

Dimensional Dependence of Electrostatic Equilibrium in Conductors: A Critical Analysis

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Abstract

The study examines the electrostatic charge distribution in n -dimensional conductors. By generalizing Gauss's law and using hyperspherical coordinates, the ratio of electrostatic potential energies between a uniform distribution in the volume (W_{\ominus}) and a uniform distribution on the surface (W_{\circ}) is calculated. For $n \leq 2$, the two distributions are energetically equivalent, whereas for $n > 2$, the surface distribution is more favorable, with a ratio $W_{\ominus}/W_{\circ} = \lambda(n) = 2n/(n+2)$. The main conclusion is that the concentration of charge on the surface constitutes an energetically favorable state only in spaces with more than two dimensions, revealing a fundamental difference in the behavior of electrostatic systems depending on spatial dimensionality. This study, providing new theoretical insight into the dimensional dependence of electrostatic equilibrium, could have applications in higher-dimensional theories.

Keywords

n -Dimensional Conductors, Charge Distribution, Potential Energy, Hyperspherical Coordinates, Gauss's Law, Electrostatic Equilibrium, Dimensional Analysis

1. Introduction

It is well known that when a conductor carries an electric charge, this charge is not distributed throughout its entire volume. On the contrary, it concentrates uniformly on its surface, with a density that depends on the local geometry [1]-[3]. This happens because mutual repulsion drives charges as far apart as possible, which corresponds to the conductor's surface. Alternatively, this phenomenon can be explained by the fact that the system achieves greater energy stability. One way to quantitatively substantiate the above observation is through the calculation

and comparison of potential energies for two extreme cases: a) uniform distribution of charge on the surface and b) uniform distribution of charge throughout the entire volume. The subject of the present work is the investigation of charge distribution in similar conductor systems with different numbers of dimensions (one-dimensional, two-dimensional, four-dimensional, etc.).

It should be noted that there is a difference between a) the analysis of the energy of a circular or linear conductor in three-dimensional space and b) the analysis of a circle in an imaginary two-dimensional space or a straight segment in an imaginary one-dimensional space. The first case has been examined by [4]-[6]. In the present work, we will examine the second case, namely each shape will be considered in its corresponding imaginary dimension. Similarly, AL-Jaber [7] also deals with the second case using the approaches of [8] and [9]. He calculated the electrostatic energy of charged hyperspherical shells and spheres in n -dimensional space. He showed that the energy presents a minimum at dimension $n = 9$. He calculates the two energies separately but does not calculate their ratio.

In the present work, a direct comparison of stability is made through the ratio of energies. The critical dimension $n=2$ is highlighted, at which the nature of the comparison changes, and a physical interpretation is provided for the significance of this particular dimension, which is related to the divergence of integrals. Finally, for $n > 2$, a closed analytical formula is proposed: $\lambda(n) = 2n/(n+2)$.

The present work is structured in seven parts, including the introduction. In part 2, the potential energies of the system are examined. In part 3, the hyperspherical coordinate system is presented. In part 4, Gauss's law is formulated in n dimensions. In part 5, the enclosed charge is analyzed in all cases. In part 6, the relative ratio is calculated for spaces of (n) dimensions. Finally, in part 7, the conclusions of the work are presented.

2. Electrostatic Potential Energy of the System

The energy of any distribution (W) is given by the following equation ([1], p. 95):

$$W = \frac{\epsilon_0}{2} \iiint E^2 d\tau \quad (1)$$

where ϵ_0 , E , and $d\tau$ are the electric permittivity of vacuum (8.85×10^{-12} F/m), the electric field, and the differential volume element, respectively.

We denote by W_{\ominus} the potential energy of the system when the charge is uniformly distributed throughout its entire volume, and by W_{\circ} the potential energy of the system when the charge is uniformly distributed over its surface. By using Equation (1) for each of the two cases and integrating over all space, we obtain Equations (2) and (3):

$$W_{\ominus} = \frac{3}{20} \frac{q^2}{\pi R \epsilon_0} \quad (2)$$

$$W_{\circ} = \frac{1}{8} \frac{q^2}{\pi R \epsilon_0} \quad (3)$$

What is sought is the ratio W_{\ominus}/W_{\circ} as a function of the number of dimensions (n). For $n = 3$, the value of the ratio is given by Equation (4) and is equal to 1.2.

$$\frac{W_{\ominus}}{W_{\circ}} = \frac{\frac{3}{8} \frac{q^2}{\pi R \epsilon_0}}{\frac{1}{8} \frac{q^2}{\pi R \epsilon_0}} = \frac{6}{5} = 1.2 \quad (4)$$

To perform the necessary calculations, the well-known formulas of electrostatics, which provide the electric field and the potential in three dimensions, will need to be generalized to n dimensions. For this purpose, a hyperspherical coordinate system in n dimensions is initially required.

3. Hyperspherical Coordinate System

The hyperspherical coordinate system will consist of n coordinates. The first will be the distance from the origin, while the remaining $n - 1$ will be angles. Of these, the first angle will range from 0 to 2π , while the remaining $n - 2$ will range from 0 to π . For the integrals, the following relation will be used [10] and [11]:

$$d^n x = \prod_{k=1}^n dx_k = r^{n-1} dr \prod_{k=1}^{n-1} (\sin \theta_k)^{n-k-1} d\theta_k \quad (5)$$

This hyperspherical coordinate system, for $n = 3$, reduces to the well-known spherical coordinate system, while for $n = 2$ it reduces to the polar coordinate system.

4. Gauss's Law in n -Dimensions

In n dimensions, the surface has $(n - 1)$ dimensions, and Gauss's law is assumed as follows [7]:

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{in}}{\epsilon_0} \quad (6)$$

where:

$$d\vec{a} = \hat{r} r^{n-1} \prod_{k=1}^{n-1} (\sin \theta_k)^{n-k-1} d\theta_k \quad (7)$$

In every n -sphere, due to symmetry, $\vec{E} \nearrow \nearrow \hat{r}$, and therefore, from Equations (6) and (7), Equation (8) follows:

$$\begin{aligned} \oint E r^{n-1} \prod_{k=1}^{n-1} (\sin \theta_k)^{n-k-1} d\theta_k &= \frac{q_{in}}{\epsilon_0} \Rightarrow ER^{n-1} \oint \prod_{k=1}^{n-1} (\sin \theta_k)^{n-k-1} d\theta_k = \frac{q_{in}}{\epsilon_0} \\ \Rightarrow E &= \frac{q_{in}}{\epsilon_0 R^{n-1} \oint \prod_{k=1}^{n-1} (\sin \theta_k)^{n-k-1} d\theta_k} \Rightarrow E = \frac{q_{in}}{\epsilon_0 R^{n-1} p}, \end{aligned} \quad (8)$$

where $p = \oint \prod_{k=1}^{n-1} (\sin \theta_k)^{n-k-1} d\theta_k = \text{constant}$

To keep the presentation as simple as possible, we do not include the limits of integration in Equation (8).

5. Enclosed Charge in All Cases

It now remains to calculate the enclosed charge q_{in} in all cases as in **Table 1**. It is

observed that, whether the charge is distributed over the volume or on the surface, q_{in} is equal to the total charge of the sphere (Q) in both cases when the distance from the center of the sphere at which we calculate the electric field (r) is greater than the radius of the sphere R . For $r < R$, when the charge is only on the surface, there is no enclosed charge ($q_{in,O(r<R)} = 0$). Similarly, when the charge is uniformly distributed throughout the volume, the enclosed charge is nonzero.

Table 1. Enclosed charge in all cases.

| | Surface Charge | Volume Charge |
|---------|---------------------|--|
| $r < R$ | $q_{in,O(r<R)} = 0$ | $q_{in,\Theta(r<R)} = Q \frac{r^n}{R^n}$ |
| $r > R$ | $q_{in,O(r>R)} = Q$ | $q_{in,\Theta(r>R)} = Q$ |

6. Calculation of the Ratio for n -Dimensions

Taking into account Equations (5) and (8) and the data from **Table 1**, the ratio λ as a function of n is obtained:

$$\begin{aligned}
 \lambda(n) &= \frac{W_{\Theta}}{W_O} = \frac{W_{\Theta(r<R)} + W_{\Theta(r>R)}}{W_{O(r<R)} + W_{O(r>R)}} = \frac{W_{\Theta(r<R)} + W_{O(r>R)}}{0 + W_{O(r>R)}} = \frac{W_{\Theta(r<R)}}{W_{O(r>R)}} + 1 \\
 &= \frac{\frac{\epsilon_0}{2} \int E_{\Theta(r<R)}^2 d^n x}{\frac{\epsilon_0}{2} \int E_{(r<R)}^2 d^n x} + 1 \\
 &= \frac{\int \left(\frac{q_{in,\Theta(r<R)}}{\epsilon_0 r^{n-1} p} \right)^2 r^{n-1} dr \prod_{k=1}^{n-1} (\sin \theta_{\kappa})^{n-k-1} d\theta_{\kappa}}{\int \left(\frac{q_{in,(r>R)}}{\epsilon_0 r^{n-1} p} \right)^2 r^{n-1} dr \prod_{k=1}^{n-1} (\sin \theta_{\kappa})^{n-k-1} d\theta_{\kappa}} + 1 \\
 &= \frac{\int_0^R \left(\frac{Q \frac{r^n}{R^n}}{\epsilon_0 r^{n-1} p} \right)^2 r^{n-1} dr \int \prod_{k=1}^{n-1} (\sin \theta_{\kappa})^{n-k-1} d\theta_{\kappa}}{\int_R^{+\infty} \left(\frac{Q}{\epsilon_0 r^{n-1} p} \right)^2 r^{n-1} dr \int \prod_{k=1}^{n-1} (\sin \theta_{\kappa})^{n-k-1} d\theta_{\kappa}} + 1 \\
 &= \frac{\int_0^R \left(\frac{r}{R^n} \right)^2 r^{n-1} dr}{\int_R^{+\infty} \left(\frac{1}{r^{n-1}} \right)^2 r^{n-1} dr} + 1 \\
 &= \frac{\frac{1}{R^{2n}} \int_0^R r^2 r^{n-1} dr}{\int_R^{+\infty} \left(\frac{1}{r^{n-1}} \right)^2 r^{n-1} dr} + 1 \\
 &= \frac{\int_0^R r^{n+1} dr}{R^{2n} \int_R^{+\infty} r^{1-n} dr} + 1
 \end{aligned}$$

The integral of the numerator, for each value of n is equal to:

$$\int_0^R r^{n+1} dr = \frac{r^{n+2}}{n+2} \Big|_0^R = \frac{R^{n+2}}{n+2}$$

At this point, it is necessary to examine two cases for the integral of the denominator given by Equation (9):

$$\int_R^{+\infty} r^{1-n} dr \quad (9)$$

Case 1

If $n > 2 \Leftrightarrow 2-n < 0 \Leftrightarrow \lim_{r \rightarrow +\infty} (r)^{2-n} = 0$, the value of the antiderivative at infinity becomes zero, thus the integral converges:

$$\int_R^{+\infty} r^{1-n} dr = \frac{r^{2-n}}{2-n} \Big|_R^{+\infty} = \frac{\lim_{r \rightarrow +\infty} (r)^{2-n}}{2-n} - \frac{R^{2-n}}{2-n} = -\frac{R^{2-n}}{2-n} = \frac{R^{2-n}}{n-2}$$

Hence,

$$\lambda(n) = \frac{\int_0^R r^{n+1} dr}{R^{2n} \int_R^{+\infty} r^{1-n} dr} + 1 = \frac{\frac{R^{n+2}}{n+2}}{R^{2n} \frac{R^{2-n}}{n-2}} + 1 = \frac{\frac{R^{n+2}}{n+2}}{\frac{R^{2n} R^{2-n}}{n-2}} + 1 = \frac{n-2}{n+2} + 1 = \frac{2n}{n+2}$$

Therefore, for $n > 2$, the ratio of the two energies depends on the number of dimensions as shown in Equation (10):

$$\lambda(n) = \frac{2n}{n+2} \quad \forall n > 2 \quad (10)$$

Case 2

If $n \leq 2 \Leftrightarrow 2-n \geq 0 \Leftrightarrow \lim_{r \rightarrow +\infty} (r)^{2-n} = +\infty$, the value of the antiderivative at infinity tends to infinity and the integral in the denominator diverges; therefore, the whole fraction tends to zero and $\lambda(n) = 1$.

$$\begin{aligned} \lambda(n) &= \lim_{r \rightarrow +\infty} \frac{\int_0^R (r')^{n+1} dr'}{R^{2n} \int_R^r (r')^{1-n} dr'} + 1 \\ &= \lim_{r \rightarrow +\infty} \frac{\frac{R^{n+2}}{n+2}}{R^{2n} \int_R^r (r')^{1-n} dr'} + 1 \\ &= \lim_{r \rightarrow +\infty} \frac{\frac{R^{2-n}}{n+2}}{\int_R^r (r')^{1-n} dr'} + 1 \\ &= \lim_{r \rightarrow +\infty} \frac{\frac{R^{2-n}}{n+2}}{\int_R^r (r')^{1-n} dr'} + 1 = 1 \end{aligned}$$

Because

$$\lim_{r \rightarrow +\infty} \int_R^r (r')^{1-n} dr' = \lim_{r \rightarrow +\infty} \frac{(r')^{2-n}}{2-n} \Big|_R^r = \lim_{r \rightarrow +\infty} \frac{(r)^{2-n}}{2-n} - \frac{R^{2-n}}{2-n} = +\infty$$

The two cases for $n \leq 2$ and $n > 2$ are illustrated in **Figure 1**.

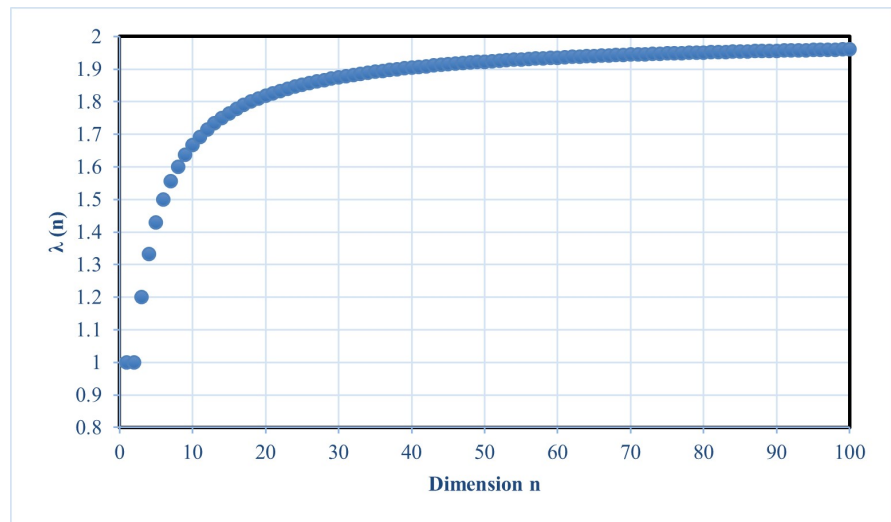


Figure 1. The ratio $\lambda(n) = W_{\Theta} / W_O$.

As mentioned in the introduction, Al-Jaber [7] studied the electrostatic energy of hyperspherical shells and uniformly charged spheres in n dimensions. He observed that both energies exhibit a minimum at $n = 9$. As already noted, the present study differs because it examines the ratio W_{Θ} / W_O for the same system (a sphere with different distributions). This explains why the critical dimension in the present analysis is $n = 2$ (where the integral changes its convergence behavior) rather than $n = 9$ (where the geometry is optimized). The theoretical implication is that the energy ratio function behaves fundamentally differently from the functions giving absolute energy values in higher dimensions.

7. Conclusions

The present study calculated the ratio of electrostatic energies $\lambda(n) = W_{\Theta} / W_O$ between a uniform volume charge distribution and a uniform surface charge distribution of a hyperspherical conductor in n -dimensional space. By generalizing Gauss's law and using hyperspherical coordinates, a two-branched analytical expression was derived, revealing a difference in the behavior of electrostatic systems depending on the dimensionality of space.

The main findings of the study can be summarized as follows:

a) For $n \leq 2$, the ratio is $\lambda(n) = 1$, indicating that the two charge distributions exhibit the same energetic stability. This result arises from the divergence of the integral when $n \leq 2$, which mathematically eliminates the difference between the two distributions, because when both total energies are infinite, their difference (which remains finite, being equal to the electrostatic energy inside the sphere) becomes negligible relative to their magnitude, leading to energetic equivalence. In one- and two-dimensional space, the concept of a "surface" distribution loses its energetic advantage.

b) For $n > 2$, the ratio is given by the closed-form expression $\lambda(n) = 2n/(n+2)$, which is always greater than unity. This demonstrates that the surface distribution is energetically more favorable in spaces with more than two dimensions.

Applying Equation (10), three characteristic cases can be distinguished:

For $n = 3$ (our physical space): $\lambda(3) = 6/5 = 1.2$

For $n = 4$: $\lambda(4) = 4/3 \approx 1.33$

For $n \rightarrow \infty$: $\lambda(n) \rightarrow 2$

The result $\lambda(n) \rightarrow 2$ in the limit of many dimensions indicates that the surface distribution remains consistently more favorable, but the difference does not grow without bound. The energetic stability of the surface distribution relative to the volume distribution reaches an asymptotic limit, regardless of how complex the space becomes.

The critical dimension $n = 2$ expresses a fundamental property of the electrostatic field: only in spaces with $n > 2$ does the field of the surface and volume distributions of the sphere, as well as the field of a point source, decay rapidly enough for the total energy to be finite at infinity. In one or two dimensions, the field decays more slowly, and the electrostatic energy diverges, rendering the two distributions energetically equivalent.

The present analysis could be extended to: a) deriving explicitly the integral form of Gauss's law, Equation (6), using generalizations of the divergence theorem and Maxwell's first equation to show how these laws remain invariant in n -dimensions; b) deriving Equations (5) and (7) using the findings of [10] and [11]; c) non-uniform charge distributions in higher dimensions; d) dynamic problems and time evolution in n -dimensional space; e) connections with quantum field theory; and f) generalization to non-Euclidean geometries (hyperbolic or spherical space).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Griffiths, D.J. (2013) Introduction to Electrodynamics. 4th Edition, Pearson.
- [2] Jackson, J.D. (1999) Classical Electrodynamics. 3rd Edition, Wiley.
- [3] Purcell, E.M. and Morin, D.J. (2013) Electricity and Magnetism. 3rd Edition, Cambridge University Press. <https://doi.org/10.1017/cbo9781139012973>
- [4] Griffiths, D.J. and Li, Y. (1996) Charge Density on a Conducting Needle. *American*

- Journal of Physics*, **64**, 706-714. <https://doi.org/10.1119/1.18236>
- [5] Friedberg, R. (1993) Solitons in Physics. *American Journal of Physics*, **61**, 1084-1091.
- [6] Good, R.H. (1997) Comment on "Charge Density on a Conducting Needle," by David J. Griffiths and Ye Li [am. J. Phys. **64**(6), 706-714 (1996)]. *American Journal of Physics*, **65**, 155-156. <https://doi.org/10.1119/1.18786>
- [7] ALJaber, S.M. (2010) Multidimensional Electrostatic Energy and Classical Renormalization. *Natural Science*, **2**, 760-763. <https://doi.org/10.4236/ns.2010.27095>
- [8] Corbò, G. (2009) Renormalization in Classical Field Theory. *European Journal of Physics*, **31**, L5-L8. <https://doi.org/10.1088/0143-0807/31/1/l02>
- [9] Tort, A.C. (2010) Another Example of Classical Renormalization in Electrostatics. *European Journal of Physics*, **31**, L49-L50. <https://doi.org/10.1088/0143-0807/31/2/l02>
- [10] Shapiro, J.A. (2012). Hyperspherical Coordinates (in N Dimensions). Rutgers University. <https://www.physics.rutgers.edu/~shapiro/618/lects/hypersph.pdf>
- [11] Blumenson, L.E. (1960) A Derivation of N-Dimensional Spherical Coordinates. *The American Mathematical Monthly*, **67**, 63-66. <https://doi.org/10.2307/2308932>