be misspecified in an event that the distribution of the error return innovations from the given GARCH model are assumed to follow a certain given distribution, whereas there exists a true distribution which is different from the assumed distribution (Richmond & Horowitz, 2015). In the case of a misspecified model, possible problems with parameter estimation and statistical model inference are highly likely to arise and may need to be further assessed and investigated.

The aim of this paper is to quantify and identify model misspecification of the GARCH (1, 1) model with Normal, Student-t and GED distributions. Therefore, analysis is done on the estimation of the key parameters of the model by evaluating the parameter estimates accuracy and consistency. The statistic $\hat{\alpha}_n$ of the parameter α is said to be consistent, if it converges in probability to the true value of the parameter, i.e. $\lim_{n\to\infty} P\left(\left|\hat{\alpha}_n - \alpha\right| > \epsilon\right) = 0$, where $\epsilon > 0$. Accuracy is referred to the difference between the observed and estimated volatilities for each of the models (Ding & Meade, 2010). These statistical measurements help in quantifying the magnitude of error, ϵ, between the statistic and a true parameter of interest. The key parameters being alpha, α, which represents how volatility acts to new information and beta, β, which indicates the persistence of volatility, in other words how fast large volatilities decay after a shock.

Model misspecification is demonstrated by conducting a simulation experiment using Normal returns on the GARCH (1, 1) model assuming that the error terms follow either a Normal Student-t or General error distribution. These assumed error term distributions are used most commonly by academics and practitioners (Haghnejad, 2013). The GARCH (1, 1) with Normal distribution is symmetrical and does not capture the heavy-tails behavior of the returns. However, to capture the heavy-tails of the returns which are asymmetrically distributed, the Student-t and GED are used. Model misspecification is further quantified and identified by using real data. The data chosen is from Samsung electronics daily stock prices, Bitcoin-USD daily cryptocurrency and Moody's seasoned Aaa corporate bond yield (BAAA).

Model misspecification is observed when the distributions of the error terms and innovations are not consistent as the sample size increases. GARCH models are useful in assessing risk and expected returns for assets that display clustered periods of volatility in returns. Therefore, financial modelers and stock market traders could benefit from these findings since working with the appropriate models and their relevant assumptions will help in making informed decisions.

The outline of this paper is given as follows: Section 2 describes the specification of the GARCH (1, 1) model and its parameter estimates. Section 3 outlines the simulation steps, simulated datasets results, and discussion of the results. In Section 4, the empirical results and findings are presented. Finally, in Section 5 we conclude the paper and give future research.

2. GARCH Model Specification

2.1. Asset Returns

The daily asset prices, P_t, are transformed into continuously compounded daily returns by taking the simple return's natural logarithm of the daily asset prices over yesterday asset price, P_{t-1}. As a result, the daily returns r_t are computed as:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right). \tag{1}$$

2.2. Preliminary Data Analysis

Preliminary data analysis is done to determine some key features and characteristics of the observed data. This is done using descriptive statistics and basic plots such as Histogram charts and Quantile-Quantile plots (QQ-plots).

2.3. Normality Test

The Normality of the return series in this study is formally tested using the Shapiro-Wilk (SW) and Jarque-Bera (JB) tests. The null and alternative hypotheses for both tests are defined as:

H_0: The distribution of the series is normal;

H_a: The distribution of the series is not normal.

The JB and SW tests are illustrated in Section 2.3.1. and Section 2.3.2., respectively. Both the tests have the decision rule that the null hypothesis is rejected if the p-value obtained for the test statistic is less than the significant level of $\alpha = 5\%$.

2.3.1. Jarque-Bera (JB)

The JB test formulated by Jarque and Bera (1987) derives tests for normality from observations and regression residuals using the Lagrange multiplier approach on the Pearson family of distributions. This test analyses how well the skewness, S, and kurtosis, K, of the data matches that of a Normal distribution. The JB test statistic is given as:

$$\text{JB} = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right), \tag{2}$$

where

$$S = \frac{\frac{1}{T}\sum_{t=1}^{T}(r_t)^3}{\left(\frac{1}{T}\sum_{t=1}^{T}(r_t)^2\right)^{\frac{3}{2}}}, \tag{3}$$

$$K = \frac{\frac{1}{T}\sum_{t=1}^{T}(r_t)^4}{\left[\frac{1}{T}\sum_{t=1}^{T}(r_t)^2\right]^2}, \tag{4}$$

T represents the sample size and r_t represents the t^{th} order statistic.

2.3.2. Shapiro-Wilk (SW) Test

The SW test proposed by Shapiro and Wilk (1965) is one of the most widely used tests for normality. The correlation between data and corresponding normal scores forms the basis of the SW test. It is especially powerful when applied to smaller sample sizes as it has more power to detect non-normality. The SW test statistic for normality is defined as:

$$\text{SW} = \frac{\left(\sum_{t=1}^{T} a_t r_t\right)^2}{\sum_{t=1}^{T} \left(r_t\right)^2}, \tag{5}$$

where a_t is given by:

$$a_t = \frac{m^{\text{T}} V^{-1}}{\sqrt{m^{\text{T}} V^{-1} V^{-1} m}}, \tag{6}$$

where $m = \left(m_1, \cdots, m_n\right)^{\text{T}}$ are known as the expected values of the order statistics of independent and identically distributed (i.i.d) random variables and V is the covariance matrix of those order statistics. In this paper we make use of the SW and JB to test whether the simulated returns and return series calculated from the closing prices fit normal distribution.

2.4. Test for Stationarity

Several time series techniques assume that a series is generated by a stationary process. In other words, the mean, variance, and autocorrelation of the series generated remain constant over time. A return time series, r_t, is said to be weakly stationary if the following properties hold:

1) $E\left(r_t\right) = \mu_t$ for all t;
2) $Var\left(r_t\right) = E\left[\left(r_t - \mu_t\right)^2\right] = \sigma^2$;
3) $Cov\left(r_t, r_{t-k}\right) = E\left[\left(r_t - \mu_t\right)\left(r_{t-k} - \mu_t\right)\right] = \gamma_k$ for all t and any k;

with γ_k being the covariance at lag k between r_t and r_{t-k}. The variance of the time series r_t is obtained as γ_0, that is if $k = 0$ (Tsay, 2005). Stationarity tests are performed in order to avoid spurious results (Hill et al., 2018). Stationarity of a series can be determined by looking at time series plots, identifying if any trends or seasonality is present in the series. Formal tests can also be performed on the series to determine stationarity. In this study the Augmented Dickey-Fuller (ADF) and Phillips Perron (PP) tests will be used.

2.4.1. Augmented Dickey-Fuller (ADF) Test

The ADF test was developed by Dickey and Fuller (1979). The purpose of the ADF test is to see if the unit root is still present in the return series. The series is said to be stationary if the unit root does not exist in the return series. Yunita (2016) and Santos (2019) both considers the ADF test as a good measure in assessing the stationarity of a series. The ADF model is used to test the following hypotheses:

H_0: The series is non-stationary;

H_a: The series is stationary.

If we do not reject the H_0 it implies that the series is non-stationary or contains a unit root. We do not reject the null hypothesis if the p-value of the test displays a higher value at a significance level of 5%.

2.4.2. Phillips Perron (PP) Test

An alternative stationarity test is the PP test developed by Phillips and Perron (1988). The estimating scheme for the PP test is the same as for the ADF test. Although these tests are similar, they differ in the way they deal with heteroscedascity and autocorrelation. Some of the advantages of the PP test are that it is extremely powerful when used for large sample sizes and it corrects heteroskedasticity in the error terms. The PP test makes use of non-parametric methods and disregards serial correlation (Santos, 2019). The hypothesis procedures for the PP test are the same as the ADF test. The calculated p-value of PP is then compared with the significance level of 5%. If the calculated p-value is less than the significance level, we reject the null hypothesis that the series have a unit root, therefore confirming that the series is stationary.

2.5. Auto Regressive Conditional Heteroscedasticity (ARCH) Effects

The presence of stationarity and ARCH effects in the residual return series should be evaluated before applying GARCH models to returns series. To test the presence of ARCH effects in the return series data the Ljung-Box and Lagrange multiplier (LM) tests are used. Discussion on these tests follows.

2.5.1. The Ljung-Box Test

The Ljung-box test formulated by Box and Pierce (1970) is used to determine the presence of serial autocorrelation up to a given lag k. The Ljung-Box hypotheses test is given by:

H_0: The series data do not exhibit serial autocorrelation;

H_a: The series data exhibit serial autocorrelation.

The formula of the Ljung-box test statistic is given by:

$$Q = T(T+2)\sum_{k=1}^{s}\frac{\hat{\rho}^2(k)}{T-k}, \tag{7}$$

where s is the number of autocorrelation lags and $\hat{\rho}(k)$ is known as the estimated autocorrelation of the series at lag k. ρ_k is defined as:

$$\rho_k = \frac{\sum_{t=1}^{T}(r_t - \bar{r})(r_{t-k} - \bar{r})}{\sum_{t=1}^{T}(r_{t-k} - \bar{r})^2},$$

where $\bar{r}$ is the mean of the return series. We reject the null hypothesis if the calculated p-value is less than the significance level of 5% which imply that the return series data display serial autocorrelation.

2.5.2. The Lagrange Multiplier Test

It is necessary to assess the residuals for the presence of Auto Regressive Conditional Heteroscedasticity (ARCH) effects before applying GARCH models on the return series data. This is done by use of the LM test defined as:

H_0: No ARCH effects in series;

H_a: Presence of ARCH effects in series;

with the test statistic used given by:

$$\text{LM} = T.r^2 \sim \chi^2(p), \tag{8}$$

where r^2 is the coefficient of determination. The number of restrictions imposed by the null hypothesis is defined by p.

2.6. The Generalized Auto Regressive Conditional Heteroskedasticity (GARCH) Model

The GARCH model was proposed by Bollerslev (1986) which is an extension of the ARCH model. The GARCH model is a common statistical modelling tool used for predicting the volatility of financial asset returns because it allows for conditional variance to be dependent upon previous own lags (Xiao et al., 2009). The GARCH (1, 1) model is specified as follows:

$$r_t = \mu_t + a_t, \tag{9}$$

$$a_t = \sigma_t \epsilon_t, \epsilon_t \sim N(0,1) \text{ or } t(0,1,\nu) \text{ or } GED(0,1,\kappa), \tag{10}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{11}$$

$$\alpha_0 > 0, \alpha_1 > 0, \beta_1 > 0, (\alpha_1 + \beta_1) < 1,$$

where α_0, α_1 and β_1 are the parameters of the conditional variance. The degrees of freedom are denoted by ν, κ denotes the shape of the distribution and σ_{t-1}^2 is considered as the volatility of yesterday. The constraints on the parameters α_0, α_1 and β_1, assure that volatility (σ_t^2) is always positive and stationary. The sum, $\alpha_1 + \beta_1$ measures how fast large volatility series decay after a shock, which is known as volatility persistence. The ϵ_t known as the innovation distribution is a sequence of i.i.d random variables. It is often assumed that the innovation follows a normal distribution. However, this is not always the case as financial time series are often leptokurtic. As a result, we make use of the Student-t and GED distributions because they consider the leptokurtic behaviour of financial time series data. The conditional density and log-likelihood function of ϵ_t under the Normal, Student-t and GED distributions are discussed as follows.

2.6.1. The Standardized Normal Distribution

$$f(\epsilon_t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-0,5\left(\frac{\epsilon_t^2}{\sigma_t^2}\right)}, -\infty < \epsilon_t < \infty \tag{12}$$

where σ_t^2 is the variance of the innovations. The corresponding log-likelihood function is given by:

$$l_T(\epsilon_t) = -0.5\left[T \times \ln(2\pi) + \sum_{t=1}^{T} \ln(\sigma_t^2) + \sum_{t=1}^{T} \frac{\epsilon_t^2}{\sigma_t^2}\right]. \tag{13}$$

2.6.2. The Standardized Student-t Distribution

$$f(\epsilon_t,\nu)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)\sigma_t^2}}\left[1+\frac{\epsilon_t^2}{(\nu-2)\sigma_t^2}\right]^{-\left(\frac{\nu+1}{2}\right)},-\infty<\epsilon_t<\infty \quad (14)$$

where ν is the degrees of freedom ($\nu>2$) also known as the shape parameter and Γ is the gamma function. The Student-t distribution is symmetric around zero. The Student-t distribution converges to the Normal distribution as the shape parameter approaches infinity. The log-likelihood function for the Student-t distribution can be written as:

$$\begin{aligned} l_T(\epsilon_t,\nu)=T\left[\ln\Gamma\left(\frac{\nu+1}{2}\right)-\ln\Gamma\left(\frac{\nu}{2}\right)-0.5\ln\left[\pi(\nu-2)\right]\right] \\ -0.5\sum_{t=1}^{T}\left[\ln\left(\sigma_t^2\right)-(1+\nu)\ln\left(1+\frac{\epsilon_t^2}{(\nu-2)\sigma_t^2}\right)\right]. \end{aligned} \quad (15)$$

2.6.3. The Standardized GED

$$f(\epsilon_t,\kappa)=\frac{\kappa e^{-0.5\left|\frac{\epsilon_t}{\lambda\sigma_t}\right|^{\kappa}}}{\lambda 2^{(\kappa+1/\kappa)}\Gamma(1/\kappa)}, \quad (16)$$

where the shape parameter $\kappa>0$, the scale parameter $\lambda=\left[\frac{\Gamma\left(\frac{1}{\kappa}\right)}{2^{(2/\kappa)}\Gamma(3/\kappa)}\right]^{0.5}$ and the error terms have values between $-\infty<\epsilon_t<\infty$.

Depending on the degrees of freedom the GED is a symmetric distribution that can be both leptokurtic and platykurtic. The GED has thicker tails than the Normal distribution when $0<\kappa<2$ (leptokurtic) and has thinner tails than the normal distribution at $\kappa>2$ (platykurtic). At $\kappa=2$ the GED distribution becomes the standard Normal distrubtion and as κ approaches infinity the GED distribution becomes a Uniform distribution on the interval $\left[-\sqrt{3},\sqrt{3}\right]$. The log-likehood function for the GED can be given by:

$$l_T(\epsilon_t,\kappa)=\sum_{t=1}^{T}\left[\ln\left(\frac{\kappa}{\lambda}\right)-0.5\left|\frac{\epsilon_t}{\lambda}\right|^{\kappa}-\left(1+\kappa^{-1}\right)\ln(2)-\ln\Gamma\left(\frac{1}{\kappa}\right)-0.5\ln\left(\sigma_t^2\right)\right]. \quad (17)$$

3. Simulated Results

In this section we present the results obtained for the simulated datasets. To show the effect of model misspecification using the simulated datasets, the statistical inference analysis and model selection criteria outlined in Section 2 are used. Simulations are conducted using RStudio with the packages fGarch (Wuertz et al., 2013), fBasics (Wuertz et al., 2017), tseries (Trapletti et al., 2020) and quantmod (Ryan et al., 2020). The true innovation distribution for the simulation studies of all three scenarios were generated from the Standard Normal distribution, i.e., $r_t \sim N(0,1)$. The different samples sizes were drawn by taking the last T data point i.e., the last 10, 50, 100 etc return series. We generated different initial values for α_0, α_1 and β_1. The different initialised parameter values used for each scenario and each model is summarised in **Table 1**.

The QQ-plots of the simulated returns for sample sizes 1000, 5000 and 10,000 may be provided upon request from the supplementary material. We observe that the returns shy away from normality with increase in α_1 and decrease in β_1. The steepness of the reference line is also affected by this change in the initialised parameters.

For Scenario 1, α_0 and α_1 had a constant increment of 0.1 and 0.2 respectively, and β_1 was assigned a constant decrease of 0.2 between 0 and 1. Additional scenarios are given in the supplementary material, where Scenario 2 initial parameter β_1 has a constant increment of 0.2, α_0 and α_1 were assigned a constant decrease of 0.1 and 0.2 respectively. Scenario 3 may also be found in the supplementary material with the initialised parameters α_0, α_1 and β_1 tied with a constant increment of 0.1 between 0 and 1. The different sample sizes that were generated from the simulations are $T = \{10, 50, 100, 200, 500, 1000, 5000, 10000\}$. The simulated data generated from the GARCH (1, 1) with Normal distribution innovations were then fitted into a GARCH (1, 1) model with innovations following Normal, Student-t and GED distributions respectively. The results in **Table 2** to **Table 5** show the estimated parameters with their related standard errors (SE) in brackets. Furthermore, the results pretended in the **Table 2** to **Table 5** are volatility persistence measure given by $\alpha_1 + \beta_1$, Goodness-of-fit measure given by the Log-likelihood and model selection criteria given by AIC and BIC.

Simulation Study: Scenario 1

Four different models were simulated from a GARCH (1, 1) process with different parameter initial values of α_0, α_1, β_1 and sample sizes as illustrated in **Table 1**. According to **Table 2-5**, there is consistency in the SE which is a measure of estimating efficiency of the parameters for all three distributions at the given same sample sizes. However, we note that the GED produced smaller SE values than the Student-t distribution. This is also evident as we observe that as the sample size increase, the GED estimates are similar compared to those

Table 1. Initial parameter values.

Scenario	Model	α_0	α_1	β_1
Scenario 1	1	0.1	0.1	0.8
	2	0.2	0.3	0.6
	3	0.3	0.5	0.4
	4	0.4	0.7	0.2

Notes: This table represents the initialised values used for each scenario and each model.

Table 2. Scenario 1 Model 1: $\sigma_t^2 = 0.1 + 0.1a_{t-1}^2 + 0.8\sigma_{t-1}^2$.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.092797 (0.312300)	0.000000 (NaN)	0.792543 (0.509600)	0.792543	−10.387690	–	–
	50	0.247205 (NaN)	0.000000 (NaN)	0.683000 (NaN)	0.683000	−64.543000	2.741720	2.894682
	100	0.515391 (0.243710)+	0.313620 (0.171160)−	0.218490 (0.250100)	0.532110	−142.382500	2.927651	3.031857
	200	0.551310 (0.276030)+	0.241840 (0.115070)+	0.219900 (0.296190)	0.461740	−281.256800	2.852568	2.918535
	500	0.280581 (0.128830)+	0.192765 (0.070340)+	0.478952 (0.183970)+	0.671717	−658.115800	2.648463	2.682180
	1000	0.266805 (0.125850)+	0.175380 (0.053660)+	0.490860 (0.192250)+	0.666240	−1287.413000	2.582825	2.602456
	5000	0.076052 (0.015230)+	0.079810 (0.010950)+	0.838810 (0.023180)+	0.918620	−6835.653000	2.735861	2.741075
	10,000	0.084648 (0.011625)+	0.083991 (0.007755)+	0.830173 (0.016455)+	0.914164	−13925.400000	2.785880	2.788764
Student-t	10	0.498260 (NaN)	0.000000 (0.298900)	0.000000 (NaN)	0.000000	−10.492690	–	–
	50	0.672763 (0.508600)	0.219563 (0.363600)	0.000000 (8.069000)	0.219563	−64.720050	2.788802	2.980004
	100	0.553297 (0.274740)+	0.353272 (0.202200)−	0.202750 (0.256300)	0.556022	−143.339700	2.966794	3.097052
	200	0.597735 (0.333200)−	0.265164 (0.135430)−	0.209946 (0.332220)	0.475110	−283.492100	2.884921	2.967379
	500	0.289317 (0.155560)−	0.197990 (0.080630)+	0.500066 (0.206780)+	0.698056	−663.669500	2.674678	2.716824
	1000	0.257727 (0.141860)−	0.176616 (0.061980)+	0.534789 (0.204610)+	0.711405	−1298.200000	2.606400	2.630939
	5000	0.081890 (0.018530)+	0.084449 (0.013050)+	0.836469 (0.026660)+	0.920918	−6862.990000	2.747196	2.753713
	10,000	0.096489 (0.014543)+	0.088165 (0.009244)+	0.822432 (0.019272)+	0.910597	−13972.050000	2.795410	2.799015
GED	10	0.310868 (0.802300)	0.000000 (NaN)	0.345600 (1.489000)	0.345600	−10.384230	–	–
	50	0.259707 (NaN)	0.000000 (NaN)	0.667073 (NaN)	0.667073	−64.520510	2.780820	2.972023
	100	0.514520 (0.238940)+	0.309120 (0.166260)−	0.222655 (0.247130)	0.531775	−142.341600	2.946832	3.077091
	200	0.547422 (0.256370)+	0.239650 (0.108330)+	0.226142 (0.275200)	0.465792	−280.949600	2.859496	2.941954

Continued

	500	0.284677 (0.116350)+	0.197720 (0.066270)+	0.469630 (0.166230)+	0.667350	−656.621700	2.646487	2.688633
	1000	0.277384 (0.116370)+	0.180498 (0.050030)+	0.472655 (0.176830)+	0.653153	−1284.796000	2.579592	2.604131
	5000	0.076030 (0.015200)+	0.079815 (0.010920)+	0.838836 (0.023130)+	0.918651	−6835.645000	2.736258	2.742775
	10,000	0.084918 (0.011736)+	0.083987 (0.007809)+	0.829889 (0.016596)+	0.913876	−13925.210000	2.786042	2.789647

Notes: This table presents the estimates of Scenario 1 Model 1 returns fitted using the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding SE. "+", "−", "." denotes significance at 1%, 5% and 10% levels respectively.

Table 3. Scenario 1 Model 2: $\sigma_t^2 = 0.2 + 0.3a_{t-1}^2 + 0.6\sigma_{t-1}^2$.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.010567 (5.584000)	0.000000 (2.076000)	0.999999 (4.037000)	0.999999	−15.072110	–	–
	50	0.030394 (0.113100)	0.000000 (NaN)	0.973356 (NaN)	0.973356	−72.035600	3.041424	3.194386
	100	0.337239 (0.302450)	0.117460 (0.122990)	0.608080 (0.276460)+	0.725540	−151.425900	3.108517	3.212724
	200	0.188907 (0.104230)−	0.203089 (0.091280)+	0.661254 (0.126880)+	0.864343	−303.794900	3.077949	3.143916
	500	0.235710 (0.082880)+	0.328460 (0.074580)+	0.530400 (0.091810)+	0.858860	−775.677500	3.118710	3.152427
	1000	0.250588 (0.065500)+	0.273907 (0.047630)+	0.570504 (0.068860)+	0.844411	−1566.367000	3.140734	3.160365
	5000	0.186149 (0.019675)+	0.287085 (0.019708)+	0.615691 (0.022762)+	0.902776	−8045.231000	3.219692	3.224960
	10,000	0.195478 (0.014640)+	0.292343 (0.014310)+	0.604074 (0.016860)+	0.896417	−16018.170000	3.204435	3.207319
Student-t	10	0.041899 (2.521000)	0.000000 (NaN)	1.000000 (2.483000)+	1.000000	−15.360760	–	–
	50	0.048275 (0.136900)	0.000000 (NaN)	0.962590 (NaN)	0.962590	−72.806500	3.112260	3.303462
	100	0.380179 (0.364950)	0.111464 (0.135180)	0.611736 (0.296710)+	0.723200	−152.889400	3.157788	3.288046
	200	0.176107 (0.123220)	0.193886 (0.102850)−	0.699870 (0.142820)+	0.893756	−305.807400	3.108074	3.190532
	500	0.241219 (0.095270)+	0.339037 (0.088100)+	0.546555 (0.100000)+	0.885592	−780.622900	3.142492	3.184638
	1000	0.273102 (0.081940)+	0.287574 (0.058180)+	0.566611 (0.081620)+	0.854185	−1573.665000	3.157330	3.181869

Continued

	5000	0.197582 (0.023408)+	0.303939 (0.023573)+	0.613669 (0.025556)+	0.917608	−8074.697000	3.231897	3.238396
	10,000	0.205062 (0.017300)+	0.309223 (0.017050)+	0.604201 (0.018810)+	0.913424	−16078.450000	3.216689	3.220295
GED	10	1.103100 (0.771600)	0.000000 (NaN)	0.000000 (0.219800)	0.000000	−13.707760	–	–
	50	0.047949 (0.090690)	0.000000 (NaN)	0.957198 (NaN)	0.957198–	71.303360	3.052134	3.243337
	100	0.325748 (0.261280)	0.131951 (0.117220)	0.605605 (0.240930)+	0.737556	−150.586200	3.111725	3.241983
	200	0.196608 (0.097550)+	0.212583 (0.089150)+	0.647040 (0.119580)+	0.859623	−303.439900	3.084399	3.166857
	500	0.238595 (0.079250)+	0.332419 (0.070950)+	0.525230 (0.087600)+	0.857649	−774.979800	3.119919	3.162065
	1000	0.248880 (0.063560)+	0.273870 (0.046440)+	0.571840 (0.066840)+	0.845710	−1566.181000	3.142361	3.166900
	5000	0.186098 (0.019569)+	0.287009 (0.019601)+	0.615786 (0.022642)+	0.902795	−8045.173000	3.220069	3.226586
	10,000	0.195530 (0.014520)+	0.292300 (0.014200)+	0.604070 (0.016730)+	0.896370	−16017.900000	3.204581	3.208186

Notes: This table presents the estimates of Scenario 1 Model 2 returns fitted using the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding SE. "+", "–", "." denotes significance at 1%, 5% and 10% levels respectively.

Table 4. Scenario 1 Model 3: $\sigma_t^2 = 0.3 + 0.5a_{t-1}^2 + 0.4\sigma_{t-1}^2$.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.000000 (0.570200)	0.000000 (0.262300)	0.957920 (0.088500)	0.957920	−10.105930	–	–
	50	0.355595 (0.212110)–	0.908470 (0.567220)	0.099880 (0.233830)	1.008350	−75.350940	3.174037	3.326999
	100	0.176660 (0.112450)	0.591960 (0.198480)+	0.433380 (0.114230)+	1.025340	−174.131300	3.562626	3.666833
	200	0.132731 (0.056190)+	0.726780 (0.168860)+	0.379410 (0.082660)+	1.106190	−325.710400	3.297104	3.363071
	500	0.199371 (0.048990)+	0.541618 (0.084920)+	0.430570 (0.061050)+	0.972188	−814.478900	3.273916	3.307632
	1000	0.234100 (0.039710)+	0.571080 (0.059930)+	0.390790 (0.041180)+	0.961870	−1622.476000	3.252952	3.272583
	5000	0.282421 (0.021652)+	0.512348 (0.026020)+	0.404431 (0.020987)+	0.916779	−8301.752000	3.322301	3.327515
	10,000	0.291385 (0.016129)+	0.503095 (0.018866)+	0.399800 (0.015961)+	0.902895	−16403.060000	3.281411	3.284296

Continued

Student-t	10	0.356143 (0.746200)	0.000000 (NaN)	0.274409 (1.342000)	0.274409	−10.320510	–	–
	50	0.371950 (0.268300)	0.636810 (0.534800)	0.230110 (0.315900)	0.866920	−75.505420	3.174037	3.326999
	100	0.211810 (0.141300)	0.598000 (0.225400)+	0.439980 (0.124700)+	1.037980	−175.200000	3.604000	3.734259
	200	0.154351 (0.071310)+	0.691809 (0.190020)+	0.400210 (0.093180)+	1.092019	−326.825000	3.318250	3.400708
	500	0.211163 (0.057500)+	0.566390 (0.100150)+	0.429000 (0.066460)+	0.995390	−816.651100	3.286604	3.328750
	1000	0.251439 (0.047340)+	0.595540 (0.071070)+	0.390030 (0.045710)+	0.985570	−1628.511000	3.267022	3.291561
	5000	0.293727 (0.025399)+	0.546251 (0.031364)+	0.406134 (0.023220)+	0.952385	−8338.962000	3.337585	3.344102
	10,000	0.303475 (0.018956)+	0.530203 (0.022519)+	0.403371 (0.017714)+	0.933574	−16468.990000	3.294799	3.298404
GED	10	0.000000 (0.216600)	0.000000 (0.276800)	0.938830 (NaN)	0.938830	−9.027367	–	–
	50	0.351430 (0.199280)−	0.967760 (0.661110)	0.081150 (0.211370)	1.048910	−75.340620	3.213625	3.404827
	100	0.165450 (0.100870)	0.616560 (0.194820)+	0.426530 (0.106840)+	1.043090	−173.848800	3.576977	3.707235
	200	0.133698 (0.057600)+	0.721260 (0.173890)+	0.381240 (0.084590)+	1.102500	−325.699500	3.306995	3.389453
	500	0.199755 (0.049660)+	0.541780 (0.086040)+	0.430187 (0.061760)+	0.971967	−814.441100	3.277764	3.319910
	1000	0.234410 (0.040060)+	0.570870 (0.060380)+	0.390680 (0.041490)+	0.961550	−1622.454000	3.254907	3.279446
	5000	0.283039 (0.021223)+	0.511903 (0.025430)+	0.404275 (0.020548)+	0.916178	−8300.768000	3.322307	3.328824
	10,000	0.291607 (0.015976)+	0.503143 (0.018674)+	0.399615 (0.015806)+	0.902758	−16402.630000	3.281526	3.285131

Notes: This table presents the estimates of Scenario 1 Model 3 returns fitted using the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding SE. "+", "-", "." denotes significance at 1%, 5% and 10% levels respectively.

from Normal distribution. In terms of the model selection criteria, the GED outperforms the Student-t distribution because it produces smaller AIC and BIC values that are close to those of the Normal distribution.

Scenario 1 illustration example: In order to illustrate the phenomena of model misspecification as a result of GARCH (1, 1) model with Normal distributed error terms, we consider a financial market analyst going through the process of building a model. The analyst was given a task to identify the model

Table 5. Scenario 1 Model 4: $\sigma_t^2 = 0.4 + 0.7a_{t-1}^2 + 0.2\sigma_{t-1}^2$.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.000000 (0.570200)	0.000000 (0.262300)	0.957920 (0.088500)	0.957920	−10.105930	–	–
	50	0.355595 (0.212110)−	0.908470 (0.567220)	0.099880 (0.233830)	1.008350	−75.350940	3.174037	3.326999
	100	0.176660 (0.112450)	0.591960 (0.198480)+	0.433380 (0.114230)+	1.025340	−174.131300	3.562626	3.666833
	200	0.132731 (0.056190)+	0.726780 (0.168860)+	0.379410 (0.082660)+	1.106190	−325.710400	3.297104	3.363071
	500	0.199371 (0.048990)+	0.541618 (0.084920)+	0.430570 (0.061050)+	0.972188	−814.478900	3.273916	3.307632
	1000	0.234100 (0.039710)+	0.571080 (0.059930)+	0.390790 (0.041180)+	0.961870	−1622.476000	3.252952	3.272583
	5000	0.282421 (0.021652)+	0.512348 (0.026020)+	0.404431 (0.020987)+	0.916779	−8301.752000	3.322301	3.327515
	10,000	0.291385 (0.016129)+	0.503095 (0.018866)+	0.399800 (0.015961)+	0.902895	−16403.060000	3.281411	3.284296
Student-t	10	0.356143 (0.746200)	0.000000 (NaN)	0.274409 (1.342000)	0.274409	−10.320510	–	–
	50	0.371950 (0.268300)	0.636810 (0.534800)	0.230110 (0.315900)	0.866920	−75.505420	3.174037	3.326999
	100	0.211810 (0.141300)	0.598000 (0.225400)+	0.439980 (0.124700)+	1.037980	−175.200000	3.604000	3.734259
	200	0.154351 (0.071310)+	0.691809 (0.190020)+	0.400210 (0.093180)+	1.092019	−326.825000	3.318250	3.400708
	500	0.211163 (0.057500)+	0.566390 (0.100150)+	0.429000 (0.066460)+	0.995390	−816.651100	3.286604	3.328750
	1000	0.251439 (0.047340)+	0.595540 (0.071070)+	0.390030 (0.045710)+	0.985570	−1628.511000	3.267022	3.291561
	5000	0.293727 (0.025399)+	0.546251 (0.031364)+	0.406134 (0.023220)+	0.952385	−8338.962000	3.337585	3.344102
	10,000	0.303475 (0.018956)+	0.530203 (0.022519)+	0.403371 (0.017714)+	0.933574	−16,468.990000	3.294799	3.298404
GED	10	0.000000 (0.216600)	0.000000 (0.276800)	0.938830 (NaN)	0.938830	−9.027367	–	–
	50	0.351430 (0.199280)−	0.967760 (0.661110)	0.081150 (0.211370)	1.048910	−75.340620	3.213625	3.404827
	100	0.165450 (0.100870)	0.616560 (0.194820)+	0.426530 (0.106840)+	1.043090	−173.848800	3.576977	3.707235
	200	0.133698 (0.057600)+	0.721260 (0.173890)+	0.381240 (0.084590)+	1.102500	−325.699500	3.306995	3.389453

Continued

500	0.199755 (0.049660)+	0.541780 (0.086040)+	0.430187 (0.061760)+	0.971967	−814.441100	3.277764	3.319910
1000	0.234410 (0.040060)+	0.570870 (0.060380)+	0.390680 (0.041490)+	0.961550	−1622.454000	3.254907	3.279446
5000	0.283039 (0.021223)+	0.511903 (0.025430)+	0.404275 (0.020548)+	0.916178	−8300.768000	3.322307	3.328824
10,000	0.291607 (0.015976)+	0.503143 (0.018674)+	0.399615 (0.015806)+	0.902758	−16,402.630000	3.281526	3.285131

Notes: This table presents the estimates of Scenario 1 Model 4 returns fitted using the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding SE. "+", "−", "." denotes significance at 1%, 5% and 10% levels respectively

that best describes 50 simulated data points. He had to simulate 50 data points using a GARCH (1, 1) model with the true innovation following a Normal distribution. The initial parameter values for α_0, α_1 and β_1 were set to 0.3; 0.5 and 0.4 respectively. The 50 simulated data points were then fitted into a GARCH (1, 1) model with innovations following Normal, Student-t and GED distributions. The output he obtained is that of **Table 4** for $T = 50$. After analysing the output, the Student-t distribution produced the lowest AIC (3.174037) and BIC (3.326999) values. The optimal model that an analyst identified was the GARCH (1, 1) model with Student-t innovations by application of the model selection criteria. This is model misspecification because the inappropriate innovation distribution is assumed, in this scenario the Student-t. When we observe **Table 5**, we note that the Student-t is a least performer among the other two distributions for all samples sizes excluding $T = 50$. Whereas the Normal distribution was consistent in producing the lowest AIC and BIC values which imply that the GARCH (1, 1) model with Normally distributed error terms is the best suited model because of its consistency. It is expected mainly due to the fitted simulated data was generated from the Normal distribution. An analyst did not consider that small sample does not change the likelihood. That is, the likelihood of GARCH model changes from being flat as the sample size becomes very large.

The following remarks are made on the results of the simulated data fitted to a GARCH (1, 1) model with innovations assumed to follow Normal, Student-t and GED respectively. When the true innovation distribution is Normal, the GARCH (1, 1) model with GED consistently outperforms the GARCH (1, 1) model with Student-t distribution. We based this conclusion on the AIC and BIC values obtained. In other words, the GED had lower AIC and BIC values in comparison to the Student-t distribution. We also see that when the true innovation distribution is Normal, the fitted GARCH (1, 1) model with GED produces similar results to the true model innovation, particularly for large sample sizes. The results obtained for all three scenarios show that when we draw data from a Normal distribution it does not show much volatility. We also found that when data is simulated from a GARCH (1, 1) model assuming that the error terms follows a Normal distribution then Student-t and GED can be fitted to the data. We expect that when we fit a GARCH (1, 1) model assuming that the error terms follow Normal, Student-t and GED distributions respectively, similar results will be obtained for our empirical dataset if the innovations are Normally distributed. We also note the importance of the goodness-of-fit and model selection criteria as one of the critical initial steps in the model building process because it helps us to minimize the likelihood of misspecifying a model.

4. Empirical Results

The empirical data were extracted from online yahoo finance platform that comprises of the daily closing prices for Bitcoin cryptocurrency over the period of 17 September 2014 to 29 July 2021 and the Samsung Electronics over the period 26 May 2011 to 30 July 2021. The Moody's seasoned Aaa corporate bond yield (BAAA) over the period 23 September 1997 to 9 May 2008 was retrieved from the Federal Reserve bank of St. Louis (FRED). To illustrate the finding from the simulation, the BAAA closing prices were limited to the given period for the confirmation of normally distributed returns. These three real-life datasets were chosen as they exhibit different distributional behaviour which is shown by the descriptive statistics. The different samples sizes were drawn the same way as the simulated data was drawn. That is, by keeping the last return data points T from the entire data, such that $T = \{10, 50, 100, 200, 500, 1000, 2000, 2500\}$. The return series of the real datasets were calculated using the continuously compounded rate of return given by Equation 1. The transformation of the closing prices and yields made it possible for the assessment of the forecast accuracy for the GARCH models. The closing prices and yields of the three real datasets as well as their return series are presented in Figure 1. The return series of Samsung Electronics, Bitcoin and BAAA were fitted into a GARCH (1, 1) model with Normal, Student-t and GED, respectively.

4.1. Descriptive Statistics Results

The results of the descriptive statistics are given in Table 6. Bitcoin returns have asymmetrical distribution. This is supported by the results obtained for the skewness of Bitcoin returns. The value of the skewness is −0.82, indicating that the returns from Bitcoin are negatively skewed distributed. The kurtosis of Bitcoin returns series is greater than 3 which indicates that the distribution of returns is leptokurtic. That is, the distribution of return series has heavier tails than that of a Normal distribution.

Samsung electronics return distribution has a skewness value close to zero indicating that the distribution of its returns is fairly symmetrical. The kurtosis of 1.77 which is less than 3 indicates that Samsung electronics return series has a distribution which is platykurtic. This implies that the distribution has thinner tails than the Normal distribution. BAAA yield returns has skewness and kurtosis

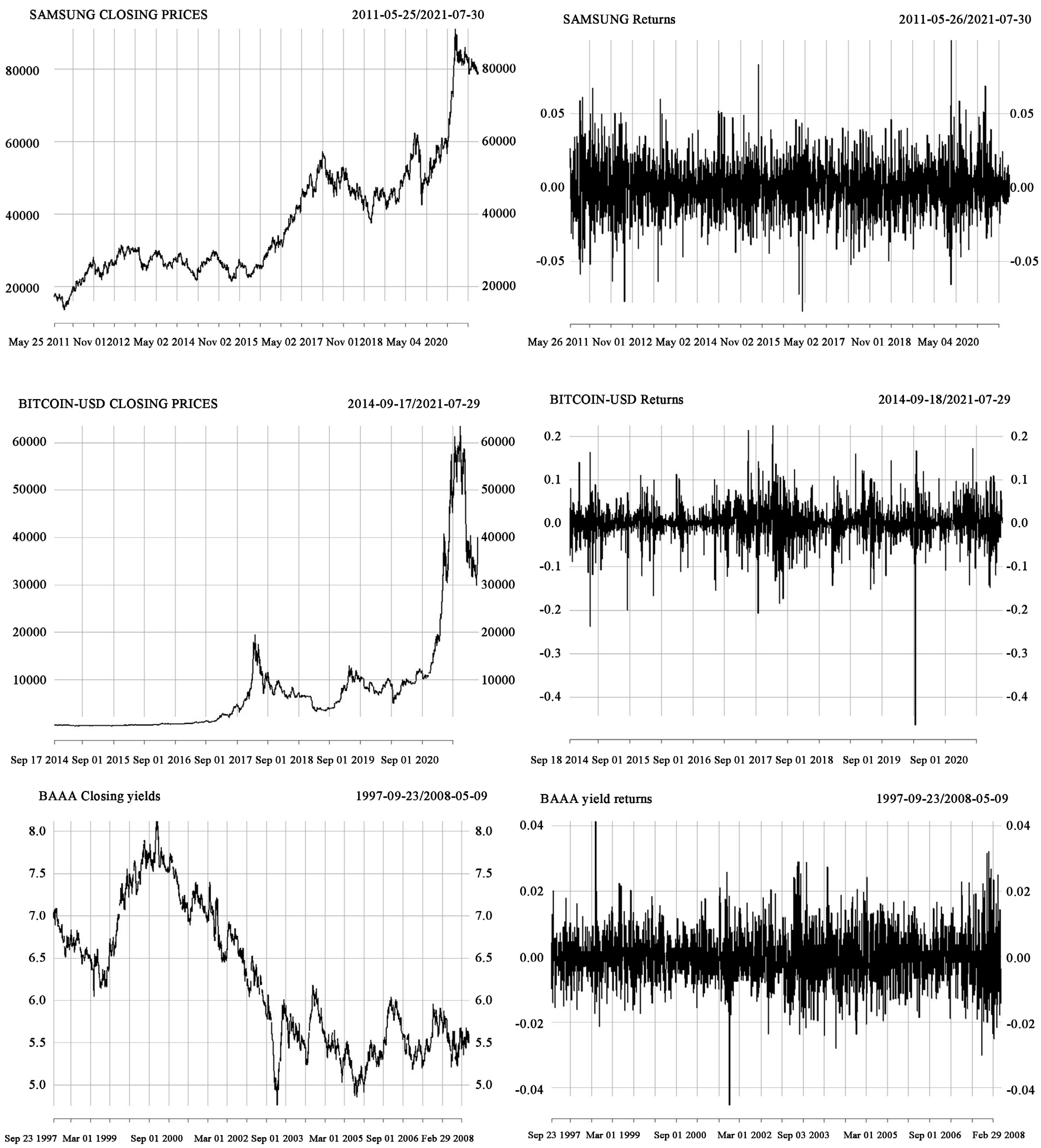

Figure 1. Closing stock prices and return series.

Table 6. Descriptive statistics of the return series.

	T	Mean	Standard deviation	Skewness	Kurtosis
Bitcoin/USD	2500	0	0.04	−0.82	11.55
Samsung electronics	2500	0	0.02	0.06	1.77
BAAA	2500	0	0.01	0.06	−0.01

values close to zero. Indicating that the distribution of its yield returns is fairly symmetric and may have thinner tails than the Normal distribution. We confirm this by formally testing the yield return series through the use of Jarque-Bera test and Shapiro-Wilk test.

4.2. Normality Test Results

Normality is formally tested for Samsung Electronics, Bitcoin/USD and BAAA using the JB and SW tests. The overall Samsung returns ($T = 2500$) do not follow a normal distribution. This is confirmed from the JB and SW probability values in Table 7 as the hypothesis tests has rejected the normality of the Samsung return series at 5% significance level. This is also apparent in Figure 2, the QQ-plot

Table 7. Normality tests of the return series.

Return series	T	JB Test		SW Test	
		Statistic	p-value	Statistic	p-value
Bitcoin/USD	10	0.929230	0.440400	0.664450	0.717300
Samsung		0.82967	0.3316	2.4888	0.288110
BAAA		0.977330	0.949300	0.079343	0.961100
Bitcoin/USD	50	0.980940	0.592200	2.696200	0.259700
Samsung		0.938130	0.011370	8.773900	0.012440
BAAA		0.961520	0.102900	4.129600	0.126800
Bitcoin/USD	100	0.974860	0.052570	4.416300	0.109900
Samsung		0.986490	0.404200	2.064000	0.356300
BAAA		0.985010	0.318100	1.306900	0.520200
Bitcoin/USD	200	0.975730	0.001537	19.284000	0.000065
Samsung		0.949180	0.000006	76.894000	2.2e−16
BAAA		0.981290	0.009091	6.109000	0.047150
Bitcoin/USD	500	0.845490	2.2e−16	13676.000000	2.2e−16
Samsung		0.968300	6.38E−09	203.200000	2.2e−16
BAAA		0.993610	0.032920	2.006400	0.366700
Bitcoin/USD	1000	0.878840	2.2e−16	15801.000000	2.2e−16
Samsung		0.982550	1.43e−09	167.610000	2.2e−16
BAAA		0.995000	0.002239	4.142500	0.126000
Bitcoin/USD	2000	0.901100	2.2e−16	12418.000000	2.2e−16
Samsung		0.982020	3.53e−15	368.600000	2.2e−16
BAAA		0.996690	0.000255	3.290100	0.193000
Bitcoin/USD	2500	0.898260	2.2e−16	14444.000000	2.2e−16
Samsung		0.984710	9.30e−16	334.960000	2.20e−16
BAAA		0.996930	0.000064	1.609700	0.447100

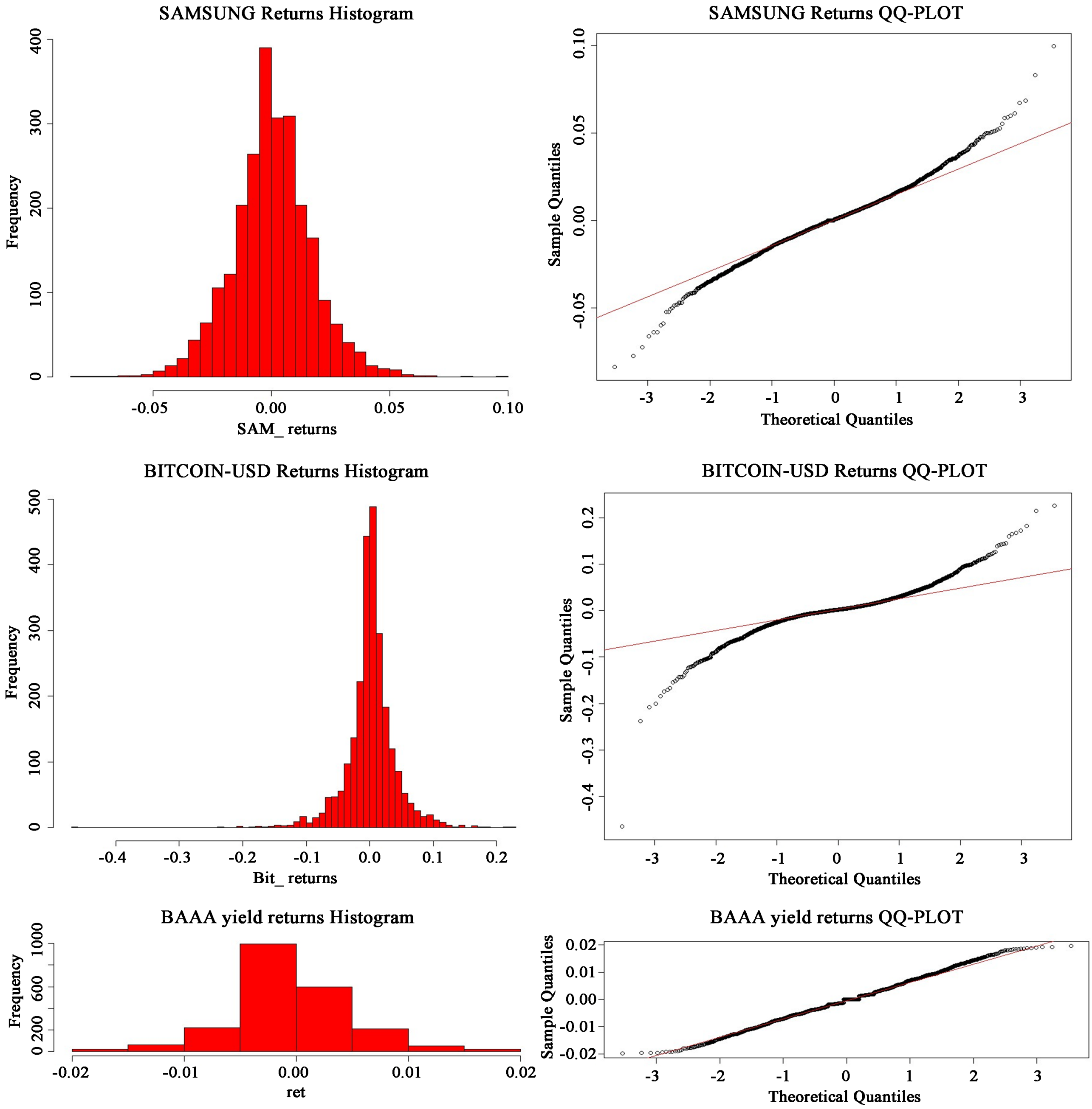

Figure 2. Histogram and QQ-plots of return series.

of Samsung electronics returns with data points outside the range of −1 and 1 have diverted from the straight line. This is an indication of the heavy-tail returns. Both JB and SW tests rejected the normality of Bitcoin returns at 5% significance level for $T = 2500$. There is a presence of extreme heavy tails from the QQ-plot of Bitcoin return series revealed by Figure 2. BAAA yield return series show that the SW probability value in Table 9 is greater than 5%. Thus, we do not reject the null hypothesis at a 5% significance level and conclude that BAAA yield return series follow a normal distribution. Supporting the conclusion made form Figure 2, whereby majority of BAAA yield returns on the QQ-plot falls directly along the reference line with only a few data points on the lower and upper extremes diverting from the straight line.

4.3. Stationarity Test

Stationarity of Samsung returns, Bitcoin returns and BAAA yield returns are formally tested by use of the ADF and PP tests. The results obtained for both tests are presented in Table 8. The probability values of both tests confirm stationarity of the three return series by testing at a 5% significance level.

Table 8. Stationarity tests of Return series

Return series	T	Augmented Dickey-Fuller (ADF)		Phillips Perron (PP)	
		Dickey-Fuller	p-value	Dickey-Fuller z(alpha)	p-value
Bitcoin/USD		−0.456590	0.976900	−10.171000	0.463800
Samsung	10	−16.861000	0.010000	−9.277700	0.523600
BAAA		−1.030300	0.916100	−12.810000	0.286900
Bitcoin/USD		−3.845900	0.023310	−52.295000	0.010000
Samsung	50	−5.175500	0.010000	−44.638000	0.010000
BAAA		−4.020700	0.015880	−50.718000	0.010000
Bitcoin/USD		−3.975300	0.013230	−122.870000	0.010000
Samsung	100	−5.723300	0.010000	−96.480000	0.010000
BAAA		−5.241600	0.010000	−99.439000	0.010000
Bitcoin/USD		−4.649200	0.010000	−243.790000	0.010000
Samsung	200	−6.069000	0.010000	−182.670000	0.010000
BAAA		−6.933400	0.010000	−191.640000	0.010000
Bitcoin/USD		−7.678900	0.010000	−592.950000	0.010000
Samsung	500	−7.540800	0.010000	−518.880000	0.010000
BAAA		−8.091400	0.010000	−466.430000	0.010000
Bitcoin/USD		−8.778900	0.010000	−1191.00000	0.010000
Samsung	1000	−10.385000	0.010000	−998.490000	0.010000
BAAA		−10.915000	0.010000	−916.760000	0.010000
Bitcoin/USD		−11.670000	0.010000	−2188.30000	0.010000
Samsung	2000	−13.188000	0.010000	−1811.60000	0.010000
BAAA		−13.051000	0.010000	−1920.00000	0.010000
Bitcoin/USD		−12.858000	0.010000	−2673.00000	0.010000
Samsung	2500	−14.476000	0.100000	−2231.60000	0.010000
BAAA		−13.590000	0.010000	−2393.50000	0.010000

4.4. ARCH Effects Test

The presence of ARCH disturbances in time series data are tested using the Ljung-Box and LM tests. In Table 9 the p-values obtained for the LM test of the three return series at different sample sizes was less than 0.05 hence we reject the null hypothesis and conclude that ARCH effect is present in the return series. In other words, the returns of all three stocks are heteroscedastic.

4.5. GARCH Model Selection Process

The three closing price returns series of Samsung Electronics, Bitcoin against US

Table 9. ARCH effects test of the return series.

Return series	T	Ljung-Box Test		LM Arch test	
		X-squared	p-value	LM	p-value
Bitcoin/USD		NA	NA	−0.540340	1.000000
Samsung	10	NA	NA	1.119700	0.290000
BAAA		NA	NA	1.087700	0.297000
Bitcoin/USD		12.457000	0.255700	36.580000	1.456e−0.9
Samsung	50	11.675000	0.307400	36.247000	1.74e−09
BAAA		7.979800	0.630800	51.788000	6.18e−13
Bitcoin/USD		14.610000	0.146900	88.882000	2.2e−16
Samsung	100	6.476300	0.773800	41.158000	1.40e−10
BAAA		2.849700	0.984700	71.572000	2.20e−16
Bitcoin/USD		15.926000	0.101800	251.050000	2.2e−16
Samsung	200	17.779000	0.058810	317.430000	2.20e−16
BAAA		6.899300	0.734900	179.250000	2.20e−16
Bitcoin/USD		25.154000	0.005061	770.750000	2.2e−16
Samsung	500	17.676000	0.060680	533.620000	2.20e−16
BAAA		7.464800	0.681000	274.910000	2.20e−16
Bitcoin/USD		27.394000	0.002255	9425.900000	2.2e−16
Samsung	1000	8.058600	0.623100	1110.400000	2.20e−16
BAAA		7.756000	0.652700	592.650000	2.20e−16
Bitcoin/USD		18.570000	0.046070	11282.000000	2.2e−16
Samsung	2000	14.903000	0.135600	2685.900000	2.20e−16
BAAA		11.891000	0.292400	969.900000	2.20e−16
Bitcoin/USD		23.879000	0.007929	12611.00000	2.2e−16
Samsung	2500	25.025000	0.005298	3178.500000	2.20e−16
BAAA		16.916000	0.076250	1213.100000	2.20e−16

dollar and the BAAA yield have small $\hat{\alpha}_1$ values with large $\hat{\beta}_1$ values as observed in **Table 10** to **Table 12**, similar to Scenario 1's results. There is consistency in the SE of the estimated parameters for all three innovations at the same sample sizes. However, the GED produced relatively smaller SE values compared to the competing distributions. The Samsung Electronics returns series results show that the time series weak stationarity condition holds for all three distributions, i.e. $\hat{\alpha}_1 + \hat{\beta}_1 < 1$. For sample sizes larger than 500 the GARCH (1, 1) model with GED innovations is identified as the best model because it produced the highest log-likelihood values, lowest AIC and BIC values. The fitted GARCH (1, 1) model with GED innovations describes Samsung Electronics return series to be the appropriate choice of model. Results indicate that the GARCH (1, 1) model with Student-t distribution is the optimal model to describe the volatility of the Bitcoin series when we consider sample sizes of 200 to 2500. The Student-t distribution is consistently producing the increasing largest log-likelihood and decreasing smallest AIC and BIC values as compared to the Normal and the GED. However, once the sample size is greater than or equal to 500 the volatility persistence measure exceeds the bound of being less than one. That is, the key assumption of a GARCH model is being violated due to the time series weak stationarity condition does not hold. Therefore, we conclude that there is model misspecification present since the optimal model chosen does not uphold all the model specifications. Models such as the Integrated Generalized Autoregressive Conditional heteroskedasticity (IGARCH) may be appropriate fit for the time series of Bitcoin/USD. The output obtained for BAAA yield returns displayed in **Table 12** indicates that the BIC model selection criterion has flat consistency in producing the smallest values for the Normal distribution. This is also the case for the simulated return series of Scenario 1. **Table 12** shows that the time series stationarity condition holds for all three distributions, i.e., $\hat{\alpha}_1 + \hat{\beta}_1 < 1$. Using the BIC criterion as our key model selection measure, the GARCH (1, 1) with Normal innovations is identified as the best model to describe the yield returns of BAAA. However, if we consider the GARCH (1, 1) with Normal, Student-t and GED innovations only at sample size 200 the inappropriate distribution for the error terms would have been assumed to be GED. The latter conclusion would be inappropriate because **Table 11** shows that the GARCH (1, 1) with innovations that are Normally distributed has flat consistently for the BIC values among the other two distributions. The observation is compared to the scenario illustrative example where the analysis misspecified the appropriate model by neglecting the consistency of model selection criterion. Similarly, model misspecification in this case would be imminent since returns that are normally distributed cannot be best described by a GED.

In this Section we presented the results obtained for the three simulated scenarios and three real datasets after application of the methods discussed in Section 3. We considered the GARCH (1, 1) with Normal, Student-t and GED innovations and varying sample sizes. We conducted a series of simulations for the

Table 10. Empirical results: Samsung Electronics return series.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.000000 (NaN)	0.000000 (NaN)	0.975480 (NaN)	0.975480	34.896250	–	–
	50	0.000012 (0.000009)	0.000000 (NaN)	0.792570 (NaN)	0.792570	173.106700	−6.764269	−6.611308
	100	0.000055 (NaN)	0.311820 (NaN)	0.000000 (NaN)	0.311820	332.901800	−6.578035	−6.473828
	200	0.000000 (0.000002)	0.072561 (0.031070)+	0.922570 (0.030020)+	0.995131	575.846100	−5.718461	−5.652495
	500	0.000002 (0.000002)	0.078830 (0.020130)+	0.916650 (0.021940)+	0.995480	1355.578000	−5.406311	−5.372594
	1000	0.000002 (0.000002)	0.047540 (0.009490)+	0.944580 (0.011870)+	0.992120	2694.591000	−5.381183	−5.361552
	2000	0.000005 (0.000002)+	0.036626 (0.006807)+	0.943870 (0.011030)+	0.980496	5405.382000	−5.401382	−5.390180
	2500	0.000005 (0.000002)+	0.038435 (0.006306)+	0.944270 (0.009722)+	0.982705	6670.077000	−5.332862	−5.323543
Student-t	10	0.000000 (NaN)	0.000000 (NaN)	0.951180 (NaN)	0.951180	35.595920	–	–
	50	0.000001 (0.000014)	0.000000 (NaN)	0.985060 (0.190600)+	0.985060	173.693300	−6.747731	−6.556529
	100	0.000057 (NaN)	0.359930 (NaN)	0.000000 (NaN)	0.359930	332.261100	−6.545221	−6.414963
	200	0.000002 (0.000003)	0.065403 (0.036690)−	0.923770 (0.038270)+	0.989173	579.266500	−5.742665	−5.660207
	500	0.000005 (0.000004)	0.094813 (0.030220)+	0.890370 (0.035440)+	0.985183	1360.767000	−5.423069	−5.380922
	1000	0.000004 (0.000003)	0.054045 (0.014250)+	0.933710 (0.018680)+	0.987755	2705.129000	−5.400259	−5.375720
	2000	0.000007 (0.000003)+	0.036095 (0.009536)+	0.936790 (0.017480)+	0.972885	5439.307000	−5.434307	−5.420304
	2500	0.000006 (0.000002)+	0.039118 (0.008281)+	0.941830 (0.013020)+	0.980948	6707.091000	−5.361673	−5.350025
GED	10	0.000000 (0.000022)	0.000000 (NaN)	0.925420 (NaN)	0.925420	36.313640	–	–
	50	0.000009 (0.000013)	0.000000 (NaN)	0.852920 (0.113000)+	0.852920	173.341300	−6.733653	−6.542451
	100	0.000056 (NaN)	0.307170 (NaN)	0.000000 (NaN)	0.307170	332.954200	−6.559084	−6.428825
	200	0.000007 (0.000002)	0.066140 (0.035110)−	0.924730 (0.036130)+	0.990870	579.213700	−5.742137	−5.659679
	500	0.000003 (0.000004)	0.084446 (0.026740)+	0.905200 (0.031220)+	0.989646	1360.993000	−5.423974	−5.381828
	1000	0.000003 (0.000003)	0.049946 (0.012610)+	0.939070 (0.016470)+	0.989016	2709.146000	−5.408292	−5.383753

Continued

	2000	0.000007 (0.000003)+	0.035653 (0.008846)+	0.939580 (0.015860)+	0.975233	5443.485000	−5.438485	−5.424483
	2500	0.000006 (0.000002)+	0.038252 (0.007915)+	0.942800 (0.012590)+	0.981052	6711.864000	−5.365491	−5.353843

Notes: This table presents the estimates of Samsung electronics returns fitted by the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding standard error. "+", "-", "." denotes significance at 1%, 5% and 10% levels respectively.

Table 11. Empirical results: Bitcoin return series.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.000000 (NaN)	0.202170 (0.427500)	0.893940 (NaN)	1.096110	23.567850	–	–
	50	0.000000 (0.000027)	0.000000 (NaN)	0.990780 (NaN)	0.990780	90.302540	−3.452102	−3.299140
	100	0.000098 (0.000169)	0.038030 (0.038730)	0.916731 (0.082010)+	0.954761	161.747800	−3.154957	−3.050750
	200	0.000091 (0.000066)	0.033292 (0.023290)	0.920790 (0.035040)+	0.954082	332.903100	−3.289031	−3.223064
	500	0.000044 (0.000016)−	0.047913 (0.01430)+	0.918040 (0.018760)+	0.965953	877.599900	−3.494399	−3.460683
	1000	0.000146 (0.000031)+	0.113047 (0.026060)+	0.812889 (0.030140)+	0.925936	1824.079000	−3.640158	−3.620527
	2000	0.000065 (0.000010)+	0.161940 (0.020330)+	0.822080 (0.017750)+	0.984020	3779.669000	−3.775669	−3.764467
	2500	0.000070 (0.000011)+	0.140940 (0.015960)+	0.831470 (0.015680)+	0.972410	4754.742000	−3.800593	−3.791275
Student-t	10	0.000000 (0.000281)	0.089503 (0.361900)	1.000000 (NaN)	1.089503	23.375740	–	–
	50	0.000000 (NaN)	0.000000 (NaN)	0.991870 (NaN)	0.991870	90.630060	−3.425202	−3.234000
	100	0.000112 (0.000192)	0.053317 (0.054160)	0.903722 (0.089829)+	0.957039	163.876100	−3.177522	−3.047263
	200	0.000080 (0.000077)	0.042904 (0.029950)	0.919720 (0.043080)+	0.962624	339.543200	−3.345432	−3.262974
	500	0.000023 (0.000020)	0.105340 (0.041580)+	0.910910 (0.022980)+	1.016250	959.210100	−3.816841	−3.774694
	1000	0.000060 (0.000095)	0.431030 (0.653500)	0.911710 (0.018110)+	1.342740	1999.320000	−3.988640	−3.964101
	2000	0.000005 (0.000005)	0.209710 (0.058640)+	0.896090 (0.014160)+	1.105800	4089.112000	−4.084112	−4.070110
	2500	0.000011 (0.000006)−	0.216090 (0.050090)+	0.879830 (0.014500)+	1.095920	5148.471000	−4.114776	−4.103128

Continued

GED	10	0.000335 (0.000526)	0.248970 (0.345200)	0.000000 (1.523000)	0.248970	25.440740	–	–
	50	0.000000 (NaN)	0.000000 (NaN)	0.991300 (NaN)	0.991300	90.794590	−3.431784	−3.240581
	100	0.000090 (0.000178)	0.042754 (0.046890)	0.916566 (0.089390)+	0.959320	164.527800	−3.190557	−3.060298
	200	0.002133 (NaN)	0.000000 (0.026870)	0.000000 (NaN)	0.000000	339.373500	−3.343735	−3.261277
	500	0.002133 (NaN)	0.000000 (0.026870)	0.000000 (NaN)	0.000000	339.373500	−3.343735	−3.261277
	1000	0.002133 (NaN)	0.000000 (0.026870)	0.000000 (NaN)	0.000000	339.373500	−3.343735	−3.261277
	2000	0.002133 (NaN)	0.000000 (0.026870)	0.000000 (NaN)	0.000000	339.373500	−3.343735	−3.261277
	2500	0.002133 (NaN)	0.000000 (0.026870)	0.000000 (NaN)	0.000000	339.373500	−3.343735	−3.261277

Notes: This table presents the estimates of Bitcoin returns fitted by the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding standard error. "+", "–", "." denotes significance at 1%, 5% and 10% levels respectively.

Table 12. Empirical results: BAAA yield return series.

Distribution	*T*	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\alpha}_1 + \hat{\beta}_1$	Log likelihood	AIC	BIC
Normal	10	0.000000 (NaN)	0.000000 (NaN)	0.959330 (NaN)	0.959330	35.031900	–	–
	50	0.000000 (NaN)	0.000000 (NaN)	0.988200 (NaN)	0.988200	193.306600	−7.572265	−7.419303
	100	0.000004 (0.000010)	0.024159 (0.057990)	0.843100 (0.319000)	0.867259	369.927800	−7.318555	−7.214348
	200	0.000003 (0.000005)	0.016135 (0.032830)	0.883390 (0.181100)	0.899525	754.229100	−7.502291	−7.436325
	500	0.000011 (0.000006)+	0.075276 (0.043600)–	0.624210 (0.167200)+	0.699486	1839.462000	−7.341848	−7.308131
	1000	0.000006 (0.000003)–	0.098018 (0.032660)+	0.756090 (0.098110)+	0.854108	3691.923000	−7.375847	−7.356216
	2000	0.000000 (0.000000)+	0.031954 (0.007505)+	0.957820 (0.010470)+	0.989774	7150.957000	−7.146957	−7.135755
	2500	0.000000 (0.000000)+	0.031207 (0.006431)+	0.959840 (0.008787)+	0.991047	8899.864000	−7.116691	−7.107373
Student-t	10	0.000049 (0.000061)	0.000000 (NaN)	0.162500 (0.714300)	0.162500	34.798390	–	–
	50	0.000000 (NaN)	0.000000 (NaN)	0.988250 (NaN)	0.988250	194.169100	−7.566764	−7.375561
	100	0.000005 (0.000019)	0.034151 (0.083060)	0.807130 (0.580800)	0.841281	370.585000	−7.311699	−7.181441

Continued

	200	0.000002 (0.000005)	0.020316 (0.037780)	0.897630 (0.187000)+	0.917946	756.935700	−7.519357	−7.436900
	500	0.000010 (0.000005)−	0.096872 (0.053320)−	0.639500 (0.158800)+	0.736372	1839.028000	−7.336113	−7.293967
	1000	0.000005 (0.000003)−	0.113670 (0.039410)+	0.754230 (0.103500)+	0.867900	3691.795000	−7.373591	−7.349052
	2000	0.000000 (0.000000)+	0.039522 (0.009806)	0.954130 (0.012030)+	0.993652	7141.531000	−7.136531	−7.122528
	2500	0.000000 (0.000000)−	0.037729 (0.008185)+	0.957400 (0.009813)+	0.995129	8887.543000	−7.106034	−7.094386
GED	10	0.000042 (0.000053)	0.000000 (NaN)	0.232200 (0.649400)	0.232200	34.931740	−	−
	50	0.000000 (NaN)	0.000000 (NaN)	0.988330 (NaN)	0.988330	194.451900	−7.578077	−7.386875
	100	0.000006 (0.000033)	0.027240 (0.098400)	0.802510 (0.980500)	0.829750	371.249400	−7.324988	−7.194729
	200	0.000003 (0.000006)	0.015561 (0.038730)	0.896850 (0.221000)+	0.912411	757.957500	−7.529575	−7.447117
	500	0.000011 (0.000006)−	0.082396 (0.048410)−	0.630780 (0.169500)+	0.713176	1840.658000	−7.342633	−7.300487
	1000	0.000005 (0.000003)−	0.101750 (0.035320)+	0.754930 (0.102700)+	0.856680	3693.603000	−7.377205	−7.352666
	2000	0.000000 (0.000000)+	0.032360 (0.007692)+	0.957520 (0.010640)+	0.989880	7151.084000	−7.132082	−7.132082
	2500	0.000000 (0.000000)+	0.031569 (0.006580)+	0.959600 (0.008915)+	0.991169	8900.022000	−7.116018	−7.104370

Notes: This table presents the estimates of BAAA returns fitted by the GARCH (1, 1) model with Normal, Student-t and GED. Values in the parentheses are the corresponding standard error. "+", "−", "." denotes significance at 1%, 5% and 10% levels respectively.

GARCH (1, 1) model assuming the error terms follow a Normal distribution. We found that when the true innovation distribution is Normal, the GARCH (1, 1) model with GED produces similar results to the true model innovation, particularly for large sample sizes. The results revealed that if the innovation assumed does not follow a similar distribution to that of the return series then model misspecification is highly imminent. The return series of Samsung Electronics daily stock prices, Bitcoin-USD daily cryptocurrency and Moody's seasoned Aaa corporate bond yield (BAAA) were fitted to the GARCH (1, 1) with Normal, Student-t and GED innovations.

Overall results revealed that these models are subject to model misspecification if the return distributions of the real data do not follow the assumed error terms distributions in the fitted model.

5. Conclusions

This paper looked at the Quantification on GARCH (1, 1) model misspecification with three assumed error term distributions. The paper illustrates and shows the importance of following quantitative analysis basic steps to curb and reduce the risk of model misspecification. The following steps were implemented for the simulated data and empirical data.

We simulated data from GARCH (1, 1) model with the error terms following a Normal distribution. Thereafter, the simulated data was fitted to GARCH (1, 1) model varying the distribution of the error terms from Normal, Student-t and GED. In order to determine the behaviour of the simulated GARCH (1, 1) model with normal distribution, the initial parameters were varied within the weak stationarity domain and different sample sizes were considered. We concluded that the GED outperforms the Student-t distribution when considering estimation accuracy, goodness of fit and the model selection criteria of the simulated returns. The simulation process was very useful to identify and understand the GARCH (1, 1) model behaviour when the conditional return distributions, initial parameters and the sample sizes were varied.

We considered three real datasets, namely: Samsung electronics daily stock prices, Bitcoin-USD daily cryptocurrency and Moody's seasoned Aaa corporate bond yield (BAAA). To assess the GARCH (1, 1) model we calculated the return series of each dataset. Preliminary analyses were first performed to determine the behaviour of the three-return series followed by normality tests. The normality test showed that only the BAAA yield returns followed a Normal distribution. In the paper of Gyamerah (2019) where the volatility of Bitcoin returns were evaluated, they also found Bitcoin return series to be skewed and heavy tailed. The returns series were tested for stationarity and ARCH effect before applying the GARCH (1, 1) models. We test for stationarity because it is a statistical property of time series analysis that needs to hold. Each return series were fitted to GARCH (1, 1) with Normal, Student-t and GED innovations and varying sample sizes. The models were analysed using the Log-likelihood, AIC and BIC with the BIC being the key model selection measure. Samsung electronics optimal model identified was the GARCH (1, 1) with GED. Bitcoin return series was best described by the GARCH (1, 1) with Student-t. However, the time series stationarity condition did not hold for sample sizes larger than 500. The GARCH (1, 1) model with normally distributed error terms was identified as the optimal model for the BAAA return series. The best GARCH model can still be subjected to model misspecification. Hence, one should ensure that the data has a distribution similar to the distribution assumed by the models error terms. The results showed that if users avoid implementing these basic steps, then the model has high chance of being subjected to model misspecification.

This paper can be replicated by simulating data with true distribution assuming the error terms follows a Student-t, GED or any other distribution. Few examples of models to consider for future studies may include models such as the

Asymmetric Power Generalized Auto Regressive Conditional Heteroskedastic (APARCH), the Component Generalized Auto Regressive Conditional Heteroskedastic (CGARCH), the Exponential Generalized Auto Regressive Conditional Heteroskedastic (EGARCH), the Fractional Integrated Generalized Auto Regressive Conditional Heteroskedastic (FIGARCH) and the Integrated Generalized Auto Regressive Conditional Heteroskedastic (IGARCH).The use of non-parametric GARCH models could also be considered as no distribution assumptions will be made similar to the work done by Er and Fidan (2013).

6. Supplementary Material

The supplementary material may be provided by the corresponding author upon request. The supplementary material for this article discusses the other scenarios where the parameters of the simulated GARCH (1, 1) model with normal distribution of the error terms are initialised differently from the behaviour of the financial market prices. The findings of the GARCH (1, 1) models produce similar results. Furthermore, the illustrate examples are presented for these scenarios and the empirical results of the three stock price returns as the sample size increases are presented.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

Barrieu, P., & Scandolo, G. (2015). Assessing Financial Model Risk. *European Journal of Operational Research, 242,* 546-556. https://doi.org/10.1016/j.ejor.2014.10.032

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics, 31,* 307-327. https://doi.org/10.1016/0304-4076(86)90063-1

Bollerslev, T. (1987). A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return. *The Review of Economics and Statistics, 69,* 542-547. https://doi.org/10.2307/1925546

Box, G. E., & Pierce, D. A. (1970). Distribution of Residual Autocorrelations in Autoregressive-Integrated Moving Average Time Series Models. *Journal of the American statistical Association, 65,* 1509-1526. https://doi.org/10.1080/01621459.1970.10481180

Dickey, D. A., & Fuller, W. A. (1979). Distribution of the Estimators for Autoregressive Time Series with a Unit Root. *Journal of the American statistical Association, 74,* 427-431. https://doi.org/10.2307/2286348

Ding, J., & Meade, N. (2010). Forecasting Accuracy of Stochastic Volatility, GARCH and EWMA Models under Different Volatility Scenarios. *Applied Financial Economics, 20,* 771-783. https://doi.org/10.1080/09603101003636188

Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica: Journal of the Econometric Society, 50,* 987-1007. https://doi.org/10.2307/1912773

Er, Ş., & Fidan, N. (2013). Modeling Istanbul Stock Exchange-100 Daily Stock Returns: A Nonparametric GARCH Approach. *Journal of Business Economics and Finance, 2,* 36-50.

Gyamerah, S. A. (2019). Modelling the Volatility of Bitcoin Returns Using GARCH Models. *Quantitative Finance and Economics, 3,* 739-753. https://doi.org/10.3934/QFE.2019.4.739

Haghnejad, A. (2013). Investigating the Asymmetry in Volatility for the Iranian Stock Market. *Iranian Journal of Economic Studies, 2,* 95-118.

Hansen, P. R., Huang, Z., & Shek, H. H. (2012). Realized GARCH: A Joint Model for Returns and Realized Measures of Volatility. *Journal of Applied Econometrics, 27,* 877-906. https://doi.org/10.1002/jae.1234

Hill, R. C., Griffiths, W. E., & Lim, G. C. (2018). *Principles of Econometrics*. John Wiley & Sons.

Jarque, C. M., & Bera, A. K. (1987). A Test for Normality of Observations and Regression Residuals. *International Statistical Review/Revue Internationale de Statistique, 55,* 163-172.

Lawrance, A. (2013). Exploratory Graphics for Financial Time Series Volatility. *Journal of the Royal Statistical Society: Series C (Applied Statistics), 62,* 669-686. https://doi.org/10.1111/rssc.12016

Nelson, D. B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica, 59,* 347-370. https://doi.org/10.2307/2938260

Phillips, P. C., & Perron, P. (1988). Testing for a Unit Root in Time Series Regression. *Biometrika, 75,* 335-346. https://doi.org/10.1093/biomet/75.2.335

Richmond, C. D., & Horowitz, L. L. (2015). Parameter Bounds on Estimation Accuracy under Model Misspecification. *IEEE Transactions on Signal Processing, 63,* 2263-2278. https://doi.org/10.1109/TSP.2015.2411222

Ryan, J., Ulrich, J., Thielen, W., Teetor, P., Bronder, S., & Ulrich, M. (2020). Quantmod: Quantitative Financial Modelling Framework for R. Quantmod. https://search.r-project.org/CRAN/refmans/quantmod/html/quantmod-package.html

Santos, A. R. G. N. V. (2019). *Forecasting Inflation in Portugal by Using Machine Learning Techniques*. Universidade de Lisboa. (In Portugal)

Shapiro, S. S., & Wilk, M. B. (1965). An Analysis of Variance Test for Normality (Complete Samples). *Biometrika, 52,* 591-611. https://doi.org/10.1093/biomet/52.3-4.591

Trapletti, A., Hornik, K., LeBaron, B., & Hornik, M. K. (2020). *Package 'Tseries'*. https://cran.r-project.org/web/packages/tseries/tseries.pdf

Tsay, R. S. (2005). *Analysis of Financial Time Series*. John Wiley & Sons. https://doi.org/10.1002/0471746193

White, H. (1982). Maximum Likelihood Estimation of Misspecified Models. *Econometrica, 50,* 1-25. https://doi.org/10.2307/1912526

Wuertz, D., Setz, T., Chalabi, Y., Boudt, C., Chausse, P., Miklovac, M., Setz, M. T., & RUnit, S. (2013). *Package 'fGarch'*. https://cran.r-project.org/web/packages/fGarch/fGarch.pdf

Wuertz, D., Setz, T., Chalabi, Y., Maechler, M., & Setz, M. T. (2017). Package 'fBasics'. *Rmetrics-Markets and Basic Statistics.* R Foundation for Statistical Computing.

Xiao, J., Brooks, R. D., & Wong, W.-K. (2009). GARCH and Volume Effects in the Australian Stock Markets. *Annals of Financial Economics, 5,* Article ID: 0950005. https://doi.org/10.1142/S2010495209500055

Yaya, O. S., Olubusoye, O. E., & Ojo, O. O. (2014). Estimates and Forecasts of GARCH Model under Misspecified Probability Distributions: A Monte Carlo Simulation Approach. *Journal of Modern Applied Statistical Methods, 13,* Article No. 28. https://doi.org/10.22237/jmasm/1414816020

Yunita, I. (2016). Volatility Modeling Using ARCH/GARCH Method: Aplication on Asia Pasific Index. *Proceedings of 3rd International Seminar and Conference on Learning Organization* (pp. 182-189.). Atlantis Press. https://doi.org/10.2991/isclo-15.2016.34.