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Analytical Solution of Nonlinear System of Fractional Differential Equations

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Abstract

In this paper, we apply the Adomian decomposition method (ADM) for solving nonlinear system of fractional differential equations (FDEs). The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed. Some applications are solved such as fractional-order rabies model.

Keywords

Fractional Differential Equations, Adomian Decomposition Method, Existence, Uniqueness, Error Analysis, Rabies Model

1. Introduction

This paper is concerned with the analytical solution of a nonlinear system of fractional differential equations. Systems of fractional differential equations (FDEs) have many applications in engineering and science, including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems ([1]-[16]). We use Adomain decomposition method ([17]-[24]) for solving this type of equations. The existence and uniqueness of the solution are proved, the convergence of ADM series solution is discussed and the error analysis is given. This method has many advantages; it is efficiently working with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization.

2. Formulation of the Problem

Consider the following nonlinear system of FDEs:

$$ {}_0\mathcal{D}_t^{\sigma_n} y_i(t) + \beta_i(t) f_i(\bar{y}) = x_i(t), \tag{1} $$

subject to the initial conditions,

$$ y_i^{(j-1)}(0) = c_{ij}, \quad i, j = 1, 2, \cdots, n. \tag{2} $$

where,

$$ \bar{y} = \{y_1(t), y_2(t), \cdots, y_n(t)\}, $$

$$ {}_0\mathcal{D}_t^{\sigma_n} \equiv {}_0D_t^{\alpha_n}\, {}_0D_t^{\alpha_{n-1}}\, {}_0D_t^{\alpha_{n-2}} \cdots {}_0D_t^{\alpha_1}, $$

$$ \sigma_n = \sum_{k=1}^{n} \alpha_k, \; n-1 \le \alpha_k \le n, $$

where $x_i(t)$ is bounded $\forall t \in J = [0, T]$, $T \in R^+$, $|\beta_i(\tau)| \le M_i$ $\forall 0 \le \tau \le t \le T$, M_i are finite constants and $f_i(\bar{y})$ satisfy Lipschitz condition with Lipschitz constants L_i such as,

$$ |f_i(\bar{y}) - f_i(\bar{z})| \le L_i |\bar{y} - \bar{z}| \tag{3} $$

and the fractional derivative in this system is of sequential Caputo sense which defined as

$$ {}_0D_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{y^{(n)}(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < \alpha < n), $$

In the applications, the Caputo sense is preferred to be used because the initial conditions of $y_i(t)$ and its derivatives will be of integer orders and have a physical meaning.

Now performing subsequently the fractional integration of order $\alpha_n, \alpha_{n-1}, \cdots, \alpha_1$, this reduces the system (1)-(2) to the system of FIEs,

$$ \begin{aligned} y_i(t) = {} & \sum_{j=1}^{n} \frac{c_{ij}}{\Gamma(\sigma_j)} t^{\sigma_j - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} x_i(\tau)\,\mathrm{d}\tau \\ & - \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} f_i(\bar{y})\,\mathrm{d}\tau \end{aligned} \tag{4} $$

and has Adomian polynomials representation,

$$ f_i(\bar{y}) = \sum_{k=0}^{\infty} A_{ik}(y_{i0}, y_{i1}, \cdots, y_{in}) \tag{5} $$

where,

$$ A_{ik} = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \left[f_i\left(\sum_{j=0}^{\infty} \lambda^j y_j \right) \right]_{\lambda=0} \tag{6} $$

Substitute from Equation (5) into Equation (3), we get

$$ \begin{aligned} y_i(t) = {} & \sum_{j=1}^{n} \frac{c_{ij}}{\Gamma(\sigma_j)} t^{\sigma_j - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} x_i(\tau)\,\mathrm{d}\tau \\ & - \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} \sum_{k=0}^{\infty} A_{ik}\,\mathrm{d}\tau. \end{aligned} \tag{7} $$

Let $y_i(t) = \sum_{k=0}^{\infty} y_{ik}(t)$ in (7) we get,

$$y_{i0}(t)=\sum_{j=1}^{n}\frac{c_{ij}}{\Gamma(\sigma_j)}t^{\sigma_j-1}+\frac{1}{\Gamma(\sigma_n)}\int_0^t(t-\tau)^{\sigma_n-1}x_i(\tau)\mathrm{d}\tau, \tag{8}$$

$$y_{ik}(t)=-\frac{1}{\Gamma(\sigma_n)}\int_0^t\beta_i(\tau)(t-\tau)^{\sigma_n-1}A_{i(k-1)}\mathrm{d}\tau, \quad k\ge 1. \tag{9}$$

Finally, the solution is,

$$y_i(t)=\sum_{k=0}^{\infty}y_{ik}(t). \tag{10}$$

3. Analysis of Convergence

3.1. The Uniqueness of Solution

In the previous section, we find the series solution (10) of the system (1)-(2) and here we want to prove the existence and uniqueness of this series solution.

Theorem 1. *If* $0<\alpha<1$ *where* $\alpha=\frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}$, *then the series* (10) *is the solution of the system* (1)-(2) *and this solution is unique, where* $L=\max\{L_1,L_2,\cdots,L_n\}$, $M=\max\{M_1,M_2,\cdots,M_n\}$.

Proof. *For existence,*

$$\begin{aligned}
y_i(t)&=\sum_{k=0}^{\infty}y_{ik}(t)=y_{i0}(t)+\sum_{k=1}^{\infty}y_{ik}(t)\\
&=y_{i0}(t)-\frac{1}{\Gamma(\sigma_n)}\sum_{k=1}^{\infty}\int_0^t\beta_i(\tau)(t-\tau)^{\sigma_n-1}A_{i(k-1)}\mathrm{d}\tau\\
&=y_{i0}(t)-\frac{1}{\Gamma(\sigma_n)}\int_0^t\beta_i(\tau)(t-\tau)^{\sigma_n-1}\sum_{k=1}^{\infty}A_{i(k-1)}\mathrm{d}\tau\\
&=y_{i0}(t)-\frac{1}{\Gamma(\sigma_n)}\int_0^t\beta_i(\tau)(t-\tau)^{\sigma_n-1}\sum_{k=0}^{\infty}A_{ik}\mathrm{d}\tau\\
&=\sum_{j=1}^{n}\frac{c_{ij}}{\Gamma(\sigma_j)}t^{\sigma_j-1}+\frac{1}{\Gamma(\sigma_n)}\int_0^t(t-\tau)^{\sigma_n-1}x_i(\tau)\mathrm{d}\tau\\
&\quad-\frac{1}{\Gamma(\sigma_n)}\int_0^t\beta_i(\tau)(t-\tau)^{\sigma_n-1}f_i(\bar{y})\mathrm{d}\tau
\end{aligned}$$

then the Adomian's series solution satisfy Equation (4) which is the reduced system of FIEs to the system (1)-(2).

For uniqueness of the solution: Assume that $\bar{y}$ and $\bar{z}$ are two different solutions to the system (1)-(2) and hence,

$$\begin{aligned}
|\bar{y}-\bar{z}|&=\left|\frac{1}{\Gamma(\sigma_n)}\int_0^t\beta_i(\tau)(t-\tau)^{\sigma_n-1}\left[f_i(\bar{y})-f_i(\bar{z})\right]\mathrm{d}\tau\right|\\
&\le\frac{1}{\Gamma(\sigma_n)}\int_0^t(t-\tau)^{\sigma_n-1}|\beta_i(\tau)||f_i(\bar{y})-f_i(\bar{z})|\mathrm{d}\tau\\
&\le\frac{L_iM_i}{\Gamma(\sigma_n)}|\bar{y}-\bar{z}|\int_0^t(t-\tau)^{\sigma_n-1}\mathrm{d}\tau
\end{aligned}$$

$$\le \frac{L_i M_i T^{\sigma_n}}{\Gamma(\sigma_n + 1)} |\bar{y} - \bar{z}|$$

$$\le \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n + 1)} |\bar{y} - \bar{z}|$$

Let $\frac{LMT^{\sigma_n}}{\Gamma(\sigma_n + 1)} = \alpha$ where, $0 < \alpha < 1$ then,

$$|\bar{y} - \bar{z}| \le \alpha |\bar{y} - \bar{z}|$$

$$(1-\alpha)|\bar{y} - \bar{z}| \le 0$$

but, $(1-\alpha)|\bar{y} - \bar{z}| \ge 0$ and since, $(1-\alpha) \ne 0$ then, $|\bar{y} - \bar{z}| = 0$ this implies that, $\bar{y} = \bar{z}$ and this completes the proof.

3.2. Proof of Convergence

Theorem 2. *The series solution* (10) *of the system* (1)-(2) *using ADM converges if* $|y_{i1}| < \infty$ *and* $0 < \alpha < 1$, $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n + 1)}$, *where* $L = \max\{L_1, L_2, \cdots, L_n\}$, $M = \max\{M_1, M_2, \cdots, M_n\}$.

Proof. Define the Banach space $(C[J], \|\cdot\|)$, the space of all continuous functions on J with the norm $\|y(t)\| = \max_{t\in J} |y(t)|$ and a sequence $\{S_{in}\}$ such that, $S_{in} = \sum_{k=0}^{n} y_{ik}(t)$. We have,

$$f(S_{in}) = \sum_{k=0}^{n} A_{ik}(y_{i0}, y_{i1}, \cdots, y_{in})$$

Let, S_{in} and S_{im} be two arbitrary partial sums with $n \ge m$. Now, we are going to prove that $\{S_{in}\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned}
\|S_{in} - S_{im}\| &= \max_{t\in J} |S_{in} - S_{im}| \\
&= \max_{t\in J} \left| \sum_{k=m+1}^{n} y_{ik}(t) \right| \\
&= \max_{t\in J} \left| \sum_{k=m+1}^{n} -\frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} A_{i(k-1)} \mathrm{d}\tau \right|
\end{aligned}$$

$$\begin{aligned}
\|S_{in} - S_{im}\| &= \max_{t\in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} \sum_{k=m+1}^{n} A_{i(k-1)} \mathrm{d}\tau \right| \\
&= \max_{t\in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} \sum_{k=m}^{n-1} A_{ik} \mathrm{d}\tau \right| \\
&= \max_{t\in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} \left[f\left(S_{i(n-1)}\right) - f\left(S_{i(m-1)}\right) \right] \mathrm{d}\tau \right| \\
&\le \frac{1}{\Gamma(\sigma_n)} \max_{t\in J} \int_0^t (t-\tau)^{\sigma_n - 1} |\beta_i(\tau)| \left| f\left(S_{i(n-1)}\right) - f\left(S_{i(m-1)}\right) \right| \mathrm{d}\tau \\
&\le \frac{L_i M_i}{\Gamma(\sigma_n)} \max_{t\in J} \left| S_{i(n-1)} - S_{i(m-1)} \right| \left| \int_0^t (t-\tau)^{\sigma_n - 1} \mathrm{d}\tau \right| \\
&\le \frac{LMT^{\sigma_n}}{\sigma_n \Gamma(\sigma_n)} \left\| S_{i(n-1)} - S_{i(m-1)} \right\| \le \alpha \left\| S_{i(n-1)} - S_{i(m-1)} \right\|
\end{aligned}$$

Let $n = m+1$ then,

$$\left\| S_{i(m+1)} - S_{im} \right\| \le \alpha \left\| S_{im} - S_{i(m-1)} \right\| \le \alpha^2 \left\| S_{i(m-1)} - S_{i(m-2)} \right\| \le \cdots \le \alpha^m \left\| S_{i1} - S_{i0} \right\|$$

Using the triangle inequality,

$$\begin{aligned} \left\| S_{in} - S_{im} \right\| &\le \left\| S_{i(m+1)} - S_{im} \right\| + \left\| S_{i(m+2)} - S_{i(m+1)} \right\| + \cdots + \left\| S_{in} - S_{i(n-1)} \right\| \\ &\le \left[\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1} \right] \left\| S_{i1} - S_{i0} \right\| \\ &\le \alpha^m \left[1 + \alpha + \cdots + \alpha^{n-m-1} \right] \left\| S_{i1} - S_{i0} \right\| \\ &\le \alpha^m \left[\frac{1-\alpha^{n-m}}{1-\alpha} \right] \left\| y_{i1}(t) \right\| \end{aligned}$$

Since, $0 < \alpha < 1$, and $n \ge m$ then, $\left(1-\alpha^{n-m}\right) \le 1$. Consequently,

$$\left\| S_{in} - S_{im} \right\| \le \frac{\alpha^m}{1-\alpha} \left\| y_{i1}(t) \right\| \le \frac{\alpha^m}{1-\alpha} \max_{t \in J} \left| y_{i1}(t) \right|$$

but, $\left| y_{i1}(t) \right| \le \infty$ and as $m \to \infty$ then, $\left\| S_{in} - S_{im} \right\| \to 0$ and hence, $\{ S_{in} \}$ is a Cauchy sequence in this Banach space so, the series $\sum_{k=0}^{\infty} y_{ik}(t)$ converges and the proof is complete.

3.3. Error Analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

Theorem 3. *The maximum absolute truncation error of the series solution* (10) *to the system* (1)-(2) *is estimated to be*,

$$\max_{t \in J} \left| y_i(t) - \sum_{k=0}^{m} y_{ik}(t) \right| \le \frac{\alpha^m}{1-\alpha} \max_{t \in J} \left| y_{i1}(t) \right|.$$

Proof. From Theorem 2 we have,

$$\left\| S_{in} - S_{im} \right\| \le \frac{\alpha^m}{1-\alpha} \max_{t \in J} \left| y_{i1}(t) \right|.$$

But, $S_{in} = \sum_{k=0}^{n} y_{ik}(t)$ as $n \to \infty$ then, $S_{in} \to y_i(t)$ so,

$$\left\| y_i(t) - S_{im} \right\| \le \frac{\alpha^m}{1-\alpha} \max_{t \in J} \left| y_{i1}(t) \right|.$$

So, the maximum absolute truncation error in the interval J is,

$$\max_{t \in J} \left| y_i(t) - \sum_{k=0}^{m} y_{ik}(t) \right| \le \frac{\alpha^m}{1-\alpha} \max_{t \in J} \left| y_{i1}(t) \right|$$

and this completes the proof.

4. Numerical Examples

Example 1. Consider the following nonlinear system of FDEs,

$$\begin{aligned} D^\alpha y_1 &= 2 y_2^2, \\ D^\alpha y_2 &= t y_1, \\ D^\alpha y_3 &= y_2 y_3, \end{aligned} \tag{11}$$

subject to the initial conditions,

$$y_1(0)=0, y_2(0)=1, y_3(0)=1,$$

where $\alpha \in (0,1)$.

This system was discussed before in [25], it is solved by using the iterative method. Now, we will solve it by using ADM. Applying ADM to system (11) leads to the following recursive relations,

$$y_{1,0}=0, \quad y_{1,j+1}=J^{\alpha}\left(2A_{1,j}\right), \tag{12}$$

$$y_{2,0}=1, \quad y_{2,j+1}=J^{\alpha}\left(t y_{1,j}\right), \tag{13}$$

$$y_{3,0}=1, \quad y_{3,j+1}=J^{\alpha}\left(A_{2,j}\right), \tag{14}$$

where $A_{1,j}$ and $A_{2,j}$ represent the Adomian polynomials of the nonlinear terms y_2^2 and $y_2 y_3$ respectively.

Using the relations (12)-(14), the first three terms of the series solution when $\alpha=1$ are,

$$\begin{aligned} y_1 &= t+\cdots, \\ y_2 &= 1+\frac{t^3}{3}+\cdots, \\ y_3 &= 1+t+\frac{t^2}{2}+\cdots. \end{aligned} \tag{15}$$

while for $\alpha=0.75$ are,

$$\begin{aligned} y_1 &= \frac{4t^{3/4}}{3\Gamma(3/4)}+\cdots, \\ y_2 &= 1+\frac{32\Gamma(11/4)t^{5/2}}{45\sqrt{\pi}\Gamma(3/4)}+\cdots, \\ y_3 &= 1+\frac{4t^{3/4}}{3\Gamma(3/4)}+\frac{4t^{3/2}}{3\sqrt{\pi}}+\cdots. \end{aligned} \tag{16}$$

and for $\alpha=0.5$ are,

$$\begin{aligned} y_1 &= \frac{2\sqrt{t}}{\sqrt{\pi}}+\cdots, \\ y_2 &= 1+\frac{3t^2}{4}+\cdots, \\ y_3 &= 1+\frac{2\sqrt{t}}{\sqrt{\pi}}+t+\cdots. \end{aligned} \tag{17}$$

Figures 1(a)-(c) show ADM solution of y_1, y_2 and y_3 at different values of α ($\alpha=1, 0.75, 0.5, 0.25$).

Example 2. Consider the following nonlinear system of FDEs,

$$\begin{aligned} D^{0.5}y_1 &= \Gamma(1.5)+y_2^2-t^4, \\ D^{1.5}y_2 &= \frac{\Gamma(3)}{\Gamma(1.5)}y_1+y_1^4-t^2, \\ D^{2.5}y_3 &= \frac{\Gamma(5)}{\Gamma(2.5)}y_1^3, \end{aligned} \tag{18}$$

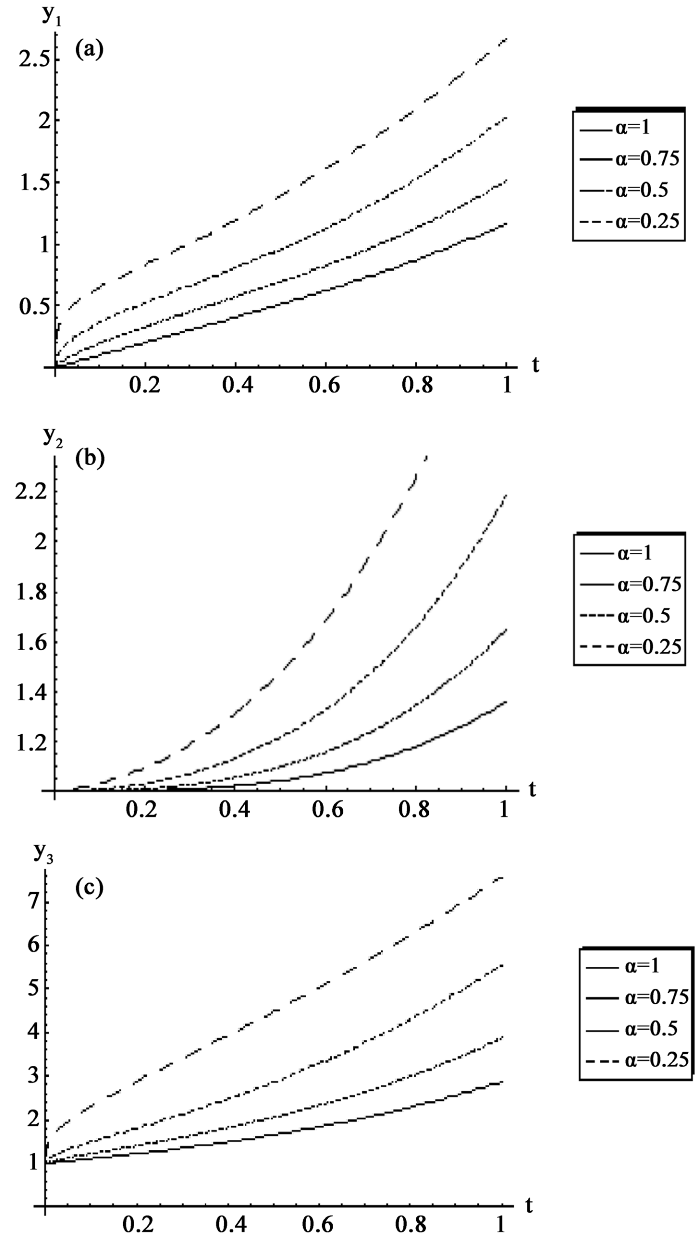

Figure 1. (a) ADM Sol. [$n = 5$]; (b) ADM Sol. [$n = 5$]; (c) ADM Sol [$n = 5$].

subject to the initial conditions,

$$y_1(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 0,$$
$$y_3(0) = 0,\ y_3'(0) = 0,\ y_3''(0) = 0,$$

which has the exact solution $y_1(t) = t^{0.5}$, $y_2(t) = t^2$ and $y_3(t) = t^4$.

Using ADM to system (18) leads to the following scheme,

$$y_{1,0} = t^{1/2} - \frac{\Gamma(5)}{\Gamma(5.5)} t^{4.5}, \quad y_{1,j+1} = J^{1/2}\left(A_{1,j}\right), \tag{19}$$

$$y_{2,0} = -\frac{\Gamma(3)}{\Gamma(4.5)} t^{3.5}, \quad y_{2,j+1} = \frac{\Gamma(3)}{\Gamma(1.5)} J^{1.5}\left(y_{1,j}\right) + J^{1.5}\left(A_{2,j}\right), \tag{20}$$

$$y_{3,0} = 0, \quad y_{3,j+1} = \frac{\Gamma(5)}{\Gamma(2.5)} J^{2.5}\left(A_{3,j}\right), \tag{21}$$

where $A_{1,j}, A_{2,j}$ and $A_{3,j}$ represent the Adomian polynomials of the nonlinear terms y_2^2, y_1^4 and y_1^3 respectively.

Using relations (19)-(21), the first few terms of the series solution are,

$$\begin{aligned} y_1 &= t^{0.5} - 0.458516t^{4.5} + 0.0106171t^{7.5} + \cdots, \\ y_2 &= -0.171943t^{3.5} + t^2 + 0.171943t^{3.5} - 0.0752253t^6 - 0.094092t^{7.5} \\ &\quad + 0.0334503t^{11.5} - 0.00647686t^{15.5} + 0.000523436t^{19.5} + \cdots, \\ y_3 &= t^4 - 0.177317t^8 + 0.0269405t^{12} - 0.00192082t^{16} + \cdots. \end{aligned} \tag{22}$$

A comparison between ADM solution and exact solution of y_1, y_2 and y_3 is given in Figures 2(a)-(c) ($n = 10$).

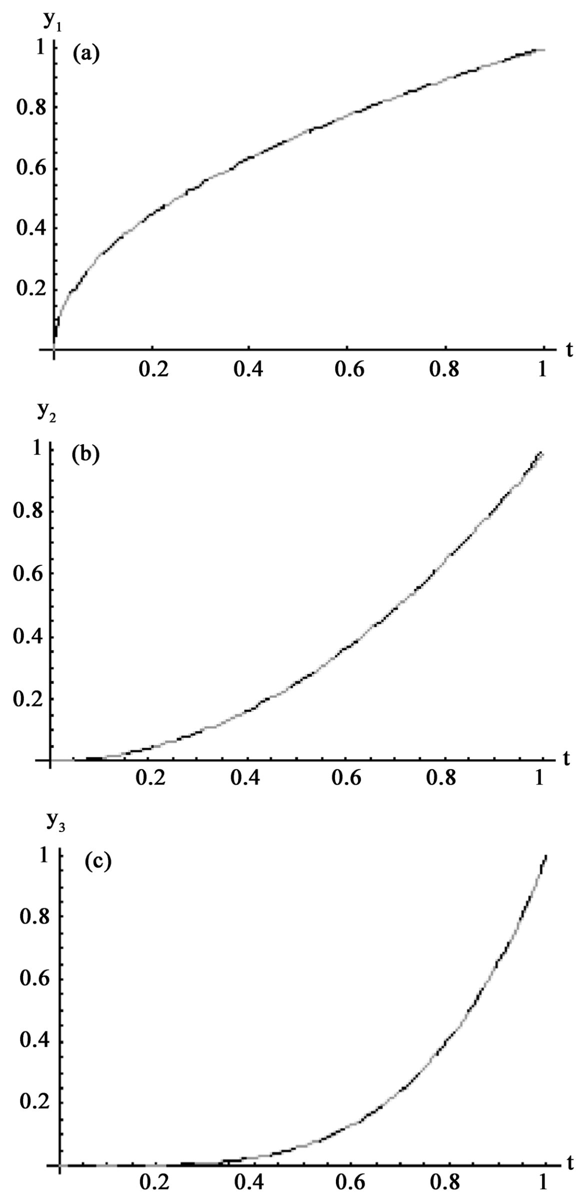

Figure 2. (a) ADM and Exact Sol; (b) ADM and Exact Sol; (c) ADM and Exact Sol.

Example 3. Consider the following nonlinear system of FDEs [26],

$$D^{\alpha} y_1 = y_1^2 + y_2, \quad D^{\alpha} y_2 = y_2 \cos y_1, \tag{23}$$

subject to the initial conditions,

$$y_1(0) = 0, y_2(0) = 1,$$

where $\alpha \in (0,1)$.

Using ADM to system (23) leads to the following scheme,

$$y_{1,0} = 0, \quad y_{1,j+1} = J^{\alpha}\left(A_{1,j}\right) + J^{\alpha}\left(y_{2,j}\right), \tag{24}$$

$$y_{2,0} = 1, \quad y_{2,j+1} = J^{\alpha}\left(A_{2,j}\right), \tag{25}$$

where $A_{1,j}$ and $A_{2,j}$ represent the Adomian polynomials of the nonlinear terms y_1^2 and $y_2 \cos y_1$ respectively.

Using the relations (24)-(25), the first four terms of the series solution are,

$$y_1 = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\left[\Gamma(\alpha+1)\right]^2 \Gamma(3\alpha+1)} + \cdots, \tag{26}$$

$$y_2 = 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\left[\Gamma(\alpha+1)\right]^2 \Gamma(3\alpha+1)} + \cdots. \tag{27}$$

Figure 3(a) and Figure 3(b) show ADM solution of y_1 and y_2 at different

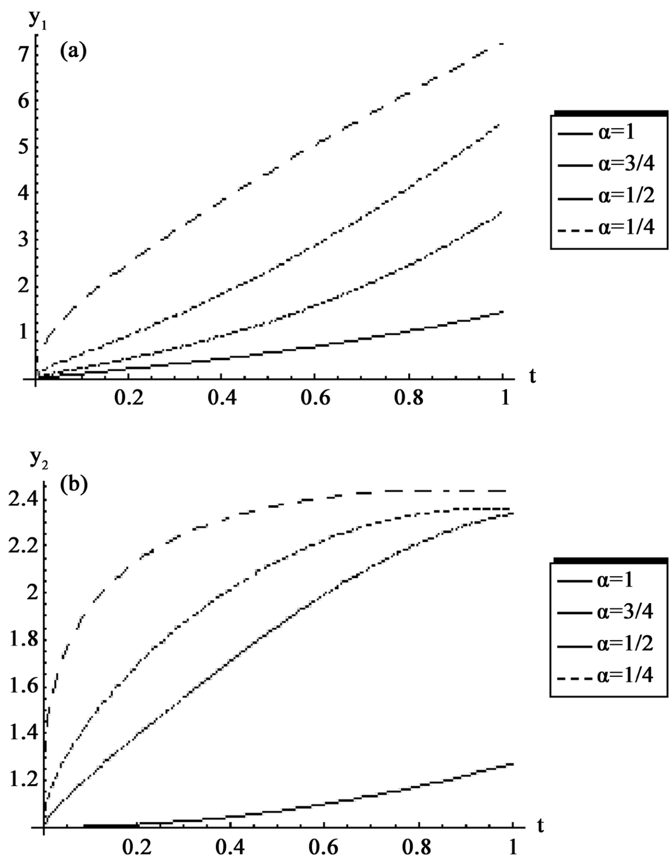

Figure 3. (a) ADM Sol. $[n=5]$; (b) ADM Sol. $[n=5]$.

values of α ($\alpha = 1, 3/4, 1/2, 1/4$).

Example 4. Consider the nonlinear system of FDEs,

$$D^{3/2} y_1 = \frac{1}{8} y_2^2 + t,$$
$$D^{3/2} y_2 = \frac{1}{4} y_1^4 + t^2, \quad 0 < t \le 1, \tag{28}$$

subject to the initial conditions,

$$y_1(0) = 0,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 0.$$

Using ADM to the system (28), we get

$$y_{1,0} = J^{3/2}(t), \quad y_{1,j+1} = \frac{1}{8} J^{3/2}(A_{1,j}),$$
$$y_{2,0} = J^{3/2}(t^2), \quad y_{2,j+1} = \frac{1}{4} J^{3/2}(A_{2,j}), \tag{29}$$

From the relations (29), the first two terms of the series solution are,

$$y_1 = \left(\frac{8t^{5/2}}{15\sqrt{\pi}}\right) + \left(\frac{262144 t^{17/2}}{1206079875\sqrt{\pi}}\right) + \cdots,$$
$$y_2 = \left(\frac{32 t^{7/2}}{105\sqrt{\pi}}\right) + \left(\frac{268435456 t^{23/2}}{1129407654375 \pi^{3/2}}\right) + \cdots. \tag{30}$$

Figure 4(a) and Figure 4(b) show ADM solution of y_1 and y_2 ($m = 5$).

Now, we will use Theorem 3 to evaluate the maximum absolute truncated error of the series solution (30). So, we evaluate the following values,

- L_1: $|f_1(y) - f_1(z)| = |y^2 - z^2| \le |y+z||y-z| \le 2|y-z| \Rightarrow L_1 = 2$.

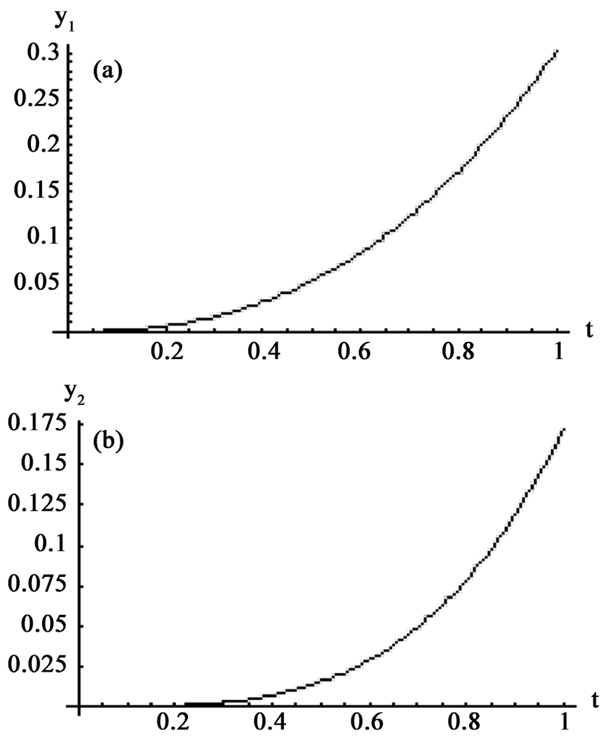

Figure 4. (a) ADM Sol; (b) ADM Sol.

- M_1: $\left|\beta_1(\tau)\right| \le \frac{1}{8} \Rightarrow M_1 = \frac{1}{8}$.
- L_2: $\left|f_2(y) - f_2(z)\right| = \left|y^4 - z^4\right| \le \left|y^2 + z^2\right|\left|y+z\right|\left|y-z\right| \le 4\left|y-z\right| \Rightarrow L_2 = 4$.
- M_2: $\left|\beta_2(\tau)\right| \le \frac{1}{4} \Rightarrow M_2 = \frac{1}{4}$.
- α: $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n + 1)} = \frac{1}{\Gamma(5/2)}$.
- $\max_{t\in J}\left|y_{11}(t)\right| = \frac{262144}{1206079875\sqrt{\pi}}$, $\max_{t\in J}\left|y_{21}(t)\right| = \frac{268435456}{1129407654375\pi^{3/2}}$.

The maximum error of y_1:

$$\max_{t\in J}\left|y_1(t) - \sum_{k=0}^{m} y_{1k}(t)\right| \le \frac{\alpha^m}{1-\alpha}\max_{t\in J}\left|y_{11}(t)\right|,$$

- For $m = 5$: $\max_{t\in J}\left|y_1(t) - \sum_{k=0}^{5} y_{1k}(t)\right| \le 0.000119234$,
- For $m = 10$: $\max_{t\in J}\left|y_1(t) - \sum_{k=0}^{10} y_{1k}(t)\right| \le 0.0000287222$,
- For $m = 15$: $\max_{t\in J}\left|y_1(t) - \sum_{k=0}^{15} y_{1k}(t)\right| \le 6.9189\times 10^{-6}$,
- For $m = 20$: $\max_{t\in J}\left|y_1(t) - \sum_{k=0}^{20} y_{1k}(t)\right| \le 1.66669\times 10^{-6}$.

The maximum error of y_2:

$$\max_{t\in J}\left|y_2(t) - \sum_{k=0}^{m} y_{2k}(t)\right| \le \frac{\alpha^m}{1-\alpha}\max_{t\in J}\left|y_{21}(t)\right|,$$

- For $m = 5$: $\max_{t\in J}\left|y_2(t) - \sum_{k=0}^{5} y_{2k}(t)\right| \le 0.0000415025$,
- For $m = 10$: $\max_{t\in J}\left|y_2(t) - \sum_{k=0}^{10} y_{2k}(t)\right| \le 9.99755\times 10^{-6}$,
- For $m = 15$: $\max_{t\in J}\left|y_2(t) - \sum_{k=0}^{15} y_{2k}(t)\right| \le 2.40831\times 10^{-6}$,
- For $m = 20$: $\max_{t\in J}\left|y_2(t) - \sum_{k=0}^{20} y_{2k}(t)\right| \le 5.80138\times 10^{-7}$.

5. Application: On Fractional-Order Rabies Model

The fractional-order rabies model,

$$\begin{aligned} D^\alpha y_1 &= -by_1y_2, \\ D^\alpha y_2 &= by_1y_2 - dy_2, \end{aligned} \tag{31}$$

subject to the initial conditions,

$$y_1(0) = 1, y_2(0) = 2,$$

was discussed before in [27], it was solved by using Adams-type predictor-corrector method. Now, we will solve it by using ADM.

Applying ADM to the system (31) leads to the following scheme,

$$y_{1,0} = 1, \quad y_{1,j+1} = -bJ^\alpha\left[A_{1,j}\right], \tag{32}$$

$$y_{2,0} = 2, \quad y_{2,j+1} = J^\alpha\left[A_{1,j} - 3y_{2,j}\right], \tag{33}$$

where $A_{1,j}$ represents the Adomian polynomials of the nonlinear term y_1y_2.

Using the relations (32)-(33) and taking $b = 1, d = 3, \alpha = 0.9$, the first five-terms of the series solution are,

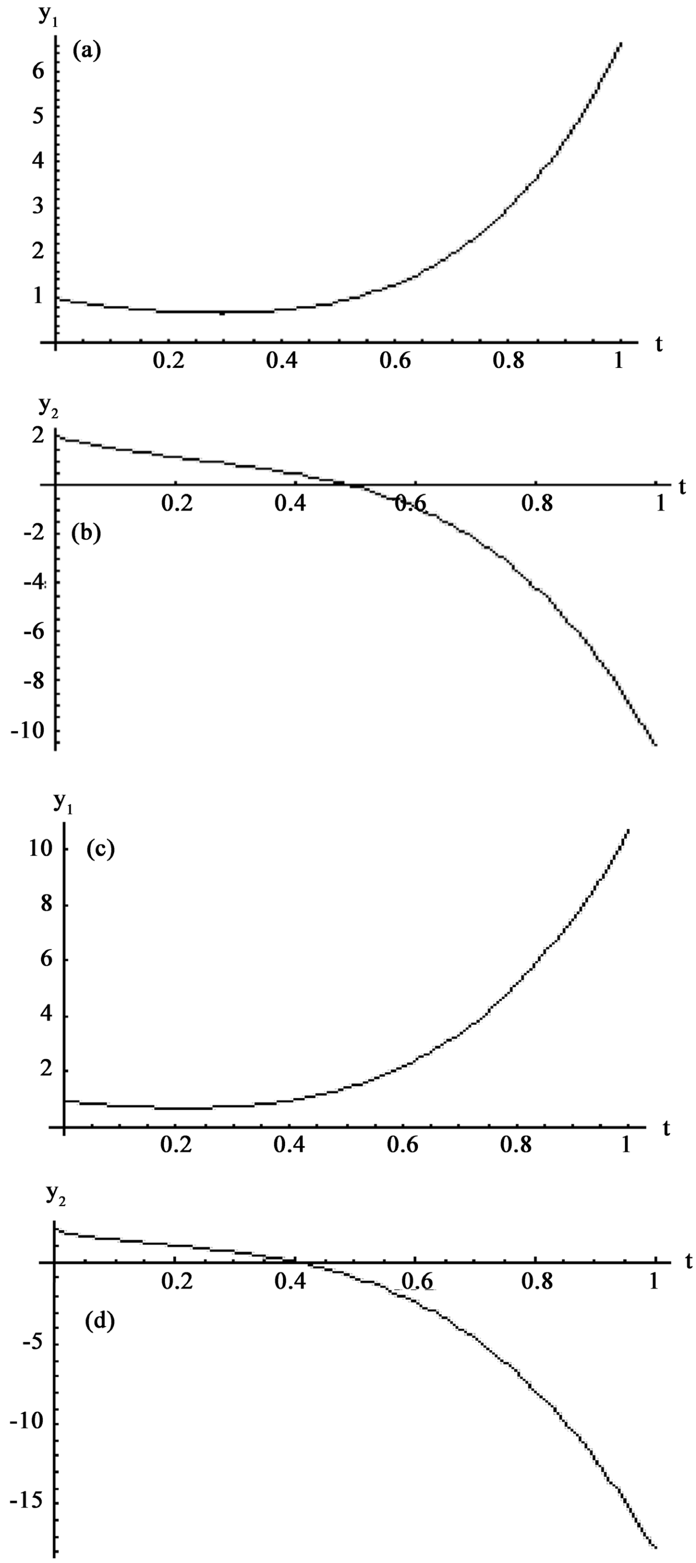

Figure 5. (a) ADM Sol. [α = 0.9]; (b) ADM Sol. [α = 0.9]; (c) ADM Sol. [α = 0.8]; (d) ADM Sol. [α = 0.8].

$$\begin{aligned} y_1 &= 1 - 2.07951t^{0.9} + 4.77187t^{1.8} - 8.27196t^{2.7} + 11.207t^{3.6} + \cdots, \\ y_2 &= 2 - 4.15902t^{0.9} + 2.38594t^{1.8} + 5.39472t^{2.7} - 16.2513t^{3.6} + \cdots. \end{aligned} \tag{34}$$

while for $\alpha = 0.8$, the series solution will be,

$$\begin{aligned} y_1 &= 1 - 2.14734t^{0.8} + 5.59587t^{1.6} - 11.1311t^{2.4} + 17.371t^{3.2} + \cdots, \\ y_2 &= 2 - 4.29469t^{0.8} + 2.79794t^{1.6} + 7.10592t^{2.4} - 25.5643t^{3.2} + \cdots. \end{aligned} \tag{35}$$

Figures 5(a)-(d) show ADM solution of y_1 and y_2 ($n = 5$) at different values of α.

6. Conclusion

In this paper, we use a simple method to solve nonlinear system of FDEs, this method gives a good approximate analytical solution of this type of equation as we compare ADM solution with the exact solution and also by evaluating the maximum absolute error which results from using partial sum of the series ADM solution.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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