

Stochastic HIV Infection Model with CTLs Immune Response Driven by Lévy Jumps

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Abstract

This paper mainly investigates the effect of the Lévy jumps on the stochastic HIV infection model with cytotoxic T lymphocytes (CTLs) immune response. First, we prove that there is a unique global positive solution in any population dynamics, then we find sufficient conditions for the extinction of the disease. For proofing the persistence in mean, a special Lyapunov function be established, we obtain that if $\tilde{R}_1 > 1$ the infected CD4⁺ T-cells and virus particles will persistence in mean. Finally, numerical simulations are carried out to illustrate the theoretical results.

Keywords

HIV-1 Infection, Lévy Jump, CTLs Immune Response, Persistence in Mean

1. Introduction

AIDS (acquired immunodeficiency syndrome) is caused by the human immunodeficiency virus (HIV). The progress of HIV infection has several different stages, including the early stage of infection, the clinical latency stage, stage of immune system becomes damaged, and the final stage that HIV progresses to acquired immunodeficiency syndrome (AIDS)-a fatal disease, for the life threatening effect of HIV that motivated numerous research to study HIV infection in different ways [1].

HIV is a retrovirus that attacks the body's immune system, CD4⁺ T-cells are the primary target cells of HIV infection [2]. CD4⁺ T-cells play a central role in immune regulation, therefore their depletion has a wide range of deleterious effects on the functioning of the entire immune system and leads to the immunodeficiency of AIDS [3].

During the process of viral infection, the host response is induced, stimulating the production of CTLs, which kill the infected $CD4^+$ T-cells by CTLs immune responses when the body is infected with HIV. CTLs are the major immune response factors in the human body that limit the virus's in vivo replication and determine the number of viruses. Recently, the dynamics of HIV infection with CTLs response have attracted much attention from scholars to conduct research. Such as, Elaiw *et al.* showed the global stability of HIV/HTLV [4]. Koenig *et al.* pointed out, there is clearest evidence for the active CTLs selection of viral variants could contribute to the pathogenesis of AIDS and that clinical progression can occur despite high levels of circulating HIV-1-specific CTLs [5]. Therefore, CTLs play a critical role in inhibiting HIV by killing virus-infected T cells, much research has been done on the dynamics of HIV infection with CTLs response [6] [7] [8], and so on. The earlier models for viral dynamics with immune response are the general form:

$$\begin{cases} \frac{dx(t)}{dt} = n(x) - h(x, v) \\ \frac{dy(t)}{dt} = h(x, v) - \delta y(t) - py(t)z(t), \\ \frac{dv(t)}{dt} = N\delta y(t) - \mu v(t), \\ \frac{dz(t)}{dt} = cy(t)z(t) - bz(t). \end{cases} \quad (1.1)$$

In ref [8], Nowak *et al.* set $n(x) = \Lambda - dx(t)$ and $h(x, v) = kx(t)v(t)$, where, the susceptible $CD4^+$ T-cells $x(t)$ are generated at a rate Λ , die at a density-dependent rate $dx(t)$, and become infected by virus particles at a rate kxv ; the infected $CD4^+$ T-cells $y(t)$ are produced at rate kxv , die at rate $\delta y(t)$, and killed by CTLs at rate $py(t)z(t)$; the free virus particles $v(t)$ released by each infected $CD4^+$ T-cells when it lyses at rate N and cleared from the system at rate μ ; the virus-specific CTLs cells $z(t)$ proliferate at a rate cy by contact with infected cells, and lost at rate $bz(t)$, which may include reversion to quiescent or memory phenotypes as well as death.

From the studies of [8], we know that the CTLs response will increase when $cy > b$. If $R_0 < 1$, the model has an infection-free equilibrium $E_0 = \left(\frac{\Lambda}{d}, 0, 0, 0\right)$ and it is asymptotically stable; while if $R_0 > 1$, the CTLs response may become only transiently activated, but eventually, the system will converge to the equilibrium $E_1 = (\bar{x}, \bar{y}, \bar{v}, 0)$, and when $R_1 < 1$ is asymptotically stable without an active CTLs response; finally, if $R_1 > 1$, the system shows damped oscillations to the equilibrium $E_2 = (x^*, y^*, v^*, z^*)$ which is asymptotically stable, where

$$R_0 = \frac{k\Lambda N}{d\mu}, \quad R_1 = \frac{ck\Lambda N}{cd\mu + bkN\delta},$$

$$(\bar{x}, \bar{y}, \bar{v}, 0) = \left(\frac{\mu}{kN}, \frac{\Lambda}{\delta} \frac{R_0 - 1}{R_0}, \frac{\Lambda N}{\mu} \frac{R_0 - 1}{R_0}, 0\right),$$

$$(x^*, y^*, v^*, z^*) = \left(\frac{c\Lambda\mu}{cd\mu + bkN\delta}, \frac{b}{c}, \frac{bN\delta}{c\mu}, \frac{\delta}{p}(R_1 - 1) \right).$$

However, Singh *et al.* have derived from the experimental data that HIV transcription is an inherently random process and produces a high stochastic variability in HIV-1 gene expression [9]. Rouzine *et al.* have pointed out that random variation of HIV can affect virus eradication in combination with antiretroviral therapy [10]. In HIV viral dynamical model, T cell proliferation proceeds through the logical growth phase, but these works do not involve random noise. Therefore, Daqing Jiang *et al.* thought it was more reasonable to include both healthy T cell proliferation and CTLs immune responses in an HIV model, and the model would have a further effect on model behavior [11] [12] [13] [14]. As a result, environmental fluctuation in the HIV viral dynamical model is more reasonable. Daqing Jiang *et al.* [14] take into account the effect of randomly fluctuating environment and incorporate the white noise with two parameters of system (1.1) because epidemic systems are often subject to environmental noise, and the deterministic models do not incorporate the effect of fluctuating environment. They replaced the parameters $\mu \rightarrow \mu + \sigma_1 \dot{B}_1(t)$, $b \rightarrow b + \sigma_2 \dot{B}_2(t)$. Hence, the stochastic version corresponding to system (1.1) takes the following form:

$$\begin{cases} dx(t) = [\Lambda - dx(t) - kx(t)v(t)] dt, \\ dy(t) = [kx(t)v(t) - \delta y(t) - py(t)z(t)] dt, \\ dv(t) = [N\delta y(t) - \mu v(t)] dt - \sigma_1 v(t) dB_1(t), \\ dz(t) = [cy(t)z(t) - bz(t)] dt - \sigma_2 z(t) dB_2(t). \end{cases} \tag{1.2}$$

The infection-free equilibrium is stochastically asymptotically stable in the large, when $R_0 < 1$; when the white noise is small, while if $R_0 > 1$ and $R_1 \leq 1$ and satisfy $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t E \left[\|X(s) - E_1\|^2 \right] ds$ (where $X(t)$ represent the solution of system (1.2) and $\|\cdot\|$ is l^2 norm) is small, we have CTLs response is only transiently activated, when $R_1 < 1$, the CTLs response is activated. But when the white noise is large, the CTLs response is still transiently activated even if $R_1 > 1$ satisfy $\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t E \left[\|X(s) - E_2\|^2 \right] ds$ is small, which never happens in system (1.1).

Due to the inherently stochastic nature of biochemical processes, the dynamic process of HIV viral infection may suffer strong fluctuation [15], such that the classical stochastic model (1.2) cannot explain the strong, occasional fluctuations of the biological environment. It is reasonable to further consider another random noise, namely the lévy jump noise, into HIV viral dynamical model. Therefore, the aim of this paper is to present a comprehensive study for stochastic system with lévy jump process to describe this strong fluctuation. The main points and novel contributions of the paper are as follows:

- 1) It described the strong fluctuation by introducing a lévy jump process into the HIV viral dynamical model, which can describe the phenomena cause a big

jump to occur occasionally. It overcome the drawbacks of the classical stochastic model which cannot explain the strong, occasional fluctuation of the biological environment.

2) Using the Khasminskii-Mao theorem and appropriate Lyapunov functions, we show that the model has a unique global positive solution.

3) By applying Itô's formula and the large number theorem for martingales, we established sufficient conditions for the extinction and persistence in mean of infected CD4⁺ T-cells and virus particles.

In this paper, we aim to introduce Lévy noise into above ecological epidemic model, as a result, model (1.1) becomes:

$$\begin{cases} dx(t) = [\Lambda - dx(t) - kx(t)v(t)]dt + \sigma_1 x(t)dB_1(t) + \int_Z \gamma_1(u)x(t-)\tilde{N}(dt, du), \\ dy(t) = [kx(t)v(t) - \delta y(t) - py(t)z(t)]dt + \sigma_2 y(t)dB_2(t) + \int_Z \gamma_2(u)y(t-)\tilde{N}(dt, du), \\ dv(t) = [N\delta y(t) - \mu v(t)]dt + \sigma_3 v(t)dB_3(t) + \int_Z \gamma_3(u)v(t-)\tilde{N}(dt, du), \\ dz(t) = [cy(t)z(t) - bz(t)]dt + \sigma_4 z(t)dB_4(t) + \int_Z \gamma_4(u)z(t-)\tilde{N}(dt, du). \end{cases} \quad (1.3)$$

here $x(t-)$, $y(t-)$, $v(t-)$ and $z(t-)$ are the left limit of $x(t)$, $y(t)$, $v(t)$ and $z(t)$, $\tilde{N}(dt, du) = N(dt, du) - \pi(du)dt$, N is a Poisson counting measure with characteristic measure π on a measurable subset Z of $(0, +\infty)$ with $\pi(Z) < \infty$. $\gamma_i(u) : Z \times \Omega \rightarrow R$ is bounded and continuous ($i = 1, 2, 3$), $B_i(t)$ ($i = 1, 2, 3$) are standard Brownian motions, and independent with N throughout the paper.

This paper is organized as follows. In Section 2, we will give some preliminaries and show there exist a global and positive solution under appropriate conditions. In Section 3, we will investigate the conditions for the extinction of infected cells and CTLs extinction. In Section 4, conditions will be derived for the persistence in mean. In Section 5, numerical simulations are carried out to illustrate the theoretical results. Finally, we give some conclusions.

2. Preliminaries and Global Positive Solution

First, we introduce the following notations, throughout this paper, let

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions (*i.e.*, it is increasing and right continuous while \mathcal{F}_0 contains all P-null sets). And we let $B_i(t)$ be defined on the probability space and $\mathbb{R}_+^d = \{x \in \mathbb{R}_+ : x_i > 0, i = 1, 2, \dots, d\}$.

Assume that $X(t) \in R^+$ is an Itô's-Lévy process of the form

$$dX(t) = F(X(t^-), t^-)dt + G(X(t^-), t^-)dB(t) + \int_Z H(X(t^-), t^-, u)\tilde{N}(dt, du),$$

where $F : R^n \times R^+ \times S \rightarrow R^n$, $G : R^n \times R_+ \times S \rightarrow R^n$ and

$H : R^n \times R_+ \times S \times Z \rightarrow R^n$ are measurable functions.

Given $V \in C^{2,1}(R^n \times R^+ \times S; R^+)$, the operator LV by

$$\begin{aligned} LV(X, t) = & V_t(X, t) + V_x(X, t)F(X, t) + \frac{1}{2}trace[G^T(X, t)V_{xx}(X, t)G(X, t)] \\ & + \int_Z \{V(X + H(X, t)) - V(X, t) - V_x(X, t)H(X, t, u)\}v(du), \end{aligned}$$

where

$$V_t(X, t) = \frac{\partial V_X(X, t)}{\partial t}, \quad V_X(X, t) = \left(\frac{\partial V_X(X, t)}{\partial X_1}, \dots, \frac{\partial V_X(X, t)}{\partial X_n} \right),$$

$$V_{XX}(X, t) = \left(\frac{\partial^2 V_X(X, t)}{\partial X_i \partial X_j} \right)_{n \times n}.$$

Then the generalized Itô's formula with Lévy jumps is given by

$$dV(X, t) = LV(X, t)dt + V_X(X, t)G(X, t)dB(t) + \int_Z \{V(X + H(X, t)) - V(X, t)\} \tilde{N}(dt, du).$$

For convenience, we introduce following notations and the assumption.

Assumption 1. We assume that $1 + \gamma_i(u) > 0, u \in Z, i = 1, 2, 3$, and there is a positive constant c such that

$$\int_Z [\ln(1 + \gamma_1(u))]^2 \pi(du) < c.$$

Lemma 2.1. [16] Suppose that $Z(t) \in C(\Omega \times [0, \infty), \mathbb{R}_+)$. Under Assumption 1,

1) If there are two positive constants T and δ_0 such that

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_{i=1}^n \alpha_i B(t) + \sum_{i=1}^n k_i \int_0^t \int_Z \ln(1 + c_i(z)) \tilde{N}(dt, dz), \text{ a.s.}$$

for all $t > T$, where α_i, δ and k_i are constants, then

$$\begin{cases} \langle Z \rangle^* \leq \frac{\delta}{\delta_0} \text{ a.s.,} & \text{if } \delta \geq 0; \\ \lim_{t \rightarrow \infty} Z(t) = 0 \text{ a.s.,} & \text{if } \delta < 0. \end{cases}$$

2) If there exist three T, δ, δ_0 such that

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_{i=1}^n \alpha_i B(t) + \sum_{i=1}^n k_i \int_0^t \int_Z \ln(1 + c_i(z)) \tilde{N}(dt, dz), \text{ a.s. (2.1)}$$

for all $t > T$, then $\langle Z \rangle_* \geq \frac{\delta}{\delta_0}$.

Lemma 2.2. We assume for some $\theta > 2, \mu^* - \frac{\theta - 1}{2} \sigma^2 - \frac{1}{\theta} \lambda^* > 0$ holds. For any initial value $(x(0), y(0), v(0), z(0)) \in \mathbb{R}_+^3$, model (1.3) has a unique positive solution $(x(t), y(t), v(t), z(t)) \in \mathbb{R}_+^3$ for any $t \geq 0$ almost surely. Furthermore, the solution $(x(t), y(t), v(t), z(t))$ of model (1.3) has the following properties:

$$\lim_{t \rightarrow \infty} \frac{x(t) + y(t) + v(t) + \frac{p}{c} z(t)}{t} = 0,$$

moreover,

$$\lim_{t \rightarrow \infty} \frac{x(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{y(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{v(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{z(t)}{t} = 0.$$

where $\sigma^2 = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2$, $\mu^* = \min\{d, \delta, \mu, b\} - N\delta$, and

$$\lambda^* = \int_Z \left[(1 + \gamma_1(u) \vee \gamma_2(u) \vee \gamma_3(u) \vee \gamma_4(u))^\theta - 1 - \gamma_1(u) \wedge \gamma_2(u) \wedge \gamma_3(u) \wedge \gamma_4(u) \right] \pi(du)$$

Proof. Define $X = x + y + v + \frac{p}{c}z$, $V = X^\theta$, applying Itô's formula, we get

$$dV(X) = LV(X)dt + \theta X^{\theta-1} \left[\sigma_1 dB_1(t) + \sigma_2 dB_2(t) + \sigma_3 dB_3(t) + \sigma_4 \frac{p}{c} dB_4(t) \right] + \int_Z \left[\left(x + \gamma_1 x + y + \gamma_2 y + v + \gamma_3 v + \frac{p}{c}z + \frac{p}{c}\gamma_4 z \right)^\theta - X^\theta \right] \tilde{N}(dt, du).$$

where

$$\begin{aligned} LV &= \theta X^{\theta-1} \left[\Lambda - dx - \delta y - \mu v - \frac{bp}{c}z + N\delta y \right] + \frac{\theta(\theta-1)}{2} X^{\theta-2} \left[\sigma_1^2 x^2 + \sigma_2^2 y^2 + \sigma_3^2 v^2 + \sigma_4^2 \frac{p^2}{c^2} z^2 \right] + \int_Z \left[\left(x + \gamma_1 x + y + \gamma_2 y + v + \gamma_3 v + \frac{p}{c}z + \gamma_4 \frac{p}{c}z \right)^\theta - X^\theta - \theta X^{\theta-1} \left(\gamma_1 x + \gamma_2 y + \gamma_3 v + \gamma_4 \frac{p}{c}z \right) \right] \pi(du) \\ &\leq \theta X^{\theta-1} \left[\Lambda - dx - \delta y - \mu v - \frac{bp}{c}z + N\delta y \right] + \frac{\theta(\theta-1)}{2} X^\theta (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) + \int_Z X^\theta \left[(1 + \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4)^\theta - 1 - (\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4) \right] \pi(du) \\ &\leq \theta X^{\theta-2} \left[\Lambda X - \left(\mu^* - \frac{\theta-1}{2} \sigma^2 - \frac{1}{\theta} \lambda^* \right) X^2 \right] \end{aligned}$$

where $\mu^* = \min\{d, \delta, \mu, b\} - N\delta$, $\theta > 2$, $b = \mu^* - \frac{\theta-1}{2} \sigma^2 - \frac{1}{\theta} \lambda^* > 0$,

$$\sigma^2 = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2,$$

$$\lambda^* = \int_Z \left[(1 + \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4)^\theta - 1 - (\gamma_1 \wedge \gamma_2 \wedge \gamma_3 \wedge \gamma_4) \right] \pi(du).$$

Thus, we have

$$dV(X) \leq \theta X^{\theta-2} (\Lambda X - bX^2) dt + \theta X^{\theta-1} \left[\sigma_1 dB_1(t) + \sigma_2 dB_2(t) + \sigma_3 dB_3(t) + \sigma_4 \frac{p}{c} dB_4(t) \right] + \int_Z X^\theta \left[(1 + \gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4)^\theta - 1 \right] \tilde{N}(dt, du).$$

The following proof is similar with [16].

Lemma 2.3. We assume for some $\theta > 2$, $\mu^* - \frac{\theta-1}{2} \sigma^2 - \frac{1}{\theta} \lambda^* > 0$ holds. For

any initial value $(x(0), y(0), v(0), z(0)) \in \mathbb{R}_+^3$, model (1.3) has a unique positive solution $(x(t), y(t), v(t), z(t)) \in \mathbb{R}_+^3$ for any $t \geq 0$ almost surely. Furthermore, the solution $(x(t), y(t), v(t), z(t))$ of model (1.3) has the following properties:

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln x(t)}{t} &\leq 0, \quad \limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} \leq 0, \\ \limsup_{t \rightarrow \infty} \frac{\ln v(t)}{t} &\leq 0, \quad \limsup_{t \rightarrow \infty} \frac{\ln z(t)}{t} \leq 0, \quad a.s. \end{aligned}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\int_0^t x(r) dB_1(r)}{t} &= 0, \quad \lim_{t \rightarrow \infty} \frac{\int_0^t y(r) dB_2(r)}{t} = 0, \\ \lim_{t \rightarrow \infty} \frac{\int_0^t v(r) dB_3(r)}{t} &= 0, \quad \lim_{t \rightarrow \infty} \frac{\int_0^t v(r) dB_3(r)}{t} = 0, \\ \lim_{t \rightarrow \infty} \frac{\int_0^t \int_Z \gamma_1(z) x(s) \tilde{N}(ds, du)}{t} &= 0, \quad \lim_{t \rightarrow \infty} \frac{\int_0^t \int_Z \gamma_2(z) y(s) \tilde{N}(ds, du)}{t} = 0, \\ \lim_{t \rightarrow \infty} \frac{\int_0^t \int_Z \gamma_3(z) v(s) \tilde{N}(ds, du)}{t} &= 0, \quad \lim_{t \rightarrow \infty} \frac{\int_0^t \int_Z \gamma_4(z) z(s) \tilde{N}(ds, du)}{t} = 0, \text{ a.s.} \end{aligned}$$

The proof of Lemma 2.3 obtained inspired by ref [17], the Burkholder-Davis-Gundy inequality and Hölder inequality.

Theorem 2.1. For any given initial value $(x(0), y(0), v(0), z(0)) \in \mathbb{R}_+^4$, there is a unique positive solution $(x(t), y(t), v(t), z(t))$ of model (1.3) on $t \geq 0$ and the solution will remain in \mathbb{R}_+^4 with probability 1, namely $(x(t), y(t), v(t), z(t)) \in \mathbb{R}_+^4$ for all $t \geq 0$ almost surely.

Proof. Since the coefficients of the equations is locally Lipschitz continuous, for any given initial value $(x(0), y(0), v(0), z(0)) \in \mathbb{R}_+^4$, there is a unique local solution $(x(t), y(t), v(t), z(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = \infty$ a.s. At first, we prove $x(t)$, $y(t)$, $v(t)$, and $z(t)$ do not explode to infinity in a finite time. Set $m_0 > 0$ be sufficiently large, so that $x(0)$, $y(0)$, $v(0)$ and $z(0)$ lie with the interval $\left[\frac{1}{m_0}, m_0\right]$. For each integer $m \geq m_0$, define the stopping time

$$\begin{aligned} \tau_m &= \inf \left\{ t \in [0, \tau_e) : \min(x(t), y(t), v(t), z(t)) \leq \frac{1}{m} \right. \\ &\quad \left. \text{or } \max(x(t), y(t), v(t), z(t)) \geq m \right\}. \end{aligned}$$

Clearly, τ_m is increasing as $m \rightarrow \infty$. Set $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, where $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ is true, then $\tau_e = \infty$ and $(x(t), y(t), v(t), z(t)) \in \mathbb{R}_+^4$ a.s. If this statement is false, then there exist a pair of constants $T > 0$ and $0 < \varepsilon < 1$, such that

$$P(\tau_\infty \leq T) \geq \varepsilon.$$

Hence, there is an integer $m_1 \geq m_0$, such that

$$P(\tau_{m_1} \leq T) \geq \varepsilon \text{ for all } m_1 \geq m_0.$$

Define a C^4 -function by

$$\begin{aligned} V(x, y, v, z) &= \left(x - a_1 - a_1 \ln \frac{x}{a_1} \right) + \left(y - a_2 - a_2 \ln \frac{y}{a_2} \right) \\ &\quad + c_1 (v - 1 - \ln v) + c_2 (z - 1 - \ln z), \end{aligned}$$

where a is a positive constant to be defined later. The nonnegativity of this function can be seen from

$$(u - 1 - \ln u) \geq 0 \text{ for } u \geq 0.$$

Let $m \geq m_0$ and $T > 0$ be arbitrary. Using Itô's formula, we get

$$\begin{aligned} dV(x, y, v, z) = & LV(x, y, v, z)dt + (x - a_1)\sigma_1 dB_1(t) + (y - a_2)\sigma_2 dB_2(t) \\ & + (v - 1)c_1\sigma_3 dB_3(t) + (z - 1)c_2\sigma_4 dB_4(t) + \int_Z [\gamma_1(u)x(t) \\ & - a_1 \ln(1 + \gamma_1(u)) + \gamma_2(u)y(t) - a_2 \ln(1 + \gamma_2(u)) + c_1\gamma_3(u)v(t) \\ & - c_1 \ln(1 + \gamma_3(u)) + c_2\gamma_4(u)z(t) - c_2 \ln(1 + \gamma_4(u))] \tilde{N}(dt, du). \end{aligned}$$

where

$$\begin{aligned} LV(x, y, v, z) \leq & (\Lambda + a_1d + a_2\delta + c_1\mu + bc_2) - dx + (c_1N\delta - c_2c - \delta)y \\ & + (ka_1 - c_1\mu)v + (pa_2 - bc_2)z + (c_2c - p)yz - \frac{a_1\Lambda}{x} \\ & - \frac{a_2kxv}{y} - \frac{c_1N\delta y}{v} + \frac{a_1}{2}\sigma_1^2 + \frac{a_2}{2}\sigma_2^2 + \frac{c_1}{2}\sigma_3^2 + \frac{c_2}{2}\sigma_4^2 \\ & + \int_Z [a_1(\gamma_1(u) - \ln(1 + \gamma_1(u))) + a_2(\gamma_2(u) - \ln(1 + \gamma_2(u))) \\ & + c_1(\gamma_3(u) - \ln(1 + \gamma_3(u))) + c_2(\gamma_4(u) - \ln(1 + \gamma_4(u)))] \pi(du) \\ \leq & \left(\Lambda + a_1d + a_2\delta + c_1\mu + bc_2 \frac{a_1}{2}\sigma_1^2 + \frac{a_2}{2}\sigma_2^2 + \frac{c_1}{2}\sigma_3^2 + \frac{c_2}{2}\sigma_4^2 \right) + 4k_2 \\ =: & M \end{aligned}$$

where $a_1 = \frac{\mu p + \mu \delta}{kN\delta}$, $a_2 = \frac{b}{c}$, $c_1 = \frac{p + \delta}{N\delta}$ and $c_2 = \frac{p}{c}$. Let

$$k_2 = \max \left\{ \int_Z a_1(\gamma_1(u) - \ln(1 + \gamma_1(u))) \pi(du), \int_Z a_2(\gamma_2(u) - \ln(1 + \gamma_2(u))) \pi(du), \int_Z c_1(\gamma_3(u) - \ln(1 + \gamma_3(u))) \pi(du), \int_Z c_2(\gamma_4(u) - \ln(1 + \gamma_4(u))) \pi(du) \right\}.$$

The proof of the remainder is similar to the proof [18], so we omitted it.

3. Extinction of the Disease

For simplicity, we introduce the following notations:

$$\bar{\sigma}_i = \frac{1}{2}\sigma_i^2 + \int_Z [\gamma_i(u) - \ln(1 + \gamma_i(u))] \pi(du), \quad i = 1, 2, 3;$$

$$K_i(t) = \int_0^t \int_Z \ln(1 + \gamma_i(u)) \tilde{N}(ds, du), \quad i = 1, 2, 3;$$

$$\langle x(t) \rangle = \frac{1}{t} \int_0^t x(s) ds.$$

$$\tilde{R}_0 = \frac{2Nks}{d \left(\mu + \frac{1}{2}\sigma_3^2 \right)^2 \wedge \frac{1}{2}\sigma_2^2}$$

Theorem 3.1. Assume that for any given initial value

$(x(0), y(0), v(0), z(0)) \in \mathbb{R}_+^4$, the solution $(x(t), y(t), v(t), z(t)) \in \mathbb{R}_+^4$ of model (1.3) has the property

$$\lim_{t \rightarrow \infty} y(t) = 0, \quad \lim_{t \rightarrow \infty} v(t) = 0 \text{ a.s.}$$

If $\tilde{R}_0 < 0$ holds, then $y(t)$ and $v(t)$ will go to zero exponentially with probability one.

Proof. Define $F(x, y, v, z) = x + y + \frac{1}{N}v + \frac{p}{c}z$. Applying Itô's formula [19],

we obtain

$$\begin{aligned}
 dV(x, y, v, z) = & \left(\Lambda - dx - \frac{\mu}{N}v - \frac{bp}{c}z \right) dt + \sigma_1 x dB_1(t) + \sigma_2 y dB_2(t) \\
 & + \frac{\sigma_3}{N}v dB_3(t) + \frac{p}{c}\sigma_4 z dB_2(t) + \int_Z \left[\gamma_1(u)x + \gamma_2(u)y \right. \\
 & \left. + \frac{1}{N}\gamma_3(u)v + \frac{p}{c}\gamma_4(u)z \right] \tilde{N}(dt, du). \tag{3.1}
 \end{aligned}$$

Integrating both sides of (3.1) from 0 to t yields,

$$\begin{aligned}
 & \frac{x(t) - x(0)}{t} + \frac{y(t) - y(0)}{t} + \frac{1}{N} \frac{v(t) - v(0)}{t} + \frac{p}{c} \frac{z(t) - z(0)}{t} \\
 = & \Lambda - d \langle x(t) \rangle - \frac{\mu}{N} \langle v(t) \rangle - \frac{pb}{cd} \langle z(t) \rangle + \frac{\sigma_1}{t} \int_0^t x(s) dB(s) \\
 & + \frac{\sigma_2}{t} \int_0^t y(s) dB(s) + \frac{\sigma_3}{Nt} \int_0^t v(s) dB(s) + \frac{p\sigma_4}{ct} \int_0^t z(s) dB(s) \\
 & + \frac{1}{t} \int_0^t \int_Z (\gamma_1(u)x(s) + \gamma_2(u)y(s)) \tilde{N}(ds, du) \\
 & + \frac{1}{Nt} \int_0^t \int_Z \gamma_3(u)v(s) \tilde{N}(ds, du) + \frac{p}{ct} \int_0^t \int_Z \gamma_4(u)z(s) \tilde{N}(ds, du).
 \end{aligned}$$

Clearly, we can derive that

$$\langle x(t) \rangle = \frac{\Lambda}{d} - \frac{\mu}{Nd} \langle v(t) \rangle - \frac{pb}{cd} \langle z(t) \rangle + \varphi(t), \tag{3.2}$$

where

$$\begin{aligned}
 \varphi(t) = & - \left[\frac{x(t) - x(0)}{t} + \frac{y(t) - y(0)}{t} + \frac{1}{N} \frac{v(t) - v(0)}{t} + \frac{p}{c} \frac{z(t) - z(0)}{t} \right. \\
 & - \frac{\sigma_1}{t} \int_0^t x(s) dB(s) - \frac{\sigma_2}{t} \int_0^t y(s) dB(s) - \frac{\sigma_3}{Nt} \int_0^t v(s) dB(s) \\
 & - \frac{p\sigma_4}{ct} \int_0^t z(s) dB(s) - \frac{1}{t} \int_0^t \int_Z (\gamma_1(u)x(s) + \gamma_2(u)y(s)) \tilde{N}(ds, du) \\
 & \left. - \frac{1}{Nt} \int_0^t \int_Z \gamma_3(u)v(s) \tilde{N}(ds, du) - \frac{p}{ct} \int_0^t \int_Z \gamma_4(u)z(s) \tilde{N}(ds, du) \right].
 \end{aligned}$$

Making use of Lemma 2.2, we can see

$$\lim_{t \rightarrow \infty} \varphi(t) = 0. \tag{3.3}$$

Define $M(t) = Ny + v$,

Applying Itô's formula, we can conclude that

$$\begin{aligned}
 d \ln M(t) & = \left[\frac{1}{Ny + v} (kNxv - pNyz - \mu v) - \frac{N^2 \sigma_2^2 y^2 + \sigma_3^2 v^2}{2(Ny + v)^2} \right. \\
 & \left. + \int_Z [\ln(1 + \gamma_2(u)) - \gamma_2(u)] \pi(du) + \int_Z [\ln(1 + \gamma_3(u)) - \gamma_3(u)] \pi(du) \right] dt \\
 & + \frac{N\sigma_2 y}{Ny + v} dB_2(t) + \frac{\sigma_3 v}{Ny + v} dB_3(t) + \int_Z [\ln(1 + \gamma_2(u)) + \ln(1 + \gamma_3(u))] N(ds, du) \tag{3.4}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left[\frac{kNxv}{v} - \frac{1}{(Ny+v)^2} \left(\frac{\sigma_2^2}{2} N^2 y^2 + \left(\mu + \frac{\sigma_3^2}{2} \right) v^2 \right) \right] dt + \frac{N\sigma_2 y}{Ny+v} dB_2(t) \\
 &+ \frac{\sigma_3 v}{Ny+v} dB_3(t) + \int_Z \left[\ln(1+\gamma_2(u)) + \ln(1+\gamma_3(u)) \right] N(ds, du) \\
 &\leq \left[kNx - \frac{1}{2} \left(\left(\mu + \frac{1}{2} \sigma_3^2 \right)^2 \wedge \frac{1}{2} \sigma_2^2 \right) \right] dt + \frac{N\sigma_2 y}{Ny+v} dB_2(t) + \frac{\sigma_3 v}{Ny+v} dB_3(t) \\
 &+ \int_Z \left[\ln(1+\gamma_2(u)) + \ln(1+\gamma_3(u)) \right] N(ds, du). \tag{3.5}
 \end{aligned}$$

Integrating from 0 to t on both sides of (3.4), we obtain

$$\begin{aligned}
 \frac{\ln M(t)}{t} &\leq kN \langle x(t) \rangle - \frac{1}{2} \left(\left(\mu + \frac{1}{2} \sigma_3^2 \right)^2 \wedge \frac{1}{2} \sigma_2^2 \right) + \frac{N\sigma_2}{t} \int_0^t \frac{y}{Ny+v} dB_2(s) \\
 &+ \frac{\sigma_3}{t} \int_0^t \frac{v}{Ny+v} dB_3(s) + \frac{1}{t} \int_0^t \int_Z \left[\ln(1+\gamma_2(u)) \right. \\
 &\left. + \ln(1+\gamma_3(u)) \right] N(ds, du) + \frac{\ln M(0)}{t} \\
 &\leq Nk \left[\frac{\Lambda}{d} - \frac{\mu}{Nd} \langle v(t) \rangle - \frac{pb}{cd} \langle z(t) \rangle + \varphi(t) \right] \\
 &- \frac{1}{2} \left(\left(\mu + \frac{1}{2} \sigma_3^2 \right)^2 \wedge \frac{1}{2} \sigma_2^2 \right) + \phi(t) \\
 &\leq \frac{Nk\Lambda}{d} - \frac{1}{2} \left(\left(\mu + \frac{1}{2} \sigma_3^2 \right)^2 \wedge \frac{1}{2} \sigma_2^2 \right) + Nk\varphi(t) + \phi(t),
 \end{aligned}$$

where,

$$\phi(t) = \frac{\sigma_2}{t} B_2(t) + \frac{\sigma_3}{t} B_3(t) + \frac{K_2(t)}{t} + \frac{K_3(t)}{t} + \frac{\ln M(0)}{t}.$$

According to Assumption 1,

$$\langle k_2, k_2 \rangle(t) = t \int_Y \left[\ln(1+\gamma_2(u)) \right]^2 \pi(du) < tc,$$

where $\langle k_2, k_2 \rangle(t)$ is k_2 's quadratic.

According to the large number theorem for martingales [20], we have

$$\lim_{t \rightarrow \infty} \frac{B_i(t)}{t} = 0, \quad i = 2, 3 \quad a.s. \tag{3.6}$$

Then, by Lemma 2 [21] [22], we obtain

$$\lim_{t \rightarrow \infty} \frac{K_i(t)}{t} = 0 \quad i = 1, 2, 3 \quad a.s. \tag{3.7}$$

According to (2.2), (3.6) and (3.7), we have

$$\lim_{t \rightarrow \infty} \phi(t) = 0.$$

Clearly, if $\tilde{R}_0 < 1$ holds, then

$$\limsup_{t \rightarrow \infty} \frac{\ln M(t)}{t} \leq 0 \quad a.s.$$

Thus, we have

$$\lim_{t \rightarrow \infty} y(t) = 0, \quad \lim_{t \rightarrow \infty} v(t) = 0 \quad a.s. \tag{3.8}$$

That is to say, the disease will die out with probability one.

4. Persistence

Definition 4.1. Model (1.3) is said to be persistent in the mean, if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds > 0, \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t v(s) ds > 0, \quad a.s.$$

$$\tilde{R}_1 = \frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right)(\mu + \bar{\sigma}_3)(d + \bar{\sigma}_1)}$$

Theorem 4.1. For any solution $(x(t), y(t), v(t), z(t))$ of model (1.3) with initial value $(x(0), y(0), v(0), z(0)) \in \mathbb{R}_+^4$, if $\lambda > 0$, then there exists

$$\lim_{t \rightarrow \infty} \langle v(t) \rangle \geq \frac{\lambda}{k}, \quad \lim_{t \rightarrow \infty} \langle y(t) \rangle \geq \frac{\mu \lambda}{k N \delta} \quad a.s.$$

Proof. Define $F(x, y, v, z) = -\ln x - c_1 \left[\ln y - \frac{1}{b}(cx + cy + pz) \right] - c_2 \ln v$, we have

$$\begin{aligned} dF(x, y, v, z) = & LF(x, y, v, z) dt + \left(\frac{c_1 c}{b} \sigma_1 x - \sigma_1 \right) dB_1(t) \\ & + \left(\frac{c_1 c}{b} \sigma_2 y - c_1 \sigma_2 \right) dB_2(t) - c_2 \sigma_3 dB_3(t) + \frac{c_1 p}{b} \sigma_4 z dB_4(t) \\ & - \int_Z \left[\ln(1 + \gamma_1(u)) + c_1 \ln(1 + \gamma_2(u)) + c_2 \ln(1 + \gamma_3(u)) \right] \tilde{N}(dt, du) \\ & + \frac{c_1}{b} \int_Z \left[c\gamma_1(u)x + c\gamma_2(u)y + p\gamma_4(u)z \right] \tilde{N}(dt, du). \end{aligned}$$

$$\begin{aligned} LF(x, y, v, z) = & -\frac{\Lambda}{x} - \frac{c_1 k x v}{y} - \frac{c_2 N \delta y}{v} + d + kv + c_1 \left(\delta + \frac{c\Lambda}{b} \right) + \frac{c_1}{2} \sigma_2^2 \\ & + \int_Z c_1 \left[\gamma_2 - \ln(1 + \gamma_2) \right] \pi(du) + c_2 \mu + \frac{c_2}{2} \sigma_3^2 \\ & + \int_Z c_2 \left[\gamma_3 - \ln(1 + \gamma_3) \right] \pi(du) + \frac{1}{2} \sigma_1^2 \\ & + \int_Z \left[\gamma_1 - \ln(1 + \gamma_1) \right] \pi(du) - \frac{c_1 c \delta}{b} y - \frac{c_1 c d}{b} x \\ \leq & -3\sqrt{\Lambda c_1 k c_2 N \delta} + d + kv + \bar{\sigma}_1 + c_1 \left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2 \right) + c_2 (\mu + \bar{\sigma}_3). \end{aligned}$$

Let,

$$c_1 \left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2 \right) = c_2 (\mu + \bar{\sigma}_3) = \frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2 \right) (\mu + \bar{\sigma}_3)}.$$

So,

$$c_1 = \frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right)^2 (\mu + \bar{\sigma}_3)}, \quad c_2 = \frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right) (\mu + \bar{\sigma}_3)^2}.$$

Therefore,

$$\begin{aligned} LF(x, y, v, z) &\leq -\frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right) (\mu + \bar{\sigma}_3)} + d + kv + \bar{\sigma}_1 \\ &= -(d + \bar{\sigma}_1) \left[\frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right) (\mu + \bar{\sigma}_3) (d + \bar{\sigma}_1)} - 1 \right] + kv \quad (4.2) \\ &= -\lambda + kv, \end{aligned}$$

where,

$$\lambda = (d + \bar{\sigma}_1) \left[\frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right) (\mu + \bar{\sigma}_3) (d + \bar{\sigma}_1)} - 1 \right].$$

Substituting (4.2) into (4.1), then we obtain,

$$\begin{aligned} dF(x, y, v, z) &\leq -\lambda + kv + \left(\frac{c_1 c}{b} \sigma_1 x - \sigma_1\right) dB_1(t) + \left(\frac{c_1 c}{b} \sigma_2 y - c_1 \sigma_2\right) dB_2(t) \\ &\quad - c_2 \sigma_3 dB_3(t) + \frac{c_1 p}{b} \sigma_4 z dB_4(t) - \int_Z [\ln(1 + \gamma_1(u)) \\ &\quad + c_1 \ln(1 + \gamma_2(u)) + c_2 \ln(1 + \gamma_3(u))] \tilde{N}(dt, du) \quad (4.3) \\ &\quad + \frac{c_1}{b} \int_Z [c\gamma_1(u)x + c\gamma_2(u)y + p\gamma_4(u)z] \tilde{N}(dt, du) \\ &\quad - \int_Z [\ln(1 + \gamma_1(u)) + c_1(1 + \gamma_2(u)) + c_2(1 + \gamma_3(u))] \tilde{N}(dt, du). \end{aligned}$$

Integrating from 0 to t on both side of (4.3), yields

$$\begin{aligned} \frac{c_2 \ln v(t)}{t} &\geq \lambda - k \langle v(t) \rangle + \frac{1}{t} \int_0^t \left(\sigma_1 - \frac{c_1 c}{b} \sigma_1 x\right) dB_1(s) + \frac{1}{t} \int_0^t \left(c_1 \sigma_2 - \frac{c_1 c}{b} \sigma_2 y\right) dB_2(s) \\ &\quad + \frac{1}{t} \int_0^t c_2 \sigma_3 dB_3(s) - \frac{1}{t} \int_0^t \frac{c_1 p}{b} \sigma_4 z dB_4(s) + \frac{K_1(t)}{t} + \frac{K_2(t)}{t} + \frac{K_3(t)}{t} \\ &\quad - \frac{1}{t} \int_0^t \int_Z \left[\frac{c_1}{b} (c\gamma_1(u)x + c\gamma_2(u)y + p\gamma_4(u)z)\right] \tilde{N}(ds, du) \\ &\quad - \frac{\ln x(t) - \ln x(0)}{t} - \frac{c_1 (\ln y(t) - \ln y(0))}{t} + \frac{c_1 c y(t) - y(0)}{b t} \\ &\quad + \frac{c_1 p z(t) - z(0)}{b t} + \frac{c_1 c x(t) - x(0)}{b t} + \frac{c_2 \ln v(0)}{t}. \end{aligned}$$

Consider of Lemma 2.1, 2.2, 2.3 and (3.7) if $\tilde{R}_1 \geq 1$, then we have

$$\lim_{t \rightarrow \infty} \langle v(t) \rangle \geq \frac{\lambda}{k}, \quad (4.4)$$

that is to say,

$$\liminf_{t \rightarrow \infty} \langle v(t) \rangle \geq \frac{\lambda}{k}.$$

On the other hand, base on the model of (1.3), we have

$$\frac{v(t) - v(0)}{t} = N\delta \langle y(t) \rangle - \mu \langle v(t) \rangle + \int_0^t \frac{\sigma_3 v dB_3(\gamma)}{t} + \frac{1}{t} \int_0^t \int_Z \gamma_3(u) v \tilde{N}(ds, du),$$

we get,

$$\langle y(t) \rangle = \frac{\mu}{N\delta} \langle v(t) \rangle + \frac{1}{N\delta} \psi(t),$$

where,

$$\psi(t) = - \left[\frac{v(0) - v(t)}{t} + \int_0^t \frac{\sigma_3 v dB_3(\gamma)}{t} + \frac{1}{t} \int_0^t \int_Z \gamma_3(u) v(ds) \right].$$

According to Lemma 2.2 and Lemma 2.3, we have

$$\lim_{t \rightarrow \infty} \psi(t) = 0,$$

According to (4.4), we obtain

$$\lim_{t \rightarrow \infty} \langle y(t) \rangle \geq \frac{\mu\lambda}{kN\delta},$$

Thus,

$$\liminf_{t \rightarrow \infty} \langle y(t) \rangle \geq \frac{\mu\lambda}{kN\delta}.$$

5. Numerical Results

In this section, we present some numerical simulations to discuss the effect of lévy noise on the viral dynamics. Euler scheme is an order $\frac{1}{2}$ approximation, be used to investigate the dynamics of stochastic differential equations driven by lévy process in some papers [23] [24] [25] [26].

In **Figure 1**, fix parameters as follows: $\Lambda = 8, k = 0.1, \delta = 0.5, d = 0.3, N = 3, \mu = 2, c = 0.3, b = 2.5, p = 0.9$, the intensity of noise $\sigma_1 = 5, \sigma_2 = 2, \sigma_3 = 6, \sigma_4 = 1, \gamma_1 = 0.1, \gamma_2 = 0.3, \gamma_3 = 0.4, \gamma_4 = 0.5$. Note that

$$\tilde{R}_0 = \frac{2Nks}{d \left(\mu + \frac{1}{2} \sigma_3^2 \right) \wedge \frac{1}{2} \sigma_2^2} = 0.67 < 1. \text{ We compared the process of deterministic}$$

model, stochastic model with white noise and stochastic model with Lévy jump, the infected CD4⁺ T-cells, virus particles and CTLs cells extinction in all cases, but strong fluctuation will accelerate the extinction of all.

In **Figure 2**, fix parameters as follows: $\Lambda = 8, k = 0.3, \delta = 0.5, d = 0.3, N = 10, \mu = 0.9, c = 0.6, b = 2.4, p = 0.9$, the intensity of noise $\sigma_1 = 1.5, \sigma_2 = 1.2, \sigma_3 = 0.5, \sigma_4 = 2, \gamma_1 = 1, \gamma_2 = 0.3, \gamma_3 = 0.4, \gamma_4 = 0.5$. Note that

$$\tilde{R}_1 = \frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2 \right) (\mu + \bar{\sigma}_3) (d + \bar{\sigma}_1)} > 1, \text{ that the infected CD4}^+ \text{ T-cells, virus}$$

particles and CTLs cells existence in all cases.

In **Figure 3**, fix parameters as follows: $\Lambda = 8$, $k = 0.3$, $\delta = 0.5$, $d = 0.3$, $N = 10$, $\mu = 0.9$, $c = 0.6$, $b = 4.2$, $p = 0.9$, the intensity of noise $\sigma_1 = 1.5$, $\sigma_2 = 1.2$, $\sigma_3 = 1.6$, $\sigma_4 = 2$, $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 0.4$, $\gamma_4 = 0.5$. Note that

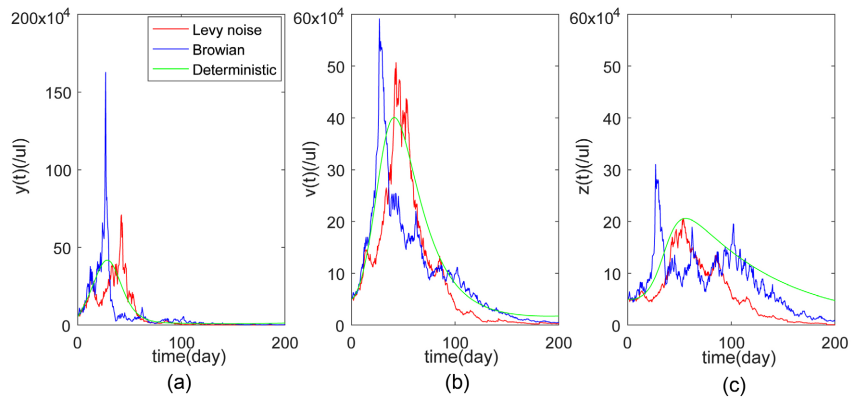


Figure 1. Numerical simulation of the path $y(t)$, $v(t)$ and $z(t)$ for the deterministic model, stochastic model with white noise and stochastic model with lévy jump. The infected $CD4^+$ cells, virus particles and CTLs cells extinction in all cases.

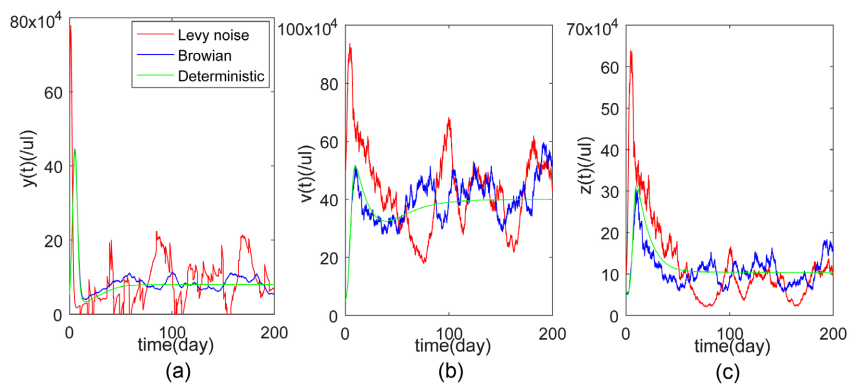


Figure 2. Numerical simulation of the path $y(t)$, $v(t)$ and $z(t)$ for the deterministic model, stochastic model with white noise and stochastic model with lévy jump. The infected $CD4^+$ cells, virus particles and CTLs cells exist in all cases.

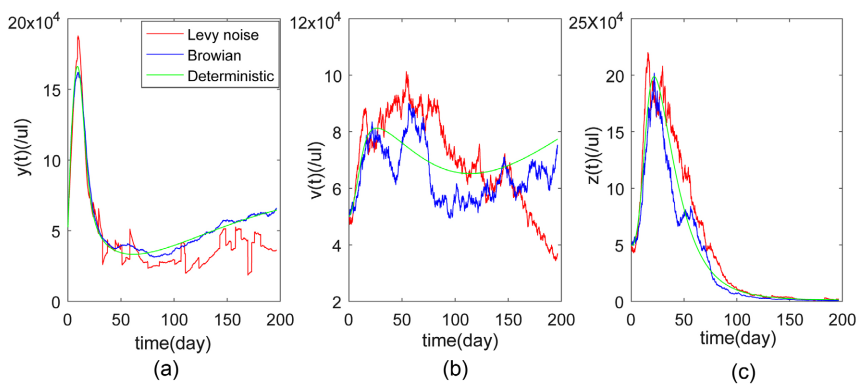


Figure 3. Numerical simulation of the path $y(t)$, $v(t)$ and $z(t)$ for the deterministic model, stochastic model with white noise and stochastic model with lévy jump. The infected $CD4^+$ cells and virus particles exist but CTLs cells extinction in all cases.

$$\tilde{R}_1 = \frac{\Lambda k N \delta}{\left(\delta + \frac{c\Lambda}{b} + \bar{\sigma}_2\right)(\mu + \bar{\sigma}_3)(d + \bar{\sigma}_1)} > 1, \text{ the infected CD4}^+ \text{ cells and virus particles}$$

exist but CTLs cells extinction in all cases, but this phenomenon can't be obtained by theoretical analysis. From **Figure 3(a)** and **Figure 3(b)**, we can see that strong fluctuation will result in the decrease of the infected CD4⁺ T-cells, virus particles.

6. Conclusions

This paper investigated the dynamics of a stochastic HIV infection model with CTLs immune response and lévy jumps. Through theoretical analysis and numerical simulations, we obtain some results about HIV infection on a cellular level driven by lévy noise. First, we investigate the global existence and unique positive solutions; then by constructing a suitable stochastic Lyapunov function, we give sufficient conditions that $\tilde{R}_0 < 1$, the infected CD4⁺ T-cells, virus particles and CTLs cells extinction in probability. Then, we adopted a special method to deal with the model, and obtained if $\tilde{R}_1 > 1$ the infected CD4⁺ T-cells, virus particles and CTLs cells persistence in the mean. By numerical simulations, it verifies the theoretical results, and we also observe some meaningful phenomena that, strong perturbation of the environment is a benefit to the extinction of the infected CD4⁺ T-cells and virus particles.

Due to the inherently stochastic nature of HIV infection, some interesting topics deserve further discussion, such as considering the effects of time delay on the stochastic system, we will go about this case subsequently.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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