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Erratum to "Positive Solutions for a Class of Quasilinear Schrödinger Equations with Nonlocal Term" [Journal of Applied Mathematics and Physics (2022) 347-359]

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The original online version of this article (Liao, P., Ping, R. and Chen, S. (2022) Positive Solutions for a Class of Quasilinear Schrödinger Equations with Nonlocal Term. *Journal of Applied Mathematics and Physics*, **10**, 347-359. https://doi.org/10.4236/jamp.2022.102027) needs some further amendments and clarification.

1. Introduction

where $N \geq 3$, $0 < \mu < N$, $1 \leq q \leq \frac{N}{N-2}$, $\frac{2N-\mu}{N} \leq p < \frac{2N-\mu}{N-2}$, the function $V \in C\left(\mathbb{R}^N, \mathbb{R}^+\right)$, g is a C^1 even function with $g'(t) \leq 0$ for all $t > 0$, $g(0) = 0$, $\lim_{t \to +\infty} g(t) = a$, $0 < a < 1$.

2. Preliminary Results

Next, we introduce some minimization with corresponding energy functional and define

$$m_b = \inf_{u \in M_b} E(u),$$

where

$$M_b = \left\{ u \in H^1\left(\mathbb{R}^N\right) : \|u\|_{L^{q+1}} = b \right\}, \quad b > 0,$$

and

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[g^2(u)|\nabla u|^2 + V(x)u^2\right] - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\left[\left(|x| * |u|^p\right)|u|^p\right].$$

We also define

$$\omega_b = \inf_{v \in W_b} F(v),$$

where

$$W_b = \left\{ v \in H_V^1(\mathbb{R}^N) : \left\| G^{-1}(v) \right\|_{L^{q+1}} = b \right\}, \quad b > 0,$$

and

$$F(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) G^{-1}(v)^2 \right) - \frac{\lambda}{2p}\int_{\mathbb{R}^N} \left[\left(|x| * |G^{-1}(v)|^p \right) |G^{-1}(v)|^p \right].$$

Proof: For any $v \in W_b$, let $u = G^{-1}(v)$, from the definition of *g*, we get

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} \frac{|\nabla v|^2}{g^2\left(G^{-1}(v)\right)} \le \frac{1}{a^2}\int_{\mathbb{R}^N} |\nabla v|^2 < +\infty,$$

and

$$\int_{\mathbb{R}^N} u^2 \le \int_{\mathbb{R}^N} V(x) G^{-1}(v)^2 < +\infty,$$

so $u \in M_b$. It follow that $F(v) = E\left(G^{-1}(v)\right) = E(u) \ge m_b$, hence $\omega_b \ge m_b$, moreover, for any $u \in M_b$, let $v = G(u)$, then $u = G^{-1}(v)$. We assume $E(u) < +\infty$, since $u \in H^1(\mathbb{R}^N)$, $2 < \frac{2Np}{2N-\mu} < 2^*$, then $u \in L^{\frac{2Np}{2N-\mu}}(\mathbb{R}^N)$. By Hardy-Little-Sobolev-inequality, we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N} \left[g^2(u)|\nabla u|^2 + V(x)u^2 \right] \\ &= E(u) + \frac{\lambda}{2p}\int_{\mathbb{R}^N} \left(|x| * |u|^p \right) |u|^p \\ &\le E(u) + \frac{\lambda C}{2p}\left(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{2N-\mu}} \right)^{\frac{2N-\mu}{N}} < +\infty. \end{aligned}$$

■

The proof of Lemma 2.4

Proof: (1) For any $v \in H_V^1(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N} |G^{-1}(v)|^{\frac{2Np}{2N-\mu}} \le C\int_{\mathbb{R}^N} |v|^{\frac{2Np}{2N-\mu}} < +\infty$, where $2 \le \frac{2Np}{2N-\mu} < 2^*$, similarly as the proof of Lemma2.3, by Hardy-Little-Sobolev-inequality, we have

$$\begin{aligned} F(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x) G^{-1}(v)^2 \right) - \frac{\lambda}{2p}\int_{\mathbb{R}^N} \left(|x| * |G^{-1}(v)|^p \right) |G^{-1}(v)|^p \\ &\le \frac{1}{2a^2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)v^2 \right) + \frac{\lambda C}{2p}\left(\int_{\mathbb{R}^N} |G^{-1}(v)|^{\frac{2Np}{2N-\mu}} \right)^{\frac{2N-\mu}{N}} \\ &\le \frac{1}{2a^2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)v^2 \right) + \frac{\lambda C}{2p}\left(\int_{\mathbb{R}^N} |v|^{\frac{2Np}{2N-\mu}} \right)^{\frac{2N-\mu}{N}} < +\infty. \end{aligned}$$

With the proof of continuity, note that F consist of three terms. By Lemma 1.1, we need to check the convolution term only. Using Hardy-Little-Sobolev-inequality

$$\begin{aligned}
&\frac{\lambda}{2p}\left|\int_{\mathbb{R}^N}\left(|x|*\left|G^{-1}(v_n)\right|^p\right)\left|G^{-1}(v_n)\right|^p-\int_{\mathbb{R}^N}\left(|x|*\left|G^{-1}(v)\right|^p\right)\left|G^{-1}(v)\right|^p\right|\\
&\le\frac{\lambda}{2p}\left(\left|\int_{\mathbb{R}^N}|x|*\left(\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right)\left|G^{-1}(v_n)\right|^p\right|\right.\\
&\quad\left.+\left|\int_{\mathbb{R}^N}\left(|x|*\left|G^{-1}(v)\right|^p\right)\left(\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right)\right|\right)\\
&\le C\left|\left(\int_{\mathbb{R}^N}\left|G^{-1}(v_n)\right|^{pr}\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}\left(\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right)^r\right)^{\frac{1}{r}}\right|\\
&\quad+C\left|\left(\int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{pr}\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}\left(\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right)^r\right)^{\frac{1}{r}}\right|,
\end{aligned}$$

and

$$\left|\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right|^r\le C\left(|v_n|^{pr}+|v|^{pr}\right),$$

where $r=\frac{2N}{2N-\mu}$. We know $\|v_n-v\|_{H_V^1(\mathbb{R}^N)}\to 0$ if $n\to+\infty$. So $\{v_n\}$ is bounded in $H_V^1(\mathbb{R}^N)$. By Sobolev embedding theorem and Lemma 3.4 [22]

$$\left|\int_{\mathbb{R}^N}\left(|x|*\left|G^{-1}(v_n)\right|^p\right)\left|G^{-1}(v_n)\right|^p-\int_{\mathbb{R}^N}\left(|x|*\left|G^{-1}(v)\right|^p\right)\left|G^{-1}(v)\right|^p\right|\to 0,\ n\to+\infty.$$

For (2) we consider the second and the third terms of the functional *F*, we see for $\phi\in H_V^1(\mathbb{R}^N)$, using Hölder inequality, we get

$$\begin{aligned}
&\left|\frac{1}{2t}\int_{\mathbb{R}^N}V(x)\left(G^{-1}(v+t\phi)^2-G^{-1}(v)^2\right)-\int_{\mathbb{R}^N}\frac{V(x)G^{-1}(v)}{g\left(G^{-1}(v)\right)}\phi\right|\\
&=\left|\int_0^1\mathrm{ds}\int_{\mathbb{R}^N}V(x)\left(\frac{G^{-1}(v+ts\phi)}{g\left(G^{-1}(v+ts\phi)\right)}-\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right)\phi\right|\\
&\le\int_0^1\mathrm{ds}\left(\int_{\mathbb{R}^N}V(x)\left|\frac{G^{-1}(v+ts\phi)}{g\left(G^{-1}(v+ts\phi)\right)}-\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^2\right)^{\frac{1}{2}}\int_0^1\mathrm{ds}\left(\int_{\mathbb{R}^N}V(x)\phi^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Using the definition of *g* and Lemma 1.1, we know

$$\begin{aligned}
\left|\frac{G^{-1}(v+ts\phi)}{g\left(G^{-1}(v+ts\phi)\right)}-\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^2&\le\left|G^{-1}(v+ts\phi)+G^{-1}(v)\right|^2\\
&\le C\left(\left|G^{-1}(v+ts\phi)\right|^2+\left|G^{-1}(v)\right|^2\right)\\
&\le C\left(|v+ts\phi|^2+|v|^2\right)\\
&\le C\left(|v|^2+|\phi|^2\right).
\end{aligned}$$

By the dominated convergence theorem

$$\left|\int_0^1 \mathrm{d}s\int_{\mathbb{R}^N} V(x)\left(\frac{G^{-1}(v+ts\phi)}{g\left(G^{-1}(v+ts\phi)\right)}-\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right)\phi\right|\to 0,\quad t\to 0.$$

For the third term, we have

$$\lambda\left|\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left|G^{-1}(v_n)\right|^p\right]\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}\phi-\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right]\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\phi\right|$$

$$\le\lambda\left|\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left(\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right)\right]\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}\phi\right|$$

$$+\lambda\left|\int_{\mathbb{R}^N}\left(|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right)\left(\frac{\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}-\frac{\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right)\phi\right|$$

$$\le C\int_{\mathbb{R}^N}\left(\left(\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right)^r\right)^{\frac{1}{r}}\int_{\mathbb{R}^N}\left(\left|\frac{\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}\phi\right|^r\right)^{\frac{1}{r}}$$

$$+C\left(\int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{pr}\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}\left(\frac{\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}-\frac{\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right)^{\frac{p}{p-1}r}\right)^{\frac{p-1}{pr}}\left(\int_{\mathbb{R}^N}|\phi|^{pr}\right)^{\frac{1}{pr}}.$$

and

$$\left|\left|G^{-1}(v_n)\right|^p-\left|G^{-1}(v)\right|^p\right|^r\le C\left(|v_n|^{pr}+|v|^{pr}\right),$$

$$\left|\frac{\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}-\frac{\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^{\frac{p}{p-1}r}\le C\left(|v_n|^{pr}+|v|^{pr}\right),$$

where $r=\dfrac{2N}{2N-\mu}$, $2\le pr<2^*$. Since $\|v_n-v\|_{H_V^1(\mathbb{R}^N)}\to 0$ if $n\to+\infty$, $H_V^1(\mathbb{R}^N)$ embedding into $L^r(\mathbb{R}^N)$ is compact and $\{v_n\}$ is bounded in $H_V^1(\mathbb{R}^N)$. Using Lemma 3.4 [22], we know

$$\left|\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left|G^{-1}(v_n)\right|^p\right]\left|G^{-1}(v_n)\right|^{p-2}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}\phi\right.$$

$$\left.-\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right]\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\phi\right|\to 0,\quad n\to+\infty.$$

By Lemma 1.1

$$\begin{aligned}\left|\left\langle F'(v),\phi\right\rangle\right| &= \left|\int_{\mathbb{R}^N}\nabla v\nabla\phi+\int_{\mathbb{R}^N}\frac{V(x)G^{-1}(v)}{g\left(G^{-1}(v)\right)}\phi\right. \\ &\quad \left. -\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^p\right]\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\phi\right| \\ &\le C\|v\|_{H_V^1(\mathbb{R}^N)}\|\phi\|_{H_V^1(\mathbb{R}^N)}+C\|v\|_{L^{pr}(\mathbb{R}^N)}^{p-1}\|\phi\|_{L^{pr}(\mathbb{R}^N)}.\end{aligned}$$

from Sobolev embedding theorem, we get $F'(v)$ is a continuous linear functional on $H_V^1\left(\mathbb{R}^N\right)$.■

3. Main Conclusion

Remark 3.1. *From assumption of V, we know* $H_V^1\left(\mathbb{R}^N\right)$ *embedding into* $L^p\left(\mathbb{R}^N\right)$ *is compact. In the process of the proof of theorem* 3.1, *it is important for us to construct auxiliary function, then by implicit function theorem to prove it and lemma* 3.4 [22] *play a great role in this paper. Moreover, when* $q\ge 2^*$ *is a open question for Equation* (1.1), *someone could do it if they are interested.*

Proof of Theorem 3.1: Step 1: By the assumptions of (V₁) or (V₂), ω_b is achieved at some $0\le v_b\le W_b$ with $v_b\ne 0$.

Let $\{v_n\}\in W_b$ be a minimizing sequence for ω_b. Set $u_n=G^{-1}(v_n)$, then $\{u_n\}\in M_b$ is a minimizing sequence for m_b. We can assume $u_n\ge 0$. It shows that $E(u_n)\to m_b$, so there exist $C>0$ such that

$$\begin{aligned}C &\ge E(u_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left[g^2(u_n)|\nabla u_n|^2+V(x)u_n^2\right]-\frac{\lambda}{2p}\int_{\mathbb{R}^N}\left[\left(|x|*|u_n|^p\right)|u_n|^p\right] \\ &\ge \frac{a}{2}\int_{\mathbb{R}^N}\left[|\nabla u|^2+V(x)u_n^2\right]-\frac{\lambda}{2p}\int_{\mathbb{R}^N}\left(|x|*|u_n|^p\right)|u_n|^p.\end{aligned}$$

By Hölder inequality and Hardy-Little-Sobolev-inequality,

$$\begin{aligned}\int_{\mathbb{R}^N}\left[\left(|x|*|u_n|^p\right)|u_n|^p\right] &\le \left(\int_{\mathbb{R}^N}|u_n|^{pr}\right)^{\frac{2}{r}} \\ &\le \left(\left(\int_{\mathbb{R}^N}|u_n|^2\right)^{\frac{\theta pr}{2}}\left(\int_{\mathbb{R}^N}|u_n|^{q+1}\right)^{\frac{(1-\theta)pr}{q+1}}\right)^{\frac{2}{r}} \\ &= \left(\int_{\mathbb{R}^N}|u_n|^2\right)^{\theta p}b^{\frac{2(1-\theta)p}{q+1}} \\ &\le b^{\frac{2p-\theta p+\theta pq-q-1}{q+1}}\left(\varepsilon\int_{\mathbb{R}^N}|u_n|^2+C(\varepsilon)b\right).\end{aligned}$$

where $\theta=\dfrac{2(q+1)-2pr}{(q-1)pr}$, $0<\theta p<1$, $r=\dfrac{2N}{2N-\mu}$, $\varepsilon>0$,

$C(\varepsilon)=\left(\dfrac{\varepsilon}{\theta p}\right)^{-\frac{\theta p}{1-\theta p}}(1-\theta p)$, then

$$
\begin{aligned}
C &\geq E\left(u_n\right) \\
&\geq \frac{a}{2}\int_{\mathbb{R}^N}\left[\left|\nabla u_n\right|^2+V(x)u_n^2\right]-\frac{\lambda b^{\frac{2p-\theta p+\theta pq-q-1}{q+1}}}{2p}\left(\varepsilon\int_{\mathbb{R}^N}V(x)u_n^2+C(\varepsilon)b\right) \\
&\geq\left(\frac{a}{2}-\frac{\lambda\varepsilon b^{\frac{2p-\theta p+\theta pq-q-1}{q+1}}}{2p}\right)\left(\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2+V(x)\left|u_n\right|^2\right)-\frac{\lambda}{2p}C(\varepsilon)b^{\frac{2p-\theta p+\theta pq}{q+1}}.
\end{aligned}
$$

Taking $\varepsilon>0$ small enough such that $\frac{a}{2}-\frac{\lambda\varepsilon b^{\frac{2p-\theta p+\theta pq-q-1}{q+1}}}{2p}>0$. It implies that $u_n(x)$ is bounded in $H_V^1\left(\mathbb{R}^N\right)$. By the compact embedding result from $H_V^1\left(\mathbb{R}^N\right)$ into $L^r\left(\mathbb{R}^N\right)$ for $2\leq r<2^*$. We may assume that $u_n\rightharpoonup u_b$ in $H_V^1\left(\mathbb{R}^N\right)$, $u_n\to u_b$ in $L^r\left(\mathbb{R}^N\right)$ for $2\leq r<2^*$ and $u_n(x)\to u_b(x)$ a.e $x\in\mathbb{R}^N$. Hence $u_b\in M_b$, since $u_n\geq 0$, $u_b\geq 0$ and $u_b\neq 0$. Similarly as the proof of Lemma 2.4 (1), we have

$$\int_{\mathbb{R}^N}\left(|x|*\left|u_n\right|^p\right)\left|u_n\right|^p\to\int_{\mathbb{R}^N}\left(|x|*\left|u_b\right|^p\right)\left|u_b\right|^p,\quad n\to+\infty.$$

Hence

$$
\begin{aligned}
m_b &= \lim_{n\to\infty}E\left(u_n\right) \\
&\geq\liminf_{n\to\infty}\left\{\frac{1}{2}\int_{\mathbb{R}^N}\left[g^2\left(u_n\right)\left|\nabla u_n\right|^2+V(x)u_n^2\right]-\frac{\lambda}{2p}\int_{\mathbb{R}^N}\left(|x|*\left|u_n\right|^p\right)\left|u_n\right|^p\right\} \\
&\geq E\left(u_b\right).
\end{aligned}
$$

Step 2: Set $h_{q+1}(v)=\frac{1}{q+1}\int_{\mathbb{R}^N}\left|G^{-1}(v(x))\right|^{q+1}$ for $2\leq q+1<2^*$, then $h_{q+1}(v)\in C^1\left(H_V^1\left(\mathbb{R}^N\right),\mathbb{R}\right)$.

In fact, for any $\varphi\in H_V^1\left(\mathbb{R}^N\right)$, by Lemma 1.1 and Hölder's inequality, we have

$$
\begin{aligned}
\left|\left\langle h_{q+1}'(v),\varphi\right\rangle\right| &= \left|\int_{\mathbb{R}^N}\frac{\left|G^{-1}(v)\right|^{q-1}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\varphi\right| \\
&\leq C\left(\int_{\mathbb{R}^N}|v|^{q+1}\right)^{\frac{q}{q+1}}\left(\int_{\mathbb{R}^N}|\varphi|^{q+1}\right)^{\frac{1}{q+1}} \\
&\leq C\|\varphi\|_{H_V^1\left(\mathbb{R}^N\right)}.
\end{aligned}
$$

then $h_{q+1}'(v)\in\left(H_V^1\left(\mathbb{R}^N\right)\right)^*$.

$$
\begin{aligned}
&\left|\left\langle h_{q+1}'\left(v_n\right)-h_{q+1}'(v),\varphi\right\rangle\right| \\
&=\left|\int_{\mathbb{R}^N}\left(\frac{\left|G^{-1}\left(v_n\right)\right|^{q-1}G^{-1}\left(v_n\right)}{g\left(G^{-1}\left(v_n\right)\right)}-\frac{\left|G^{-1}(v)\right|^{q-1}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right)\varphi\right| \qquad (1)\\
&\leq\left(\int_{\mathbb{R}^N}\left|\frac{\left|G^{-1}\left(v_n\right)\right|^{q-1}G^{-1}\left(v_n\right)}{g\left(G^{-1}\left(v_n\right)\right)}-\frac{\left|G^{-1}(v)\right|^{q-1}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^{\frac{q+1}{q}}\right)^{\frac{q}{q+1}}\left(\int_{\mathbb{R}^N}|\varphi|^{q+1}\right)^{\frac{1}{q+1}}.
\end{aligned}
$$

and

$$\left|\frac{\left|G^{-1}(v_n)\right|^{q-1}G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}-\frac{\left|G^{-1}(v)\right|^{q-1}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right|^{\frac{q+1}{q}}\le C\left(\left|v_n\right|^{q+1}+\left|v\right|^{q+1}\right),$$

Since $v_n\to v$ in $H_V^1\left(\mathbb{R}^N\right)$, $H_V^1\left(\mathbb{R}^N\right)$ embedding into $L^r\left(\mathbb{R}^N\right)$ is compact and $\{v_n\}$ is bounded in $H_V^1\left(\mathbb{R}^N\right)$, where $2\le r<2^*$. By $2\le q+1<2^*$ and Lemma 3.4 [22], we have

$$\left|\left\langle h'_{q+1}(v_n)-h'_{q+1}(v),\varphi\right\rangle\right|\to 0,\quad n\to+\infty.$$

then $h_{q+1}(v)\in C^1\left(H_V^1\left(\mathbb{R}^N\right),\mathbb{R}\right)$ for $2\le q+1<2^*$.

Step 3: For any $b\ge 0$, there exist $\beta(b)\in\mathbb{R}$ such that $0<u_b=G^{-1}(v_b)\in M_b$ is a weak solution of Equation (1.1) with $\lambda=\lambda(b)$. In fact, by lemma2.4,

$$\left\langle F'(v),\varphi\right\rangle=\int_{\mathbb{R}^N}\nabla v\nabla\varphi+\int_{\mathbb{R}^N}\frac{V(x)G^{-1}(v)}{g\left(G^{-1}(v)\right)}\varphi$$

$$-\lambda\int_{\mathbb{R}^N}\frac{\left[|x|^{-\mu}*\left|G^{-1}(v)\right|^{p}\right]\left|G^{-1}(v)\right|^{p-2}G^{-1}(v)}{g\left(G^{-1}(v)\right)}\varphi.$$

Take limit $t\to 0$, we get $\left\langle F'(v_b),v\right\rangle\ge 0$, by arbitrariness of *v*, one has $\left\langle F'(v_b),-v\right\rangle\ge 0$. It follows that $\left\langle F'(v_b),v\right\rangle=0$, for every $v\in\mathcal{N}\left(h'_{q+1}(v_b)\right)$. Set $v'\in H_V^1\left(\mathbb{R}^N\right)$ be such that $\left\langle h'_{q+1}(v_b),v'\right\rangle=1$, for every $\varphi\in H_V^1\left(\mathbb{R}^N\right)$, let

$$\psi=\varphi-\left\langle h'_{q+1}(v_b),\varphi\right\rangle v'.$$

Then $\psi\in\mathcal{N}\left(h'_{q+1}(v_b)\right)$, it means $\left\langle F'(v_b),\psi\right\rangle=0$, i.e.

$$\left\langle F'(v_b),\varphi\right\rangle=\left\langle F'(v_b),v'\right\rangle\left\langle h'_{q+1}(v_b),\varphi\right\rangle.$$

Put $\beta=\beta(b)=\left\langle F'(v_b),v'\right\rangle$, we have

$$\left\langle F'(v_b),\varphi\right\rangle=\beta\left\langle h'_{q+1}(v_b),\varphi\right\rangle,$$

■