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Simplicial Complexes Which Are Minimal Cohen-Macaulay

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Abstract

Let Δ be a $(d-1)$-dimensional pure f-simplicial complex over vertex set $[n]$. In this paper, it is proved that Δ being minimal CM implies $d \geq 3$ and $n = 2d$. It is also indicated that shellable condition on a pure simplicial complex Δ is identical with existence of a full series of CM subcomplexes of Δ.

Keywords

Simplicial Complex, Cohen-Macaulay, Pure Shellable, f-Simplicial Complex

1. Introduction

Throughout, for a natural number n and $1 < d < n$, let

$$[n] =: \{1, 2, \dots, n\},\ [n]_d =: \{A \in 2^{[n]} \mid \ |A| = d\},$$

where $2^{[n]}$ is the power set of $[n]$. A simplicial complex Δ over a vertex set $[n]$ is a subset of $2^{[n]}$, which has the hereditary property under inclusion and is such that $\{i\} \in \Delta$ holds for all $i \in [n]$. Recall that a facet of Δ is a maximal element with respect to inclusion, and the facet set of Δ is denoted as $\mathcal{F}(\Delta)$. The dimension $\dim \Delta$ of Δ is the maximal number $|F| - 1$, where $F \in \mathcal{F}(\Delta)$ runs over all facets of Δ. If $\dim \Delta$ equals to $|F| - 1$ for each facet F, then Δ is said to be *pure*. Let $\Delta^{(i)} = \{F \in \Delta \mid |F| \leq i + 1\}$ be the i'th skeleton of Δ.

Cohen-Macaulay (abbreviated as CM) property is one of the central research topics in commutative algebra and the rich and deep homological achievements have fruitful applications in combinatorial aspects of commutative rings. In combinatorial commutative algebra, shellable and pure simplicial complexes are the main source of CM simplicial complexes. In a most recent work, Dao, Doolittle and Lyle in reference [1] discovered a new important combinatorial property of a CM simplicial complex Δ, i.e., $\Delta_F \cap \langle F \rangle$ *is pure of dimension* $|F| - 2$ for any facet F of Δ, where $\mathcal{F}(\Delta_F) = \mathcal{F}(\Delta) \smallsetminus \{F\}$. Based on the property, the notion of a minimal CM simplicial complex Δ is introduced and studied. To be more precisely, Δ is called minimal CM if Δ is CM but no Δ_F is CM for any facet F of Δ. Acyclic behavior of a minimal CM Δ is studied and, sufficient conditions are provided for a complex to be minimal CM . Many interesting examples of minimal CM complexes are also exhibited. Recall also from Zheng in reference [2] the other important combinatorial property of a CM simplicial complex, i.e., *CM simplicial complexes are connected in codimension one*, i.e., for any distinct facets F and G, there is a sequence $F = F_0, F_1, \ldots, F_r = G$ of facets such that $|F_i \cap F_{i+1}| = |F_{i+1}| - 1$ for all $i = 0, 1, \ldots, r-1$. For other properties of CM simplicial complexes as well as CM rings, refer also to references [3, 4, 5, 6, 7, 8].

In this paper, we use Lemma 3.1 of reference [1], to study the exact relation of shellable and CM properties for a pure complex Δ, and we study the condition for a minimal CM f-simplicial complex to be acyclic. In Section 2, we recall some work of [1] and, give a brief survey on f-simplicial complexes. In Section 3, we first indicate that shellable condition on a pure complex Δ is identical with CM properties of a full series of subcomplexes of Δ, and then we use this observation to construct nontrivial examples of pure shellable complexes by taking advantage of CoCoA in an algorithmic approach, after applying Eagon-Reiner theorem. In Section 4, we compute the dimension of the subspace $\ker(\partial_r)$ in a reduced chain complex of the simplex $\langle [n] \rangle$, and apply it to deduce that a minimal CM f-simplicial complex exists in set $[n]_d$ implies $n = 2d$.

2. Preliminaries

For a $(d-1)$-dimensional simplicial complex Δ, there is a related chain complex of $\mathfrak{K}$-spaces:

$$\mathcal{C}: \ 0 \longrightarrow C_{d-1} \xrightarrow{\partial_{d-1}} C_{d-2} \xrightarrow{\partial_{d-2}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where C_i is a free $\mathfrak{K}$-module with basis set $\{\sigma \in \Delta \mid |\sigma| = i+1\}$, while for any $1 \leq k_1 < k_2 < \cdots < k_{r+1} \leq n$,

$$\partial_r(k_1 k_2 \ldots k_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i-1} k_1 \ldots k_{i-1} \hat{k_i} k_{i+1} \ldots k_{r+1}.$$

For each i, recall that $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$ holds, and the quotient $\mathfrak{K}$-space

$$\tilde{H}_i(\Delta) =: \ker \partial_i / \operatorname{im} \partial_{i+1}$$

is called the i^{th} homology group of Δ. If $\tilde{H}_i(\Delta) = 0$ holds for all i, then Δ is said to be *acyclic*. Clearly, Δ is acyclic if and only if the corresponding chain complex $\mathcal{C}$ is an exact sequence. Recall that a cone is always acyclic (see, e.g., [6]), where Δ is called a cone if there exists a vertex such that all facets contain it as an element, and note that $\dim \tilde{H}_0(\Delta) + 1$ is the number of connected components of Δ, see, e.g., Proposition 5.2.3 of reference [6].

Throughout, let $\mathfrak{K}$ be a field and let $S = \mathfrak{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathfrak{K}$. Throughout, unless otherwise specifically stated, let Δ be a $(d-1)$-dimensional pure simplicial complex with vertex set $[n]$, where $\Delta \neq \emptyset, \{\emptyset\}$ and Δ is not a simplex.

We first recall some work of Proposition 5.2.3 [1] on minimal Cohen-Macaulay simplicial complexes. For a facet F of a simplicial complex Δ, let

$$\Delta_F = \langle G \mid G \in \mathcal{F}(\Delta) \setminus \{F\} \rangle.$$

Δ is called a *shelling move* of Δ_F if $\Delta_F \cap \langle F \rangle$ is pure of codimension 1, i.e., $\Delta_F \cap \langle F \rangle$ is generated by some nonempty subset of ∂F.

Lemma 1. ([1, Lemma 3.1]) If a simplicial complex Δ is CM with $|\mathcal{F}(\Delta)| \geq 2$, then Δ is a shelling move of Δ_F for any facet F of Δ.

A CM simplicial complex Δ is called *minimal* CM, if either it is a simplex, or else $|\mathcal{F}(\Delta)| \geq 3$ and, no Δ_F is CM for any facet F of Δ. Here the definition of minimal CM is slightly different from that of the work [1], since we are mainly interested in the nonempty simplicial complexes. Then, the shelling move property implies the following:

Theorem 2. ([1, Theorem 3.2]) Let Δ be a $(d-1)$-dimensional CM simplicial complex, which is not minimal. Then there exists a minimal CM subcomplex Γ and a series of facets $F_j, \ldots, F_1$ of Δ, such that each $\Gamma \cup \langle F_i, \ldots, F_1 \rangle$ is CM and, each $\Gamma \cup \langle F_1, \ldots, F_{i+1} \rangle$ is a shelling move of $\Gamma \cup \langle F_1, \ldots, F_i \rangle$.

In reference [1], Δ is said to be *shelled over* Γ. Clearly, shelled over is a kind of generalization of shellable for a simplicial complex.

Next, we record the following result, which is needed in this paper:

Theorem 3. ([1, Theorem 3.4]) Let Δ be a simplicial complex with dimension $d-1$, and assume $\tilde{H}_{d-1}(\Delta) \neq 0$. Then there is a maximal facet F of Δ such that the following hold:

(1) $\dim \tilde{H}_{i-1}(\Delta_F) = \begin{cases} \dim \tilde{H}_{i-1}(\Delta), & \text{if } 0 \leq i < d \\ \dim \tilde{H}_{i-1}(\Delta) - 1, & \text{if } i = d \end{cases}$.

(2) $f_{k-1}(\Delta_F) = \begin{cases} f_{k-1}(\Delta), & \text{if } 0 \leq i < d \\ f_{k-1}(\Delta) - 1, & \text{if } i = d \end{cases}$.

(3) $\operatorname{depth} \Delta = \operatorname{depth} \Delta_F$.

Surely, this theorem together with Reisner theorem imply that a minimal CM simplicial complex is acyclic.

Corollary 1. Let Δ be a minimal CM complex over vertex set $[n]$ with dimension $d-1$. Then $\tilde{H}_{d-1}(\Delta_F) = 0$ holds true for any facet F of Δ.

Proof. This is an immediate consequence of Lemma 1, Theorem 3 and, [9, Theorem 25.1] on page 142. □

Now we give a brief survey on some related established results on f-simplicial complexes. For any square-free monomial ideal I of S, let $G(I)$ be set of minimal monomial generators, and let $\operatorname{sm}(I)$ be set of square-free monomials. For the ideal I, recall that there exist two related simplicial complexes, i.e., the nonface simplicial complex

$$\delta_{\mathcal{N}}(I) =: \{\, F \in 2^{[n]} \mid X_F \in \operatorname{sm}(S) \setminus \operatorname{sm}(I) \,\}$$

of I and the facet simplicial complex $\delta_{\mathcal{F}}(I) =: \langle\, F \in 2^{[n]} \mid X_F \in \mathrm{G}(I) \,\rangle$ of the clutter $G(I)$. If they possess a same f-vector, then the ideal I is called an *f-ideal*. For a simplicial complex Δ, if its facet ideal $I(\Delta) =: \langle\, \{X_F \mid F \in \mathcal{F}(\Delta)\} \,\rangle$ is an f-ideal, then Δ is called an *f-simplicial complex*. A graph G is said to be an f-graph, if the edge ideal $I(G)$ is an f-ideal. Note that in defining an f-graph G, G is regarded as a simplicial complex of dimension no more than 1, although we do have $I(G) = I_{\mathrm{Ind}(G)}$, where $\mathrm{Ind}(G)$ is the independence simplicial complex of the graph G. Refer to references [10, 11, 12, 13, 14] for further related studies.

For a simplicial complex Δ on the vertex set $[n]$, let

$$\Delta^c =: \langle\, \{F \mid F^c \in \mathcal{F}(\Delta)\} \,\rangle \overset{i.e.}{=} \langle\, \{[n] \setminus G \mid G \in \mathcal{F}(\Delta)\} \,\rangle.$$

The definition of an f-simplicial complex seems to be reasonable with hindsight, due to the following two theorems on f-ideals.

Theorem 4. ([15, Theorem 2.3]) Let $S = K[x_1, \ldots, x_n]$, and let I be a square-free monomial ideal of S with the minimal generating set $G(I)$, where all monomials of $G(I)$ have a same homogeneous degree d. Then I is an f-ideal if and only if, set $G(I)$ is an LU-set and, $|G(I)| = \frac{1}{2}\binom{n}{d}$ holds true.

Note that $G(I)$ is said to be an L-set (U-set, respectively) if set of all degree $d-1$ factors of elements of $G(I)$ has exactly $\binom{n}{d-1}$ elements (respectively, set of degree $d+1$ square-free monomials extended from elements of $G(I)$ has cardinality $\binom{n}{d+1}$). If $G(I)$ is both a U-set and an L-set, then $G(I)$ is an LU-set. With the bijection from X^α to $\{i \in [n] \mid i \in \alpha\}$ (e.g., $x_1x_3x_4 \mapsto \{1, 3, 4\}$), one obtains the notion of an L-set (U-set, LU-set respectively) for the facet set $\mathcal{F}(\Delta)$.

Recall also the following recently discovered result:

Theorem 5. ([16, Theorem 4.1]) Δ is an f-simplicial complex, if and only if Δ^c is an f-simplicial complex.

Equivalently, a square-free monomial ideal I of S is an f-ideal if and only if the Newton complement dual ideal $\hat{I} = \langle x_1x_2\cdots x_n/u \mid u \in G(I)\rangle$ of I is an f-ideal.

It is clear that Theorem 5 follows easily from Theorem 4 for a *pure* simplicial complex Δ.

Recall that for an f-graph G, it is proved that the complement graph $\overline{G}$ is bipartite, thus $\mathrm{Ind(G)} = \overline{G}$. Recall that all f-graphs are pure shellable as a graph, i.e., the independence complex Ind(G) is pure and shellable ([15, Theorem 6.5]), while the definition of an f-graph is actually an f-simplicial complex of dimension less than or equal to 1. Thus , Theorem 6.5 of [15] may be re-stated as the following:

If Δ is an f-simplicial complex of dimension less than or equal to 1, then the homogeneous complement simplicial complex $\Delta' =: \langle\, [n]_2 \setminus \mathcal{F}(\Delta)\, \rangle$ is pure shellable.

Based on this observation, it is natural to ask the following question:

Question 6. For a pure f-simplicial complex Δ of dimension $d-1$, is the homogeneous complement simplicial complex $\Delta' =: \langle\, \sigma \mid \sigma \in [n]_d \setminus \mathcal{F}(\Delta)\, \rangle$ of Δ shellable?

We do not know counterexample in set $[5]_3$ and in $[6]_3$. But it fails in set $[8]_4$. We will give a negative answer in Example 11.

We remark that there exist a lot of pure f-simplicial complexes which are not CM when $d - 1 \geq 2$.

Finally, we claim that there exist f-simplicial complexes which are minimal CM:

Example 7. ([17, 14]) Consider a simplicial complex Δ with facet set

$$\mathcal{F}(\Delta) = \{123, 125, 136, 145, 146, 234, 246, 256, 345, 356\}$$

constructed in reference [17]. It is noticed in reference [14] that Δ (hence, Δ^c) is an f-simplicial complex. Then we take advantage of Eagon-Reiner theorem (see, e.g., [8, Theorem 8.1.9]) and CoCoA ([18]) to check that both simplicial complex Δ and its complement Δ^c are minimal CM. In particular, neither Δ nor Δ^c is shellable, which is hard to check without the notion of minimal CM (refer to [17, Example 7.7] for a general theoretical treatment).

Note that *a permutation on set* $[6] =: \{n \mid 1 \leq n \leq 6\}$ *may produce a new simplicial complex* Δ_1, which has the same property with Δ. For example, the cyclic permutation $(1, 2, 3, 4, 5, 6)$ acts on $\mathcal{F}(\Delta)$ and produces

$$\Delta_1 = \langle\, 234, 236, 124, 256, 125, 345, 135, 136, 456, 146\, \rangle.$$

Clearly, both Δ_1 and Δ_1^c are f-simplicial complexes and minimal CM.

3. Pure Shellable versus Cohen-Macaulay

We begin with the following immediate consequence of Theorem 4:

Corollary 2. Let Δ be a $(d-1)$-dimensional pure f-simplicial complex with vertex set $[n]$. Then we have

$$\operatorname{depth}(\Delta) = \begin{cases} d, & \text{if } \Delta \textit{ is } CM \\ d-1, & \text{if } \Delta \textit{ is not } CM. \end{cases}$$

Proof. Assume that Δ is not CM. Since Δ is a $(d-1)$-dimensional pure f-simplicial complex, $\mathcal{F}(\Delta)$ is an L-set, thus $\Delta^{(d-2)} = [n]_{d-1}$ holds. Clearly, $[n]_{d-1}$ is pure shellable, thus it is CM. Then the result follows from the fact that

$$\operatorname{depth}(\Delta) = 1 + \max\{i \mid \text{the i'th skeleton } \Delta^{(i)} \text{ is CM.}\}. \qquad \square$$

If the minimal subcomplex Γ in Theorem 2 is a simplex, say, $\Gamma = \langle F_0 \rangle$, then it follows by Lemma 1 that the following is a shelling of Δ, thus Δ is shellable:

$$F_0, F_1, F_2, \ldots, F_j.$$

To be more precisely, we have the following observation, which essentially should be dedicated to the authors of reference [1]:

Theorem 8. For a pure simplicial complex Δ, the following statements are equivalent:

(1) Δ is shellable.

(2) There exists a full sequence of subcomplexes Δ_i such that all Δ_i are CM, i.e., there is a total order $F_j, F_{j-1}, \ldots, F_1, F_0$ of all facets of Δ such that each $\Delta_i =: \langle F_0, F_1, \ldots, F_i \rangle$ is CM for $j \ge i \ge 1$, or equivalently, each ideal $I(\Delta_i^c)$ has a linear resolution.

Proof. $(1) \Longrightarrow (2)$: Let $F_0, F_1, \ldots, F_j$ be a shelling of Δ and let $\Delta_i = \langle F_0, F_1, \ldots, F_i \rangle$. Then for any i with $1 \le i \le j$, $\Delta_i =: \langle F_0, F_1, \ldots, F_i \rangle$ is pure and shellable, thus is CM.

$(2) \Longrightarrow (1)$: Let $F_j, F_{j-1}, \ldots, F_1, F_0$ be a full sequence of facets of Δ such that each $\Delta_i =: \langle F_0, F_1, \ldots, F_i \rangle$ is CM for $j \ge i \ge 1$. Then by Lemma 1,

$$\Delta_{j-1} \cap \langle F_j \rangle, \Delta_{j-2} \cap \langle F_{j-1} \rangle, \ldots \Delta_1 \cap \langle F_1 \rangle$$

are all pure of dimension $\dim \Delta - 1$. By definition, $F_0, F_1, \ldots, F_j$ is a shelling of Δ, thus Δ is a shellable simplicial complex. The rest statement follows from Eagon-Reiner theorem, and is convenient for checking by applying CoCoA. $\square$

Clearly, Theorem 8 shows the exact relation between the conditions of shellable and CM for a pure simplicial complex. It also exhibits the importance of Lemma 1.

As is well-known, it is in general a hard work to check if a pure simplicial complex is shellable. It seems that the new concept shelled over could open an algorithmic gate on attacking this problem, based on the algebraic characterization of a CM simplicial complex by Eagon-Reiner theorem. Refer to Examples 7, 10 and 11 for concrete operations and calculations.

When considering the condition of connected in codimension 1 (see, e.g., Proposition 1.12 of reference [2]), we have the following easy observation:

Proposition 9. Let Δ be a pure simplicial complex of dimension $d-1$, which is not a simplex. Consider the following conditions:

(1) For each face σ of Δ such that $\dim \operatorname{lk}_\Delta(\sigma) > 0$, $\operatorname{lk}_\Delta(\sigma)$ is connected.

(2) For each facet F of Δ, Δ is a shelling move of Δ_F.

Then (1) implies (2).

Proof. This follows from the proof of Lemma 3.1 of reference [1]. □

The converse does not hold true in general. For example, the simplicial complex $\Delta =: \langle 1234, 1235, 1278, 1279\rangle$ is a shelling move over Δ_F for each facet F, but $\mathrm{lk}_\Delta(12) = \langle 34, 35, 78, 79\rangle$ and it is disconnected.

For a simplicial complex Δ, recall from Lemma 1.5.3 of reference [8] that $I_{\Delta^\vee} = I(\Delta^c)$ holds, where $\Delta^c = \langle\, [n] \setminus F \mid F \in \mathcal{F}(\Delta)\,\rangle$ and, $\Delta^\vee$ is the Alexander dual complex of Δ. Recall that Δ is said to be CM, if the Stanley-Reisner ideal I_Δ of Δ is CM. Recall also the Eagon-Reiner theorem (see, e.g., Theorem 8.1.9 of reference [8]), i.e., a simplicial complex Δ is CM if and only if the Stanley-Reisner ideal $I_{\Delta^\vee}$ of $\Delta^\vee$ has a linear resolution . Thus Δ is CM if and only if the monomial ideal $I(\Delta^c)$ has a linear resolution. These results together with CoCoA are crucial to our next work.

In the following, we consider simplicial complexes of kind $(8, 4)$ and apply Theorem 8 and Example 7 to the following construction:

Example 10. We start from set

$$A = \{1345, 1347, 1358, 1367, 1368, 1456, 1468, 1478, 1567, 1578\}.$$

By Example 7, it generates a minimal CM simplicial complex Γ, where

$$\Gamma = \langle\, \{F \mid F \in A\}\,\rangle.$$

(1) Let

$$B = \{1234, 1235, 1246, 1258, 1357, 1458, 1568, 2345, 2347, 2346, 2356, 2457,$$

$$2468, 2578, 2678, 2467, 2456, 2567, 2367, 3456, 3478, 3678, 4567, 4578, 4678\},$$

and let $\Delta_1 = \langle \{F \mid F \in A \cup B\rangle$. Then it is checked that Δ_1 is a CM simplicial complex via CoCoA (reference [18]), and the following sequence

$$F_{25} = 1246, 1258, 1235, 1357, 3478, 1568, 1234, 2345, 2356, 2457, 2468, 2578,$$

$$2678, 2567, 2456, 3678, 3456, 2346, 1458, 4578, 4567, 2367, 2467, 4678, 2347$$

of facets are found to make the CM simplicial complex Δ_1 shelled over the minimal CM simplicial complex Γ, where each of the 24 monomial ideals

$$I(\langle(\Delta_1^c) \setminus \{F_{25}^c\}\rangle), I(\langle \mathcal{F}(\Delta_1^c) \setminus \{F_{25}^c, F_{24}^c\}\rangle), \ldots, I(\langle \mathcal{F}(\Delta_1^c) \setminus \{F_{25}^c, \ldots, F_2^c\}\rangle)$$

is tested via CoCoA to have linear resolution. Note that $F_1 = 2347, F_2 = 4678$, and so on.

Note that Δ_1 is not an f-simplicial complex since 127 is not in the lower set of $\mathcal{F}(\Delta_1)$, i.e., $\mathcal{F}(\Delta_1)$ is not an L-set. We do not know if Δ_1 is shellable.

(2) Inspired by the previous construction, we now construct an f-simplicial complex

$$\Delta_2 = \langle\, \{F \mid F \in A \cup C\}\,\rangle,$$

where

$$C = \{1235, 1236, 1237, 1247, 1268, 1258, 1357, 1458, 1568, 2345, 2347, 2348,$$

$$2356, 2457, 2467, 2468, 2578, 2678, 3456, 3467, 3478, 3678, 4567, 4578, 4678\}.$$

We checked that $A \cup C$ is an LU-set, thus Δ_2 is indeed an f-simplicial complex. We checked that Δ_2 is CM via CoCoA. Furthermore, we claim that the complex Δ_2 is shellable with the shelling $F_0, F_1, \ldots, F_{34}$, where

$$F_{34} = 1247, F_{33} = 1237,\ 1357, 1358, 1368, 2347, 2348, 2356, 1236, 1235, 1268,$$

$$1258, 1345, 1347, 1367, 1567, 1568, 1578, 1458, 1456, 1468, 1478, 3478, 2345,$$

$$2457, 2467, 2468, 2578, 2678, 3456, 3467, 3678, 4567, F_1 = 4678, F_0 = 4578.$$

In fact, we use CoCoA to show that each $I(\langle F_0^c, \ldots, F_i^c \rangle)$ has a linear resolution ($\forall\, 34 \geq i \geq 1$), thus

$$\langle F_0, F_1, \ldots, F_i \rangle$$

is a Cohen-Macaulay simplicial complex for each i. Then it follows from Theorem 8 that Δ_2 is shellable with

$$F_0, F_1, \ldots, F_{34}$$

as a shelling.

It is natural to ask if Δ_2 is shelled over Γ constructed in (1) of this example? We tried this via CoCoA, and the answer is yes. The following sequence

$$G_{25} = 1247, G_{24} = 1357, 1237, 2347, 2348, 2345, 2356, 1235, 1236,$$

$$1268, 1258, 2578, 2678, 2468, 2467, 2457, 3478, 3678,$$

$$1568, 4678, 3456, 3467, 4567, G_2 = 4578, G_1 = 1458$$

of facets is found to make the CM simplicial complex Δ_2 shelled over the minimal CM simplicial complex Γ, where all the 24 facet monomial ideals $I(\langle \mathcal{F}(\Delta_2^c) \setminus \{G_{25}^c, \ldots, G_r^c\} \rangle)$ are tested to have linear free resolutions.

Note that for the same CM simplicial complex Δ_2, the first minimal CM subcomplex is $\langle F_0 \rangle$ and it has only *one* facet, while the second minimal CM subcomplex is Γ and it has *ten* facets. This is the end of Example 10.

Finally, note that Example 7 provides a *very well-distributed* simplicial complex, i.e., each number r in set $[6]$ appears 5 times in the facets. Motivated by this observation, we now construct a very well-distributed simplicial complex whose facet set contains 34 elements in set $[8]_4$.

Example 11. Let $\mathcal{F}(\Delta^c)$ be set D consisting of 34 elements, where

$$D = \{1234, 1235, 1246, 1247, 1258, 1345, 1358, 1367, 1368, 1378, 1456,$$

$$1457, 1467, 1478, 1568, 1578, 1678, 2346, 2348, 2356, 2358, 2367, 2378,$$

$$2456, 2467, 2468, 2478, 2567, 2678, 3456, 3457, 3568, 3578, 4578\}.$$

We checked the following:

(1) D is very well-distributed, i.e., it has type $1^{17}2^{17}3^{17}4^{17}5^{17}6^{17}7^{17}8^{17}$.

(2) D is an LU-set over set $[8]$, so that adding any element from set $[8]_4 \setminus D$ can generate a pure f-simplicial complex Γ^c.

(3) The ideal $I(\Delta^c)$ has the following linear resolution, thus Δ is not CM:

$$0 \longrightarrow R(-8)^2 \longrightarrow R(-7)^{21} \oplus R(-8) \longrightarrow R(-6)^{68} \oplus R[-7]^2$$

$$\longrightarrow R(-5)^{81} \oplus R[-6] \longrightarrow R(-4)^{34} \longrightarrow R.$$

Note that Δ^c has the same properties.

Among the 36 f-simplicial complexes Γ obtained in (2), 7 are CM. In fact, F_{35}^c can be chosen as anyone of the following:

$$1236, 1238, 1268, 1346, 1348, 1468, 2368$$

such that $I(\langle D^c \cup \{F_{35}^c\}\rangle)$ has a linear resolution . Note that unfortunately, none of the 7 CM simplicial complexes is minimal CM. Furthermore, all Γ are shelling moves of Γ_{F_2}, where $F_2^c = 1235$.

Finally, we consider set $D' =: [8]_4 \setminus D$, which consists of 36 elements, as follows:

$$D' = \{1236, 1237, 1238, 1245, 1248, 1256, 1257, 1267, 1268, 1278, 1346, 1347, \}$$

$$1348, 1356, 1357, 1458, 1468, 1567, 2345, 2347, 2357, 2368, 2457, 2458,$$

$$2568, 2578, 3458, 3467, 3468, 3478, 3567, 3678, 4567, 4568, 4678, 5678\}.$$

Deleting any element will result in a homogeneous complement of some Γ^c in (2). We use CoCoA to calculate the 36 $I((\Gamma^c)')$ and, find 15 CM simplicial complexes Γ'. The following are all elements when one of which is deleted, the corresponding $I((\Gamma^c)')$ has a linear resolution , thus Γ' is CM:

$$1238, 1257, 1267, 1356, 1357, 1458, 1567,$$

$$2345, 2357, 2578, 3567, 3678, 4567, 4678, 5678.$$

Note that $(\Gamma^c)' = (\Gamma')^c$ always holds true. It also gives a negative answer to Question 6. This is the end of Example 11.

Note that in many examples of CM simplicial complexes Δ, we have $\Delta_F \cap \langle F\rangle = \langle\, G \mid G \in \partial F\,\rangle$ holds true for most of the facets F, but not in all cases, as the following example shows:

Example 12. Let $\mathcal{F}(\Delta_3^c) = D \cup \{1236\}$, in which D is taken as in Example 11. Let $F^c = 4578$ and consider $\Gamma =: \Delta_F \cap \langle F\rangle$. Then we have $123 \notin \mathcal{F}(\Gamma)$, thus $\mathcal{F}(\Gamma)$ is a proper subset of $\partial F =: \{123, 126, 136, 236\}$.

Notice the following

$$30 = 5 \times 6 = 10 \times 3,\ 34 \times 4 = 8 \times 17 = 136 < 140 = 35 \times 4.$$

Notice the following fact:

(1) For any odd number $d \geq 3$, $d \times \frac{1}{2}\binom{2d}{d}$ is divided by $2d$, i.e., $4 \mid \binom{2d}{d}$ holds true.

Based on the examples and Theorem 4, we now pose the following:

Conjecture. (a) There exist in set $[2d]_d$ very well-distributed f-simplicial complexes which are minimal CM, if one of the following conditions holds true:

(1) $d \geq 3$ and d is an odd number.

(2) $d \geq 4$ and d is an even number such that $4 \mid \binom{2d}{d}$.

(b) In set $[8]_4$, there exists no very well-distributed f-simplicial complex which is minimal CM.

Note that $4 \nmid \binom{8}{16}$ also holds. Thus in set $[16]_8$, the pure simplicial complexes generated by $\frac{1}{2}\binom{8}{16}$ subsets may perhaps behave just like the pure simplicial complexes in set $[8]_4$.

4. Minimal Cohen-Macaulay f-Simplicial Complexes

In this section, we study properties of f-simplicial complexes which are minimal CM. For this, we need the following:

Lemma 13. Let

$$\mathfrak{C}: 0 \longrightarrow C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

be the chain complex of the simplex $\langle [n] \rangle$ over a field $\mathfrak{K}$. Then the $\mathfrak{K}$-subspace $\ker(\partial_r)$ has dimension $\binom{n-1}{r+1}$.

Proof. For $1 < r \leq n-1$, let $\sum_{1\leq i_1<i_2<\cdots<i_r\leq n} x_{i_1 i_2\dots i_r} \cdot i_1 i_2 \dots i_r \in \ker \partial_{r-1}$, where the second $i_1 i_2 \dots i_r$ denotes the subset $\{i_1, i_2, \dots, i_r\}$ of set $[n]$ and $x_{i_1 i_2 \dots i_r}$ are elements of the base field $\mathfrak{K}$. Then

$$\sum_{1\leq i_1<i_2<\cdots<i_r\leq n} x_{i_1 i_2\dots i_r} \sum_{j=1}^{r} (-1)^{j-1} i_1 i_2 \dots i_{j-1} \hat{i_j} i_{j+1} \dots i_r = 0$$

holds true. Since C_{r-2} is a free $\mathfrak{K}$-module with all elements of $[n]_{r-1}$ as a basis, we have got a system of homogeneous linear equations, which consists of $\binom{n}{r-1}$ equations with $\binom{n}{r}$ variable $x_{i_1 i_2 \dots i_r}$. We write these $x_{i_1 i_2\dots i_r}$ as well as $i_1 i_2 \dots i_{r-1}$ in lexicographic order, and consider the rank of the coefficient matrix $M_{\binom{n}{r-1}\times\binom{n}{r}}$. Clearly, the first $\binom{n-1}{r-1}$ columns, i.e., the coefficients of $x_{1 i_2 \dots i_r}$, are linearly independent. It can be checked that each other column is a linear combination of them. Furthermore, for $i_1 i_2 \dots i_r$ with $1 \notin \{i_1, \dots, i_r\}$, note that

$$\partial_{r-1}(i_1 i_2 \dots i_r) = i_2 \dots i_r - i_1 i_3 \dots i_r + \cdots + (-1)^{r-1} i_1 \dots i_{r-1},$$

in the $i_1 \dots i_r$-th column vector $v_{i_1 \dots i_r}$ of M, the $i_1 \dots \hat{i_j} \dots i_r$-th component is $(-1)^{j-1}$ $(1 \leq j \leq r)$ and, all other components are zero. Thus we have

$$v_{i_1 \dots i_r} = v_{1 i_2 \dots i_r} - v_{1 i_1 i_3 \dots i_r} + \cdots + (-1)^{r-1} v_{1 i_1 \dots i_{r-1}}. \tag{1}$$

The exact details are essentially the same with the verification of the fact that a cone is acyclic, refer to Proposition 5.2.5 of reference [6].

Finally, we proved that the dimension of the vector space $\ker \partial_{r-1}$ is the following

$$\binom{n}{r} - \binom{n-1}{r-1} = \binom{n-1}{r}.$$

Note that the key to calculate the kernel of general ∂_{i-1} is the equality

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}.$$

This is the end of the verification. □

Remark. We illustrate the proof in computational way in two particular cases.

The first case is set $[6]_3$, and we check that $\ker \partial_1$ has dimension $\binom{n-1}{2}$, where $n = 6$. In fact, let $\sum_{1\leq i<j\leq 6} x_{ij}\{i, j\} \in \ker \partial_1$, we have

$$0 = \sum_{1\leq i<j\leq 6} x_{ij}(\{j\} - \{i\}).$$

Since C_0 in the chain complex is a free $\mathfrak{K}$-module with basis $\{1\}, \{2\}, \dots, \{6\}$, we get the following system of linear equations:

$$\begin{cases} \sum_{i=1}^{6} -x_{1i} = 0 \\ x_{12} - \sum_{i=2}^{6} x_{2i} = 0 \\ x_{13} + x_{23} - \sum_{i=3}^{6} x_{3i} = 0 \\ \sum_{i=1}^{3} x_{i4} - x_{45} - x_{46} = 0 \\ \sum_{i=1}^{4} x_{i5} - x_{56} = 0 \\ \sum_{i=1}^{5} x_{i6} = 0. \end{cases}$$

The coefficient matrix is

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & & & & & & & & & & \\ 1 & & & & & -1 & -1 & -1 & -1 & & & & & & \\ & 1 & & & & 1 & & & & -1 & -1 & -1 & & & \\ & & 1 & & & & 1 & & & 1 & & & -1 & -1 & \\ & & & 1 & & & & 1 & & & 1 & & 1 & & -1 \\ & & & & 1 & & & & 1 & & & 1 & & 1 & 1 \end{pmatrix},$$

hence, the solution to the system has exactly

$$10 = \binom{5}{2} = \binom{6}{2} - \binom{5}{1}$$

free variables, and in fact they are all k_{ij} except these $\{i, j\}$ including 1. Note that for a general n, $\dim \ker(\partial_1) = \binom{n-1}{2}$ is verified in an exactly same way. Note also that in the coefficient matrix, we have column vector relation $v_{23} = v_{12} - v_{13}$.

	12	13	14	15	16	17	18	23	24	25	26	27	28	34	35	36	37	38	45	46	47	48	56	57	58	67	68	78
123	1	-1						1																				
124	1		-1						1																			
125	1			-1						1																		
126	1				-1						1																	
127	1					-1						1																
128	1						-1						1															
134		1	-1											1														
135		1		-1											1													
136		1			-1											1												
137		1				-1											1											
138		1					-1											1										
145			1	-1															1									
146			1		-1															1								
147			1			-1															1							
148			1				-1															1						
156				1	-1																		1					
157				1		-1																		1				
158				1			-1																		1			
167					1	-1																				1		
168					1		-1																				1	
178						1	-1																					1
234								1	-1					1														
235								1		-1					1													
236								1			-1					1												
237								1				-1					1											
238								1					-1					1										
245									1	-1									1									
246									1		-1									1								
247									1			-1									1							
248									1				-1									1						
256										1	-1												1					
257										1		-1												1				
258										1			-1												1			
267											1	-1														1		
268											1		-1														1	
278												1	-1															1
345														1	-1				1									
346														1		-1				1								
347														1			-1				1							
348														1				-1				1						
356															1	-1							1					
357															1		-1							1				
358															1			-1							1			
367																1	-1									1		
368																1		-1									1	
378																	1	-1										1
456																			1	-1			1					
457																			1		-1			1				
458																			1			-1			1			
467																				1	-1					1		
468																				1		-1					1	
478																					1	-1						1
567																							1	-1		1		
568																							1		-1		1	
578																								1	-1			1
678																										1	-1	1

Table 1. Transpose C^T of the coefficient matrix C

The second case is set $[8]_4$, and we check that $\ker \partial_2$ has dimension $\binom{n-1}{3}$, where $n = 8$. In fact, let $\sum_{1 \le i < j < k \le 8} x_{ijk}\{i, j, k\} \in \ker \partial_2$, and then we have

$$0 = \sum_{1 \le i < j < k \le 8} x_{ijk}(\{j, k\} - \{i, k\} + \{i, j\}).$$

Since C_1 in the chain complex is a free $\mathfrak{K}$-module with basis set

$$\{ij \mid 1 \le i < j \le 8\},$$

we get a system of linear equations, which consists of $\binom{8}{2}$ linear equations with $\binom{8}{3}$ variables x_{ijk}. We write out the coefficient matrix M in Table 1 and it is clear that the row (in the table) rank of the matrix is not less than $\binom{7}{2}$.

Actually, after doing Gaussian elimination via excel, it is calculated that the matrix has rank $\binom{7}{2}$ =: 21. We also checked the rank by taking advantage of CoCoA version 5.3.3 (reference [18]). Besides, all x_{ijk} except x_{1jk}'s can be chosen as free variables. Certainly, there is an alternative explanation as appeared in Table 1. This shows

$$\dim\ \ker(\partial_2) = \binom{8}{3} - \binom{7}{2} = \binom{7}{3},$$

as is claimed. Note that for a general n, $\dim\ \ker(\partial_2) = \binom{n-1}{3}$ is verified in a completely same way.

Note also that in Table 1, we have row vector relation

$$v_{234}^T = v_{123}^T - v_{124}^T + v_{134}^T,$$

which is a particular case of the general formula (1) appeared in front of the remark. This ends the remark.

Corollary 3. Let $\Delta = \langle \{\sigma \mid \sigma \in [n]_r\} \rangle$. Then we have

$$\dim \tilde{H}_i(\Delta) = \begin{cases} \binom{n-1}{r}, & \text{if } \ i = n-1 \\ 0, & \text{otherwise} . \end{cases}$$

Theorem 14. Let Δ be a simplicial complex over vertex set $[n]$ with $\dim \Delta = d - 1 \geq 0$. If Δ is an f-simplicial complex and it is minimal CM, then $d \geq 3$ and $n = 2d$.

Proof. It is known that connected simplicial complexes of dimension 1 are shellable and pure. On the other hand, if Δ is not connected, then it is not CM. So, there exists no minimal CM f-simplicial complexes of dimension 1.

Now let Δ be an f-simplicial complex of dimension $d-1$, which is minimal CM. Then $\tilde{H}_{d-1}(\Delta) = 0$ by Theorem 3, which means that ∂_{d-1} is injective. Since Δ is an f-simplicial complex, $\mathcal{F}(\Delta)$ is an L-set, hence $\Delta^{(d-2)} = [n]_{d-1}$ holds true, thus, $\tilde{H}_i(\Delta) = 0, \forall 0 \leq i \leq d-3$. Hence Δ is acyclic if and only if $\tilde{H}_{d-2}(\Delta) = 0$, and the latter holds true if and only if $\dim_{\mathfrak{K}} \ker(\partial_{d-2}) = \frac{1}{2}\binom{n}{d}$ by Theorems 3 and 4.

By Lemma 13, we have $\dim_{\mathfrak{K}} \ker(\partial_{d-2}) = \binom{n-1}{d-1}$. We get

$$\frac{n(n-1)\cdots(n-d+1)}{2 \cdot d!} = \frac{(n-1)(n-2)\cdots(n-d+1)}{(d-1)!},$$

since a minimal CM simplicial complex is always acyclic by reference [1]. Thus we have $n = 2d$. □

By Mayer-Vietoris long exact sequence theorem, we get

Corollary 4. If Δ is an f-simplicial complex generated by a subset of set $[2d]_d$ and it is minimal CM, then $\tilde{H}_i(\Delta_F) \cong \tilde{H}_i(\langle F \rangle \cap \Delta_F)$ holds for all integer i, where F is any facet of Δ and, $\langle F \rangle \cap \Delta_F$ is pure of dimension $d-2$.

Proof. For any facet F of Δ, let

$$\Delta_1 = \Delta_F,\ \Delta_2 = \langle F \rangle,\ \Delta_3 = \Delta_1 \cap \Delta_2 =: \Delta_F \cap \langle F \rangle.$$

Then $\Delta = \Delta_1 \cup \Delta_2$. By Theorem 25.1 of [9] on page 142, we have the following long exact sequence of $\mathfrak{K}$-spaces:

$$0 \longrightarrow \tilde{H}_{d-1}(\Delta_3) \longrightarrow \tilde{H}_{d-1}(\Delta_1) \oplus \tilde{H}_{d-1}(\Delta_2) \longrightarrow \tilde{H}_{d-1}(\Delta)$$

$$\xrightarrow{\partial_{d-1}} \tilde{H}_{d-2}(\Delta_3) \longrightarrow \tilde{H}_{d-2}(\Delta_1) \oplus \tilde{H}_{d-2}(\Delta_2) \longrightarrow \tilde{H}_{d-2}(\Delta)$$
$$\xrightarrow{\partial_{d-2}} \cdots\cdots\cdots$$
$$\xrightarrow{\partial_2} H_1(\Delta_3) \longrightarrow H_1(\Delta_1) \oplus H_1(\Delta_2) \longrightarrow H_1(\Delta)$$
$$\xrightarrow{\partial_1} \tilde{H}_0(\Delta_3) \longrightarrow \tilde{H}_0(\Delta_1) \oplus \tilde{H}_0(\Delta_2) \longrightarrow \tilde{H}_0(\Delta) \longrightarrow 0.$$

Note that Lemma 1 implies that Δ_3 is pure of dimension $d-2$, while it follows from Theorem 14 that $\tilde{H}_i(\Delta) = 0$ holds for all i, and $\tilde{H}_i(\Delta_2) = 0$ holds true clearly. Then

$$0 \longrightarrow \tilde{H}_i(\Delta_3) \longrightarrow \tilde{H}_i(\Delta_1) \longrightarrow 0$$

is an exact sequence for every i. □

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