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An Approach to Calculate a Call Option Value on A Nontraded Underlying Asset Considering Its Risk Measures

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Abstract

It is proposed an approach to calculate the suitable parameters for a binomial tree with a specific structure that allow us to calculate a call or put option on a nontraded underlying asset based on the risk perception for the beginning of the evaluation, time $t0$, and for the expiration time $t1$. Although there is not enough information to define with certainty the value of a non-traded underlying at $t0$, as it is usual, the specific structure of the binomial tree allows us to consider the risk involved at $t0$ and it effectively replicates the expected value and variance of the underlying asset at $t1$. An optimization model calculates the suitable parameters for the binomial tree in order to minimize the difference between the probability mass function (pmf) of the underlying asset at the future time modeled by the binomial tree and the one considered for the underlying asset. The results of the proposed method confirm the feasibility of its application no matter the shape of the pmf at $t0$ and $t1$. Also, the result of the model is checked with an academic case where the risk of the underlying asset at $t0$ is included.

Keywords

Volatility, Real Options, Binomial Trees, Binary Approximations, Risk Measure

1. Introduction

The finance theory provides several tools for valuing assets with risk. The discounted cash flow (DCF) techniques have been useful in valuing some kinds of assets, like bonds, preferred stocks, other fixed securities and relatively safe stocks. But it could be less helpful for valuing companies or assets with significant growth opportunities where the options that the asset could account for a considerable part of its value (Myers, 2001).

In those cases where the options can be relevant, such as financial options, real options (including internationalization strategies (da Silva & Gomes, 2023)) and the financial side of strategic planning (Myers, 2001), the use of DCF techniques has some difficulties in valuing them, although those techniques can also be supported with Monte Carlo simulation and Decision Tree Analysis (DTA).

Indeed, since the referred options usually have different risk characteristics in comparison to the underlying asset, a different discount rate would be involved in its valuation (Brandão et al., 2005; Trigeorgis, 1997). It can be reminded that the expected return of an option over the riskless rate equals the option's elasticity times the expected return of the underlying asset over the riskless rate (Cox & Rubinstein, 1985). So the discount rate of an option is not necessarily the same of the underlying asset. This means that although the flexibility can be modeled by a DTA, the appropriate discount rate will require additional background to define the option value. In this context, it is clear that the very useful tool provided by the theory of option valuation due to its risk neutral valuation environment to discount the option value from the future.

Initially, the theory of option valuation has proposed an option value based on a known value for the underlying asset at the current time, in the sense that the asset is traded in the market. If the asset is non-traded, some methods have been proposed to duplicate the value of the non-traded asset or to consider that the market exhibits incompleteness. Some of them consider a twin asset as a proxy for the non-tradable asset (Mason & Merton, 1985; Kasanen & Trigeorgis, 1994; Cochrane & Saá-Requejo, 2000; Hubalek & Shashermayer, 2001) for obtaining a lower and upper bound for the price of the call option. Other methods maximize the expected utility function of the wealth to determine the price at most to pay for a call option (Liu, 2010). All these methods assume a unique value for the underlying asset at the current time.

However, since a call option value is calculated as the expected value of the positive excess of the asset value over an exercise price, a relatively high value of the asset, although its probability of occurrence could be low, would contribute to the expected option value with a significant part of the expected value. This justifies considering directly the several potential asset values (density function for the underlying asset value) at the current time and in the future. Consequently, an extremely high asset value at the current time and its potential evolution could explain a relevant part of an option value in a more decisive way than just to consider different values for the expected current underlying asset value. In other words, an extremely high value of the underlying asset value should be essential for anyone interested in the value of the option.

In this sense, we understand that the proposed approach could contribute positively to the development of options, in a context where most of the underlying assets are non-tradable (firms, patents, and any case of non-listed asset) and an option can create wealth or help manage risks.

The approach proposed in this paper to calculate a call option value on a non-traded underlying asset is based on: i) Suitable parameters to reflect its risk measures in the future; ii) A particular binomial tree to allow us to model the risk measures at the current time and at the expiration date. The idea is not to ignore that there is risk in the current price of the underlying asset but to consider explicitly that risk in the option value, instead of assuming a unique price for the underlying asset at the current time. These two criteria are justified in the following paragraphs.

In relation to the first criterion, in case of a non-traded underlying asset, the current price of the asset has uncertainty and there may not be enough information from the market for evaluating the risk involved in the future, which is typically considered through the volatility of the underlying asset. It is known the relevant effect of a good estimate for the volatility of the underlying asset on the value of an option (Brealey et al., 2008; Cox & Rubinstein, 1985; Godinho, 2006). The aim is not to affect the strength of the risk neutral valuation with a non-proper estimate for the volatility.

In the usual application of the option theory, there are some recommendations to estimate the volatility to be used in case of a traded asset (Hull, 2006) and in case of an underlying nontraded asset. In these last cases, if the nontraded value depends strongly on a commodity price, such as a crude distiller or an oil field depending both on crude oil price (Bjerksund & Ekern, 2001; Kemma, 2001) or an industrial steam boiler depending on a relative oil-gas price (Kulatilaka, 2001), the volatility of the rate of return of the commodity has been considered to value options on project (abandonment option or operating flexibility) and the results are sensitized for a range of volatilities. In case of a project of natural-resource investment producing a certain mineral, the volatility has been estimated as the volatility of the stock of the holding traded in financial markets (Trigeorgis, 2001). The general recommendation is to look for "comparables", mainly traded stocks with similar business risk (Brealey et al., 2008).

However, due to the relevance of the volatility in the value of the option, using "comparables" could distort the option value. In effect, the volatility would not depend only on the price of the outcome but also on the other factors affecting the value (Schwartz & Trigeorgis, 2001), such as the demand for the outcome, the different kinds of costs, dividends, among others.

In this context, it is clear that, if the volatility determines the risk involved at any time, the better estimate of volatility, the better estimate of the value of the corresponding option. However, in the proposed method, instead of the assumed volatility defining the risk of the underlying asset value at any time (expiration time included), it is assumed that there is an estimate for the density function or probability mass function (pmf) of the underlying asset value for the current time and for any "dividend date" (any date where there is a change in the asset value equivalent to dividend payment). In this way, the volatility is determined in order to minimize the difference between said pmf and the pmf assumed by the considered binomial tree at time expiration (or at any dividend date).

Although there are several methods to value options (for instance, Abraham, 2024) in the usual method for option valuation, it is assumed a volatility and a Brownian motion for the asset value or the corresponding binomial modeling for a future situation. In this way, it is supposed to implicitly certain density function or pmf for the asset value in the future. In the proposed method, it is assumed explicitly the density function for the future situation and given the uncertainty in the current situation, it is calculated the volatility that would explain the evolution between both situations. It is argued that in a non-traded case there is not enough information to define directly a volatility, so that the perception of risk at $t0$ and $t1$, through the corresponding pmf, permits to calculate the suitable parameters for the specific binomial model in order to replicate the risk measures at $t0$ and $t1$. In this way, those parameters for the binomial model will allow us to calculate an option value for $t0$, $t1$ and any other time.

In relation to the second criterion, the proposed specific binomial tree allow us to consider the risk in the current underlying asset value and to model said risk in the future. In addition, a binomial tree allows us to consider the risk neutral valuation environment to discount the option value from the future.

The goal of making it easier to harness the option pricing theory is explained by its advantage to suitable quantify the option value. In this context, the DTA and/or DCF can provide the inputs from a situation with no option value which would be complemented with the expected option value provided by option pricing theory. In this way, the total NPV of an asset can be estimated with the following three components: the traditional static NPV; the expected cash flow and variance of the asset value, both obtained for instance with Monte Carlo; and the flexibility value coming from the value of the options on the underlying asset that the holder of the option could have (Brealey et al., 2008; Čulík, 2016).

In this context, although there is no certainty about the current and future value of the underlying asset, some possible asset values and its probabilities of occurrence could be explicitly estimated. So the expected value and variance at the current time ($t0$) and at a future time ($t1$) are supposed to be known. In this way, the binomial tree of the option pricing theory is used to model the risk through a stochastic process (for instance geometric Brownian or mean-reverting process) and this process is approximated by such binomial tree.

It could be assumed that the risk in the asset value at the current time can be treated by sensitizing the current asset value (Kemma, 2001). However, that risk and the assumed volatility should be considered simultaneously in order to model properly the risk measures at the expiration time and not to overvalue or undervalue the corresponding option. In this context, the risk on the underlying asset value at the current time and at the future should be properly modeled by considering the suitable volatility and structure of the binomial tree.

It is clear that in the context of valuing options in case of non-traded assets, one of the main problems is that secondary market prices are not observable. In the approach posed here, it is understood that this issue is related to the risk assessment of the corresponding non-traded asset. In this sense, prices are related to their risk assessment. In case of a traded asset, there is implicitly a risk assessment by the market. If there is a price observable from the market, it is due to there is a risk assessment by the market. If this assessment does not exist, each person interested in evaluating an option (the buyer and the seller) could develop their own asset risk assessment for the current time ($T = t0$) and for the future ($T = t1$). With the risk assessment, this document proposes a way to get the required parameters to value an option.

Note that in any case of a nontraded underlying asset it would not be easy to test the result of an option value calculated with the proposed method since there would not be generally a competitive market for that asset or for the option. Of course, if there is a market the method for estimating an option value could not be useful in many cases. But if not, the method could provide a useful point of view in order to make a transaction easy through proposing a room for the demand and offer to negotiate.

2. Theoretical Model

In the usual option theory approach, it is assumed that a tradable asset in the current situation ($t0$) can modify its value in the future. Given a share value, or in this case an asset value, S, if the growth per time interval of the log-normal distribution for the profitability of the asset at time t, $R_t = \text{Ln}(S_t/S)$, is μ, and the variance (Var) of the profitability of the asset price per a time interval (Hull, 2006) is σ^2, the profitability at time t is shown in **Table 1** (Fernández, 1989; Hull, 2006).

Table 1. Expected value and variance in binomial tree.

	$R_t = \text{Ln}(S_t/S)$
Expected value	$\mu * t$
Variance	$\sigma^2 * t$

If the profitability of the asset has the expected value and variance shown in **Table 1**, the asset at time t, S_t, will have the expressions for the expected value and variance that are shown in (1) and (2), where S_0 is the asset value at $t = 0$ (Fernández, 1989; Hull, 2006).

$$\text{Expected Value of } S_t = E(S_t) = S_0 * \exp(\mu * t + \sigma^2 * t/2) \quad (1)$$

$$\text{Variance of } S_t = Var(S_t) = S_0^2 * (\exp(\sigma^2 * t) - 1) * \exp\left(2 * t * \left(\mu + \frac{\sigma^2}{2}\right)\right) \quad (2)$$

In case of the stochastic process of each random variable R_t and S_t, they are modeled by a binomial tree in which each period t is divided by the number n of binomial periods, each of them of length $\Delta t = t/n$.

In this context, the increase, u, and decrease, d, in the value of S in each binomial step and the probability h of the increase u are (Cox & Rubinstein, 1985):

$$u = e^{\sigma\sqrt{t/n}},\ d = e^{-\sigma\sqrt{t/n}},\ h = \frac{1}{2} + \frac{1}{2} * \frac{\mu}{\sigma} * \sqrt{t/n} \quad (3)$$

The variance of R_t in each binomial step, $Vars$, is used to estimate the variance for the total of n binomial steps (Cox & Rubinstein, 1985):

$$Vars * n = \left(\sigma^2 - \mu^2 * t/n\right) * t \quad (4)$$

In this way to the extent $n \to \infty$, $Vars * n \to \sigma^2 * t$. It is clear that the variance accuracy depends on μ and n, in relation to the variance σ^2.

In this context, the expected value and variance of an asset value in the DCF world is modeled by a binomial tree (the "binomial tree world") through the parameters, μ, σ and the probability h and, in its turn, the value of a derivative can be calculated in the "risk neutral world" using the following probability p (Cox & Rubinstein, 1985), where rf is risk-free rate.

$$p = \frac{1 + rf - d}{u - d} \quad (5)$$

2.1. Current Situation Model

In a similar way to any specific step of a binomial tree, a current estimate of the value of a nontraded asset, S, at time $t0$ could be represented within a range of values since there is risk due to it is not a traded asset. It is assumed that at time $t0$ there are nd values (nodes) of S, each of them with any probability $q_{t0\,k}$. So its expected value is $E(S_{t0})$ and its variance is $Var(S_{t0})$.

$$E\left(S_{t0}\right) = \sum_{k=1}^{nd} S_{t0\,k} * q_{t0\,k} \quad (6)$$

$$Var\left(S_{t0}\right) = \sum_{k=1}^{nd} S_{t0\,k}^2 * q_{t0\,k} - E\left(S_{t0}\right)^2 \quad (7)$$

2.2. Future Situation Model

In a similar way to the current situation, a future estimate of an asset's value, S at time $t1$, could be represented within a range of values due to its risk. It is assumed that at time $t1$ there are nd1 values of S, each of them with any probability q_{t1}. So its expected value is $E(S_{t1})$ and its variance is $Var(S_{t1})$.

$$E\left(S_{t1}\right) = \sum_{k=1}^{nd1} S_{t1\,k} * q_{t1\,k} \quad (8)$$

$$Var\left(S_{t1}\right) = \sum_{k=1}^{nd1} S_{t1\,k}^2 * q_{t1\,k} - E\left(S_{t1}\right)^2 \quad (9)$$

2.3. Model for Future Situation Associated with Binomial Tree

The movement of the asset value between time $t0$ and $t1$ could be modeled by a non-recombining tree which is formed by a set of subtrees, each of them of *m* steps and starting at each node with a $S_{t0\,j}$ value ($j=1,\cdots,nd$) and ending at *t*1. There would be a total of $n=nd*(m+1)$ values of *X*, each of them a simulated value for S_{t1}, as it is shown in Figure 1.

Figure 1 shows only one of the trees starting at *t*0, the second one from the top of the nodes at time *t*0. Also, the values of $S_{t0\,j}$ at some nodes at *t*0 and their probabilities in square parenthesis are shown. In this way, the uncertainty at *t*0 is represented by a pmf consisting of several $S_{t0\,j}$ values (*j* = 1, *nd*) and the corresponding probability of occurrence ($q_{t0\,j}$).

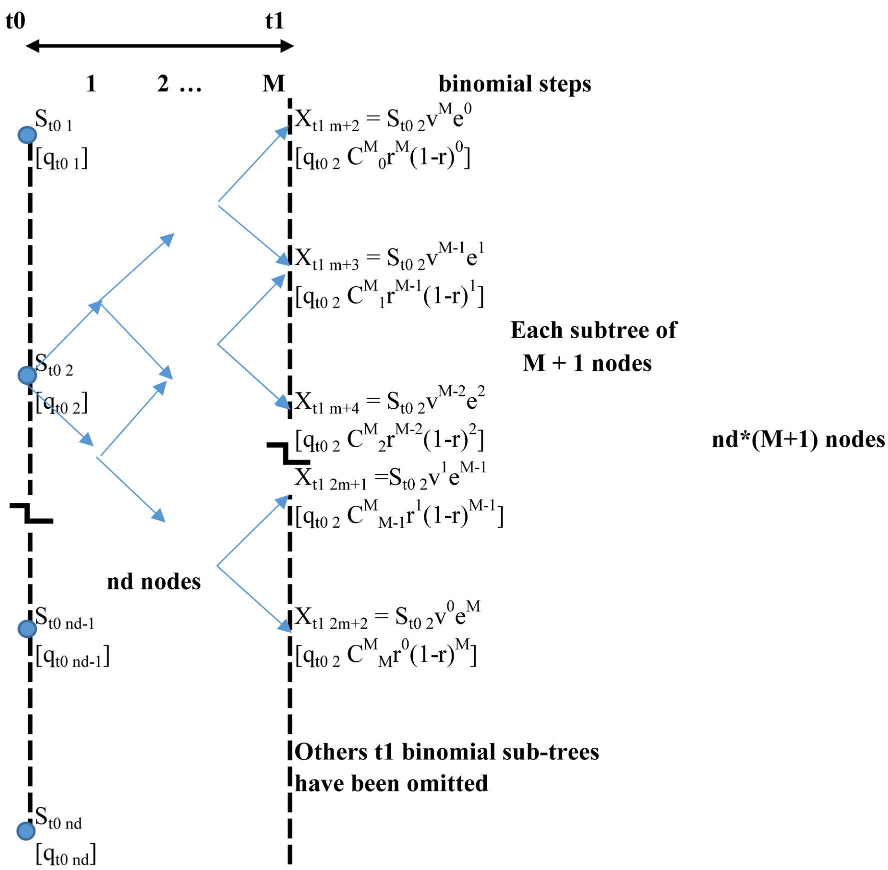

Figure 1. Binomial trees of *M* steps at nd nodes at time *t*0.

In the same Figure are shown the simulated values $X_{i\,k}$ ($k=1,\cdots,n$) at *t*1 and its probability in square parenthesis, according to the considered "*i*" non recombining binomial tree starting at node "*i*" at *t*0. In this way, the pmf for S_t at *t*0 and the set of subtrees starting at each node at *t*0 and ending at *t*1 define several values for S_t at *t*1. Analogously, the probability $q_{t0\,j}$ at *t*0 and said set of subtrees determine the probability of occurrence of the corresponding values for S_t at *t*1 and, therefore, the pmf for the asset value S_t at *t*1.

The times *t*0 and *t*1 in the binomial tree could correspond to dates for delivery of dividends, so the binomial tree goes from *t*0 to *t*1 with no interruption in their nodes. In this context it is usual that in nontraded assets the delivery of dividends depends on the availability of cash flow more than the share value or than a predetermined amount as it is the typical assumption in a binomial tree. So, the DCF supported by Monte Carlo may allow us to define the required information for $t0$, $t1$ and other similar times in a binomial tree.

It is assumed that in each one of the subtrees starting at each one of the nd nodes at time $t0$, the value of the asset at the node k ($k=0,\cdots,m$) in the i subtree at time $t1$ increases by the factor v_i^k with probability q_i^k (q_i is the probability r in the subtree "i" in Figure 1) and diminishes by the factor $e^{m-k}=1/v_i^{m-k}$ with probability $(1-q_i)^{m-k}$. Only as reference, the variables v_i and q_i of the subtree "i" are equivalent to the variables u and h of (3). The expected value of the simulated values for the asset, X_{ik}, at time $t1$ should be:

$$E(X_{ik})=\sum_{i=1}^{nd}S_{t0\,i}*q_{t0\,i}*\sum_{k=0}^{m}(v_i*q_i)^k*\left(\frac{1-q_i}{v_i}\right)^{m-k}*Combin(m,k) \quad (10)$$

In this context, a simulated value for S_{t1}, X_{ik}, corresponds to a value S_{t0}, in the node i of the total of nd nodes at $t0$, which ends up in the node k of the $m+1$ nodes of the tree of m steps starting at node i. So, the simulated value at $t1$ and its probability $Prob(X_{ik})$ are:

$$X_{ik}=S_{t0\,i}*v_i^k/v_i^{m-k} \quad (11)$$

$$Prob(X_{ik})=q_{t0\,i}*q_i^k*(1-q_i)^{m-k}*Combin(m,k) \quad (12)$$

The second sum in (10) corresponds to Newton's binomial theorem. So the previous expression for the expected value is:

$$E(X_{ik})=\sum_{i=1}^{nd}S_{t0\,i}*q_{t0\,i}*\left(v_i*q_i+\frac{1-q_i}{v_i}\right)^m \quad (13)$$

And the variance of random variable X_{ik} at time $t1$ should be:

$$Var(X_{ik})=\left(\sum_{i=1}^{nd}S_{t0\,i}^2*q_{t0\,i}*\sum_{k=0}^{m}(v_i^2*q_i)^k*\left(\frac{1-q_i}{v_i^2}\right)^{m-k}*Combin(m,k)\right)-E(X_{ik})^2 \quad (14)$$

Analogously,

$$Var(X_{ik})=\left(\sum_{i=1}^{nd}S_{t0\,i}^2*q_{t0\,i}*\left(v_i^2*q_i+(1-q_i)/v_i^2\right)^m\right)-E(X_{ik})^2 \quad (15)$$

Lastly, it is known the expression for the third moment of a stochastic variable X:

$$E\left((X-E(X))^3\right)=E(X^3)-3*E(X)*Var(X)-E(X)^3 \quad (16)$$

And the Skewness Coefficient (SC) of the X variable is:

$$SC=\frac{E\left((X-E(X))^3\right)}{Var(X)^{\frac{3}{2}}} \quad (17)$$

The expected value of S_{t1}^3 is modeled by the binomial trees and its value should be $E(X_{ik}^3)$:

$$E\left(X_{i\,k}^{3}\right)=\sum_{i=1}^{nd}S_{t0\,i}^{3}*q_{t0\,i}*\left(v_i^3*q_i+\left(1-q_i\right)/v_i^3\right)^m \quad (18)$$

Since the expected value, variance and $E\left(S_{t1}^{3}\right)$ at time $t1$ are estimated by (13), (15), (16) and (17), the differences between those values and the values of the Available Data (AD, the assumed information) for $t1$ can be represented by the following absolute value functions.

$$f1\left(v_i,q_i,m\right)=abs\left(\sum_{i=1}^{nd}S_{t0\,i}*q_{t0\,i}*\left(v_i*q_i+\frac{1-q_i}{v_i}\right)^m-E\left(S_{t1}\right)\right) \quad (19)$$

$$f2\left(v_i,q_i,m\right)=abs\left(\left(\sum_{i=1}^{nd}S_{t0\,i}^{2}*q_{t0\,i}*\left(v_i^2*q_i+\frac{1-q_i}{v_i^2}\right)^m\right)-\left(\sum_{i=1}^{nd}S_{t0\,i}*q_{t0\,i}*\left(v_i*q_i+\frac{1-q_i}{v_i}\right)^m\right)^2-Var\left(S_{t1}\right)\right) \quad (20)$$

$$f3\left(v_i,q_i,m\right)=abs\left(\sum_{i=1}^{nd}S_{t0\,i}^{3}*q_{t0\,i}*\left(v_i^3*q_i+\frac{1-q_i}{v_i^3}\right)^m-3*E\left(S_{t1}\right)*Var\left(S_{t1}\right)-E\left(S_{t1}\right)^3-SC*Var\left(S_{t1}\right)^{\frac{3}{2}}\right) \quad (21)$$

In this context and assuming $v_i=v$, $q_i=q$, the model could be defined by solving the following optimization model (the Preliminary Model, PM):

Objective Function:

$$F1=\min f3\left(v,q,m\right) \quad (22)$$

Subject to:

$$f1\left(v,q,m\right)\le\beta_1 \quad (23)$$

$$f2\left(v,q,m\right)\le\beta_2 \quad (24)$$

Where the indexes, sets and parameters are:

$i\in I$: Index and set of nodes of the binomial tree at time $t0$, $i=1,\cdots,nd$

$S_{t0\,i}$: Value at time $t0$ in node i.

$E(S_{t1})$: Expected asset value at time $t1$.

$Var(S_{t1})$: Expected asset value at time $t1$.

$q_{t0\,i}$: Probability of the asset value at node i at time $t0$, $S_{t0\,i}$.

β_1 : Allowed error between the expected value estimated by the PM and $E(S_{t1})$, for instance 1% of this value.

β_2 : Allowed error between the variance value estimated by the PM and $Var(S_{t1})$, for instance 1% of this value.

And the variables are:

v: The factor by which the asset value increases in each step of the binomial tree.

q: The probability by which the asset value increases in each step of the binomial tree.

m: The amount of steps of the binomial tree between time $t0$ and $t1$.

Since in each of the functions $f1$, $f2$, $f3$ in (19), (20) and (21) the variables v, q and m are independent of the sum ($v_i = v$, $q_i = q$), the Newton's binomial theorem is multiplied by the respective sum of each function. Unfortunately the Hessian matrix of functions $f1$, $f2$ and $f3$, not considering the absolute value function, is not positive definite. Therefore, it is not possible to assure that those functions are convex or that the result of the PM is a global minimum. However, (23) and (24), if they were an active constraint, can be solved analytically for a given amount of steps of the binomial tree (the analytical solution or AS, further details in Appendix 1) and that solution can be a reference for the starting point for a PM solution. So, the variable m could be suggested to the model and the result should be rounded to the closest integer.

In Appendix 1 is shown that in some cases the constraints (23) and (24) cannot be used with β_1 and β_2 equal to zero simultaneously but probably this should not be necessary without losing accuracy.

In the PM, the expected value and variance at $t0$ and $t1$ are provided to the model as input information. So, the variables v and q are calculated in order for the model to consider effectively those expected and variance values. Further, said model tries to diminish the difference in the skewness between the modeled pmf and the real pmf.

The PM for the movements of S_t from $t0$ to $t1$ through the set of binomial subtrees could have enough accuracy for replicating at $t1$, the information about S_t, such as the expected value, variance, skewness. It could be thought that the variables calculated by the PM could be improved to minimize the objective function by the Adjusting Model (AM). This model adjusts the values v_i and q_i of each subtree from the values v, q obtained by the PM, in order to minimize the sum of the square root of the square differences between the estimated pmf with $X_{i\,k}$ values and the given pmf of S_{t1}, weighted by the corresponding value of given pmf of S_{t1} (further details in Appendix 3). The idea behind the AM is that the variables, v_i and q_i, of each binomial tree between time $t0$ and $t1$, starting from each node at time $t0$, can be adjusted in order to get flexibility for improving the adjustment of the binomial tree model and pmf at time $t1$. The amount of steps of the binomial model in the AM, m, would be defined from the PM or given directly as a parameter.

In this context, in the proposed method, the expected value and variance at two times ($T = t0$ and $T = t1$) define the parameters v and q for one or for a set of binomial subtrees. In other methods based on a binomial tree those parameters are usually calculated through an estimated volatility but the expected value and variance at those times assumed by the model do not correspond necessarily to those risk measures estimated for those times. In addition, since it deals with a nontraded asset, the estimate for the volatility could not be done by statistics from a function $\mathrm{Ln}(S_t/S)$ because there is not enough information. This is the main issue of the proposed method.

If there are estimates for the expected value and variance of the asset value at time $t0$ and $t1$, the expressions (1) and (2) for Black and Sholes (B&S) do allow us to estimate the volatility and probability q in order to reproduce said risk measures at $t1$ (equivalent to one binomial tree) given there is certainty on the asset value at $t0$. In a non-traded underlying asset case, it would not be possible to determine directly the volatility because there is not enough information. The proposed method, given the structure of the proposed binomial tree and the risk measures at time $t0$ and $t1$, determines directly the parameters v and q for a set of binomial subtrees either by the AS or the PM and then the volatility is determined by the calculation of such parameters.

2.4. Variance at Time $t1$ and at Time $t0$

In Appendix 2 it is shown that, in case of the **Figure 1** and the same parameters v and q for the subtrees between time $t0$ and $t1$, the variance of the asset value at time $t1$, $Var(S_{t1})$, corresponds to the following:

$$Var\left(S_{t1}\right)=E\left(S_{t0}^{2}\right)*E\left(Y^{2}\right)-E\left(S_{t1}\right)^{2} \tag{25}$$

Where $E\left(S_{t0}^{2}\right)$ is the expected value of the square of the variable S at time $t0$, $t0 < t1$; $E\left(Y^{2}\right)$ is the expected value at $t1$ of the variable Y whose expression depends on v_i and q_i as it is shown in (2) of the Appendix 1. The variable Y starts with the value of 1 at $t0$ and ends up at $t1$ with some value of a binomial tree of m steps, with an increase in the node k ($k=0,\cdots,m$) by the factor v_i^k with probability q_i and/or a decrease by the factor $1/v_i^{m-k}$ with probability $\left(1-q_i\right)^{m-k}$; and $E(S_{t1})^2$ is the square of the expected value of the asset at time $t1$.

It is known that $E\left(S_{t0}^{2}\right)$ is related to the variance at $t0$ in the following way:

$$E\left(S_{t0}^{2}\right)=Var\left(S_{t0}\right)+E\left(S_{t0}\right)^{2} \tag{26}$$

By replacing (26) in (25) this expression becomes:

$$Var\left(S_{t1}\right)=Var\left(S_{t0}\right)*E\left(Y^{2}\right)+E\left(S_{t0}\right)^{2}*E\left(Y^{2}\right)-E\left(S_{t1}\right)^{2} \tag{27}$$

In case of certainty on the value of the underlying asset at $t0$, $E\left(S_{t0}\right)^{2}$ corresponds to the square of the current value of the underlying asset, S_{t0}^{2}. This means that the variance at time $t1$ could be explained by the variance coming from the risk at $t0$, $Var\left(S_{t0}\right)*E\left(Y^{2}\right)$, plus the variance because of the future after $t0$, $E\left(S_{t0}^{2}\right)*E\left(Y^{2}\right)-E\left(S_{t1}\right)^{2}$, assuming that $E\left(Y^{2}\right)$ is the same with or without risk at $t0$ (this assumption could be called "continuity of the risk"). The expression (27) shows that the risk in the current value of the underlying asset should affect an option value in comparison to the option value with no risk in the current value of the underlying asset ($Var(S_{t0})=0$). The greater the variance of S_{t0}, the greater the variance of S_{t1}.

In this context, if there is risk at $t0$ on the current underlying asset value and there is an estimate about the expected and variance values of the underlying asset at $t0$ and $t1$, the PM and AM could be used to estimate the parameters v and q for then calculating an option value on that underlying asset.

2.5. Estimate Error

The goal of the PM and the AM has been to estimate the suitable parameters of a set of binomial subtrees between time $t0$ and $t1$ in order to value a derivative with a proper model for the possible asset values between $t0$ and $t1$.

The pmf assumed by the AM to define the binomial tree can be verified or disproved statistically by goodness of fit tests in comparison to the Available Data (AD, the data corresponding to the future situation at $t1$), such as Kolmogorov-Smirnov test (KS test). This test compares the maximum difference between the cumulative density functions given by the model and the theoretical (given) data, D_{nd}, at t1 with the critical value D_{nd}^{δ} which is defined for significance level δ by:

$$P\left(D_{nd} \leq D_{nd}^{\delta}\right) = 1 - \delta \tag{28}$$

Where nd is the sample size. If the difference D_{nd} is less than the critical value, D_{nd}^{δ}, the proposed distribution is acceptable at the specified significance level δ. For $\delta = 0.01$, $D_{nd}^{\delta} = 1.63/\sqrt{nd}$.

3. Example

3.1. General Example

In a general case it will be possible to have a pmf for time $t0$ and $t1$ with enough information about the underlying asset value and its probabilities of occurrence, for instance obtained from a DCF and Monte Carlo process.

In Table 2, it is considered the following information about the expected value, variance and skewness coefficient of an underlying asset value for the current time $t0$.

Table 2. Assumptions for the example time $t0$.

Parameter		Value
Expected value $E(S_{t0})$		2,927.7
Variance (S_{t0})		65,800.6
Skewness coefficient (S_{t0})		-0.41
	S_{t0}	q_{t0}
1	1.707	0.1%
2	1.803	0.0%
3	1.899	0.0%
4	1.995	0.2%
5	2.091	0.1%
6	2.186	0.7%
7	2.282	0.9%
8	2.378	1.9%
9	2.474	2.6%

Continued

10	2.570	4.9%
11	2.666	7.8%
12	2.762	10.1%
13	2.858	15.1%
14	2.954	16.5%
15	3.050	13.1%
16	3.146	12.1%
17	3.242	8.3%
18	3.338	3.6%
19	3.433	1.5%
20	3.529	0.9%
21	3.625	0.2%

In Table 3, it is considered the information about the expected value, variance and skewness coefficient of an underlying asset value for the future time $t1$ (AD).

Table 3. Assumptions for the example time $t1$.

Parameter		Value
Expected value $E(S_{t1})$		3,161.9
Variance (S_{t1})		76,752.1
Skewness coefficient (S_{t1})		−0.41
	S_{t1}	q_{t1}
1	1.843	0.1%
2	1.947	0.0%
3	2.051	0.0%
4	2.154	0.2%
5	2.258	0.1%
6	2.361	0.7%
7	2.465	0.9%
8	2.569	1.9%
9	2.672	2.6%
10	2.776	4.9%
11	2.879	7.8%
12	2.983	10.1%
13	3.087	15.1%
14	3.190	16.5%
15	3.294	13.1%

Continued

16	3.397	12.1%
17	3.501	8.3%
18	3.605	3.6%
19	3.708	1.5%
20	3.812	0.9%
21	3.916	0.2%

The results of the PM are shown in **Table 4** for $m = 4$ steps in a binomial tree, using the AS solution for v, $m = 4$ and $q = 0.900$ as the starting point of PM. The skewness coefficient result is −0.39 according to (13), (15), (16), (17) and the result for the variables of the PM.

Table 4. Results of the PM.

Variable	Value
v	1.014
q	0.982
m	4

Skewness coefficient = −0.39.

The resulting pmf with the AM is shown in **Table 5** and its skewness coefficient is −0.39 according to (13), (15), (16), (17) and the result for the variables of the AM. The results for the variables v_i of the AM are very similar to the result for variable $v = 1.014$ of the PM. The differences are at the fourth decimal in the most of cases. The differences between the variables q_i of the AM and the variable $q = 0.982$ of the PM are at the third decimal. Since the variables of the PM and AM are very similar, the graph of both models are practically coincident in **Figure 2**.

The resulting pmf (n_{xj} in **Table 5**) and its comparison with the pmf of the AD and the pmf resulting from the PM are shown in the following **Figure 2** for $m = 4$ steps.

Table 5. Results of the AM.

	S_{t1}	q_{t1}	n_{xj}
1	1.843	0.1%	0.1%
2	1.947	0.0%	0.0%
3	2.051	0.0%	0.0%
4	2.154	0.2%	0.2%
5	2.258	0.1%	0.1%
6	2.361	0.7%	0.7%
7	2.465	0.9%	0.9%
8	2.569	1.9%	2.0%

Continued

9	2.672	2.6%	2.7%
10	2.776	4.9%	5.1%
11	2.879	7.8%	8.0%
12	2.983	10.1%	10.5%
13	3.087	15.1%	15.2%
14	3.190	16.5%	16.2%
15	3.294	13.1%	13.0%
16	3.397	12.1%	11.7%
17	3.501	8.3%	7.9%
18	3.605	3.6%	3.4%
19	3.708	1.5%	1.4%
20	3.812	0.9%	0.8%
21	3.916	0.2%	0.1%

Skewness coefficient = −0.41.

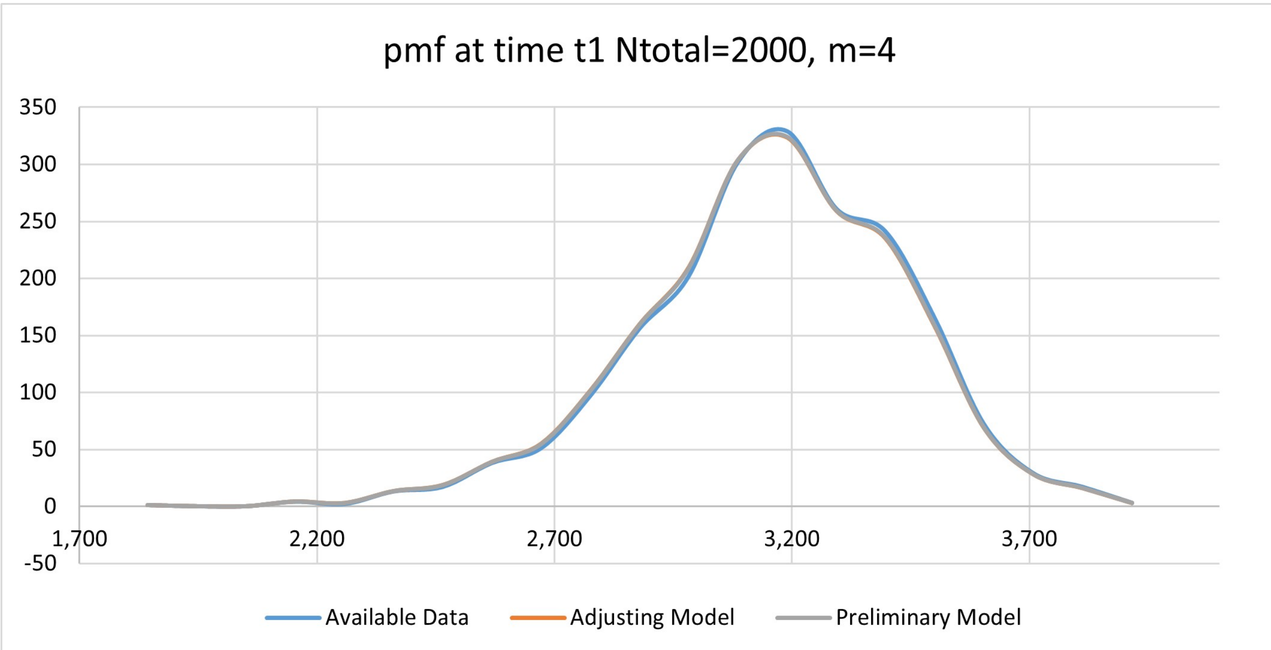

Figure 2. Pmf for $m = 4$.

According to **Figure 2** the PM and AM model can adjust the pmf modeled by the binomial model to the pmf at $t1$, practically with no error. In **Figure 3**, it is shown the situation with $m = 16$.

The error in the expected value estimate for S_{t1} ($E(S_{t1})$) between the binomial model and the pmf at $t1$ is 2.3% for $m = 4$, 0.7% for $m = 8$ and 12 and 0.0% for $m = 16$, in PM and AM.

Table 6 shows the value of the objective function of the corresponding model,

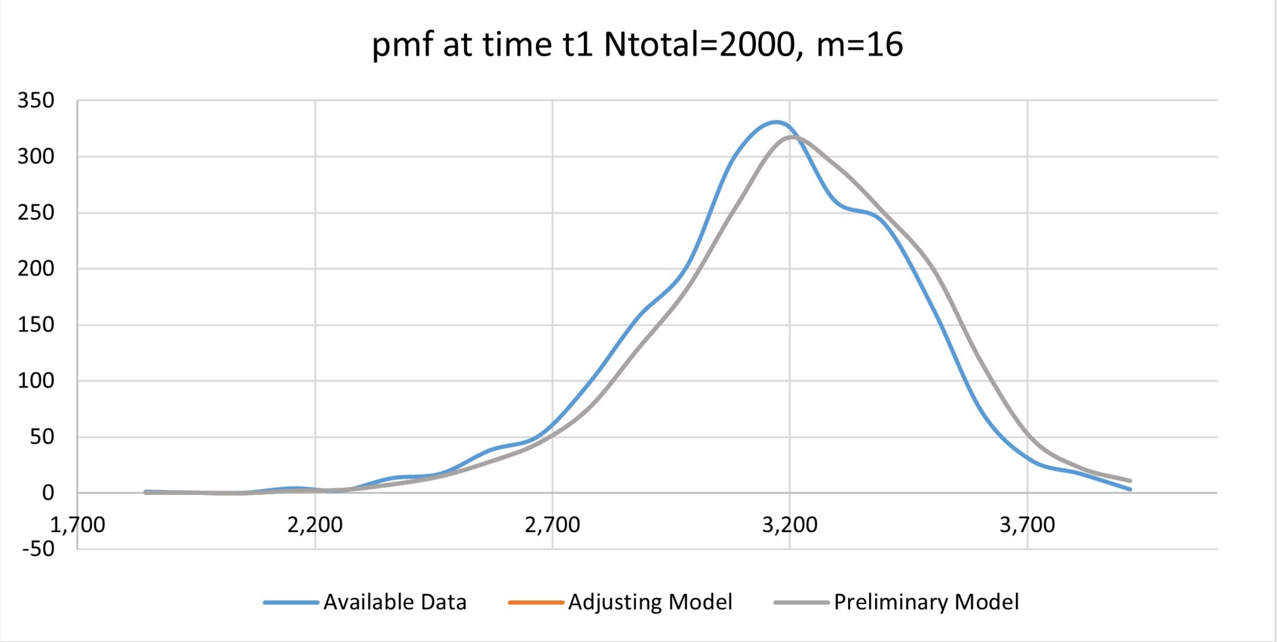

Figure 3. Pmf for $m = 16$.

PM or AM, which is the weighted average error between the model of the binomial tree and the pmf at $t1$, the value of the call option (exercise price 3000), the error in the modeled standard deviation (SD) and the result of the test fit according to the difference $D_{21}^{1\%}$ referred in (28), depending on m value. The considered acceptable error referred in (23), (24), (3.2) and (3.4) is 1%.

The error of the adjustment of PM and AM is shown multiplied by 1000, so the value 1 means 0.1%. So, the weighted average error in the estimated pmf is very low (smaller than 1.2% in all the cases), as well as the SD error and the fit test, for all the cases of m values. Although the error is greater with $m = 16$, the modeled pmf with $m = 16$ is still very close to the pmf at $t1$ of the AD with a very small error in SD. This good adjustment is confirmed by the similar result for the call option.

Specifically in relation to the PM with $m = 4$, the goodness of fit test gives a

Table 6. Square error in PM and AM , call option value, standard deviation error and Fit Test.

	Square error value (1)		Call option Value (2)		SD error	Fit Test
m	PM	AM	PM	AM	AM	$D_{21}^{1\%}$
4	2	2	3.85%	3.83%	2.53%	0.01
8	1	0	3.84%	3.84%	0.50%	0.00
12	0	0	3.80%	3.80%	0.50%	0.00
16	12	12	3.80%	3.80%	0.50%	0.08

1) Weighted average error of the sum of square root of the square difference between the pmf modeled by binomial tree and pmf at $t1$.

2) Exercise price 3,000

maximum difference D_{21} of 0.01 which is less than $D_{21}^{0.01}$ of 0.36 according to (28). All the cases with different m values have an acceptable test fit. It seems to there be a correlation between the fit test and the square error value.

3.2. Different Kinds of pmf at Time *t*0 from Time *t*1, Uniform Distribution Case

The pmf considered at *t*0 and *t*1 in section 3.1 have a shape similar to normal or lognormal distribution. Since the binomial tree tends to look as a normal o lognormal distribution it is reasonable to think how useful would be to use the proposed method if the pmf at time *t*0 and *t*1 are not similar to a normal or lognormal pmf.

In Appendix 4 is shown the pmf for time *t*0 and *t*1, with a uniform distribution (**Tables A1** and **Tables A2**). The considered acceptable error in expressions (3.2) and (3.4) in Appendix 3 is 5% instead of the 1% in section 3.1. This change is necessary to permit a solution in AM. The idea of AM it is to improve the adjustment of the modeled pmf by PM to the pmf of the AD, what could eventually worsen the variance estimate.

In **Figure 4** is shown, as frequencies, the uniform distribution of the AD and the pmf modeled by PM and AM with m = 4. In a similar way to the previous example, there are not differences between the result of the variable v and q of the PM and the variables v_i and q_i of the AM before the fifth decimal. This means that the graph of pmf for S_t with the PM and AM solution are practically coincident as it can be seen from **Figure 4.**

As it was expected, since the binomial tree tends to model a pmf similar to a normal or lognormal pmf, the modeled pmf by PM and AM are greater at the center and smaller at the extremes. **Figure 5** shows the case with m = 16, which has the same feature in the shape.

The error in the expected value estimate for S_{t1} ($E(S_{t1})$) between the binomial model and the pmf at *t*1 is 0.0% for $m = 4$, 8 and 12, and 0.4% for $m = 16$, in PM and AM.

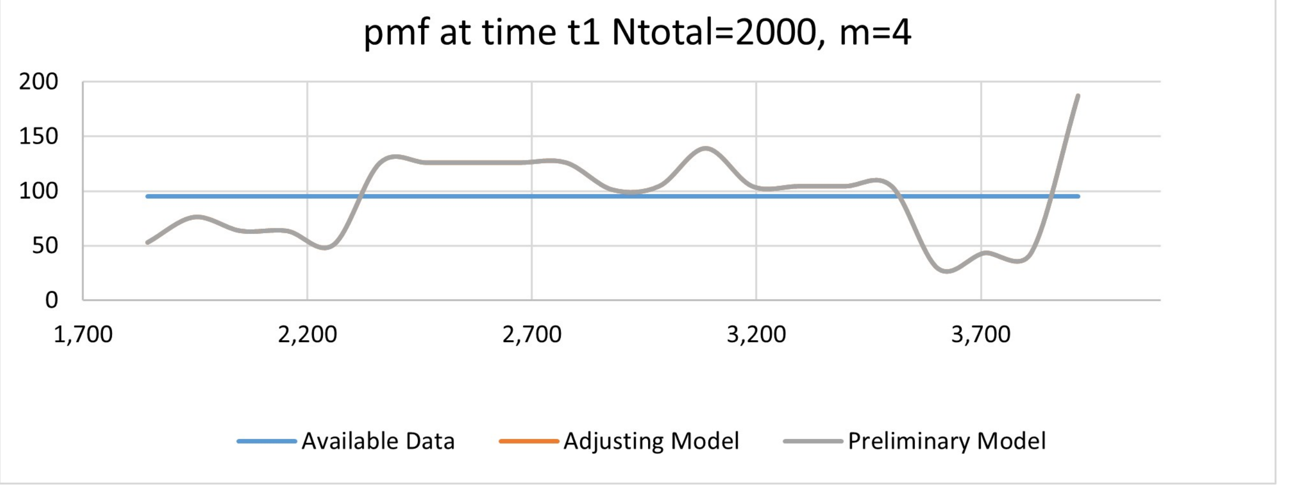

Figure 4. Pmf for $m = 4$, uniform distribution at *t*0 and *t*1.

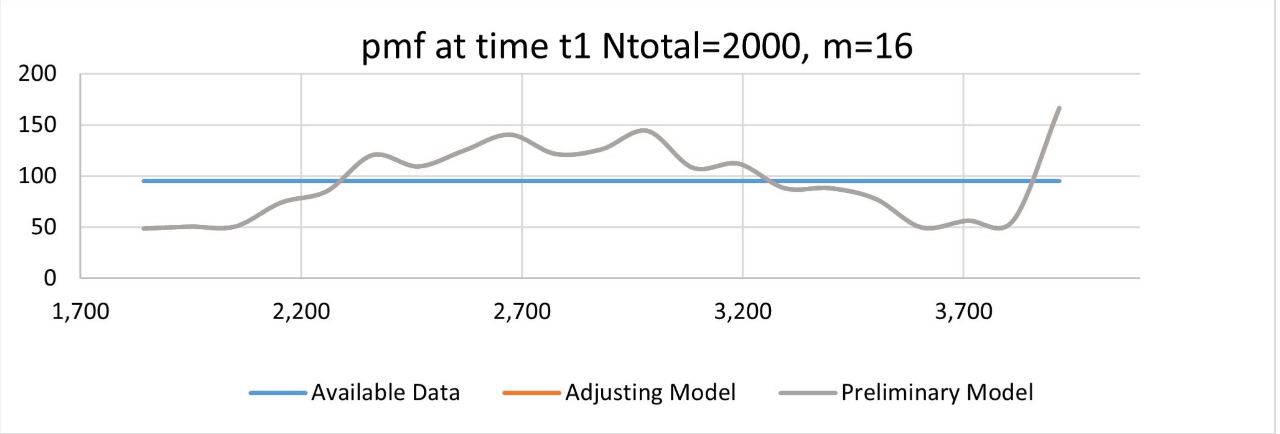

Figure 5. Pmf for $m = 16$, uniform distribution at $t0$ and $t1$.

In **Table 7**, the weighted average error is shown, the call option values calculated by PM and AM in terms of percentage of the expected value at $t0$, the standard deviation error and the fit test. The weighted average error in each value of the pmf is between 1.5% to 1.8% in all the cases.

Table 7. Case uniform pmf at $t0$ and $t1$, call value $K = 3000$.

	Square error value (1)		Call option Value (2)		SD error	Fit Test
m	PM	AM	PM	AM	AM	$D_{21}^{1\%}$
4	15	16	4.03%	4.03%	0.01%	0.08
8	16	16	3.90%	3.90%	1.67%	0.08
12	18	18	3.66%	3.66%	0.08%	0.08
16	15	15	3.74%	3.74%	1.91%	0.08

1) Weighted average error of the sum of square root of the square difference between the pmf modeled by binomial tree and pmf at $t1$.

2) Exercise price 3000

Although the call option values have more difference between each other than the case shown in section 3.1, they are in the same range of values and the maximum difference between the extreme value and the average (3.83%) is 5%, which could be considered as not too relevant.

There is not a relevant difference between the result of the PM and AM what it means that the adjustment of some of the binomial parameters have not been required for a better match with AM.

3.3. Different Kind of pmf at Time T0 from Time $T1$, Uniform-Normal Distribution Case

It is reviewed the case of a uniform pmf at $t0$, similar to the example in 3.2, and a pmf at $t1$ of the case 3.1, similar to a normal or lognormal distribution. In **Figure 6** is shown, as frequencies, the pmf of the AD and the pmf modeled by PM with $m = 4$.

The error in the expected value estimate for S_{t1} ($E(S_{t1})$) between the binomial model and the pmf at $t1$ is 0.0% for m = 4, 8, 12 and 16, in PM and AM.

The differences between the result for variable v and q of the PM and the variables v_i and q_i of the AM are at the third and fourth decimal respectively. So the graph of the resulting pmf at t1 of the PM and AM are similar but different from each other.

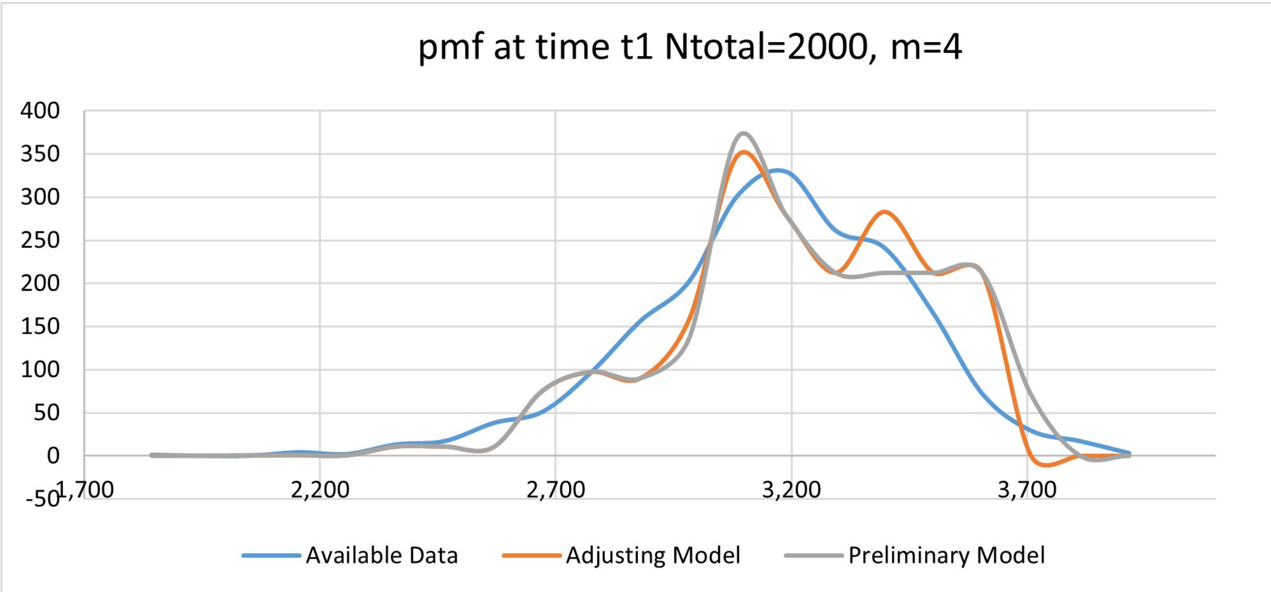

Figure 6. Pmf for $m = 4$, uniform at $t0$ and "normal" pmf at $t1$

As it can be seen from **Figure 6**, in this case the AM has improved a little the adjustment of the binomial tree model to the pmf at $t1$. In **Table 8**, it is shown that the weighted average error between the pmf of the AD and the one modeled by PM and AM, the call option values calculated by PM and AM in terms of percentage of the expected value at $t0$, the error on the standard deviation estimate and the result of the test. The results for the call option have more difference between each other than the case shown in section 3.1 and 3.2, although the results for m between 8 and 16 are in the same range of values. The result of m = 4 case is in other level from the other cases, almost double, and the same happens with the weighted average error. The error at the right side in **Figure 6** can be relevant but as it is weighted by the pmf in the objective function, which value is relatively small in comparison with the average, the effect in the objective function is not too great but it is close to the double of the pmf in the other cases.

Figure 7 shows the case with the smaller weighted average error, with m = 8. In this case AM improves the adjustment to pmf of the AD at $t1$ and the weighted average error in each estimated value for the pmf is 1%, which can be considered not relevant. In all the cases for m, the standard deviation is 2,53% what it means that (3.4) is an active constraint.

Table 8. call value K = 3000.

	Square error value (1)		Call option Value (2)		SD error	Fit Test
m	PM	AM	PM	AM	AM	$D_{21}^{1\%}$
4	26	24	1.21%	1.18%	2.53%	0.09
8	15	10	0.66%	0.64%	2.53%	0.06
12	13	12	0.49%	0.47%	2.53%	0.06
16	11	11	0.41%	0.39%	2.53%	0.07

1) Weighted average error of the sum of square root of the square difference between the pmf modeled by binomial tree and pmf at $t1$.

2) Exercise price 3000

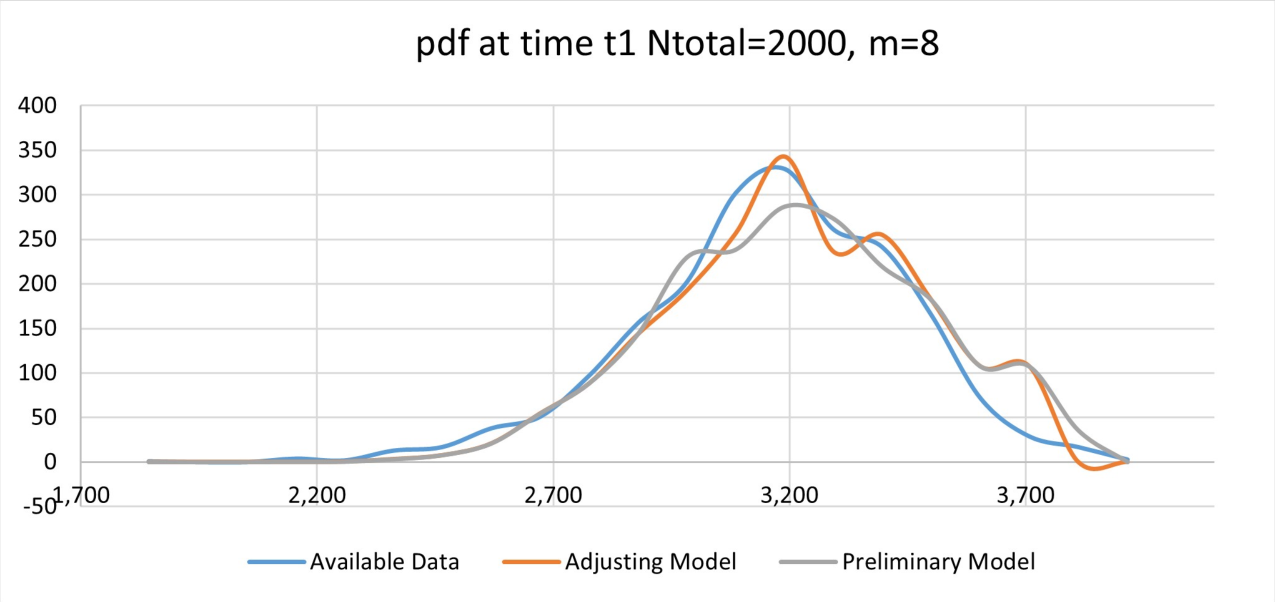

Figure 7. Pmf for $m = 8$, uniform at t0 and "normal" pmf at $t1$.

In all the cases for m the fit test is under the critical value required to accept the modeled pmf. In this way, the parameters for the binomial tree of the Figure 1 have been calculated in order to reproduce properly the underlying asset value at $t1$ in relation to the expected value, variance and skewness of the underlying asset, in addition to the suitable parameters to calculate a call option value.

In this context, according to the accuracy of the results for the expected and variance values and the available goodness of fit tests, the proposed analytical method, PM and AM, should be enough reliable. Indeed if the pmf at $t0$ and at $t1$ look like a normal distribution or if the pmf at $t0$ and $t1$ have a "similar" shape in the pmf, PM is enough for a good estimate of the binomial tree. If the pmf at $t0$ at $t1$ are different AM can be more flexible than PM for a better adjustment between the binomial model and the pmf at $t1$.

3.4. Example without and with Risk at Time $t0$

The main problems for cases dealing with non-traded underlying assets is the lack of information for the current asset value and for the volatility the asset value could have. The propose method deals with these problems. However, in order to check the validity of the proposed approach a known academic example is reviewed. In the example is assumed a volatility as the it were a traded asset. Since there is a relation between the variance at time $t0$ and the variance at time $t1$, in case of a variance at time $t0$, if the volatility between $t0$ and $t1$ is known, the variance at $t1$ could be known according to (27).

So, the idea is to consider an example as it were a non-traded underlying asset case and to calculate a call option value by B&S method and compare the result with the one given by the proposed method.

In the **Table 1** 7.2 of the reference (Hull, 2006), it is calculated an European call option with $S_0 = 50$, $K = 50$, growth per time interval of the underlying asset = 0.05, Volatility or variance per a time interval of $\text{Ln}(S_i/S) = 0.3$, $T = 0.5$ and risk-free interest rate = 5% annual. Since there is no risk in the current underlying asset value, the call option value price (B&S) is 4.817. The provided information means a variance of 121 at $t1$ due to the changes on the asset value from $t0$ to $t1$, which is calculated according to (2).

If there is risk in the current underlying asset and regarding the pmf of the **Table 9**, the variance at $t0$ is $Var(S_{t0}) = 225$.

In **Table 9**, it is shown the call option value calculated by Monte Carlo, 7.959, which is calculated with the expected value of each case of S_i.

Table 9. Pmf of the asset value at $t0$ and call option value.

i	1	2	3	4	5	6
pmf	0.05	0.15	0.30	0.30	0.15	0.05
S_i	20	30	45	55	70	80
$E(C_i)$	0.000	0.030	2.376	8.175	21.502	31,270
$E(C)$	7.959					

In a binomial tree, similar to **Figure 1** with $m = 10$, the AS is $v = 1.069$, $q = 0.502$. With these parameters and the risk on the asset value at $t0$ depicted in **Table 7**, the value of the European call is calculated through a binomial tree similar to **Figure 1**. The result is 7.964. This value is equivalent to 165% of the value with certainty. According to (27) the variance at $t1$ is 368 and the 67% of the variance is explained by the risk from the current underlying asset value since $E(Y^2)$ is equal to 1.0987 according to (2) of Appendix 1.

The goal of the example is only to explain the difference in a call option value for certainty and risk cases in relation to the current asset value.

The value calculated with a binomial tree is similar to the Monte Carlo result. This result helps understand the great difference between the certainty and risk cases for the current asset value. From **Table 9**, it can be seen that the middle value for $i = 3, 4$, whose expected asset value is 50, the expected call option value is 5.276, similar to the B&S solution. However, the call option value contributed by $i = 5$, 6, although the pmf are relatively small (0.15 and 0.05), is 4.789 and it explains the great difference between certainty and risk cases. The variance at time t0 provides higher asset values than the case with certainty what explains greater values for the call option.

This feature means that the risk involved in the current asset value should not be assessed only by considering different current asset values at time *t*0 but, for instance, by using a structure of binomial tree which can reflect the risk at time *t*0 and can estimate suitable parameters of the binomial subtrees in order to the model effectively reproduces the risk measures of the asset value at *t*1.

In this context, the considered binomial tree structure and the optimization models PM and AM could be enough for calculating a call option value for a non-traded underlying asset.

4. Conclusion

In the case of nontraded assets, their value at any time could be estimated with some level of risk through some specific process, for instance, DCF or Monte Carlo, DTA or a mix, in the sense that the risk on the underlying asset value can be expressed in terms of its expected value, variance and skewness.

A suitable way to value financial or real options on an underlying asset is option pricing theory which can include a stochastic process to model the behavior of the option and the underlying asset.

This paper proposes an approach to calculate a call or put option on a non-traded underlying asset (for instance a real option) in order to effectively reproduce the expected value and variance of the underlying asset value at the current and expiration time. The proposed approach is based on, on the one hand, suitable parameters for a binomial model of a stochastic process and, on the other, on a specific structure for a binomial tree.

The parameters for the binomial tree are calculated by a set of models, PM and AM, both supported by an analytical expression, so that the stochastic process of the binomial tree is effectively consistent with the expected value, the variance and skewness of the underlying asset to be modeled for any two times (*t*0 and *t*1). In the proposed approach, it is included the risk of the current asset value estimate (time *t*0) and its effect on the asset value at the time of expiration. The usual methods for option value estimate, such as B&S and the known Monte Carlo, assume certainty for the current underlying asset value which does not correspond to the real situation in nontraded underlying asset cases (not listed company, infrastructure projects, patents, etc.). Since the proposed method defines the parameters required in a binomial tree model, it allows us to calculate an option value on the underlying asset.

It is shown an example where the parameters of the binomial tree are calculated with the proposed method, given the expected value, variance and skewness of a nontraded underlying asset, for the current situation and for the future situation. The goodness of the proposed method can be evaluated with the comparison between the given information for the referred measures of risk and the corresponding values estimated by the proposed method. The examples register an error between 0.0% and 2.3% for the expected value of the asset at $t1$; and an error between 0.0% and 2.5% for the standard deviation of the asset modeled by the AM.

The values obtained in the example for a call option are similar for binomial trees with different amounts of steps from the current time ($t0$) to expiration time ($t1$) when the weighted average error between the pmf estimated by the binomial tree and the one considered for time $t1$ is less than 1.8% of the corresponding value of the pmf.

The PM, more simple than AM, should be reliable enough for a good estimate of the parameters for the binomial tree if the pmf at $t0$ and $t1$ have a similar shape. The AM could be a little bit more flexible than PM for a better adjustment between the binomial model and the pmf at t1 if this pmf is very different from pmf at $t0$, but PM could still be reliable in this case.

Although a binomial model represents better a lognormal or normal than a uniform distribution at time expiration ($t1$) the posed method could give reliable results in case of a different pmf distribution at $t1$, uniform distribution, for instance.

Depending on the level of the variance of the current asset value at time $t0$, a call value at time $t1$ could be substantially greater than the result with no risk for the current asset value. Therefore, in case of a nontraded asset, where there is a risk on its current value, the risk consideration at the current situation should be relevant in any derivative valuation, particularly because of the relatively high values of the pmf function. In these cases, with risk on the current asset value it is recommended a way to assess the risk through a suitable model like the proposed binomial tree.

Using an approximation for the volatility in a call option estimate by a usual method instead of the right value (for instance, the volatility of the outcome instead of the underlying asset) can lead to an approximation for the call value. The proposed method does not have that volatility accuracy problem since the risk measures are used directly for estimating the binomial tree parameters.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix 1

Comparison Between Risk and Certainty on the Current Asset Value

It will be reviewed how the risk of the underlying asset value at $t0$ affects the increase factor v and the probability q of a binomial subtree between time $t0$ an $t1$ according to Figure 1. An analytical solution for v and q (the AS) is obtained in order to the model of Figure 1 reproduces the expected value and variance at $t0$.

The risk at $t0$ is represented by the binomial tree shown in Figure 1 where all the subtrees going from $t0$ to $t1$ have the same factor v for the upward move, with probability q, and also they have the factor $1/v$ for the downward move with probability (1-q). The idea is to relate the variance and expected value at $t1$ to variance and expected value at $t0$ and then to analyze the effect in factor v.

According to Appendix 2, the variance at $t1$ of Figure 1 could be expressed in the following way:

$$Var\left(X_{t1}\right)=E\left(X_{t0}^{2}\right)*E\left(Y^{2}\right)-E\left(X_{t1}\right)^{2} \tag{1.1}$$

Where $E\left(X_{t0}^{2}\right)$ is expected square value at $t0$ of the tree of Figure 1, $E\left(Y^{2}\right)$ is at $t1$, the expected square value of the binomial subtree starting at $t0$ and ending up at $t1$ with value of 1 at the beginning node.

It is known that the expected square value $E\left(Y^{2}\right)$ in a binomial tree of m steps is:

$$E\left(Y^{2}\right)=\left(v^{2}*q+\frac{1-q}{v^{2}}\right)^{m} \tag{1.2}$$

Solving for $E\left(Y^{2}\right)$ from (1.1) and regarding (1.2) it results the following expression for probability q:

$$\frac{W-\frac{1}{v^{2}}}{v^{2}-\frac{1}{v^{2}}}=q \tag{1.3}$$

Where:

$$W=\left(\frac{Var\left(X_{t1}\right)+E\left(X_{t1}\right)^{2}}{Var\left(X_{t0}\right)+E\left(X_{t0}\right)^{2}}\right)^{1/m}=\left(\frac{\left(1+\rho_{t1}^{2}\right)*E\left(X_{t1}\right)^{2}}{\left(1+\rho_{t0}^{2}\right)*E\left(X_{t0}\right)^{2}}\right)^{1/m} \tag{1.4}$$

And ρ_{t} is the coefficient of variation at time t. Analogously, the expected value at $t1$ of Figure 1 could be expressed in the following way:

$$E\left(X_{t1}\right)=E\left(X_{t0}\right)*E\left(Y\right) \tag{1.5}$$

And,

$$\frac{Z-\frac{1}{v}}{v-\frac{1}{v}}=q \tag{1.6}$$

Where:

$$Z=\left(\frac{E\left(X_{t1}\right)}{E\left(X_{t0}\right)}\right)^{1/m} \tag{1.7}$$

From (1.4) and (1.7), the expression for W is:

$$W = \alpha * Z^2 \quad (1.8)$$

Where α is:

$$\alpha = \left(\frac{1+\rho_{t1}^2}{1+\rho_{t0}^2}\right)^{1/m} \quad (1.9)$$

Since (1.3) is equal to (1.6) and simplifying denominator of (1.6) with the denominator of (1.3), and using (1.6), the expression for v is:

$$Z * v^2 - \left(1 + \alpha * Z^2\right) * v + Z = 0 \quad (1.10)$$

The solution for v is:

$$v = \frac{\left(1+\alpha * Z^2\right) + \sqrt{\left(1+\alpha * Z^2\right)^2 - 4 * Z^2}}{2 * Z} \quad (1.11)$$

The case with the sign plus gives a value of v greater than 1 which it is consistent with the upward movement. The root of the solution for v should be positive, what it means:

$$\alpha^2 * Z^4 + \left(2 * \alpha - 4\right) * Z^2 + 1 \geq 0 \quad (1.12)$$

For the solution for Z always exists (parabola always positive), the expression for the root of the parabola should be negative, which means that the existence of v requires $\alpha > 1$. This condition seems to be reasonable in the sense that the coefficient of variation in time $t1$ (future) should be greater than in time $t0$ (past).

If the coefficient of variation at $t0$ and at $t1$ are equal ($\rho_{t0} = \rho_{t1}$, $\alpha = 1$), what could be very common in case of the asset value using net present value and Monte Carlo, it means that:

$$v = \frac{W}{Z} = Z \quad (1.13)$$

If W (or $\alpha * Z^2$) increases, it can be seen from (1.11) that v increases what it is equivalent to a greater exponential factor in (3) of the section 2 for the case with certainty on the asset value at $t0$. This issue tends to estimate greater asset value at time $t1$ for the same binomial tree, not for all the binomial trees in case of the risk case at time $t0$ (uncertainty on the asset value).

Regarding (1.4) there are several reasons that make W increase. One of them is a greater variance at $t1$. Another one it is a zero variance at $t0$, in which case $E\left(X_{t0}\right)^2$ is equal to the square value at the beginning of the binomial tree.

Appendix 2

The variance of an asset S at time $t1$ can be calculated from the following way.

$$Var\left(S_{t1}\right)=E\left(S_{t1}^{2}\right)-E\left(S_{t1}\right)^{2} \tag{2.1}$$

For each subtree starting at time $t0$ in **Figure 1** with M binomial steps, the upward move uses the factor v_i and the probability q_i, while the downward move uses the factor $e = 1/v_i$ and the probability $(1 - q_i)$. Only for the conclusion of this Appendix 2 it is assumed the same factor $v = v_i$ and $q = q_i$ for all the subtrees.

Due to form of the whole tree, the ending value of the asset S_{t1} and its probability of occurrence can be calculated. Some expressions are shown in the diagram for S at some nodes in $t = t1$ and for the corresponding probability, in square parenthesis.

The expected value of $E\left(S_{t1}^{2}\right)$ can be calculated multiplying each ending square value, S_{t1k}^{2} with its probability, f_k, with $k=1,\cdots,nd*\left(M+1\right)$.

$$E\left(S_{t1}^{2}\right)=\sum_{k=1}^{nd*(M+1)} S_{t1k}^{2}*f_{k} \tag{2.2}$$

Each ending value for S_{t1}^{2} and for f_k can be expressed in terms of the value at $T = t0$ and in terms of the parameters depending on each subtree for time $t1$. As the parameters at $T = t0$ and at $T = t1$ are independent each other, the sum in (2.2) can be separated in two independent sum. In **Figure 1**, it is shown the subtree $n = 2$. For the general case, the two independent expressions for the equation (2.2) are:

$$E\left(S_{tj}^{2}\right)=\sum_{n=1}^{nd}\left(S_{on}\right)^{2}*q_{t0n}*\sum_{m=1}^{M+1}\left(v^{M-m}*v^{-m}\right)^{2}*\left(C_{m}^{M}*q^{M-m}*\left(1-q\right)^{m}\right) \tag{2.3}$$

Each sum is independent of each other. The expression in the first sum is the expected value of S_{t0}^{2} at $T = t0$. The expression in the second sum is the expected value of the square of a stochastic value Y_{t1j} of each subtree starting at $T = t0$ and ending at $T = t1$ with a starting value of 1, all of them equal. The expression for this expected value is written as Y_{t1}^{2} in the following expression for the expected value $E\left(S_{t1}^{2}\right)$.

$$E\left(S_{t1}^{2}\right)=E\left(S_{t0}^{2}\right)*E\left(Y_{t1}^{2}\right) \tag{2.4}$$

By replacing (2.4) in (2.1), the variance at $T = t1$ can be expressed by the following way which depends on the expected value of the square value of the asset at $T = t0$ and at $T = t1$ and on the expected value of the square of the stochastic value Y_{t1}, with a starting value of 1 at $t0$.

$$Var\left(S_{t1}\right)=E\left(S_{t0}^{2}\right)*E\left(Y_{t1}^{2}\right)-E\left(S_{t1}\right)^{2}$$

Appendix 3: Adjusting Model (AM)

The AM is the following model:

Objective Function:

$$F2 = \min f4\left(v_1,\cdots,v_{nd},q_1,\cdots,q_{nd}\right) = \min \sum_{j=1}^{nd1} \sqrt{\left(nx_j - q_{t1\,j}\right)^2} * q_{t1\,j} \quad (3.1)$$

Subject to:

$$f4\left(v_1,\cdots,v_{nd},q_1,\cdots,q_{nd}\right) \le \beta_1 \quad (3.2)$$

$$\sum_{i=1}^{nd} S_{t0\,i}^2 * q_{t0\,i} * \left(v_i^2 * q_i + \frac{1-q_i}{v_i^2}\right)^m - \left(\sum_{i=1}^{nd} S_{t0\,i} * q_{t0\,i} * \left(v_i * q_i + \frac{1-q_i}{v_i}\right)^m\right)^2 \ge 0 \quad (3.3)$$

$$Abs\left(\sum_{i=1}^{nd} S_{t0\,i}^2 * q_{t0\,i} * \left(v_i^2 * q_i + \frac{1-q_i}{v_i^2}\right)^m - \left(\sum_{i=1}^{nd} S_{t0\,i} * q_{t0\,i} * \left(v_i * q_i + \frac{1-q_i}{v_i}\right)^m\right)^2 - Var\left(S_{t1}\right)\right) \le \beta_2 \quad (3.4)$$

$$q_i \le 0.99 \quad \text{with} \quad i = 1,\cdots,nd \quad (3.5)$$

$$nx_j = p_j - p_{j-1} \quad \text{with} \quad p_0 = 0; \quad j = 1,\cdots,nd1 \quad (3.6)$$

$$p_j = \sum_{i,k\in Q} q_{t0\,i} * q_i^k * \left(1-q_i\right)^{m-k} * Combin\left(m,k\right) \quad (3.7)$$

With:

Q is the set of pairs composed of a node i at time $t0$ and a node k at time $t1$ which correspond to the value X_{ik} at time $t1$, and X_{ik} is less or equal to the value $S_{t1\,j}$ of the accumulated density function for the underlying asset function at time $t1$, which is the jth value ($j = 1,\cdots,nd1$) of the pmf for S_{t1}, So,

$$i,k \in Q \Leftrightarrow X_{i\,k} \le S_{t1\,j} \quad \text{with } S_{t10} \text{ not applicable; } j = 1,\cdots,nd1$$

The objective function (3.1) minimizes the sum of the square root of the square error between the modeled pmf and the real pmf at $t1$, weighted by the pmf at $t1$. Therefore, the objective function is the weighted average error between the pmf modeled by the model and the pmf of the AD at $t1$.

The constraint (3.2) is a limit β_1 to the function $f5$ which is the difference between the modeled expected value, (13), and the given value $E\left(S_{t1}\right)$. The constraint (3.3) demands to have a positive value for the estimated variance. The constraint (3.4) is a superior limit for the difference between the Variance calculated by the model and the real value. The constraint (3.5) is a superior limit for the probability of increasing the value of the asset. The constraints (3.6) and (3.7) explain how to calculate the estimated pmf for S_{t1}.

The result for variables v and q of the PM could be modified by AM for each subtree in order to diminish the difference between the modeled pmf and the real pmf at $t1$. The AM could have a unique value for v_i or not.

Appendix 4

The following Table A1 and Table A2 show the available data for the example in sections 2.2 and 2.3.

Table A1. Assumptions for the example time $t0$ uniform distribution.

Parameter		Value
Expected value $E(S_{t0})$		2525.7
Variance (S_{t0})		23,141.6
Skewness coefficient (S_{t0})		0.07
	S_{t1}	q_{t1}
1	2.282	4.8%
2	2.305	4.8%
3	2.328	4.8%
4	2.352	4.8%
5	2.375	4.8%
6	2.399	4.8%
7	2.423	4.8%
8	2.447	4.8%
9	2.471	4.8%
10	2.496	4.8%
11	2.521	4.8%
12	2.546	4.8%
13	2.572	4.8%
14	2.598	4.8%
15	2.624	4.8%
16	2.650	4.8%
17	2.676	4.8%
18	2.703	4.8%
19	2.730	4.8%
20	2.757	4.8%
21	2.785	4.8%

Table A2. Assumptions for the example time $t1$ uniform distribution.

Parameter	Value	
Expected value $E(S_{t0})$	2879.4	
Variance (S_{t0})	393,622.4	
Skewness coefficient (S_{t0})	0.00	
	S_{t2}	q_{t2}
1	1.843	4.8%
2	1.947	4.8%
3	2.051	4.8%
4	2.154	4.8%
5	2.258	4.8%
6	2.361	4.8%
7	2.465	4.8%
8	2.569	4.8%
9	2.672	4.8%
10	2.776	4.8%
11	2.879	4.8%
12	2.983	4.8%
13	3.087	4.8%
14	3.190	4.8%
15	3.294	4.8%
16	3.397	4.8%
17	3.501	4.8%
18	3.605	4.8%
19	3.708	4.8%
20	3.812	4.8%
21	3.916	4.8%