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Optimal Insurance Design under Different Dependency Structures of Insurable Risk and Background Risk

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Abstract

In this paper, we consider the optimal insurance by minimizing the total loss of the insured under stop-loss order when the insured faces insurable risk as well as background risk. To depict the relationship between insurable risk and background risk, we apply stochastic orders proposed by Shaked and Shanthikumar (2007) to model positive or negative dependence. We derive the optimal contract forms under different dependence structures between insurable risk and background risk. We give the optimal insurance strategies when the sum of the insurable risk and background risk is stochastically decreasing or increasing in background risk. Furthermore, we consider the optimal insurance strategy under different dependence structures under two insurable risks and a single background risk.

Keywords

Optimal Insurance, Background Risk, Stop-Loss Order, Stochastic Order

1. Introduction

Everyone always faces with unpredictable risks in the real world, the policyholder tends to purchase insurance in order to reduce losses. The process of seeking the optimal ceded loss function for policyholders through certain optimization objectives is called the optimal insurance design.

Since the pioneering work of (Arrow, 1963), the field of the optimal insurance design has attracted considerable attentions in both research and practice, such as Raviv (1992), Huberman, Mayers, and Smith Jr. (1983) and Cheung, Chong, and Yam (2015). However, these papers are only confined to the single-risk framework. In practice, there always exist incomplete markets, which means that the insured is always confronted with multiple sources of risk. In addition to insurable risks, almost everyone faces risks that cannot be insured, such as war, floods and market valuation of stocks, which are called background risks. The policyholders will bear less or more residual wealth after assuming insurable risk due to the existence of background risk, which is determined by the positive or negative correlation between insurable risk and background risk. For instance, the occurrence of flood will destroy agricultural infrastructure and bring property losses to farmers. If farmers buy insurance for their own agricultural facilities, then the insurable risk and background risk are positively correlated, the policyholder's remaining wealth value will be lower than expected after assuming insurable risk. However, the silt from the floods will also make the soil more fertile and thus boost crop yields, reducing the insurable risk of crop yields. At this time, the harvest of the insurable risk crops is negatively correlated with the background risk of rainstorm, which means that the background risk can hedge the loss caused by the insurable risk. Thus, the policyholder's remaining wealth value will be higher than expected after assuming insurable risk.

It is noteworthy that different relationships between insurable risk and background risk will influence the optimal insurance contract. Shaked and Shanthikumar (2007) provided powerful tools to describe the dependence structures between insurable risk and background risk, such as the structures of positive (negative) quadrant dependence, stochastic increasing (decreasing) dependence and positive (negative) expectation dependence.

The optimal insurance design with background risk can be traced back to Doherty and Schlesinger (1983a), they proposed that the optimality of deductible insurance depends on the correlations between insurable risk and background risk. Then, the studies in this issue are generalized mainly in two directions, the studies of the first direction attempt to investigate the optimal insurance contract under specific types of insurance. For instance, Doherty and Schlesinger (1983b) have considered the optimal deductive when the insurance contract is deductible insurance under the condition that initial wealth is random, and it can be shown that Mossin's Theorem holds under negative structures of initial wealth and insurable risk. Jeleva (2000) considered the optimal level of proportional insurance under uncertainty framework. Another direction is to determine the optimal contractual forms of insurance under specific optimal conditions, Viala and Briys (1995) added the background risk into the design of the optimal insurance policy proposed by (Raviv, 1992), and they approached that the optimal contract is deductive insurance when the relationship between insurable risk and background risk is positive. Vercammen (2001) described the relationships between insurable risk and background risk by the positive and negative derivatives of the income and loss of prudent risk managers on the value of insurable risk. It is concluded that when insurable risk and background risk are positively interdependent, the optimal insurance strategy is deductible insurance. Lu, Meng, Liu, and Han (2018) discussed the optimal insurance strategy by minimizing the total loss of the insured under the stop-loss order when the structures of dependence between the two risks are positive or negative. Chi and Wei (2020) revisited the optimal insurance with background risk under general dependence structures and mixed dependence structures of insurable risk and background risk, which significantly supplements the research findings of background risk.

However, the optimization criterion of the insured will also influence the optimal form and the quantity of the insurance contract, the main optimization criterion can be divided into two categories, the first category is to maximize the expected wealth utility of policyholders, such as Aase (2009), Gerber and Pafum (1998) and Dionne and Harrington (2013), who discuss the optimal insurance strategy under different set of loss functions for separation. Another category is to minimize the loss of risks under specific metrics, such as Cai, Lemieux, and Liu (2015) and Jiang, Ren, and Zitikis (2017), they consider the optimal insurance under VaR risk measure. Denuit and Vermandele (1998) derived the optimal solution by minimizing the retained risk with respect to the stop-loss order, which was first proposed by Van Heerwaarden, Kaas, and Goovaerts (1989). It can be inferred that the goal of minimizing the total loss under the stop-loss order is also a weaker form of expected-utility maximizing criterion. Thus, our study will further discuss the optimal insurance strategy of minimizing the total loss under the stop-loss order.

The contributions of this article are as follows. First, we obtain the optimal insurance strategy. And we also investigate the optimal insurance strategy when the insured faced with two insurable risks and a single background risk under different dependence structures of the three risks.

This literature proceeds as follows. First, we introduce some different definitions of stochastic orders and outline the model in Section 2. Then, we derive the optimal insurance strategy forms under different dependence structures of insurable risk and background risk in Section 3, which extends the studies of Lu, Meng, Liu, and Han (2018). Furthermore, the optimal insurance strategy between two insurable risks and a single background risk is given in Section 4. Section 5 concludes the paper.

2. Model Design

Assume that the insured faced with insurable risk X and background risk Y, in order to alleviate the risk mitigation, the insured will choose to purchase insurance. The insured facing with insurable risk X allocates the risk $I(X)$ to the insurer and bears the risk $R_I(X) = X - I(X)$ himself. The purpose of our article is to seek the optimal formal function expression of the ceded loss function $I(x)$ while minimizing the total loss of the insured. In order to exclude ex post moral hazard, as it is shown in Huberman, Mayers, and Smith Jr. (1983), we assume both $I(x)$ and $R_I(x) = x - I(x)$ are increasing in $x(0 \le x < \infty)$, which is called the incentive-compatible constraint. In order to further optimize the set of ceded loss functions, we use the principle of indemnity suggested in Arrow (1963) and Raviv (1992), which can be given as $0 \le I(x) \le x$. Thus, we have $I(x) \in \mathcal{F}$, $\mathcal{F} = \{I \mid 0 \le I' \le x, I(x)$ and $x - I(x)$ are increasing in $x\}$.

In exchange, the premium paid by the insured to the insurer is recorded as $\Pi_{I(X)}$. There are various premium principles for premium selection, such as the Wang's premium principle, the expected value premium, and the mean value principle. In this paper, we use the expected value premium because of its excellent properties and convenient for further study. Thus, we have $\Pi_{I(X)} = (1+\alpha)E[I(X)]$ with $\alpha > 0$, then the total loss of the insured can be expressed as $T_I = X + Y + (1+\alpha)E[I(X)] - I(X)$.

Now, we proceed to find a criterion that can measure the overall loss T_I. Among various measurement standards of minimizing the total loss, stop-loss order has been applied in many studies because of its universality and universal applicability, such as Denuit and Vermandele (1998). Thus, we proceed to discuss the optimal insurance under minimizing the total loss of the insured in the stop-loss order. We first give the difinition of the stop-loss order.

Definition 2.1 *If* $E[X-d]_+ \le E[Y-d]_+$ *holds true for all* $d \in (0,\infty)$, *then X is said to be smaller than Y in the stop-loss order, denoted as* $X \le_{SL} Y$.

Then, the optimization objective can be formally stated as follows:

$$\min_{I(x)\in\mathcal{F}} E\left[X + Y + \Pi_{I(X)} - I(X) - l\right]_+ \tag{2.1}$$

with any $l \in (0,\infty)$. In the following discussions, it can be shown that the optimal insurance contract heavily relies on the dependence between the background risk and the insurable risk. And in order to more intuitively describe the correlations between *X* and *Y* in the following, we introduce a series of definitions and inferences of random orders which are stated in Shaked and Shanthikumar (2007) and Chi and Wei (2018).

Definition 2.2 *Let X and Y be two random variables.*

1) *If* $P(X > x | Y = y)$ *is increasing in* y *for any* x, *or equivalently,* $E[u(x)|Y = y]$ *is increasing in* y *for any* x *and for any increasing function* $u(x)$, *then random variable X is said to be stochastically increasing in random variable Y, denoted as* $X \uparrow_{st} Y$.

2) *If* $P(X > x | Y > y)$ *is increasing in* y *for any* x, *then X is asked to be right tail increasing in Y, denoted as* $X \uparrow_{rt} Y$.

3) *Define the distribution functions of X and Y as* $F_X(x)$ *and* $F_Y(x)$, *X is said to be smaller than Y in the stochastic order if* $F_X(x) \le F_Y(x)$, *denoted as* $X \le_{st} Y$.

4) *Y is thicker-tailed than X, written* $Y \ge_{tt} X$, *if they satisfy the following properties.*

a) $E[X] = E[Y]$.

b) *A real number* x_0 *exists with* $P(X \le x) \le P(Y \le x)$ *when* $x < x_0$, *but* $P(X \le x) \ge P(Y \le x)$ *with* $x > x_0$.

Corollary 2.1 *If* $Y \uparrow_{st} X$, *then we can conclude that* $Y \uparrow_{rt} X$.

Corollary 2.2 *If* $X_1 \le_{st} X_2$, *then we can conclude that* $E[u(X_1)] \le E[u(X_2)]$ *for any increasing function* $u(x)$.

Corollary 2.3 *If* $X \le_{tt} Y$, *then we have* $X \le_{SL} Y$.

3. The Optimal Insurance Strategy under Single Source of Insurable Risk and Background Risk

In this section, we derive the optimal insurance strategy to problem (2.1) by applying the property of stochastic orders proposed by Section 2.

Proposition 3.1 *Under the optimization objective* (2.1), *we can conclude that for any given* $I(x)\in\mathcal{F}$, *there always exists* $g_1(x)=(x-l)_+$ *and* $h_1(x)=x-(x-l)_+$ *which satisfy* $E[g_1(X)]=E[I(X)]=E[h_1(X)]$, *denote that* $T_{g_1}=X+Y-g_1(X)+\Pi_{g_1(X)}$ *and* $T_{h_1}=X+Y-h_1(X)+\Pi_{h_1(X)}$, *then the following conclusions will hold.*

1) *If* $Y\downarrow_{st}X$ *and* $X+Y\downarrow_{st}X$, *then we have* $E[T_{h_1}-l]_+\le E[T_I-l]_+$ *with any* $l\in(0,\infty)$.

2) *If* $Y\downarrow_{st}X$ *and* $X+Y\uparrow_{st}X$, *then it can be inferred that* $E[T_{g_1}-l]_+\le E[T_I-l]_+$ *with any* $l\in(0,\infty)$.

Figure 1 *and* Figure 2 *provide the function graph of* $g_1(x)$ *and* $h_1(x)$, *respectively.* $I(x)\in\mathcal{F}$, $\mathcal{F}=\{I\mid 0\le I'\le 1, I(x)$ *and* $x-I(x)$ *is increasing in* $x\}$.

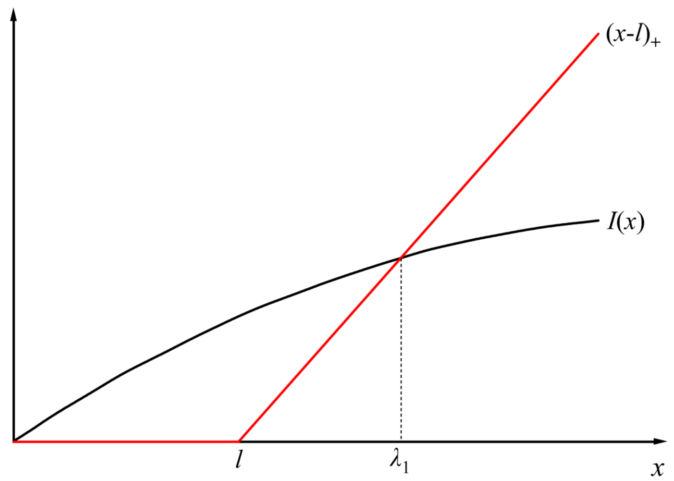

Figure 1. The figure of $g_1(x)$.

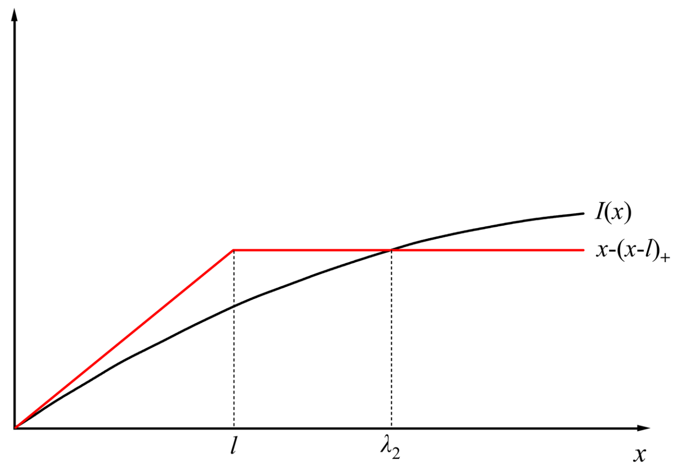

Figure 2. The figure of $h_1(x)$.

Proof. By applying (A.1) proposed by Lu, Meng, Liu, and Han (2018), it is straightforward to see that $E\left[g_1(X)\right]=E\left[I(X)\right]=E\left[h_1(X)\right]$ holds with $I(x)\in\mathcal{F}$.

1) Consider:

$$\begin{aligned}&E\left[T_I-l\right]_+-E\left[T_{h_1}-l\right]_+\\&=E\left[\int_{l-X+I(X)-\Pi_{I(X)}}^{\infty}S_{Y|X}(y)\mathrm{d}y-\int_{l-X+h_1(X)-\Pi_{h_1(X)}}^{\infty}S_{Y|X}(y)\mathrm{d}y\right]\\&=E\left[\int_{l-X+I(X)-\Pi_{I(X)}}^{l-X+h_1(X)-\Pi_{h_1(X)}}S_{Y|X}(y)\mathrm{d}y\right].\end{aligned}\tag{3.1}$$

Then, (3.1) can be simplified as follows because $S_{Y|X}(y)$ is decreasing in *y*.

$$E\left[\int_{l-X+I-\Pi_{I(X)}}^{l-X+h_1-\Pi_{h_1(X)}}S_{Y|X}(y)\mathrm{d}y\right]\geq E\left[\eta_{h_1}(X,l)\left(h_1(X)-I(X)\right)\right]\tag{3.2}$$

with $\eta_{h_1}(x,l)=P\left(X+Y-h_1(X)+\Pi_{h_1(X)}>l\mid X=x\right)$.

And based on the image of $h_1(x)$, (3.2) can be written as:

$$\begin{aligned}&\int_0^{\lambda_2}\eta_{h_1}(x,l)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x+\int_{\lambda_2}^{\infty}\eta_{h_1}(x,l)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&=\int_0^{\lambda_2}\eta_{h_1}(x,l)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x+\eta_{h_1}(\lambda_2,t)\int_{\lambda_2}^{\zeta}\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&=\int_0^{\lambda_2}\eta_{h_1}(x,l)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x-\int_{\zeta}^{\infty}\eta_{h_1}(\lambda_2,t)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&\quad+\int_{\lambda_2}^{\infty}\eta_{h_1}(\lambda_2,t)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&=\int_0^{\lambda_2}\eta_{h_1}(x,l)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x-\int_{\zeta}^{\infty}\eta_{h_1}(\lambda_2,t)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&\quad-\int_0^{\lambda_2}\eta_{h_1}(\lambda_2,t)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&=\int_0^{\lambda_2}\left(\eta_{h_1}(x,l)-\eta_{h_1}(\lambda_2,t)\right)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\\&\quad-\int_{\zeta}^{\infty}\eta_{h_1}(\lambda_2,t)\left(h_1(x)-I(x)\right)p_X(x)\mathrm{d}x\end{aligned}\tag{3.3}$$

with $\zeta\in(\lambda_2,\infty)$.

Furthermore, we obtain $\eta_{h_1}(x,l)$ is decreasing in *x* due to the condition that $X+Y\downarrow_{st}X$, which implies $\eta_{h_1}(x,l)-\eta_{h_1}(\lambda_2,t)\geq 0$, and due to the image of $h_1(x)$, the original formula is always positive, thus it can be obtained that:

$$E\left[T_I-l\right]_+-E\left[T_{h_1}-l\right]_+\geq 0,$$

which means that $h_1(x)$ is the optimal insurance strategy.

2) First, we have:

$$\begin{aligned}&E\left[T_I-l\right]_+-E\left[T_{g_1}-l\right]_+\\&=E\left[\int_{l-X+I(X)-\Pi_{I(X)}}^{\infty}S_{Y|X}(y)\mathrm{d}y-\int_{l-X+g_1(X)-\Pi_{g_1(X)}}^{\infty}S_{Y|X}(y)\mathrm{d}y\right]\\&=E\left[\int_{l-X+I(X)-\Pi_{I(X)}}^{l-X+g_1(X)-\Pi_{g_1(X)}}S_{Y|X}(y)\mathrm{d}y\right].\end{aligned}\tag{3.4}$$

Similar to the previous proof, we can simplify (3.4) as follows:

$$E\left[\int_{l-X+I(X)-\Pi_{I(X)}}^{l-X+g_1(X)-\Pi_{g_1(X)}} S_{Y|X}(y)\mathrm{d}y\right] \geq E\left[\eta_{g_1}(X,l)\left(g_1(X)-I(X)\right)\right] \tag{3.5}$$

with $\eta_{g_1}(x,l) = P\left(X+Y-g_1(X)+\Pi_{g_1(X)} > l \mid X=x\right)$, and $\eta_{g_1}(x,l)$ is increasing in x due to the condition $X+Y \uparrow_{st} X$.

Together with the image of $g_1(x)$, we conclude there exists λ_1 who satisfied $g_1(x) \leq I(x)$ if and only if $x \in (0,\lambda_1)$ and $g_1(x) \geq I(x)$ if and only if $x \in (\lambda_1,\infty)$. Reuse the second mean value theorem of integration, we obtain (3.5) can be written as:

$$\begin{aligned}
&\int_0^{\lambda_1} \eta_{g_1}(x,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x + \int_{\lambda_1}^{\infty} \eta_{g_1}(x,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x \\
&= \int_{\zeta_2}^{\lambda_1} \eta_{g_1}(\lambda_1,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x + \int_{\lambda_1}^{\infty} \eta_{g_1}(x,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x \\
&= \int_0^{\lambda 1} \eta_{g_1}(\lambda_1,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x - \int_0^{\zeta_2} \eta_{g_1}(\lambda_1,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x \\
&\quad + \int_{\lambda_1}^{\infty} \eta_{g_1}(x,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x \\
&= \int_{\lambda_1}^{\infty} \left(\eta_{g_1}(x,l)-\eta_{g_1}(\lambda_1,l)\right)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x \\
&\quad - \int_0^{\zeta_2} \eta_{g_1}(\lambda_1,l)\left(g_1(x)-I(x)\right)p_X(x)\mathrm{d}x
\end{aligned}$$

with $\zeta_2 \in (0,\lambda_1)$. Thus, we obtain $\eta_{g_1}(x,l)-\eta_{g_1}(\lambda_1,l) \geq 0$ due to the monotonicity of $\eta_{g_1}(x,l)$, and together with the property of $g_1(x)$, the original formula is always positive, then we obtain $E\left[T_I-l\right]_+ - E\left[T_{g_1}-l\right]_+ \geq 0$, which means that $g_1(x)$ is the optimal insurance strategy. ■

The above conclusion shows that when insurable risk and background risk are negatively correlated, the positive or negative relationship between the sum of insurable risk and background risk with respect to insurable risk will influence the adoption of the optimal insurance strategy.

4. The Optimal Insurance Strategy under Two Insurable Risks and Background Risk

The insurance market in reality is very complex, and we may not only face one insurable risk and one background risk, but also face situations where multiple insurable risks are influenced and interact with each other by background risk. Based on this, this section consider the optimal insurance strategy by combining cases and theorems when background risk Y, insurable risks X_1 and X_2 meet different interdependence relationships.

For example, in the medical insurance, there are many insurable diseases and many uninsurable chronic diseases. When a person suffers from a chronic disease, over time, it may trigger the production of multiple insurable diseases, and the insurable diseases will reduce his resistance, thus further aggravating the harm of chronic diseases. Here, the risks of insurable diseases can be recorded as X_1 and X_2, the risk of chronic diseases can be recorded as Y. Consequently, in this process, the relationships among X_1, Y, and X_2 can be described as $X_1 \uparrow_{st} Y$, $X_2 \uparrow_{st} Y$ and $X_1 + X_2 \uparrow_{st} Y$.

Thus, we consider the insured faced with background risk Y and insurable risks denoted as X_1 and X_2, record the insurance strategy of X_1 as $I_1(x_1)$ with $x_1 \in [0,\infty)$ and record the insurance strategy of X_2 as $I_2(x_2)$ with $x_2 \in [0,\infty)$.

Record the premiums for X_1 and X_2 as P_1 and P_2. So the total loss of the insured can be demonstrated as $T_I = X_1 + X_2 + Y - I_1(X_1) - I_2(X_2) + P_1 + P_2$ with $P_1 = (1+\alpha)E[I_1(X_1)]$ and $P_2 = (1+\alpha)E[I_2(X_2)]$, and the optimal insurance strategy can be shown as:

$$\min_{I_1, I_2 \in \mathcal{F}} E\left[X_1 + X_2 + Y + P_1 + P_2 - I_1(X_1) - I_2(X_2) - l\right]_+ \tag{4.1}$$

with any $l \in (0,\infty)$. In the following text, we proceed to investigate the optimal insurance strategy under the different dependence structures among X_2, X_2 and Y. Thus, we give the following propositions.

Proposition 4.1 *If* $Y \uparrow_{st} X_1$, $Y \uparrow_{st} X_1$, $X_1 + X_2 \uparrow_{st} Y$, *then under the optimization objective* (4.1), *the optimal insurance strategies can be given as* $I_1^*(x_1) = (x_1 - l)_+$ *and* $I_2^*(x_2) = (x_2 - l)_+$.

Proposition 4.2 *If* $Y \downarrow_{st} X_2$, $Y \downarrow_{st} X_1$, $X_1 + X_2 \uparrow_{st} Y$, *then under the optimization* (4.1), *the optimal insurance strategies are* $I_1^*(x_1) = x_1 - (x_1 - l)_+$ *and* $I_2^*(x_2) = x_2 - (x_2 - l)_+$.

The proof of Propositions 4.1 and 4.2 is presented in **Appendix**.

5. Conclusion

In this paper, we discuss the optimal insurance problem in the presence of background risk when the total loss is required to meet its minimum value under stop-loss order. We give the optimal insurance strategy while the insurable risk and background risk satisfy different dependence structures. We conclude that the optimal insurance strategy takes different forms when the total risk $X + Y$ is stochastic increasing or decreasing in X under the assumption that $Y \downarrow_{st} X$, which extends the conclusions in Lu, Meng, Liu, and Han (2018). We further consider the optimal insurance problem under different interrelationships between two insurable risks and a single background risk, thus generalizing the conclusions in Section 3. However, in exploring the interdependence between insurable risk and background risk, the dependency relationship we are discussing is still relatively simple, there are more general and representative dependency relationships between insurable risk and background risk that deserve further discussion.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

The Proof of Proposition 4.1

First record $P_1 = \Pi_{I_1(X)} = \Pi_{I_1^*(X)}$, $P_2 = \Pi_{I_2(X)} = \Pi_{I_2^*(X)}$. For any $I_1(x_1)$, $I_2(x_2) \in \mathcal{F}$, we only need to prove that $E\left[T_{I_1^*} - l\right]_+ - E\left[T_I - l\right]_+ \le 0$ for any $l \in (0, \infty)$ with $T_{I_1^*} = X_1 + X_2 + Y - I_1^*(X_1) - I_2^*(X_2) + P_1 + P_2$ and $T_I = X_1 + X_2 + Y - I_1(X_1) - I_2(X_2) + P_1 + P_2$. Thus, we have:

$$
\begin{aligned}
&E\left[X_1 + X_2 + Y - I_1^*(X_1) - I_2^*(X_2) + P_1 + P_2 - l\right]_+ - E\left[X_1 + X_2 + Y - I_1(X_1) - I_2(X_2) + P_1 + P_2 - l\right]_+ \\
&= E\left[E\left[\left\{X_1 + X_2 + Y - I_1^*(X_1) - I_2^*(X_2) + P_1 + P_2 - l\right\}_+ \mid X_1 = x_1\right]\right] \\
&= -E\left[E\left[\left\{X_1 + X_2 + Y - I_1(X_1) - I_2(X_2) + P_1 + P_2 - l\right\}_+ \mid X_1 = x_1\right]\right] \\
&= E\left[E\left\{E\left(X_1 + y + X_2 - I_1^*(X_1) - I_2^*(X_2) + P_1 + P_2 - l\right)_+ \mid Y = y\right\} \mid X_1 = x_1\right] \\
&\quad - E\left[E\left\{E\left(X_1 + y + X_2 - I_1(X_1) - I_2(X_2) + P_1 + P_2 - l\right)_+ \mid Y = y\right\} \mid X_1 = x_1\right] \\
&= E\left[E\left\{\int_{l-P_1-P_2-X_1-y+I_1^*(X_1)+I_2^*(X_2)}^{\infty} S_{X_2|Y}(x_2)\,\mathrm{d}x_2 \mid X_1 = x_1\right\}\right] - E\left[E\left\{\int_{l-P_1-P_2-X_1-y+I_1(X_1)+I_2(X_2)}^{\infty} S_{X_2|Y}(x_2)\,\mathrm{d}x_2 \mid X_1 = x_1\right\}\right] \\
&= E\left[E\left\{\int_{l-P_1-P_2-X_1-y+I_1^*(X_1)+I_2^*(X_2)}^{l-P_1-P_2-X_1-y+I_1(X_1)+I_2(X_2)} S_{X_2|Y}(x_2)\,\mathrm{d}x_2 \mid X_1 = x_1\right\}\right] \\
&\le E\left[E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right)\left[\left(I_1(x_1) - I_1^*(x_1)\right) + \left(I_2(X_2) - I_2^*(X_2)\right)\right] \mid X_1 = x_1\right\}\right] \\
&= E\left[E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right)\left(I_1(x_1) - I_1^*(x_1)\right) \mid X_1 = x_1\right\}\right] \\
&\quad + E\left[E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right)\left(I_2(x) - I_2^*(x)\right) \mid X_1 = x_1\right\}\right] \\
&= E\left[E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right)\left(I_1(x_1) - I_1^*(x_1)\right) \mid X_1 = x_1\right\}\right] \\
&\quad + E\left[E\left\{P\left(X_1 > l - P_1 - P_2 - X_2 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right)\left(I_2(x_2) - I_2^*(x_2)\right) \mid X_2 = x_2\right\}\right] \\
&= \int_0^{\infty} E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right) \mid X_1 = x_1\right\}\left(I_1(x_1) - I_1^*(x_1)\right) P_{X_1}(x_1)\,\mathrm{d}x_1 \\
&\quad + \int_0^{\infty} E\left\{P\left(X_1 > l - P_1 - P_2 - X_2 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right) \mid X_2 = x_2\right\}\left(I_2(x_2) - I_2^*(x_2)\right) P_{X_2}(x_2)\,\mathrm{d}x_2.
\end{aligned}
$$

From the image of $I_1^*(x)$ and $I_2^*(x)$, it can be inferred that there must exist λ_1 and λ_2, $I_1(x_1) \ge I_1^*(x_1)$ if and only if $x_1 \in (0, \lambda_1)$, $I_1(x_1) \le I_1^*(x_1)$ if and only if $x_1 \in (\lambda_1, \infty)$. $I_2(x_2) \ge I_2^*(x_2)$ if and only if $x_2 \in (0, \lambda_2)$, $I_2(x_2) \le I_2^*(x_2)$ if and only if $x_2 \in (\lambda_2, \infty)$. Then, the last equality can be written as:

$$
\begin{aligned}
&\int_0^{\lambda_1} E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right) \mid X_1 = x_1\right\}\left(I_1(x_1) - I_1^*(x_1)\right) p_{X_1}\,\mathrm{d}x_1 \\
&+ \int_{\lambda_1}^{\infty} E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right) \mid X_1 = x_1\right\}\left(I_1(x_1) - I_1^*(x_1)\right) p_{X_1}\,\mathrm{d}x_1 \\
&+ \int_0^{\lambda_2} E\left\{P\left(X_1 > l - P_1 - P_2 - X_2 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right) \mid X_2 = x_2\right\}\left(I_2(x_2) - I_2^*(x_2)\right) p_{X_2}\,\mathrm{d}x_2 \\
&+ \int_{\lambda_2}^{\infty} E\left\{P\left(X_1 > l - P_1 - P_2 - X_2 - Y + I_1^*(X_1) + I_2^*(X_2) \mid Y = y\right) \mid X_2 = x_2\right\}\left(I_2(x_2) - I_2^*(x_2)\right) p_{X_2}\,\mathrm{d}x_2.
\end{aligned}
$$

Then, we obtain $P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) | Y = y\right)$ is increasing in y due to the condition $X_1 + X_2 \uparrow_{st} Y$, then by using corollary 2.2, $E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) | Y = y\right) | X_1 = x_1\right\}$ is increasing in x_1 because $Y \uparrow_{st} X_1$. Similarly, $P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) | Y = y\right)$ is increasing in y together with $X_1 + X_2 \uparrow_{st} Y$, thus we have $E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) | Y = y\right) | X_2 = x_2\right\}$ is increasing in x_2 because $Y \uparrow_{st} X_2$. For a more concise expression, we record $E\left\{P\left(X_2 > l - P_1 - P_2 - X_1 - Y + I_1^*(X_1) + I_2^*(X_2) | Y = y\right) | X_1 = x_1\right\}$ as $g_1(x_1)$, and record $E\left\{P\left(X_1 > l - P_1 - P_2 - X_2 - Y + I_1^*(X_1) + I_2^*(X_2) | Y = y\right) | X_2 = x_2\right\}$ as $g_2(x_2)$. Due to the monotonicity of $g_1(x_1)$, we use the second mean value theorem for integrals, the first two terms of the formula can be given as:

$$\begin{aligned}
&\int_0^{\lambda_1} g_1(x_1)\left(I_1(x_1) - I_1^*(x_1)\right) p_{X_1}(x_1)\mathrm{d}x_1 \\
&+\int_{\lambda_1}^{\infty} g_1(x_1)\left(I_1(x_1) - I_1^*(x_1)\right) p_{X_1}(x_1)\mathrm{d}x_1 \\
&=\int_{\xi}^{\lambda_1} g_1(\lambda_1)\left(I_1(x_1) - I_1^*(x_1)\right) p_{X_1}(x_1)\mathrm{d}x_1 \\
&\quad+\int_{\lambda_1}^{\infty} g_1(x_1)\left(I_1(x_1) - I^*(x_1)\right) p_{X_1}(x_1)\mathrm{d}x_1, \xi \in (0, \lambda_1).
\end{aligned} \tag{5.1}$$

With a similar discussions as Proposition 3.1, we have $E\left[I_1^*(X_1)\right] = E\left[I_1(X_1)\right]$, which means that $\int_0^{\infty}\left(I_1(x_1) - I_1^*(x_1)\right) p_{X_1}(x_1)\mathrm{d}x_1 = 0$ and $\int_0^{\lambda_1}\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 = -\int_{\lambda_1}^{\infty}\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1$, thus (5.1) can be written as follows:

$$\begin{aligned}
&\int_0^{\lambda_1} g_1(x_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 + \int_{\lambda_1}^{\infty} g_1(\lambda_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 \\
&-\int_0^{\xi} g_1(\lambda_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 \\
&=\int_0^{\lambda_1} g_1(x_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 - \int_0^{\lambda_1} g_1(\lambda_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 \\
&\quad-\int_0^{\xi} g_1(\lambda_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 \\
&=\int_0^{\lambda_1}\left(g_1(x_1) - g_1(\lambda_1)\right)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1 \\
&\quad-\int_0^{\xi} g_1(\lambda_1)\left(I_1(x_1) - I_1^*(x_1)\right)\mathrm{d}x_1.
\end{aligned} \tag{5.2}$$

Then, we have $g_1(x_1) - g_1(\lambda_1) \le 0$ because $g_1(x_1)$ is increasing in x_1. Together with $I_1(x_1) - I_1^*(x_1) \ge 0$ when $x_1 \in (0, \lambda_1)$, we obtain (5.2) is negative. Then, we can similarly obtain that the last two terms of the formula are negative, which completes the proof. ■

The Proof of Proposition 4.2

Proof.
$T_{I^*} = X_1 + X_2 + Y - I_1^*(X_1) - I_2^*(X_2) + P_1 + P_2$ and $T_I = X_1 + X_2 + Y - I_1(X_1) - I_2(X_2) + P_1 + P_2$. Similar to Proposition 4.1, we only need to proof $E\left[T_{I^*} - l\right]_+ - E\left[T_I - l\right]_+ \le 0$ for all $l \in (0, \infty)$ with $I_1(x_1)$, $I_2(x_2) \in \mathcal{F}$.

$$
\begin{aligned}
&E\left[X_1+X_2+Y-I_1^*(X_1)-I_2^*(X_2)+P_1+P_2-l\right]_+ - E\left[X_1+X_2+Y-I_1(X_1)-I_2(X_2)+P_1+P_2-l\right]_+\\
&=E\left[E\left[\left\{X_1+X_2+Y-I_1^*(X_1)-I_2^*(X_2)+P_1+P_2-l\right\}_+ \mid X_1=x_1\right]\right]\\
&-E\left[E\left[\left\{X_1+X_2+Y-I_1(X_1)-I_2(X_2)+P_1+P_2-l\right\}_+ \mid X_1=x_1\right]\right]\\
&=E\left[E\left\{E\left(X_1+y+X_2-I_1^*(X_1)-I_2^*(X_2)+P_1+P_2-l\right)_+ \mid Y=y\right\}\mid X_1=x_1\right]\\
&\quad -E\left[E\left\{E\left(X_1+y+X_2-I_1(X_1)-I_2(X_2)+P_1+P_2-l\right)_+ \mid Y=y\right\}\mid X_1=x_1\right]\\
&=E\left[E\left\{\int_{l-P_1-P_2+X_1+y-I_1^*(X_1)-I_2^*(X_2)}^{\infty} S_{X_2|Y}(x_2)\mathrm{d}x_2 \mid X_1=x_1\right\}\right]-E\left[E\left\{\int_{l-P_1-P_2+X_1+y-I_1(X_1)-I_2(X_2)}^{\infty} S_{X_2|Y}(x_2)\mathrm{d}x_2 \mid X_1=x_1\right\}\right]\\
&=E\left[E\left\{\int_{l-P_1-P_2+X_1+y-I_1^*(X_1)-I_2^*(X_2)}^{l-P_1-P_2+X_1+y-I_1(X_1)-I_2(X_2)} S_{X_2|Y}(x_2)\mathrm{d}x_2 \mid X_1=x_1\right\}\right]\\
&\le E\left[E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\left[\left(I_1(x)-I_1^*(x)\right)+\left(I_2^*(X_2)-I_2(X_2)\right)\right]\mid X_1=x_1\right\}\right]\\
&=E\left[E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\left(I_1-I_1^*\right)\mid X_1=x_1\right\}\right]\\
&\quad +E\left[E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\left(I_2-I_2^*\right)\mid X_1=x_1\right\}\right]\\
&=E\left[E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\left(I_1-I_1^*\right)\mid X_1=x_1\right\}\right]\\
&\quad +E\left[E\left\{P\left(X_1>l-P_1-P_2-X_2-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\left(I_2-I_2^*\right)\mid X_2=x_2\right\}\right]\\
&=\int_0^{\infty}E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_1=x_1\right\}\left(I_1(x_1)-I_1^*(x_1)\right)p_{X_1}(x_1)\mathrm{d}x_1\\
&\quad +\int_0^{\infty}E\left\{P\left(X_1>l-P_1-P_2-X_2-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_2=x_2\right\}\left(I_2(x_2)-I_2^*(x_2)\right)p_{X_2}(x_2)\mathrm{d}x_2
\end{aligned}
$$

From the image of function $I_1^*(x)$ and $I_2^*(x)$, it can be seen that there must exist λ_1 and λ_2, $I_1(x_1)\le I_1^*(x_1)$ if and only if $x_1\in(0,\lambda_1)$, $I_1(x_1)\ge I_1^*(x_1)$ if and only if $x_1\in(\lambda_1,\infty)$. $I_2(x_2)\le I_2^*(x_2)$ if and only if $x_2\in(0,\lambda_2)$, $I_2(x_2)\ge I_2^*(x_2)$ if and only if $x_2\in(\lambda_2,\infty)$. Then, the last equation can be written as:

$$
\begin{aligned}
&\int_0^{\lambda_1}E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_1=x_1\right\}\left(I_1(x_1)-I_1^*(x_1)\right)\mathrm{d}x_1\\
&+\int_{\lambda_1}^{\infty}E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_1=x_1\right\}\left(I_1(x_1)-I_1^*(x_1)\right)\mathrm{d}x_1\\
&+\int_0^{\lambda_1}E\left\{P\left(X_1>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_2=x_2\right\}\left(I_2(x_2)-I_2^*(x_2)\right)\mathrm{d}x_2\\
&+\int_{\lambda_1}^{\infty}E\left\{P\left(X_1>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_2=x_2\right\}\left(I_2(x_2)-I_2^*(x_2)\right)\mathrm{d}x_2.
\end{aligned}
\tag{5.3}
$$

By observing (5.3), we obtain $P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)$ is increasing in y because $X_2+X_1\uparrow_{st}Y$, then by using corollary 2.2, $E\left\{P\left(X_2>l-P_1-P_2-X_1-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_1=x_1\right\}$ is decreasing in x_1 because $Y\downarrow_{st}X_1$. Similarly, $P\left(X_1>l-P_1-P_2-X_2-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)$ is increasing in y because $X_2+X_1\uparrow_{st}Y$, thus we have $E\left\{P\left(X_1>l-P_1-P_2-X_2-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_2=x_2\right\}$ is decreasing in x_2 together with $Y\downarrow_{st}X_2$. For a more concise expression, we record $E\left\{P\left(X_2>l-P_1-P_2-X_2-Y+I_1^*(X_1)+I_2^*(X_2)\mid Y=y\right)\mid X_1=x_1\right\}$ as $g_1(x_1)$, and record $E\left\{P\left(X_1 > l - P_1 - P_2 - X_1 - Y + I_1^*\left(X_1\right) + I_2^*\left(X_2\right) \mid Y = y\right) \mid X_2 = x_2\right\}$ as $g_2\left(x_2\right)$. Then, due to the monotonicity of $g_1\left(x_1\right)$, we use the second mean value theorem for integrals, the first two terms of (5.3) can be written as:

$$\begin{aligned}
&\int_0^{\lambda_1} g_1\left(x_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1+\int_{\lambda_1}^{\infty} g_1\left(x_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&=\int_0^{\lambda_1} g_1\left(x_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&\quad+\int_{\lambda_1}^{\xi_1} g_1\left(\lambda_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1,\ \xi_1\in\left(\lambda_1,\infty\right).
\end{aligned} \tag{5.4}$$

And by using $E\left[I_1^*\left(X_1\right)\right]=E\left[I_1\left(X_1\right)\right]$, we obtain $\int_0^{\infty}\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x=0$ and $\int_0^{\lambda_1}\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1=-\int_{\lambda_1}^{\infty}\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1$, thus (5.4) can be written as follows:

$$\begin{aligned}
&\int_0^{\lambda_1} g_1\left(\lambda_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1-\int_{\xi_1}^{\infty} g_1\left(\lambda_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&+\int_{\lambda_1}^{\infty} g_1\left(x_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&=-\int_{\lambda_1}^{\infty} g_1\left(\lambda_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1-\int_{\xi_1}^{\infty} g_1\left(\lambda_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&\quad+\int_{\lambda_1}^{\infty} g_1\left(x_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&=\int_{\lambda_1}^{\infty}\left(g_1\left(x_1\right)-g_1\left(\lambda_1\right)\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1\\
&\quad-\int_{\xi_1}^{\infty} g_1\left(\lambda_1\right)\left(I_1\left(x_1\right)-I_1^*\left(x_1\right)\right)\mathrm{d}x_1.
\end{aligned} \tag{5.5}$$

Thus, we have $g_1\left(x_1\right)-g_1\left(\lambda_1\right)\le 0$ because $g_1\left(x_1\right)$ is decreasing. Also, because $I_1\left(x_1\right)-I_1^*\left(x_1\right)\le 0$ when $x_1\in\left(0,\lambda_1\right)$ and $I_1\left(x_1\right)-I_1^*\left(x_1\right)\ge 0$ when $x_1\in\left(\lambda_1,\xi_1\right)$, we can conclude that (5.5) is negative. Then, continue to consider the last two terms of (5.3), similarly, we use the second mean value theorem for integrals because of the monotonicity of $g_2\left(x\right)$, so these two terms can be written as:

$$\begin{aligned}
&\int_0^{\lambda_2} g_2\left(x_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2+\int_{\lambda_2}^{\infty} g_2\left(x_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&=\int_0^{\lambda_2} g_2\left(x_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&\quad+\int_{\lambda_2}^{\xi_2} g_2\left(\lambda_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2,\ \xi_2\in\left(\lambda_2,\infty\right).
\end{aligned} \tag{5.6}$$

Then, by using the condition that $\int_0^{\lambda_2}\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2=-\int_{\lambda_2}^{\infty}\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2$, we obtain (5.6) is equal to:

$$\begin{aligned}
&\int_0^{\lambda_2} g_2\left(\lambda_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2-\int_{\xi_2}^{\infty} g_2\left(\lambda_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&+\int_{\lambda_2}^{\infty} g_2\left(x_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&=-\int_{\lambda_2}^{\infty} g_2\left(\lambda_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2-\int_{\xi_2}^{\infty} g_2\left(\lambda_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&\quad+\int_{\lambda_2}^{\infty} g_2\left(x_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&=\int_{\lambda_2}^{\infty}\left(g_2\left(x_2\right)-g_2\left(\lambda_2\right)\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2\\
&\quad-\int_{\xi_2}^{\infty} g_2\left(\lambda_2\right)\left(I_2\left(x_2\right)-I_2^*\left(x_2\right)\right)\mathrm{d}x_2.
\end{aligned} \tag{5.7}$$

Then, we have $g_2(x_2)-g_2(\lambda_2)\le 0$ because $g_2(x_2)$ is decreasing. And because $I_2(x_2)-I_2^*(x_2)\le 0$ when $x_2\in(0,\lambda_2)$, $I_2(x_2)-I_2^*(x_2)\ge 0$ when $x_2\in(\lambda_2,\infty)$, we can conclude that (5.7) is negative. So, through the discussions above, we draw the conclusion that (5.3) is negative, which means that $E\left[T_I^*-l\right]_+-E\left[T_I-l\right]_+\le 0$, therefore, when we adopt insurance strategies I_1^* and I_2^*, the total loss of the insurer reaches the minimum in the sense of stop-loss order, it means that I_1^* and I_2^* are our optimal insurance strategies under (4.1). ■