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Global Dynamics of a Stochastic Two-Prey One-Predator Model with *S*-Type Distributed Delays and Lévy Noises

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Abstract

In this paper, a stochastic two-prey one-predator model with *S*-type distributed time delays and Lévy noises is considered. Using the comparison theorem and Ito's formula, sufficient conditions of persistence in the mean and extinct for each population are established. Then, conditions of global attractivity and stability in distribution by Barbalat's conclusion are also obtained. Furthermore, Euler numerical simulation method is given to demonstrate our conclusions.

Keywords

Distributed Delays, Lévy Noises, Global Attractivity, Stability in Distribution

1. Introduction

For a long time in the past, many scholars have been working on various biological models. The dynamic relationship between predator and prey has always been one of the most important and interesting topics in biological mathematics. There are many related works and literature [1] [2] [3]. Initially only two species were considered in the model, but this is often not the case in real world. The three-species system is more responsive to the real world. More recently, some authors [4] [5] claimed that the two-species model does not describe a dynamic relationship in the real world. So some scholars began to expand their research into three or more population models [6] [7].

In the real world, the behavior between predator and prey is not always continuous. In some cases, young predators can't engage in predation, or young prey can't be preyed on. These phenomena are called time delays. Similar time delays phenomena include hibernation, pregnancy, and migration and so on. Therefore, time delays are commonly thought to be taken into account in population system. At present, there are a lot of research results on time delay models [8] [9] [10] [11]. On the other hand, in an ecosystem, relationships between species include predator-prey, competition, and cooperation. But among these relationships, predator-prey is the most common and complex [12]. The general two-prey one-predator delayed model is as follows:

$$\begin{cases} dy_1(t) = y_1(t)\left[r_1 - a_{11}y_1(t) - a_{12}y_2(t-\tau_{12}) - a_{13}y_3(t-\tau_{13})\right]dt, \\ dy_2(t) = y_2(t)\left[r_2 - a_{21}y_1(t-\tau_{21}) - a_{22}y_2(t) - a_{23}y_3(t-\tau_{23})\right]dt, \\ dy_3(t) = y_3(t)\left[-r_3 + a_{31}y_1(t-\tau_{31}) + a_{32}y_2(t-\tau_{32}) - a_{33}y_3(t)\right]dt, \end{cases} \tag{1}$$

with initial data,

$$y_i(\theta) = \xi_i(\theta), \theta \in [-\tau, 0], \tau = \max\{\tau_{ij}\}, i, j = 1,2,3, i \ne j, \tag{2}$$

where $y_i(t)$ is the population of the prey, $i = 1,2$, and $y_3(t)$ is the population of predator. r_i is the growth rate of prey-*i*, $i = 1,2$, and r_3 is the death rate of the predator. $a_{ii} > 0, i = 1,2,3$ is the intra-specific competition coefficients of populations y_i; $a_{13} > 0, a_{23} > 0$ are the capture rates of predator; $a_{31} > 0, a_{32} > 0$ denote the efficiency of food conversion; $a_{12} > 0, a_{21} > 0$ are the competition rates between population y_1 and y_2. $\tau_{ij} > 0 (i \ne j), i, j = 1,2,3$ denotes the time delays.

In addition, the dynamic of the population is always affected by stochastic perturbations. It is necessary to takes tochastic perturbations into account since perturbations are inevitable in studying population dynamics [13]-[22]. Li and Mao [21] researched a non-autonomous competitive system with white noise. Liu and Qiu [22] studied an autonomous stochastic predator-prey delay model with white noise. And they established sufficient and necessary criteria for persistence in the mean and extinction of predator and prey. Usually, we assume that the stochastic perturbations mainly affect the growth rate or death rate of species. That is to say, we suppose that $r_i \to r_i + \sigma_i dB_i(t), i = 1,2,3$ in the model, where $B_i(t), i = 1,2,3$ stand for standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$, and $\sigma_i^2, i = 1,2,3$ are the intensity of the white noises. Then we obtain the delayed model with stochastic perturbations:

$$\begin{cases} dy_1(t) = y_1(t)\left[r_1 - a_{11}y_1(t) - a_{12}y_2(t-\tau_{12}) - a_{13}y_3(t-\tau_{13})\right]dt \\ \qquad + \sigma_1 y_1(t) dB_1(t), \\ dy_2(t) = y_2(t)\left[r_2 - a_{21}y_1(t-\tau_{21}) - a_{22}y_2(t) - a_{23}y_3(t-\tau_{23})\right]dt \\ \qquad + \sigma_2 y_2(t) dB_2(t), \\ dy_3(t) = y_3(t)\left[-r_3 + a_{31}y_1(t-\tau_{31}) + a_{32}y_2(t-\tau_{32}) - a_{33}y_3(t)\right]dt \\ \qquad + \sigma_3 y_3(t) dB_3(t), \end{cases} \tag{3}$$

with initial data (2). Geng Jand Liu M [23] have studied dynamics of model (3). By using the comparison theorem, they established sufficient criteria for the extinction and persistence of prey and predator under certain assumptions, and studied the stability in distribution of the stochastic model.

However, models with discrete delays and continuous delays don't include each other. Some scholars pointed that the $\mathcal{S}$-type distributed delays can include both delays [24] [25]. Therefore, $\mathcal{S}$-type delays should be considered in the model. On the other hand, there are some environmental perturbations such as earthquakes, epidemics, hurricanes and so on. These perturbations differ from white noise because of its sudden and destructive. Many authors [26]-[31] pointed out that these perturbations cannot be replaced by white noise, and a process that can simulate these sudden perturbations is needed. Several authors thought that the Lévy noises can represent these sudden perturbations. Liu and Wang [28] studied the persistence and extinction of two-species model with Lévy noises. Liu and Bai [31] investigated the dynamic of a stochastic model with Lévy noises and studied the stability in distribution of the solutions (SDS) of model by Lyapunov function approach.

Motivated by the above analysis, we will add the $\mathcal{S}$-type distributed time delays and Lévy noises into the model to study the dynamic of the real population more accurately. Considering $\mathcal{S}$-type distributed time delays and Lévy noises into model (3) yields,

$$\begin{cases} \mathrm{d}y_1(t) = y_1(t^-)\Big[r_1 - a_{11}y_1(t^-) - a_{12}\int_{-\tau_{12}}^{0} y_2(t^- + \theta)\mathrm{d}\mu_{12}(\theta) \\ \qquad - a_{13}\int_{-\tau_{13}}^{0} y_3(t^- + \theta)\mathrm{d}\mu_{13}(\theta)\Big]\mathrm{d}t + \sigma_1 y_1(t^-)\mathrm{d}B_1(t) \\ \qquad + y_1(t^-)\int_{\mathbb{Z}} \gamma_1(u)\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}y_2(t) = y_2(t^-)\Big[r_2 - a_{21}\int_{-\tau_{21}}^{0} y_1(t^- + \theta)\mathrm{d}\mu_{21}(\theta) - a_{22}y_2(t^-) \\ \qquad - a_{23}\int_{-\tau_{23}}^{0} y_3(t^- + \theta)\mathrm{d}\mu_{23}(\theta)\Big]\mathrm{d}t + \sigma_2 y_2(t^-)\mathrm{d}B_2(t) \\ \qquad + y_2(t^-)\int_{\mathbb{Z}} \gamma_2(u)\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}y_3(t) = y_3(t^-)\Big[-r_3 + a_{31}\int_{-\tau_{31}}^{0} y_1(t^- + \theta)\mathrm{d}\mu_{31}(\theta) \\ \qquad + a_{32}\int_{-\tau_{32}}^{0} y_2(t^- + \theta)\mathrm{d}\mu_{23}(\theta) - a_{33}y_3(t^-)\Big]\mathrm{d}t \\ \qquad + \sigma_3 y_3(t^-)\mathrm{d}B_3(t) + y_3(t^-)\int_{\mathbb{Z}} \gamma_3(u)\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u), \end{cases} \tag{4}$$

with initial data (2), where $y_i(t^-)$ is the left limit of $y_i(t)$, $\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u) = \Gamma(\mathrm{d}t,\mathrm{d}u) - \lambda(\mathrm{d}u)\mathrm{d}t$ is a compensated Poisson process, Γ is a Poisson counting measure, λ is the characteristic measure of Γ on a measurable subset $\mathbb{Z}$ in $(0,+\infty)$ with $\lambda(\mathbb{Z}) < +\infty$, $\lambda(\mathrm{d}u)$ is the measure of $\mathbb{Z}$. $\gamma_i(u), i = 1,2,3$ are bounded functions with $\gamma_i(u) > -1, u \in \mathbb{Z}$. And $\int_{-\tau_{ij}}^{0} y_j(t^- + \theta)\mathrm{d}\mu_{ij}(\theta), i, j = 1,2,3, i \neq j$ are Lebesgue-Stieltjes integrals, $\mu_{ij}(\theta)$ denote nonnegative variation functions that defined on $[-\tau, 0]$, $\tau = \max\{\tau_{ij}\}, i, j = 1,2,3; i \neq j$ satisfying $\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta) = 1$.

In this paper, we consider a stochastic three-species model with $\mathcal{S}$-type distributed delays and Lévy noises. Unlike the deterministic model, the stochastic exists no traditional positive equilibrium state. Therefore, we need to study the convergence in distribution of solutions. Because of time delays, it's difficult to apply the traditional methods like solving the corresponding Fokker–Planck equation to get the explicit solution. So, the asymptotic approach is what we are going to use.

The organization of this paper is as follows. In Section 2, we give some notations and important lemmas which are necessary to our discussion. In Section 3, main results are obtained by using Ito's formula and comparison theorem, such as persistence, extinction of species, and stability in distribution of model. Then, some numerical simulation results are presented to verify our conclusions in Section 4. In Section 5, there are some conclusions of this paper and some ideas for future work.

2. Preliminaries

Firstly, for the simplicity, we make the following notations

$$b_i = r_i - 0.5\sigma_i^2 - \int_{\mathbb{Z}}\left[\gamma_i(u) - \ln\left(1+\gamma_i(u)\right)\right]\lambda(\mathrm{d}u),\ \ i = 1,2,$$

$$b_3 = -r_3 - 0.5\sigma_3^2 - \int_{\mathbb{Z}}\left[\gamma_3(u) - \ln\left(1+\gamma_3(u)\right)\right]\lambda(\mathrm{d}u),$$

$$\delta_i = 0.5\sigma_i^2 + \int_{\mathbb{Z}}\left[\gamma_i(u) - \ln\left(1+\gamma_i(u)\right)\right]\lambda(\mathrm{d}u),\ i = 1,2,3,$$

$$R_i(t) = \int_0^t\int_{\mathbb{Z}} \ln\left(1+\gamma_i(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u),\ i = 1,2,3,$$

$$\overline{f(t)} = t^{-1}\int_0^t f(s)\mathrm{d}s,\quad f^* = \limsup_{t\to\infty} f(t),\quad f_* = \liminf_{t\to\infty} f(t),$$

$$\overline{f(t)}^* = \limsup_{t\to\infty} t^{-1}\int_0^t f(s)\mathrm{d}s,\quad \overline{f(t)}_* = \liminf_{t\to\infty} t^{-1}\int_0^t f(s)\mathrm{d}s,$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{vmatrix},\ \Delta_1 = \begin{vmatrix} r_1 & a_{12} & a_{13} \\ r_2 & a_{22} & a_{23} \\ -r_3 & -a_{32} & a_{33} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_{11} & r_1 & a_{13} \\ a_{21} & r_2 & a_{23} \\ -a_{31} & -r_3 & a_{33} \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & r_1 \\ a_{21} & a_{22} & r_2 \\ -a_{31} & -a_{32} & -r_3 \end{vmatrix},\ \tilde{\Delta}_1 = \begin{vmatrix} \delta_1 & a_{12} & a_{13} \\ \delta_2 & a_{22} & a_{23} \\ \delta_3 & -a_{32} & a_{33} \end{vmatrix},\ \tilde{\Delta}_2 = \begin{vmatrix} a_{11} & \delta_1 & a_{13} \\ a_{21} & \delta_2 & a_{23} \\ -a_{31} & \delta_3 & a_{33} \end{vmatrix},$$

$$\tilde{\Delta}_3 = \begin{vmatrix} a_{11} & a_{12} & \delta_1 \\ a_{21} & a_{22} & \delta_2 \\ -a_{31} & -a_{32} & \delta_3 \end{vmatrix},\quad \Pi = \begin{vmatrix} r_1 & \delta_1 & a_{13} \\ r_2 & \delta_2 & a_{23} \\ -r_3 & \delta_3 & a_{33} \end{vmatrix}.$$

Moreover, let

$$\eta = \Delta_3 - \tilde{\Delta}_3,\ \kappa_1 = b_1 a_{22} - b_2 a_{12},\ \kappa_2 = b_2 a_{11} - b_1 a_{21},$$

and Δ_{ij} be the complement minor of c_{ij} in Δ, where c_{ij} is the *i*-th row and *j*-th column element of Δ.

In order to state our results, we assume that

Assumption 1: There is a positive constant L such that
$\int_{\mathbb{Z}}\left(\ln\left(1+\gamma_i(u)\right)\right)^2\lambda(\mathrm{d}u) < L$.

Assumption 2: $\Delta_{23} > 0, \Delta_{32} > 0, \Delta_{33} > 0, \Delta_{13} < 0, \Delta_{13} < 0, \Delta_{31} < 0, b_1 > b_2$.

Suppose that $\Pi > 0$, which mean that y_1 has a stronger survival ability than y_2 (see, e.g. [32]).

The following lemmas are necessary for our discussion.

Lemma 2.1: For any initial value $(\xi_1(\theta), \xi_2(\theta), \xi_3(\theta)) \in C([-\gamma, 0], R_+^3), -\gamma \le \theta \le 0$, model (4) has a unique global solution $(y_1(t), y_2(t), y_3(t)) \in R_+^3$ *a.s.* for all $t \ge 0$.

Proof. Since the coefficients in (4) satisfies the local Lipschitz condition, for any initial value $(\xi_1(\theta), \xi_2(\theta), \xi_3(\theta)) \in C([-\gamma, 0], R_+^3)$, model (4) has a unique positive $(y_1(t), y_2(t), y_3(t)) \in R_+^3$ *a.s.* for $t \in [-\gamma, \tau_e]$, where τ_e is the explosion time. We only need to verify that $\tau_e = \infty$ *a.s.*. Let $m_0 > 0$ be sufficiently large such that $\xi_1(0), \xi_2(0), \xi_3(0) \in [1/m_0, m_0]$, for each integer $m \ge m_0$. We define the stopping time,

$$\tau_m = \inf\left\{t \in [0, \tau_e) : y_1(t) \notin \left(\frac{1}{m}, m\right), y_2(t) \notin \left(\frac{1}{m}, m\right), y_3(t) \notin \left(\frac{1}{m}, m\right)\right\}.$$

Obviously, τ_m is strictly increasing with *m*. Let $\tau_\infty = \lim_{m\to\infty} \tau_m$ *a.s.*. There is $\tau_\infty \le \tau_e$ *a.s.*, so we need to prove $\tau_\infty = \infty$ *a.s.*. If the statement is not true, then there exist $T > 0$ and $0 < \varepsilon < 1$ such that $P(\tau_\infty \le T) > \varepsilon$, and there exist an integer $m_1 > m_0$, for any $m > m_1$ such that $P(\tau_m \le T) > \varepsilon$.

We define

$$V(y) = \beta V_1(y_1) + V_2(y_2) + V_3(y_3) + V_4(t),$$

where

$$y = (y_1, y_2, y_3),\ V_1(y_1) = y_1 - 1 - \ln y_1,\ V_2(y_2) = y_2 - 1 - \ln y_2,$$
$$V_3(y_3) = y_3 - 1 - \ln y_3, \text{ and}$$

$$\begin{aligned} V_4(t) = {} & \frac{\beta}{2n^2} a_{12} \int_{-\tau_{12}}^{0} \int_{t+\theta}^{t} y_2^2(s^-) \mathrm{d}s \mathrm{d}\mu_{12}(\theta) + \frac{\beta}{2n^2} a_{13} \int_{-\tau_{13}}^{0} \int_{t+\theta}^{t} y_3^2(s^-) \mathrm{d}s \mathrm{d}\mu_{13}(\theta) \\ & + \frac{a_{21}}{2n^2} \int_{-\tau_{13}}^{0} \int_{t+\theta}^{t} y_1^2(s^-) \mathrm{d}s \mathrm{d}\mu_{21}(\theta) + \frac{a_{23}}{2n^2} \int_{-\tau_{23}}^{0} \int_{t+\theta}^{t} y_3^2(s^-) \mathrm{d}s \mathrm{d}\mu_{23}(\theta) \\ & + \frac{n^2}{2} a_{31} \int_{-\tau_{31}}^{0} \int_{t+\theta}^{t} y_1^2(s^-) \mathrm{d}s \mathrm{d}\mu_{31}(\theta) + \frac{n^2}{2} a_{32} \int_{-\tau_{32}}^{0} \int_{t+\theta}^{t} y_2^2(s^-) \mathrm{d}s \mathrm{d}\mu_{32}(\theta), \end{aligned} \tag{5}$$

for constant $\beta > 0$ and integer $n > 0$, can be chosen such that

$$\begin{aligned} & -\beta a_{11} + \frac{a_{21}}{2n^2} + \frac{n^2}{2} a_{31} < 0,\ \frac{\beta}{2n^2} a_{12} - a_{22} + \frac{n^2}{2} a_{32}, \\ & \frac{\beta}{2n^2} a_{13} + \frac{a_{23}}{2n^2} - a_{33} + \frac{a_{31}}{2n^2} + \frac{a_{32}}{2n^2} < 0. \end{aligned} \tag{6}$$

For model (4), by Ito's formula [33], we can obtain

$$\begin{aligned} \mathrm{d}V(y) = {} & \left[\beta LV_1(y_1) + LV_2(y_2) + LV_3(y_3) + \frac{\mathrm{d}}{\mathrm{d}t} V_4(t)\right] \mathrm{d}t \\ & + \beta\sigma_1(y_1 - 1) \mathrm{d}B_1(t) + \beta \int_Z \left(y_1\gamma_1(u) - \ln(1 + \gamma_1(u))\right) \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\ & + \sigma_2(y_2 - 1) \mathrm{d}B_2(t) + \int_Z \left(y_2\gamma_2(u) - \ln(1 + \gamma_2(u))\right) \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\ & + \sigma_3(y_3 - 1) \mathrm{d}B_3(t) + \int_Z \left(y_3\gamma_3(u) - \ln(1 + \gamma_3(u))\right) \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{7}$$

where

$$LV_1(y_1)=(y_1-1)\Big(r_1-a_{11}y_1(t^-)-a_{12}\int_{-\tau_{12}}^{0}y_2(t^-+\theta)\mathrm{d}\mu_{12}(\theta)$$
$$-a_{13}\int_{-\tau_{13}}^{0}y_3(t^-+\theta)\mathrm{d}\mu_{13}(\theta)\Big)+0.5\sigma_1^2$$
$$+\int_Z\big(\gamma_1(u)-\ln(1+\gamma_1(u))\big)\lambda(\mathrm{d}u),$$

$$LV_2(y_2)=(y_2-1)\Big(r_2-a_{22}y_2(t^-)-a_{21}\int_{-\tau_{21}}^{0}y_1(t^-+\theta)\mathrm{d}\mu_{21}(\theta)$$
$$-a_{23}\int_{-\tau_{23}}^{0}y_3(t^-+\theta)\mathrm{d}\mu_{23}(\theta)\Big)+0.5\sigma_2^2$$
$$+\int_Z\big(\gamma_2(u)-\ln(1+\gamma_2(u))\big)\lambda(\mathrm{d}u),$$

$$LV_3(y_3)=(y_3-1)\Big(-r_3-a_{33}y_3(t^-)+a_{31}\int_{-\tau_{31}}^{0}y_1(t^-+\theta)\mathrm{d}\mu_{31}(\theta)$$
$$+a_{32}\int_{-\tau_{32}}^{0}y_2(t^-+\theta)\mathrm{d}\mu_{32}(\theta)\Big)+0.5\sigma_3^2$$
$$+\int_Z\big(\gamma_3(u)-\ln(1+\gamma_3(u))\big)\lambda(\mathrm{d}u).$$

For any integer $n>0$, we can get following results by basic inequality $a^2+b^2\ge 2ab$,

$$a_{12}\int_{-\tau_{12}}^{0}y_2(t^-+\theta)\mathrm{d}\mu_{12}(\theta)\le\frac{1}{2}a_{12}\left(n^2+\frac{1}{n^2}\int_{-\tau_{12}}^{0}y_2^2(t^-+\theta)\mathrm{d}\mu_{12}(\theta)\right),\quad (8)$$

$$a_{13}\int_{-\tau_{13}}^{0}y_3(t^-+\theta)\mathrm{d}\mu_{13}(\theta)\le\frac{1}{2}a_{13}\left(n^2+\frac{1}{n^2}\int_{-\tau_{13}}^{0}y_3^2(t^-+\theta)\mathrm{d}\mu_{13}(\theta)\right).\quad (9)$$

Then, using (8) and (9) in $LV_1(y_1),LV_2(y_2),LV_3(y_3)$, we obtain

$$LV_1(y_1)\le r_1y_1-a_{11}y_1^2-r_1+a_{11}y_1+0.5\sigma_1^2+\int_Z\big(\gamma_1(u)-\ln(1+\gamma_1(u))\big)\lambda(\mathrm{d}u)$$
$$+a_{12}\int_{-\tau_{12}}^{0}y_2(t^-+\theta)\mathrm{d}\mu_{12}(\theta)+a_{13}\int_{-\tau_{13}}^{0}y_3(t^-+\theta)\mathrm{d}\mu_{13}(\theta)$$
$$\le r_1y_1-a_{11}y_1^2-r_1+a_{11}y_1+0.5\sigma_1^2+\int_Z\big(\gamma_1(u)-\ln(1+\gamma_1(u))\big)\lambda(\mathrm{d}u)\quad (10)$$
$$+\frac{n^2}{2}(a_{12}+a_{13})+\frac{a_{12}}{2n^2}\int_{-\tau_{12}}^{0}y_2^2(t^-+\theta)\mathrm{d}\mu_{12}(\theta)$$
$$+\frac{a_{13}}{2n^2}\int_{-\tau_{13}}^{0}y_3^2(t^-+\theta)\mathrm{d}\mu_{13}(\theta),$$

$$LV_2(y_2)\le r_2y_2-a_{22}y_2^2-r_2+a_{22}y_2+0.5\sigma_2^2+\int_Z\big(\gamma_2(u)-\ln(1+\gamma_2(u))\big)\lambda(\mathrm{d}u)$$
$$+a_{21}\int_{-\tau_{21}}^{0}y_1(t^-+\theta)\mathrm{d}\mu_{21}(\theta)+a_{23}\int_{-\tau_{23}}^{0}y_3(t^-+\theta)\mathrm{d}\mu_{23}(\theta)$$
$$\le r_2y_2-a_{22}y_2^2-r_2+a_{22}y_2+0.5\sigma_2^2+\int_Z\big(\gamma_2(u)-\ln(1+\gamma_2(u))\big)\lambda(\mathrm{d}u)\quad (11)$$
$$+\frac{n^2}{2}(a_{21}+a_{23})+\frac{a_{21}}{2n^2}\int_{-\tau_{21}}^{0}y_1^2(t^-+\theta)\mathrm{d}\mu_{21}(\theta)$$
$$+\frac{a_{23}}{2n^2}\int_{-\tau_{23}}^{0}y_3^2(t^-+\theta)\mathrm{d}\mu_{23}(\theta).$$

Similarly, there is

$$y_3a_{31}\int_{-\tau_{31}}^{0}y_1(t^-+\theta)\mathrm{d}\mu_{31}(\theta)\le\frac{1}{2}a_{31}\left(n^2\int_{-\tau_{31}}^{0}y_1^2(t^-+\theta)\mathrm{d}\mu_{31}(\theta)+\frac{1}{n^2}y_3^2\right),$$

$$y_3 a_{32} \int_{-\tau_{32}}^{0} y_2 \left(t^- + \theta\right) \mathrm{d}\mu_{32}(\theta) \le \frac{1}{2} a_{32} \left(n^2 \int_{-\tau_{32}}^{0} y_2^2 \left(t^- + \theta\right) \mathrm{d}\mu_{32}(\theta) + \frac{1}{n^2} y_3^2 \right).$$

Then, we get

$$\begin{aligned} LV_3(y_3) \le & -r_3 y_3 - a_{33} y_3^2 + r_3 + a_{33} y_3 + 0.5\sigma_3^2 + \int_Z \left(\gamma_3(u) - \ln\left(1 + \gamma_3(u)\right)\right) \lambda(\mathrm{d}u) \\ & + y_3 a_{31} \int_{-\tau_{31}}^{0} y_1 \left(t^- + \theta\right) \mathrm{d}\mu_{31}(\theta) + y_3 a_{32} \int_{-\tau_{32}}^{0} y_2 \left(t^- + \theta\right) \mathrm{d}\mu_{32}(\theta) \\ \le & -r_3 y_3 - a_{33} y_3^2 + r_3 + a_{33} y_3 + 0.5\sigma_3^2 + \int_Z \left(\gamma_3(u) - \ln\left(1 + \gamma_3(u)\right)\right) \lambda(\mathrm{d}u) \\ & + \frac{y_3^2}{2n^2}\left(a_{31} + a_{32}\right) + \frac{n^2}{2} a_{31} \int_{-\tau_{31}}^{0} y_1^2 \left(t^- + \theta\right) \mathrm{d}\mu_{31}(\theta) \\ & + \frac{n^2}{2} a_{32} \int_{-\tau_{32}}^{0} y_2^2 \left(t^- + \theta\right) \mathrm{d}\mu_{32}(\theta). \end{aligned} \tag{12}$$

From (5), (10), (11) and (12), there is

$$\begin{aligned} & \beta LV_1(y_1) + LV_2(y_2) + LV_3(y_3) + \frac{\mathrm{d}}{\mathrm{d}t} V_4(t) \\ & \le \beta \left[r_1 y_1 - a_{11} y_1^2 - r_1 + a_{11} y_1 + 0.5\sigma_1^2 + \int_Z \left(\gamma_1(u) - \ln\left(1 + \gamma_1(u)\right)\right) \lambda(\mathrm{d}u) \right. \\ & \left. + \frac{n^2}{2}\left(a_{12} + a_{13}\right) \right] + \frac{\beta}{2n^2} a_{12} y_2^2 + \frac{\beta}{2n^2} a_{13} y_3^2 + r_2 y_2 - a_{22} y_2^2 - r_2 + a_{22} y_2 \\ & + 0.5\sigma_2^2 + \int_Z \left(\gamma_2(u) - \ln\left(1 + \gamma_2(u)\right)\right) \lambda(\mathrm{d}u) + \frac{a_{21}}{2n^2} y_1^2 + \frac{a_{23}}{2n^2} y_3^2 \\ & - r_3 y_3 - a_{33} y_3^2 + r_3 + a_{33} y_3 + 0.5\sigma_3^2 + \int_Z \left(\gamma_3(u) - \ln\left(1 + \gamma_3(u)\right)\right) \lambda(\mathrm{d}u) \\ & + \frac{y_3^2}{2n^2} a_{31} + \frac{y_3^2}{2n^2} a_{32} + \frac{n^2}{2} a_{31} y_1^2 + \frac{n^2}{2} a_{32} y_2^2. \end{aligned} \tag{13}$$

Therefore, using (13) and (6) in (7), there exists a constant $K > 0$ such that

$$\begin{aligned} \mathrm{d}V(y) \le & K\mathrm{d}t + \beta\sigma_1\left(y_1 - 1\right)\mathrm{d}B_1(t) + \beta \int_Z \left(y_1\gamma_1(u) - \ln\left(1 + \gamma_1(u)\right)\right) \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\ & + \sigma_2\left(y_2 - 1\right)\mathrm{d}B_2(t) + \int_Z \left(y_2\gamma_2(u) - \ln\left(1 + \gamma_2(u)\right)\right) \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\ & + \sigma_3\left(y_3 - 1\right)\mathrm{d}B_3(t) + \int_Z \left(y_3\gamma_3(u) - \ln\left(1 + \gamma_3(u)\right)\right) \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

Then, from this result, according to argument in [34], we obtain

$$\infty \le KT + V_0\left(y_1(0), y_2(0), y_3(0)\right) \le \infty,$$

which leads a contradiction, so there is $\tau_\infty = \infty$, therefore $\tau_e = \infty$ *a.s.*. The proof of Lemma 2.1 is completed.

Lemma 2.2: For any initial value $\left(\xi_1(\theta), \xi_2(\theta), \xi_3(\theta)\right) \in C\left(\left[-\gamma, 0\right], R_+^3\right)$, $\left(y_1(t), y_2(t), y_3(t)\right)$ is a positive solution of model (4). Then for any $p > 0$, there exist constants $K_i(p) > 0$ such that

$$\limsup_{t\to\infty} E\left[y_i^p(t)\right] \le K_i(p), \ i = 1, 2, 3.$$

Proof. We only prove that $\limsup_{t\to\infty} E\left[y_1^p(t)\right] \le K_1(p)$. The proof of $\limsup_{t\to\infty} E\left[y_2^p(t)\right] \le K_2(p)$ and $\limsup_{t\to\infty} E\left[y_3^p(t)\right] \le K_3(p)$ is standard and similar.

Define $G_1(t) = e^t y_1^p(t)$, by Ito's formula, we get

$$
\begin{aligned}
\mathrm{d}G_1(t) &= \mathrm{e}^t y_1^p \Bigg\{ \frac{p(p-1)\sigma_1^2}{2} + \int_{\mathbb{Z}} \Big(\big(1+\gamma_1(u)\big)^p - p\gamma_1(u) \Big) \lambda(\mathrm{d}u) + p\Big[r_1 - a_{11} y_1(t^-) \\
&\quad - a_{12} \int_{-\tau_{12}}^{0} y_2(t^- + \theta) \mathrm{d}\mu_{12}(\theta) - a_{13} \int_{-\tau_{13}}^{0} y_3(t^- + \theta) \mathrm{d}\mu_{13}(\theta) + 1 \Big] \Bigg\} \\
&\quad + p\mathrm{e}^t y_1^p \sigma_1 \mathrm{d}B_1(t) + \int_{\mathbb{Z}} \mathrm{e}^t y_1^p \big(1+\gamma_1(u)\big)^p \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\
&= LG_1(t) + p\mathrm{e}^t y_1^p \sigma_1 \mathrm{d}B_1(t) + \int_{\mathbb{Z}} \mathrm{e}^t y_1^p \big(1+\gamma_1(u)\big)^p \tilde{\Gamma}(\mathrm{d}s, \mathrm{d}u).
\end{aligned} \tag{14}
$$

There is

$$
\begin{aligned}
LG_1(t) \le \mathrm{e}^t y_1^p \Bigg\{ & \frac{p(p-1)\sigma_1^2}{2} + \int_{\mathbb{Z}} \Big(\big(1+\gamma_1(u)\big)^p - p\gamma_1(u) \Big) \lambda(\mathrm{d}u) \\
& + p\Big[r_1 - a_{11} y_1(t^-) \Big] + 1 \Bigg\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
K_1(p) = \max_{y_1 \ge 0} \Bigg\{ & y_1^p \Bigg[\frac{p(p-1)\sigma_1^2}{2} + \int_{\mathbb{Z}} \Big(\big(1+\gamma_1(u)\big)^p - p\gamma_1(u) \Big) \lambda(\mathrm{d}u) + p r_1 + 1 \Bigg] \\
& - p a_{11} y_1^{p+1} \Bigg\},
\end{aligned}
$$

then, we get

$$
LG_1(t) \le K_1(p) \mathrm{e}^t.
$$

Integrating both sides of (14), and taking expectations lead to

$$
E\left(\mathrm{e}^t y_1^p\right) - \xi_1^p(0) \le K_1(p)\left(\mathrm{e}^t - 1\right).
$$

This means that $\limsup_{t\to\infty} E\left[y_1^p(t)\right] \le K_1(p)$. Hence, the proof is completed.

Lemma 2.3: [28] Suppose that $g(t) \in C\left[\Omega \times [0,+\infty), R_+\right]$, and Assumption 1 holds,

(I) If there exist positive constants T, λ, λ_0 and λ_i for all $t \ge T$,

$$
\ln g(t) \le \lambda t - \lambda_0 \int_0^t g(s)\mathrm{d}s + \sum_{i=1}^{3} \sigma_i B_i(t) + \sum_{i=1}^{3} \lambda_i R_i(t),
$$

then,

$$
\begin{cases}
\overline{g}^* \le \lambda/\lambda_0 & a.s., \ \text{if } \lambda \ge 0; \\
\lim_{t\to+\infty} g(t) = 0 & a.s., \ \text{if } \lambda < 0.
\end{cases}
$$

(II) If there exist positive constants T, λ, λ_0 and λ_i for all $t \ge T$,

$$
\ln g(t) \ge \lambda t - \lambda_0 \int_0^t g(s)\mathrm{d}s + \sum_{i=1}^{3} \sigma_i B_i(t) + \sum_{i=1}^{3} \lambda_i R_i(t),
$$

then,

$$
\overline{g}^* \ge \lambda/\lambda_0 \quad a.s.
$$

Consider the following auxiliary model,

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t^-)\left[r_1 - a_{11}x_1(t^-)\right]\mathrm{d}t + \sigma_1 x_1(t^-)\mathrm{d}B_1(t) + x_1(t^-)\int_Z \gamma_1(u)\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}x_2(t) = x_2(t^-)\left[r_2 - a_{22}x_2(t^-)\right]\mathrm{d}t + \sigma_2 x_2(t^-)\mathrm{d}B_2(t) + x_2(t^-)\int_Z \gamma_2(u)\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}x_3(t) = x_3(t^-)\Big[-r_3 + a_{31}\int_{-\tau_{31}}^{0} x_1(t^- + \theta)\mathrm{d}\mu_{31}(\theta) + a_{32}\int_{-\tau_{32}}^{0} x_3(t^- + \theta)\mathrm{d}\mu_{23}(\theta) \\ \qquad - a_{33}x_3(t^-)\Big]\mathrm{d}t + \sigma_3 x_3(t^-)\mathrm{d}B_3(t) + x_3(t^-)\int_Z \gamma_3(u)\tilde{\Gamma}(\mathrm{d}t,\mathrm{d}u). \end{cases} \quad (15)$$

with initial value $x_i(\theta) = \xi_i(\theta), \theta \in [-\tau, 0], i = 1,2,3$.

According to the comparison theorem [35], we obtain

$$y_i(t) \le x_i(t),\ a.s., i = 1,2,3. \quad (16)$$

Lemma 2.4: Suppose that $(x_1(t), x_2(t), x_3(t))$ is any positive solution of (15). Then

(i) $\lim_{t\to\infty}\sup \ln x_i(t)/t \le 0, a.s. i = 1,2,3$;

(ii) For any positive constant τ, $\lim_{t\to\infty} t^{-1}\int_{t-\tau}^{t} x_i(s)\mathrm{d}s = 0\ a.s., i = 1,2,3$.

The proof of this lemma is standard, hence it is omitted.

Lemma 2.5: For model (15)

(i) If $b_1 < 0$, then

$$\lim_{t\to\infty} x_i(t) = 0, a.s.\ i = 1,2,3.$$

(ii) If $b_1 > 0, b_2 > 0, b_3 + a_{31}b_1/a_{11} + a_{32}b_2/a_{22} < 0$, then

$$\lim_{t\to\infty} \overline{x_1(t)} = \frac{b_1}{a_{11}},\ \lim_{t\to\infty} \overline{x_2(t)} = \frac{b_2}{a_{22}},\ \lim_{t\to\infty} x_3(t) = 0.$$

(iii) If $b_1 > 0, b_2 > 0, b_3 + a_{31}b_1/a_{11} + a_{32}b_2/a_{22} \ge 0$, then

$$\lim_{t\to\infty} \overline{x_1(t)} = \frac{b_1}{a_{11}},\ \lim_{t\to\infty} \overline{x_2(t)} = \frac{b_2}{a_{22}},\ \lim_{t\to\infty} \overline{x_3(t)} = \frac{b_3 + a_{31}b_1/a_{11} + a_{32}b_2/a_{22}}{a_{33}}.$$

(iv) If $b_1 > 0, b_2 < 0, b_3 + a_{31}b_1/a_{11} \ge 0$, then

$$\lim_{t\to\infty} \overline{x_1(t)} = \frac{b_1}{a_{11}},\ \lim_{t\to\infty} x_2(t) = 0,\ \lim_{t\to\infty} \overline{x_3(t)} = \frac{b_3 + a_{31}b_1/a_{11}}{a_{33}}.$$

(v) If $b_1 > 0, b_2 < 0, b_3 + a_{31}b_1/a_{11} < 0$, then

$$\lim_{t\to\infty} \overline{x_1(t)} = \frac{b_1}{a_{11}},\ \lim_{t\to\infty} x_2(t) = 0,\ \lim_{t\to\infty} x_3(t) = 0.$$

Proof. Using Ito's formula to model (15), we get

$$\begin{aligned} &\ln x_1(t) - \ln x_1(0) \\ &= \left[r_1 - 0.5\sigma_1^2 - \int_Z \left(\gamma_1(u) - \ln(1+\gamma_1(u))\right)\lambda(\mathrm{d}u)\right]t - a_{11}\int_0^t x_1(s)\mathrm{d}s \\ &\quad + \sigma_1 B_1(t) + \int_0^t\int_{\mathbb{Z}} \ln(1+\gamma_1(u))\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u), \end{aligned}$$

$$\begin{aligned} &\ln x_2(t) - \ln x_2(0) \\ &= \left[r_2 - 0.5\sigma_2^2 - \int_Z \left(\gamma_2(u) - \ln(1+\gamma_2(u))\right)\lambda(\mathrm{d}u)\right]t - a_{22}\int_0^t x_2(s)\mathrm{d}s \\ &\quad + \sigma_2 B_2(t) + \int_0^t\int_{\mathbb{Z}} \ln(1+\gamma_2(u))\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u), \end{aligned}$$

$$\begin{aligned}&\ln x_3(t)-\ln x_3(0)\\&=\left[-r_3-0.5\sigma_3^2-\int_Z\left(\gamma_3(u)-\ln\left(1+\gamma_3(u)\right)\right)\lambda(\mathrm{d}u)\right]t-a_{33}\int_0^t x_3(s)\mathrm{d}s\\&\quad+\sigma_3B_3(t)+\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_3(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u)+a_{31}\int_0^t\int_{-\tau_{31}}^0 x_1\left(s^-+\theta\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s\\&\quad+a_{32}\int_0^t\int_{-\tau_{32}}^0 x_2\left(s^-+\theta\right)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s.\end{aligned}$$

Dividing both sides of above equations by t, we obtain

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)}=b_1-a_{11}\overline{x_1(t)}+t^{-1}\sigma_1B_1(t)+t^{-1}R_1(t), \tag{17}$$

$$t^{-1}\ln\frac{x_2(t)}{x_2(0)}=b_2-a_{22}\overline{x_2(t)}+t^{-1}\sigma_2B_2(t)+t^{-1}R_2(t), \tag{18}$$

$$t^{-1}\ln\frac{x_3(t)}{x_3(0)}=b_3-a_{33}\overline{x_3(t)}+t^{-1}\sigma_3B_3(t)+t^{-1}R_3(t)+t^{-1}\varphi(t), \tag{19}$$

where

$$\varphi(t)=a_{31}\int_0^t\int_{-\tau_{31}}^0 x_1\left(s^-+\theta\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s+a_{32}\int_0^t\int_{-\tau_{32}}^0 x_2\left(s^-+\theta\right)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s. \tag{20}$$

For (i), according to (17) and (18), by Lemma 2.3, since $b_2<b_1<0$, we have

$$\lim_{t\to\infty}x_1(t)=0,\quad \lim_{t\to\infty}x_2(t)=0. \tag{21}$$

On the other hand, since

$$\begin{aligned}&\int_{-\tau_{31}}^0\int_{t+\theta}^t x_1\left(s^-\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s\le\int_{-\tau_{31}}^0\mathrm{d}\mu_{31}(\theta)\int_{t-\tau_{31}}^t x_1\left(s^-\right)\mathrm{d}s,\\&\int_{-\tau_{31}}^0\int_{\theta}^0 x_1\left(s^-\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s\le\int_{-\tau_{31}}^0\mathrm{d}\mu_{31}(\theta)\int_{-\tau_{31}}^0 x_1\left(s^-\right)\mathrm{d}s.\end{aligned}$$

By Lemma 2.4, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{31}}^0\int_{t+\theta}^t x_1\left(s^-\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s=0,\ \lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{31}}^0\int_{\theta}^0 x_1\left(s^-\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s=0. \tag{22}$$

Similarly,

$$\lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{32}}^0\int_{t+\theta}^t x_2\left(s^-\right)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s=0,\ \lim_{t\to\infty}\frac{1}{t}\int_{-\tau_{32}}^0\int_{\theta}^0 x_2\left(s^-\right)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s=0. \tag{23}$$

Notice that $\int_{-\tau_{ij}}^0\mathrm{d}\mu_{ij}(\theta)=1$. Simplify (20), then we get

$$\begin{aligned}\varphi(t)=a_{31}&\left[\int_0^t x_1\left(s^-\right)\mathrm{d}s-\int_{-\tau_{31}}^0\int_{t+\theta}^t x_1\left(s^-\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s\right.\\&\left.+\int_{-\tau_{31}}^0\int_{\theta}^0 x_1\left(s^-\right)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s\right]+a_{32}\left[\int_0^t x_2\left(s^-\right)\mathrm{d}s\right.\\&\left.-\int_{-\tau_{32}}^0\int_{t+\theta}^t x_2\left(s^-\right)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s+\int_{-\tau_{32}}^0\int_{\theta}^0 x_2\left(s^-\right)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s\right].\end{aligned} \tag{24}$$

Substituting (21), (22), and (23) into (24), we obtain

$$\lim_{t\to\infty}\frac{1}{t}\varphi(t)=0\ a.s.$$

Note that $b_3<0$. According to (19) and Lemma 2.3, we get

$$\lim_{t\to\infty} x_3(t) = 0.$$

For (ii), since $\lim_{t\to\infty} \ln \frac{x_1(0)}{t} = 0$, then

$$(b_1 - \varepsilon)t - a_{11}\int_0^t x_1(s^-)\mathrm{d}s + \sigma_1 B_1(t) + R_1(t) \le \ln x_1(t)$$
$$\le (b_1 + \varepsilon)t - a_{11}\int_0^t x_1(s^-)\mathrm{d}s + \sigma_1 B_1(t) + R_1(t).$$

According to Lemma 2.3, there is

$$\overline{x_1}^* = \limsup_{t\to\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le \frac{b_1}{a_{11}},$$

and

$$\overline{x_1}_* = \liminf_{t\to\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \ge \frac{b_1}{a_{11}}.$$

This means

$$\lim_{t\to\infty} \overline{x_1} = \frac{b_1}{a_{11}} \quad a.s..$$

Similarly, there is

$$\lim_{t\to\infty} \overline{x_2} = \frac{b_2}{a_{22}} \quad a.s.$$

Therefore, for any $\varepsilon > 0$, there exist a positive T, we have that for $t > T$,

$$|\varphi(t)| \le a_{31}\frac{b_1}{a_{11}} + a_{32}\frac{b_2}{a_{22}} + \varepsilon. \tag{25}$$

Substituting (25) into (19), for enough large t, we obtain

$$t^{-1}\ln\frac{x_3(t)}{x_3(0)} \le b_3 + a_{31}\frac{b_1}{a_{11}} + a_{32}\frac{b_2}{a_{22}} + \varepsilon - a_{33}\overline{x_3(t)} + t^{-1}\sigma_3 B_3(t) + t^{-1}R_3(t).$$

Because of $b_3 + a_{31}b_1/a_{11} + a_{32}b_2/a_{22} \le 0$, we have

$$\lim_{t\to\infty} x_3(t) = 0 \; a.s.$$

The proof of (iii) is similar to above by Lemma 2.3.

For (iv) and (v), according to (17) and (18), by $b_1 > 0, b_2 < 0$ and Lemma 2.3, then

$$\lim_{t\to\infty} \overline{x_1(t)} = \frac{b_1}{a_{11}}, \quad \lim_{t\to\infty} x_2(t) = 0.$$

Then, from (24) we obtain

$$|\varphi(t)| \le a_{31}\frac{b_1}{a_{11}} + \varepsilon, \tag{26}$$

for any $\varepsilon > 0$. Substituting (26) into (19), there exists enough large t such that

$$\left| t^{-1}\ln\frac{x_3(t)}{x_3(0)} - b_3 - a_{31}\frac{b_1}{a_{11}} + a_{33}\overline{x_3(t)} - t^{-1}\sigma_3 B_3(t) - t^{-1}R_3(t) \right| \le \varepsilon.$$

Consequently, according to Lemma 2.3, we can get that

$$\lim_{t\to\infty}\overline{x_3(t)}=\frac{b_3+a_{31}b_1/a_{11}}{a_{33}}.$$

In the same way, there is

$$t^{-1}\ln\frac{x_3(t)}{x_3(0)}\le b_3+a_{31}\frac{b_1}{a_{11}}+\varepsilon-a_{33}\overline{x_3(t)}+t^{-1}\sigma_3B_3(t)+t^{-1}R_3(t).$$

Because of $b_3+a_{31}b_1/a_{11}<0$, we obtain

$$\lim_{t\to\infty}x_3(t)=0.$$

This proof is completed.

3. Main Results

3.1. Extinction and Persistence in Mean

Theorem 3.1: If all Assumptions hold, for model (4), one has

(I) If $b_1<0$, then

$$\lim_{t\to\infty}y_i(t)=0,\ a.s.i=1,2,3.$$

(II) If $\eta>0$, then

$$\lim_{t\to\infty}y_i(t)=\frac{\Delta_i-\overline{\Delta_i}}{\Delta}\quad a.s.i=1,2,3.$$

(III) If $b_2>0,\eta<0,\kappa_1<0$, then

$$\lim_{t\to\infty}y_1(t)=0,\lim_{t\to\infty}\overline{y_2(t)}=\frac{b_2}{a_{22}},\lim_{t\to\infty}y_3(t)=0.a.s.$$

(IV) If $b_1>0,\eta<0,\kappa_2<0$, then

$$\lim_{t\to\infty}\overline{y_1(t)}=\frac{b_1}{a_{11}},\lim_{t\to\infty}y_2(t)=0,\lim_{t\to\infty}y_3(t)=0.a.s.$$

(V) If $\eta<0,\kappa_1>0,\kappa_2>0$, then

$$\lim_{t\to\infty}\overline{y_1(t)}=\frac{\kappa_1}{\Lambda_{33}},\lim_{t\to\infty}\overline{y_2(t)}=\frac{\kappa_2}{\Lambda_{33}},\lim_{t\to\infty}y_3(t)=0.a.s.$$

Proof. By applying Ito's formula to model (4),

$$\begin{aligned}
&t^{-1}\ln\frac{y_1(t)}{y_1(0)}\\
&=b_1-a_{11}\overline{y_1(t)}+t^{-1}\sigma_1B_1(t)+t^{-1}\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_1(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u)\\
&\quad-t^{-1}a_{12}\int_0^t\int_{-\tau_{12}}^0 y_2(s+\theta)\mathrm{d}\mu_{12}(\theta)\mathrm{d}s-t^{-1}a_{13}\int_0^t\int_{-\tau_{13}}^0 y_3(s+\theta)\mathrm{d}\mu_{13}(\theta)\mathrm{d}s\\
&=b_1-a_{11}\overline{y_1(t)}-a_{12}\overline{y_2(t)}-a_{13}\overline{y_3(t)}+t^{-1}\sigma_1B_1(t)\\
&\quad+t^{-1}\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_1(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u)\\
&\quad+t^{-1}\left\{a_{12}\int_{-\tau_{12}}^0\left[\int_{t+\theta}^t y_2(s)\mathrm{d}s-\int_\theta^0 y_2(s)\mathrm{d}s\right]\mathrm{d}\mu_{12}(\theta)\right. &(27)\\
&\quad\left.+a_{13}\int_{-\tau_{13}}^0\left[\int_{t+\theta}^t y_3(s)\mathrm{d}s-\int_\theta^0 y_3(s)\mathrm{d}s\right]\mathrm{d}\mu_{13}(\theta)\right\}\\
&=b_1-a_{11}\overline{y_1(t)}-a_{12}\overline{y_2(t)}-a_{13}\overline{y_3(t)}+t^{-1}\sigma_1B_1(t)+t^{-1}R_1(t)+t^{-1}\Phi_1(t),
\end{aligned}$$

$$
\begin{aligned}
& t^{-1}\ln\frac{y_2(t)}{y_2(0)} \\
& = b_2 - a_{22}\overline{y_2(t)} + t^{-1}\sigma_2 B_2(t) + t^{-1}\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_2(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u) \\
& \quad - t^{-1}a_{21}\int_0^t\int_{-\tau_{21}}^0 y_1(s+\theta)\mathrm{d}\mu_{21}(\theta)\mathrm{d}s - t^{-1}a_{23}\int_0^t\int_{-\tau_{23}}^0 y_3(s+\theta)\mathrm{d}\mu_{23}(\theta)\mathrm{d}s \\
& = b_2 - a_{22}\overline{y_2(t)} - a_{21}\overline{y_1(t)} - a_{23}\overline{y_3(t)} + t^{-1}\sigma_2 B_2(t) \\
& \quad + t^{-1}\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_2(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u) \\
& \quad + t^{-1}\left\{a_{21}\int_{-\tau_{21}}^0\left[\int_{t+\theta}^t y_1(s)\mathrm{d}s - \int_\theta^0 y_1(s)\mathrm{d}s\right]\mathrm{d}\mu_{21}(\theta)\right. \\
& \quad \left. + a_{23}\int_{-\tau_{23}}^0\left[\int_{t+\theta}^t y_3(s)\mathrm{d}s - \int_\theta^0 y_3(s)\mathrm{d}s\right]\mathrm{d}\mu_{23}(\theta)\right\} \\
& = b_2 - a_{22}\overline{y_2(t)} - a_{21}\overline{y_1(t)} - a_{23}\overline{y_3(t)} + t^{-1}\sigma_2 B_2(t) + t^{-1}R_2(t) + t^{-1}\Phi_2(t),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
& t^{-1}\ln\frac{y_3(t)}{y_3(0)} \\
& = b_3 - a_{33}\overline{y_3(t)} + t^{-1}\sigma_3 B_3(t) + t^{-1}\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_3(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u) \\
& \quad - t^{-1}a_{31}\int_0^t\int_{-\tau_{31}}^0 y_1(s+\theta)\mathrm{d}\mu_{31}(\theta)\mathrm{d}s - t^{-1}a_{32}\int_0^t\int_{-\tau_{32}}^0 y_{32}(s+\theta)\mathrm{d}\mu_{32}(\theta)\mathrm{d}s \\
& = b_3 - a_{33}\overline{y_3(t)} + a_{31}\overline{y_1(t)} + a_{32}\overline{y_2(t)} + t^{-1}\sigma_3 B_3(t) \\
& \quad + t^{-1}\int_0^t\int_{\mathbb{Z}}\ln\left(1+\gamma_3(u)\right)\tilde{\Gamma}(\mathrm{d}s,\mathrm{d}u) \\
& \quad + t^{-1}\left\{a_{31}\int_{-\tau_{31}}^0\left[\int_t^{t+\theta} y_1(s)\mathrm{d}s + \int_\theta^0 y_1(s)\mathrm{d}s\right]\mathrm{d}\mu_{31}(\theta)\right. \\
& \quad \left. + a_{32}\int_{-\tau_{32}}^0\left[\int_t^{t+\theta} y_2(s)\mathrm{d}s + \int_\theta^0 y_2(s)\mathrm{d}s\right]\mathrm{d}\mu_{32}(\theta)\right\} \\
& = b_3 - a_{33}\overline{y_3(t)} + a_{31}\overline{y_1(t)} + a_{32}\overline{y_2(t)} + t^{-1}\sigma_3 B_3(t) + t^{-1}R_3(t) + t^{-1}\Phi_3(t).
\end{aligned}
\tag{29}
$$

Since

$$
\int_{-\tau_{12}}^0\int_{t+\theta}^t y_2(s^-)\mathrm{d}\mu_{12}(\theta)\mathrm{d}s \le \int_{-\tau_{12}}^0\mathrm{d}\mu_{12}(\theta)\int_{t-\tau_{12}}^t y_2(s^-)\mathrm{d}s,
$$

$$
\int_{-\tau_{12}}^0\int_\theta^0 y_2(s^-)\mathrm{d}\mu_{12}(\theta)\mathrm{d}s \le \int_{-\tau_{12}}^0\mathrm{d}\mu_{12}(\theta)\int_{-\tau_{12}}^0 y_2(s^-)\mathrm{d}s,
$$

by Lemma 2.4, we get

$$
\lim_{t\to\infty} t^{-1}\int_{-\tau_{12}}^0\int_{t+\theta}^t y_2(s^-)\mathrm{d}\mu_{12}(\theta)\mathrm{d}s = 0,\ \lim_{t\to\infty}\int_{-\tau_{12}}^0\int_\theta^0 y_2(s^-)\mathrm{d}\mu_{12}(\theta)\mathrm{d}s = 0.
$$

Similarly, there is

$$
\lim_{t\to\infty} t^{-1}\int_{-\tau_{13}}^0\int_{t+\theta}^t y_3(s^-)\mathrm{d}\mu_{13}(\theta)\mathrm{d}s = 0,\ \lim_{t\to\infty}\int_{-\tau_{13}}^0\int_\theta^0 y_3(s^-)\mathrm{d}\mu_{13}(\theta)\mathrm{d}s = 0.
$$

Therefore, for $\Phi_1(t)$, we have

$$
\lim_{t\to\infty} t^{-1}\Phi_1(t) = 0\ a.s. \tag{30}
$$

In the same way, the following conclusions can be deduced,

$$
\lim_{t\to\infty} t^{-1}\Phi_2(t) = 0,\ \lim_{t\to\infty} t^{-1}\Phi_3(t) = 0,\ a.s. \tag{31}
$$

For (I), according to (i) in Lemma 2.5, there is

$$\lim_{t\to\infty} x_i(t) = 0,\ i = 1,2,3.$$

In view of (16), we obtain

$$\lim_{t\to\infty} y_i(t) = 0,\ i = 1,2,3.$$

Now, let us prove (II).

Let β_1, β_2 be the solution of the equations:

$$\begin{cases} a_{11}\beta_1 + a_{21}\beta_2 = -a_{31} \\ a_{12}\beta_1 + a_{22}\beta_2 = -a_{32}. \end{cases}$$

By Cramer's Rule, then

$$\beta_1 = \frac{-\Delta_{13}}{\Delta_{33}} > 0,\ \beta_2 = \frac{\Delta_{23}}{\Delta_{33}} > 0.$$

Computing $(27)\times(-\beta_1)+(28)\times(-\beta_2)+(29)$, we get

$$\begin{aligned} t^{-1}\ln\frac{y_3(t)}{y_3(0)} &= \left(b_3 - \sum_{i=1,2}\beta_i b_i\right) - \left(a_{33} - a_{13}\beta_1 - a_{23}\beta_2\right)\overline{y_3(t)} + t^{-1}\sigma_3 B_3(t) \\ &\quad + t^{-1}R_3(t) + t^{-1}\sum_{i=1,2}\beta_i \ln\frac{y_i(t)}{y_i(0)} - t^{-1}\sum_{i=1,2}\beta_i\sigma_i B_i(t) \\ &\quad - t^{-1}\sum_{i=1,2}\beta_i R_i(t) + t^{-1}\Phi_3(t) - t^{-1}\sum_{i=1,2}\beta_i\Phi_i(t) \\ &= \frac{\Delta_3 - \tilde{\Delta}_3}{\Delta_{33}} - \frac{\Delta}{\Delta_{33}}\overline{y_3(t)} + t^{-1}\sigma_3 B_3(t) + t^{-1}R_3(t) \\ &\quad + t^{-1}\sum_{i=1,2}\beta_i \ln\frac{y_i(t)}{y_i(0)} - t^{-1}\sum_{i=1,2}\beta_i\sigma_i B_i(t) - t^{-1}\sum_{i=1,2}\beta_i R_i(t) \\ &\quad + t^{-1}\Phi_3(t) - t^{-1}\sum_{i=1,2}\beta_i\Phi_i(t). \end{aligned} \tag{32}$$

According to Lemma 2.4, there exists a positive T_1, for any $\varepsilon > 0$ and all $t > T_1$, such that

$$\limsup_{t\to\infty} x_i(t)/t \le 0,\ i = 1,2,3.$$

In view of (16), there is

$$\limsup_{t\to\infty} y_i(t)/t \le 0,\ i = 1,2,3. \tag{33}$$

Therefore, from (30), (31) and (33), we obtain

$$\begin{aligned} &t^{-1}\sum_{i=1,2}\beta_i \ln\frac{y_i(t)}{y_i(0)} \le 0.5\varepsilon, \\ &t^{-1}\Phi_3(t) - t^{-1}\sum_{i=1,2}\beta_i\Phi_i(t) \le 0.5\varepsilon. \end{aligned} \tag{34}$$

Using (34) in (32), for enough large *t*, one can observe that

$$\begin{aligned} t^{-1}\ln\frac{y_3(t)}{y_3(0)} &\le \frac{\Delta_3 - \tilde{\Delta}_3}{\Delta_{33}} + \varepsilon - \frac{\Delta}{\Delta_{33}}\overline{y_3(t)} + t^{-1}\sigma_3 B_3(t) + t^{-1}R_3(t) \\ &\quad - t^{-1}\sum_{i=1,2}\beta_i\sigma_i B_i(t) - t^{-1}\sum_{i=1,2}\beta_i R_i(t). \end{aligned}$$

By Lemma 2.3, we get

$$\overline{y_3}^* = \limsup_{t\to\infty} t^{-1}\int_0^t y_3(s)\mathrm{d}s \le \frac{\Delta_3 - \tilde{\Delta}_3}{\Delta}\ a.s. \tag{35}$$

Let ρ_1, ρ_3 be the solution of the equations:

$$\begin{cases} a_{11}\rho_1 - a_{31}\rho_3 = a_{21} \\ a_{13}\rho_1 + a_{33}\rho_3 = a_{23}. \end{cases}$$

Consequently,

$$\rho_1 = \frac{\Delta_{12}}{\Delta_{22}} > 0,\ \rho_3 = \frac{\Delta_{32}}{\Delta_{22}} > 0.$$

Compute $(27)\times(-\rho_1)+(28)+(29)\times(-\rho_3)$, we get

$$\begin{aligned} t^{-1}\ln\frac{y_2(t)}{y_2(0)} &= \left(b_2 - \sum_{i=1,3}\rho_i b_i\right) - (a_{22} - a_{12}\rho_1 + a_{32}\rho_3)\overline{y_2(t)} + t^{-1}\sigma_2 B_2(t) \\ &\quad + t^{-1}R_2(t) + t^{-1}\sum_{i=1,3}\rho_i \ln\frac{y_i(t)}{y_i(0)} - t^{-1}\sum_{i=1,3}\rho_i\sigma_i B_i(t) \\ &\quad - t^{-1}\sum_{i=1,3}\rho_i R_i(t) + t^{-1}\Phi_2(t) - t^{-1}\sum_{i=1,3}\rho_i\Phi_i(t) \\ &= \frac{\Delta_2 - \tilde{\Delta}_2}{\Delta_{22}} - \frac{\Delta}{\Delta_{22}}\overline{y_2(t)} + t^{-1}\sigma_2 B_2(t) + t^{-1}R_2(t) \\ &\quad + t^{-1}\sum_{i=1,3}\rho_i \ln\frac{y_i(t)}{y_i(0)} - t^{-1}\sum_{i=1,3}\rho_i\sigma_i B_i(t) \\ &\quad - t^{-1}\sum_{i=1,3}\rho_i R_i(t) + t^{-1}\Phi_2(t) - t^{-1}\sum_{i=1,3}\rho_i\Phi_i(t). \end{aligned} \tag{36}$$

Similarly, there exists a positive T_2, for any $\varepsilon > 0$ and all $t > T_2$, such that

$$\begin{aligned} &t^{-1}\sum_{i=1,3}\rho_i \ln\frac{y_i(t)}{y_i(0)} \le 0.5\varepsilon, \\ &t^{-1}\Phi_2(t) - t^{-1}\sum_{i=1,3}\rho_i\Phi_i(t) \le 0.5\varepsilon. \end{aligned} \tag{37}$$

Substituting (37) into (36), we get

$$\begin{aligned} t^{-1}\ln\frac{y_2(t)}{y_2(0)} \le \frac{\Delta_2 - \tilde{\Delta}_2}{\Delta_{22}} + \varepsilon - \frac{\Delta}{\Delta_{22}}\overline{y_2(t)} + t^{-1}\sigma_2 B_2(t) + t^{-1}R_2(t) \\ + t^{-1}\sum_{i=1,3}\rho_i\sigma_i B_i(t) - t^{-1}\sum_{i=1,3}\rho_i R_i(t). \end{aligned}$$

By $\Delta_2/\tilde{\Delta}_2 > \Delta_3/\tilde{\Delta}_3 > 1$, then we obtain

$$\overline{y_2}^* = \limsup_{t\to\infty} t^{-1}\int_0^t y_2(s)\mathrm{d}s \le \frac{\Delta_2 - \tilde{\Delta}_2}{\Delta}\ a.s. \tag{38}$$

From (35) and (38), there exists a constant $K > 0$, for enough small ε, and $t > K$, such that

$$a_{12}\overline{y_2(t)} \le a_{12}\overline{y_2(t)}^* + \varepsilon \le \frac{a_{12}(\Delta_2 - \tilde{\Delta}_2)}{\Delta} + \varepsilon\ a.s. \tag{39}$$

$$a_{13}\overline{y_3(t)} \le a_{13}\overline{y_3(t)}^* + \varepsilon \le \frac{a_{13}(\Delta_3 - \tilde{\Delta}_3)}{\Delta} + \varepsilon\ a.s. \tag{40}$$

Using (39) and (40) in (27), for enough large t

$$t^{-1}\ln\frac{y_1(t)}{y_1(0)} \ge b_1 - \frac{a_{12}\left(\Delta_2-\tilde{\Delta}_2\right)}{\Delta} - \frac{a_{13}\left(\Delta_3-\tilde{\Delta}_3\right)}{\Delta} - 3\varepsilon - a_{11}\overline{y_1(t)} + t^{-1}\sigma_1 B_1(t) + t^{-1}R_1(t).$$

According to Lemma 2.3, we obtain

$$\overline{y_1(t)}_* = \liminf_{t\to\infty} t^{-1}\int_0^t y_1(s)\,\mathrm{d}s \ge \frac{\Delta_1-\tilde{\Delta}_1}{\Delta}\ a.s. \tag{41}$$

Let ζ_2,ζ_3 be the solution of the equations:

$$\begin{cases} a_{22}\zeta_2 + a_{32}\zeta_3 = a_{12} \\ -a_{23}\zeta_2 + a_{33}\zeta_3 = a_{13}. \end{cases}$$

Then,

$$\zeta_2 = \frac{\Delta_{21}}{\Delta_{11}} > 0,\ \zeta_3 = \frac{-\Delta_{31}}{\Delta_{11}} > 0.$$

Compute $(27)+(28)\times(-\zeta_2)+(29)\times\zeta_3$, we get

$$\begin{aligned} t^{-1}\ln\frac{y_1(t)}{y_1(0)} &= \left(b_1 - \sum_{i=2,3}\zeta_i b_i\right) - \left(a_{11} - a_{21}\zeta_1 + a_{31}\zeta_3\right)\overline{y_1(t)} + t^{-1}\sigma_1 B_1(t) \\ &\quad + t^{-1}R_1(t) + t^{-1}\sum_{i=2,3}\zeta_i \ln\frac{y_i(t)}{y_i(0)} - t^{-1}\sum_{i=2,3}\zeta_i\sigma_i B_i(t) \\ &\quad - t^{-1}\sum_{i=2,3}\zeta_i R_i(t) + t^{-1}\Phi_1(t) - t^{-1}\sum_{i=2,3}\zeta_i\Phi_i(t) \\ &= \frac{\Delta_1-\tilde{\Delta}_1}{\Delta_{11}} - \frac{\Delta}{\Delta_{11}}\overline{y_1(t)} + t^{-1}\sigma_1 B_1(t) + t^{-1}R_1(t) \\ &\quad + t^{-1}\sum_{i=2,3}\zeta_i \ln\frac{y_i(t)}{y_i(0)} - t^{-1}\sum_{i=2,3}\zeta_i\sigma_i B_i(t) \\ &\quad - t^{-1}\sum_{i=2,3}\zeta_i R_i(t) + t^{-1}\Phi_1(t) - t^{-1}\sum_{i=2,3}\zeta_i\Phi_i(t). \end{aligned} \tag{42}$$

Similarly, we can get that

$$\begin{aligned} &t^{-1}\sum_{i=2,3}\zeta_i \ln\frac{y_i(t)}{y_i(0)} \le 0.5\varepsilon, \\ &t^{-1}\Phi_1(t) - t^{-1}\sum_{i=2,3}\zeta_i\Phi_i(t) \le 0.5\varepsilon. \end{aligned} \tag{43}$$

Substituting (43) into (42), for any $\varepsilon > 0$, we can obtain that

$$\begin{aligned} t^{-1}\ln\frac{y_1(t)}{y_1(0)} &\le \frac{\Delta_1-\tilde{\Delta}_1}{\Delta_{11}} + \varepsilon - \frac{\Delta}{\Delta_{11}}\overline{y_1(t)} + t^{-1}\sigma_1 B_1(t) + t^{-1}R_1(t) \\ &\quad - t^{-1}\sum_{i=2,3}\zeta_i\sigma_i B_i(t) - t^{-1}\sum_{i=2,3}\zeta_i R_i(t), \end{aligned}$$

then, we get

$$\overline{y_1(t)}^* \le \frac{\Delta_1-\tilde{\Delta}_1}{\Delta}\ a.s. \tag{44}$$

Combining (41) with (44), we obtain

$$\lim_{t\to\infty}\overline{y_1(t)}=\frac{\Delta_1-\tilde{\Delta}_1}{\Delta}\,a.s.$$

In the same way, substituting (35) and (44) into (28), and together with (38), we get

$$\lim_{t\to\infty}\overline{y_2(t)}=\frac{\Delta_2-\tilde{\Delta}_2}{\Delta}\,a.s. \tag{45}$$

Substituting (41) and (45) into (29), and together with (35), we have

$$\lim_{t\to\infty}\overline{y_3(t)}=\frac{\Delta_3-\tilde{\Delta}_3}{\Delta}\,a.s.$$

For (III), there is $\frac{\Delta_3-\tilde{\Delta}_3}{\Delta_{33}}<0$, therefore, we can get that from (29)

$$\lim_{t\to\infty}y_3(t)=0\,a.s.$$

Then, we can easily compute that

$$\begin{aligned}&a_{22}\ln y_1(t)-a_{12}\ln y_2(t)\\&=(b_1a_{22}-b_2a_{12})t-(a_{11}a_{22}-a_{12}a_{21})\overline{y_1(t)}+t^{-1}\Phi_4(t)+t^{-1}\Phi_5(t)\\&=\kappa_1t-\Delta_{33}\overline{y_1(t)}+t^{-1}\Phi_4(t)-t^{-1}\Phi_5(t),\end{aligned}$$

where

$$\Phi_4(t)=a_{22}\left(\sigma_1B_1(t)+\ln y_1(0)+R_1(t)+\Phi_1(t)\right),$$

and

$$\Phi_5(t)=a_{12}\left(\sigma_2B_2(t)+\ln y_2(0)+R_2(t)+\Phi_2(t)\right).$$

We can also get $\lim_{t\to\infty}t^{-1}\Phi_4(t)=0,\ \lim_{t\to\infty}t^{-1}\Phi_5(t)=0,\ a.s.$. From Lemma 2.4, we have that for enough large *t*, and any $\varepsilon>0$, there is

$$a_{22}\ln y_1(t)\le(\kappa_1+\varepsilon)t-\Delta_{33}\overline{y_1(t)}+t^{-1}\Phi_4(t)-t^{-1}\Phi_5(t).$$

It follows from Lemma 2.3 in [12] that,

$$\lim_{t\to\infty}y_1(t)=0\quad\text{if}\quad\kappa_1\le0,$$

and

$$\overline{y_1(t)}^{*}=\limsup_{t\to\infty}t^{-1}\int_0^t y_1(s)\,\mathrm{d}s\le\frac{\kappa_1}{\Delta_{33}}\quad\text{if}\quad\kappa_1>0. \tag{46}$$

Therefore,

$$t^{-1}\ln\frac{y_2(t)}{y_2(0)}=b_2-a_{22}\overline{y_2(t)}+t^{-1}\sigma_2B_2(t)+t^{-1}R_2(t)+t^{-1}\Phi_2(t)$$

According to Lemma 2.3, we obtain

$$\overline{y_2(t)}^{*}\le\frac{b_2}{a_{22}}\,a.s.,$$

and,

$$\overline{y_2(t)}_* \geq \frac{b_2}{a_{22}} a.s..$$

Then, $\lim_{t\to\infty} y_2(t) = \frac{b_2}{a_{22}} a.s.$.

Now, we prove that (IV) is also correct.

There is still that $\lim_{t\to\infty} y_3(t) = 0$ *a.s.*, if $\eta < 0$.

Then, computing

$$\begin{aligned} &a_{11} \ln y_2(t) - a_{21} \ln y_1(t) \\ &= (b_2 a_{11} - b_1 a_{21})t - (a_{11}a_{22} - a_{12}a_{21})\overline{y_2(t)} + t^{-1}\Phi_6(t) + t^{-1}\Phi_7(t) \\ &= \kappa_2 t - \Delta_{33}\overline{y_2(t)} + t^{-1}\Phi_6(t) - t^{-1}\Phi_7(t), \end{aligned}$$

where

$$\Phi_6(t) = a_{11}\left(\sigma_2 B_2(t) + \ln y_2(0) + R_2(t) + \Phi_2(t)\right),$$

and

$$\Phi_7(t) = a_{21}\left(\sigma_1 B_1(t) + \ln y_1(0) + R_1(t) + \Phi_1(t)\right).$$

Then we get $\lim_{t\to\infty} t^{-1}\Phi_6(t) = 0, \lim_{t\to\infty} t^{-1}\Phi_7(t) = 0,$ *a.s.*. Similarly, for any ε, we get

$$a_{11} \ln y_2(t) \leq (\kappa_2 + \varepsilon)t - \Delta_{33}\overline{y_2(t)} + t^{-1}\Phi_6(t) - t^{-1}\Phi_7(t).$$

From Lemma 2.3 in [12], we obtain

$$\lim_{t\to\infty} y_2(t) = 0 \text{ if } \kappa_2 \leq 0,$$

and

$$\overline{y_2(t)}^* \leq \frac{\kappa_2}{\Delta_{33}} \text{ if } \kappa_2 > 0. \quad (47)$$

Thus, there is

$$t^{-1} \ln \frac{y_1(t)}{y_1(0)} = b_1 - a_{11}\overline{y_1(t)} + t^{-1}\sigma_1 B_1(t) + t^{-1}R_1(t) + t^{-1}\Phi_1(t).$$

Noticing $b_1 > b_2 > 0$, from Lemma 2.3, we get

$$\lim_{t\to\infty} y_1(t) = \frac{b_1}{a_{11}} a.s..$$

Finally, we consider the most complicated case, the case (V).

Here we still have

$$\lim_{t\to\infty} y_3(t) = 0 \; a.s. \quad (48)$$

From (46), there exists a enough large *t*, for any $\varepsilon > 0$ such that

$$a_{21}\overline{y_1(t)} \leq a_{21}\overline{y_1(t)}^* \leq \frac{a_{21}\kappa_1}{\Delta_{33}} + \varepsilon. \quad (49)$$

Substituting (49) into (28), we obtain

$$t^{-1}\ln y_2(t)\ge\left(b_2-\frac{a_{21}\kappa_1}{\Delta_{33}}-\varepsilon\right)-a_{22}\overline{y_2(t)}+t^{-1}\sigma_2B_2(t)+t^{-1}R_2(t)+t^{-1}\Phi_2(t).$$

Therefore, for the arbitrariness of ε, we get that from Lemma 2.3,

$$\overline{y_2(t)}_*\ge\frac{\kappa_2}{\Delta_{33}}\ a.s..$$

Combining with (47), we obtain

$$\lim_{t\to\infty}\overline{y_2(t)}=\frac{\kappa_2}{\Delta_{33}}\ a.s.. \tag{50}$$

Using (48) and (50) into (27) gives

$$t^{-1}\ln\frac{y_1(t)}{y_1(0)}=b_1-\frac{a_{12}\kappa_2}{\Delta_{33}}-a_{11}\overline{y_1(t)}+t^{-1}\sigma_1B_1(t)+t^{-1}R_1(t)+t^{-1}\Phi_1(t).$$

From Lemma 2.3, we have

$$\overline{y_1(t)}_*=\liminf_{t\to\infty}t^{-1}\int_0^t y_1(s)\mathrm{d}s\ge\frac{b_1-a_{12}\kappa_2/\Delta_{33}}{a_{11}}=\frac{\kappa_1}{\Delta_{33}}\ a.s.,$$

and

$$\overline{y_1(t)}^*=\limsup_{t\to\infty}t^{-1}\int_0^t y_1(s)\mathrm{d}s\le\frac{\kappa_1}{\Delta_{33}}\ a.s..$$

Obviously, there is

$$\lim_{t\to\infty}\overline{y_1(t)}=\frac{\kappa_1}{\Delta_{33}}\ a.s..$$

The proof of Theorem 1 is completed.

3.2. Global Attractivity

Theorem 3.2: Let $y(t)=(y_1(t;\phi),y_2(t;\phi),y_3(t;\phi))$, $y^*(t)=(y_1(t;\phi^*),y_2(t;\phi^*),y_3(t;\phi^*))$ be the solution of the model (4), respectively, with initial values $\phi,\phi^*\in C([-\gamma,0],R_+^3)$, then

$$\lim_{t\to+\infty}E\sqrt{\left|y_1(t;\phi)-y_1(t;\phi^*)\right|^2+\left|y_2(t;\phi)-y_2(t;\phi^*)\right|^2+\left|y_3(t;\phi)-y_3(t;\phi^*)\right|^2}=0$$

Proof. Denote $\tilde{y}_i(t)=y_i(t;\phi)-y_i(t;\phi^*),i=1,2,3$, we only need to verify that

$$\lim_{t\to+\infty}E\left|\tilde{y}_i(t)\right|=\lim_{t\to+\infty}E\left|y_i(t;\phi)-y_i(t;\phi^*)\right|=0,i=1,2,3. \tag{51}$$

Define $V_i(t;\phi,\phi^*)=\left|\ln y_i(t;\phi)-\ln y_i(t;\phi^*)\right|,i=1,2,3$, by Ito's formula, we obtain

$$\begin{aligned}&LV_1(t;\phi,\phi^*)\\&=sign(\tilde{y}_1(t))\Big[-a_{11}(y_1(t;\phi)-y_1(t;\phi^*))\\&\quad-a_{12}\int_{-\tau_{12}}^0(y_2(t+\theta;\phi)-y_2(t+\theta;\phi^*))\mathrm{d}\mu_{12}(\theta)\\&\quad-a_{13}\int_{-\tau_{13}}^0(y_3(t+\theta;\phi)-y_3(t+\theta;\phi^*))\mathrm{d}\mu_{13}(\theta)\Big]\\&\le-a_{11}\left|\tilde{y}_1(t)\right|+a_{12}\int_{-\tau_{12}}^0\left|\tilde{y}_2(t+\theta)\right|\mathrm{d}\mu_{12}(\theta)+a_{13}\int_{-\tau_{13}}^0\left|\tilde{y}_3(t+\theta)\right|\mathrm{d}\mu_{13}(\theta),\end{aligned}$$

$$LV_2\left(t;\phi,\phi^*\right)$$
$$\le -a_{22}\left|\tilde{y}_2(t)\right| + a_{21}\int_{-\tau_{21}}^{0}\left|\tilde{y}_1(t+\theta)\right|\mathrm{d}\mu_{21}(\theta) + a_{23}\int_{-\tau_{23}}^{0}\left|\tilde{y}_3(t+\theta)\right|\mathrm{d}\mu_{23}(\theta),$$
$$LV_3\left(t;\phi,\phi^*\right)$$
$$= sign\left(\tilde{y}_3(t)\right)\Big[-a_{33}\left(y_1(t;\phi)-y_1\left(t;\phi^*\right)\right)$$
$$+a_{31}\int_{-\tau_{31}}^{0}\left(y_1(t+\theta;\phi)-y_1\left(t+\theta;\phi^*\right)\right)\mathrm{d}\mu_{31}(\theta)$$
$$+a_{32}\int_{-\tau_{32}}^{0}\left(y_2(t+\theta;\phi)-y_2\left(t+\theta;\phi^*\right)\right)\mathrm{d}\mu_{32}(\theta)\Big]$$
$$\le -a_{33}\left|\tilde{y}_3(t)\right| + a_{31}\int_{-\tau_{31}}^{0}\left|\tilde{y}_1(t+\theta)\right|\mathrm{d}\mu_{31}(\theta) + a_{32}\int_{-\tau_{32}}^{0}\left|\tilde{y}_2(t+\theta)\right|\mathrm{d}\mu_{32}(\theta).$$

Define $V\left(t;\phi,\phi^*\right)=\sum_{i=1}^{3}V_i\left(t;\phi,\phi^*\right)+V_4\left(t;\phi,\phi^*\right)$, where

$$V_4\left(t;\phi,\phi^*\right)$$
$$=\sum_{i=2,3}a_{1i}\int_{-\tau_{1i}}^{0}\int_{t+\theta}^{t}\left|\tilde{y}_i(s)\right|\mathrm{d}\mu_{1i}(\theta)\mathrm{d}s+\sum_{i=1,3}a_{2i}\int_{-\tau_{2i}}^{0}\int_{t+\theta}^{t}\left|\tilde{y}_i(s)\right|\mathrm{d}\mu_{2i}(\theta)\mathrm{d}s$$
$$+\sum_{i=1,2}a_{3i}\int_{-\tau_{3i}}^{0}\int_{t+\theta}^{t}\left|\tilde{y}_i(s)\right|\mathrm{d}\mu_{3i}(\theta)\mathrm{d}s.$$

From Ito's formula, it is easy to compute that

$$LV\left(t;\phi,\phi^*\right)$$
$$=LV_1\left(t;\phi,\phi^*\right)+LV_2\left(t;\phi,\phi^*\right)+LV_3\left(t;\phi,\phi^*\right)+\frac{\mathrm{d}V_4\left(t;\phi,\phi^*\right)}{\mathrm{d}t}$$
$$\le-\left(a_{11}-\sum_{i=2,3}a_{i1}\int_{-\tau_{i1}}^{0}\mathrm{d}\mu_{1i}(\theta)\right)\left|\tilde{y}_i(t)\right|-\left(a_{22}-\sum_{i=1,3}a_{i2}\int_{-\tau_{i2}}^{0}\mathrm{d}\mu_{1i}(\theta)\right)\left|\tilde{y}_2(t)\right| \tag{52}$$
$$-\left(a_{33}-\sum_{i=1,2}a_{i3}\int_{-\tau_{i3}}^{0}\mathrm{d}\mu_{i3}(\theta)\right)\left|\tilde{y}_3(t)\right|.$$

According to (52), we get

$$E\left(V\left(t;\phi,\phi^*\right)\right)$$
$$\le V\left(0;\phi,\phi^*\right)-\left(a_{11}-\sum_{i=2,3}a_{i1}\int_{-\tau_{i1}}^{0}\mathrm{d}\mu_{1i}(\theta)\right)\int_0^t E\left|\tilde{y}_1(s)\right|\mathrm{d}s$$
$$-\left(a_{22}-\sum_{i=1,3}a_{i2}\int_{-\tau_{i2}}^{0}\mathrm{d}\mu_{1i}(\theta)\right)\int_0^t E\left|\tilde{y}_2(s)\right|\mathrm{d}s$$
$$-\left(a_{33}-\sum_{i=1,2}a_{i3}\int_{-\tau_{i3}}^{0}\mathrm{d}\mu_{i3}(\theta)\right)\int_0^t E\left|\tilde{y}_3(s)\right|\mathrm{d}s.$$

This means

$$E\left(V\left(t;\phi,\phi^*\right)\right)+\left(a_{11}-\sum_{i=2,3}a_{i1}\int_{-\tau_{i1}}^{0}\mathrm{d}\mu_{1i}\left(\theta\right)\right)\int_0^t E\left|\tilde{y}_1\left(s\right)\right|\mathrm{d}s$$
$$+\left(a_{22}-\sum_{i=1,3}a_{i2}\int_{-\tau_{i2}}^{0}\mathrm{d}\mu_{1i}\left(\theta\right)\right)\int_0^t E\left|\tilde{y}_2\left(s\right)\right|\mathrm{d}s$$
$$+\left(a_{33}-\sum_{i=1,2}a_{i3}\int_{-\tau_{i3}}^{0}\mathrm{d}\mu_{i3}\left(\theta\right)\right)\int_0^t E\left|\tilde{y}_3\left(s\right)\right|\mathrm{d}s\le V\left(0;\phi,\phi^*\right)<+\infty.$$

Consequently,

$$E\left|\tilde{y}_i\left(t\right)\right|\in L^1\left[0,+\infty\right),\ i=1,2,3. \tag{53}$$

Now, we consider the continuity of $E\left(y_i\left(t\right)\right)$, $i=1,2,3$. In view of model (4), we have

$$E\left(B_i\left(t\right)\right)=0,\ \ E\left(R_i\left(t\right)\right)=0,\ i=1,2,3,$$

and

$$E\left(y_1\left(t\right)\right)=y_1\left(0\right)+\int_0^t E\Big[r_1y_1\left(s\right)-a_{11}y_1^2\left(s\right)-a_{12}y_1\left(s\right)\int_{-\tau_{12}}^{0}y_2\left(s+\theta\right)\mathrm{d}\mu_{12}\left(\theta\right)$$
$$-a_{13}y_1\left(s\right)\int_{-\tau_{13}}^{0}y_3\left(s+\theta\right)\mathrm{d}\mu_{13}\left(\theta\right)\Big]\mathrm{d}s.$$

This is means that $E\left(y_1\left(t\right)\right)$ is differential. From Lemma 2.2, we get

$$\frac{\mathrm{d}E\left(y_1\left(t\right)\right)}{\mathrm{d}t}\le E\left(y_1\left(t\right)\right)r_1\le r_1K_1,$$

where K_1 is a positive constant. Therefore, we know that $E\left(y_1\left(t\right)\right)$ is uniformly continuous. Similarly, we can also get that $E\left(y_2\left(t\right)\right)$ and $E\left(y_3\left(t\right)\right)$ are uniformly continuous. By the Barbalat's conclusion in [36] and (53), we get that

$$\lim_{t\to+\infty}E\left|\tilde{y}_i\left(t\right)\right|=0,i=1,2,3.$$

3.3. Stability in Distribution

Theorem 3.3: If all assumptions hold, model (4) is stable in distribution.

Proof. For any $\phi\in C([-\gamma,0],R_+^3)$, denote by $p(t,\phi,dz)$ the transition probability of the process $z(t)$, denote by $P(t,\phi,R_+^3)$ the probability of $(y_1(t;\phi),y_2(t;\phi),y_3(t;\phi))^{\mathrm{T}}\in R_+^3$. Denote by $\mathcal{P}([-\gamma,0],R_+^3)$ the space of all probability measures on $C([-\gamma,0],R_+^3)$. For any $P_1,P_2\in \mathcal{P}([-\gamma,0],R_+^3)$, we define

$$\mathrm{d}_G(P_1,P_2)=\sup_{g\in G}\left|\int_{R_+^3}g(z)P_1(dz)-\int_{R_+^3}g(z)P_2(dz)\right|,$$

where $G=\{g:C([-\gamma,0],R_+^3)\to R:|g(z_1)-g(z_2)|\leqslant\|z_1-z_2\|,|g(\cdot)\leqslant 1|\}$. Then, according to Lemma 2.2 and Chebyshe's inequality, for any $\phi\in C([-\gamma,0],R_+^3)$, the family $p(t,\phi,\cdot)$ is tight. That is, for any $\varepsilon\in(0,1)$, there exist a compact subset $D\subseteq R_+^3$, such that, for any $t\ge 0$,

$$P(t,\phi,D)\geqslant 1-\varepsilon.$$

Computing

$$\begin{aligned}
& \mathrm{d}_G(p(t+s,\phi,\cdot),p(t,\phi,\cdot)) \\
&= \sup_{\mathrm{g}\in G}\left|\int_{R_+^3}\mathrm{g}(z(t+s;\phi))p(t+s,\phi,dz)-\int_{R_+^3}\mathrm{g}(z(t;\phi))p(t,\phi,dz)\right| \\
&= \sup_{\mathrm{g}\in G}\left|E[\mathrm{g}(z(t+s;\phi))]-E[\mathrm{g}(z(t;\phi))]\right| \\
&= \sup_{\mathrm{g}\in G}\left|E[E[\mathrm{g}(z(t+s;\phi))|\mathcal{F}_s]]-E[\mathrm{g}(z(t;\phi))]\right| \\
&= \sup_{\mathrm{g}\in G}\left|\int_{R_+^3}E[\mathrm{g}(z(t;\psi))]p(s,\phi,d\psi)-E[\mathrm{g}(z(t;\phi))]\right| \\
&= \sup_{\mathrm{g}\in G}\left|\int_{R_+^3}E[\mathrm{g}(z(t;\psi))-\mathrm{g}(z(t;\phi))]p(s,\phi,d\psi)\right| \\
&\leqslant \sup_{\mathrm{g}\in G}\int_{R_+^3}E[|\mathrm{g}(z(t;\psi))-\mathrm{g}(z(t;\phi))|]p(s,\phi,d\psi) \\
&\leqslant \sup_{\mathrm{g}\in G}\int_{U_B}E[|\mathrm{g}(z(t;\psi))-\mathrm{g}(z(t;\phi))|]p(s,\phi,d\psi) \\
&\quad+\sup_{\mathrm{g}\in G}\int_{R_+^3\setminus U_B}E[|\mathrm{g}(z(t;\psi))-\mathrm{g}(z(t;\phi))|]p(s,\phi,d\psi),
\end{aligned}$$

where $U_B=\left\{(x,y,z)^T\in R_+^3:\sqrt{x^2+y^2+z^2}\leqslant B\right\}$. There exists a constant $B>0$,

$$\sup_{\mathrm{g}\in G}\int_{R_+^3\setminus U_B}E[|\mathrm{g}(z(t;\psi))-\mathrm{g}(z(t;\phi))|]p(s,\phi,d\psi)\leqslant 2P(s,\phi,R_+^3\setminus U_B)\leqslant 2\varepsilon.$$

By Theorem 3.2, for any $\varepsilon\in(0,1)$, and enough large *t*, we get

$$\begin{aligned}
&\sup_{\mathrm{g}\in G}\int_{U_B}E[|g(z(t;\psi))-g(z(t;\phi))|]p(s,\phi,d\psi) \\
&\leqslant \sup_{\mathrm{g}\in G}\int_{U_B}E[\|z(t;\psi)-z(t;\phi)\|]p(s,\phi,d\psi) \\
&\leqslant \sup_{\mathrm{g}\in G}\int_{U_B}\varepsilon p(s,\phi,d\psi)\leqslant\varepsilon.
\end{aligned}$$

Therefore, for enough large *t* and any $\varepsilon>0$, we can derive that

$$d_G(p(t+s,\phi,\cdot),p(t,\phi,\cdot))\leqslant 3\varepsilon.$$

That is to say, $\{p(t,\phi,\cdot):t\geqslant 0\}$ is Cauchy in $\mathcal{P}([-\gamma,0],R_+^3)$ with any initial value $\phi\in C([-\gamma,0],R_+^3)$. Then, for $\phi_0\in C([-\gamma,0],R_+^3)$, $\{p(t,\phi_0,\cdot):t\geqslant 0\}$ is Cauchy in $\mathcal{P}([-\gamma,0],R_+^3)$.

There exists a unique $\nu(\cdot)$ such that

$$\lim_{t\to+\infty}d_G(p(t,\phi_0,\cdot),\nu(\cdot))=0 \tag{54}$$

By virtue of Theorem 3.2, we derive

$$\begin{aligned}\lim_{t\to+\infty} d_G\left(p(t,\phi,\cdot),p(t,\phi_0,\cdot)\right) &= \sup_{f\in G}\left|E\left[f\left(z(t;\phi)\right)\right]-E\left[f\left(z(t;\phi_0)\right)\right]\right| \\ &\leqslant \sup_{f\in G} E\left[\left|f\left(z(t;\phi)\right)-f\left(z(t;\phi_0)\right)\right|\right] \\ &\leqslant \lim_{t\to+\infty} E\left[\left\|z(t;\phi)-z(t;\phi_0)\right\|\right]=0.\end{aligned} \tag{55}$$

By the triangle inequality, we get

$$d_G\left(p(t,\phi,\cdot),\nu(\cdot)\right)\leqslant d_G\left(p(t,\phi,\cdot),p(t,\phi_0,\cdot)\right)+d_G\left(p(t,\phi_0,\cdot),\nu(\cdot)\right). \tag{56}$$

Substituting (54) and (55) into (56) gives

$$\lim_{t\to+\infty} d_G\left(p(t,\phi,\cdot),\nu(\cdot)\right)=0.$$

This completes the proof.

4. Numerical Simulations

In this section, we use MATLAB to verify our results, we choose the parameters that: $a_{11}=0.84$, $a_{12}=0.15$, $a_{13}=0.44$, $a_{21}=0.19$, $a_{22}=1.01$, $a_{23}=1.04$, $a_{31}=0.08$, $a_{32}=0.91$, $a_{33}=0.93$, $r_1=1.30$, $r_2=1.69$, $r_3=0.35$, $\gamma_1(u)=0.6261$, $\gamma_2(u)=0.7468$, $\gamma_3(u)=0.2040$, $\tau_{12}=0.1$, $\tau_{13}=0.15$, $\tau_{21}=0.1$, $\tau_{23}=0.2$, $\tau_{31}=0.25$, $\tau_{32}=0.2$. Obviously, these coefficients satisfy all the assumptions in this paper.

(1) Set $\sigma_1^2=2.43$, $\sigma_2^2=3.38$, $\sigma_3^2=0.51$, it is easy to count that $\Delta=1.5045$ and $b_1=-0.0549<0$. Then in view of (1) of Theorem 3.1, all species are extinction. Figure 1 has verified this result.

(2) Set $\sigma_1^2=0.44$, $\sigma_2^2=1.13$, $\sigma_3^2=0.26$, it is easy to count that $\Delta=1.5045$, $\Delta_1=1.6399$, $\Delta_2=0.4324$, $\Delta_3=0.8649$, $\tilde{\Delta}_1=0.2288$, $\tilde{\Delta}_2=0.2297$, $\tilde{\Delta}_3=0.6558$ and $\eta=0.2091>0$. In view of (II) of Theorem 3.1, we obtain that

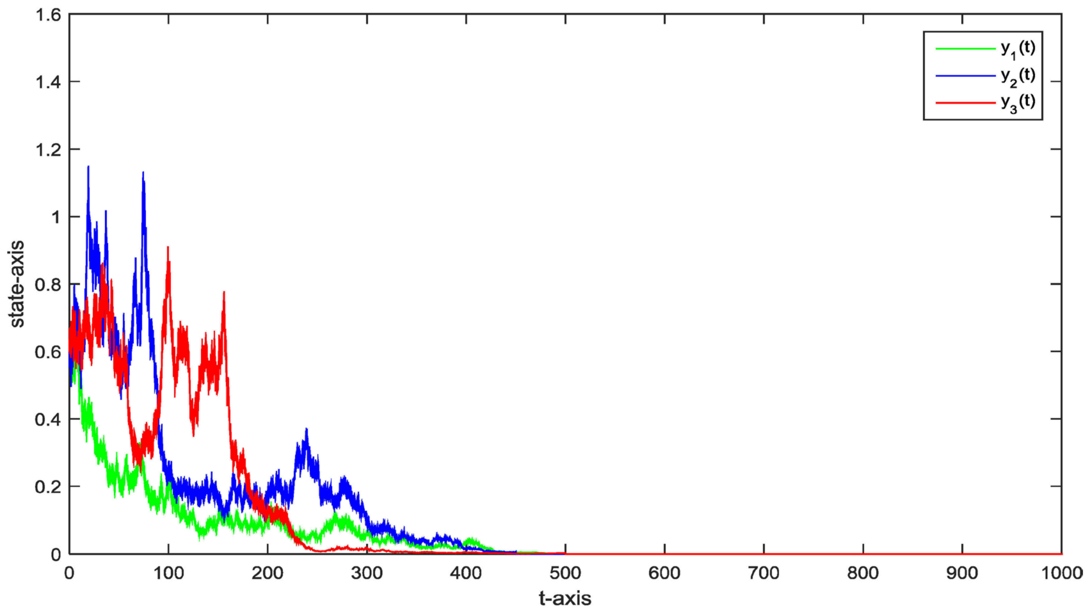

Figure 1. Is the path of $y_1(t), y_2(t), y_3(t)$ with $\sigma_1^2=2.43, \sigma_2^2=3.38, \sigma_3^2=0.51$.

$$\lim_{t\to\infty} y_1(t) = \frac{\Delta_1 - \tilde{\Delta}_1}{\Delta} = 0.9379, \lim_{t\to\infty} y_2(t) = \frac{\Delta_2 - \tilde{\Delta}_2}{\Delta} = 0.1347,$$

$$\lim_{t\to\infty} y_3(t) = \frac{\Delta_3 - \tilde{\Delta}_3}{\Delta} = 0.1390.$$

See Figure 2.

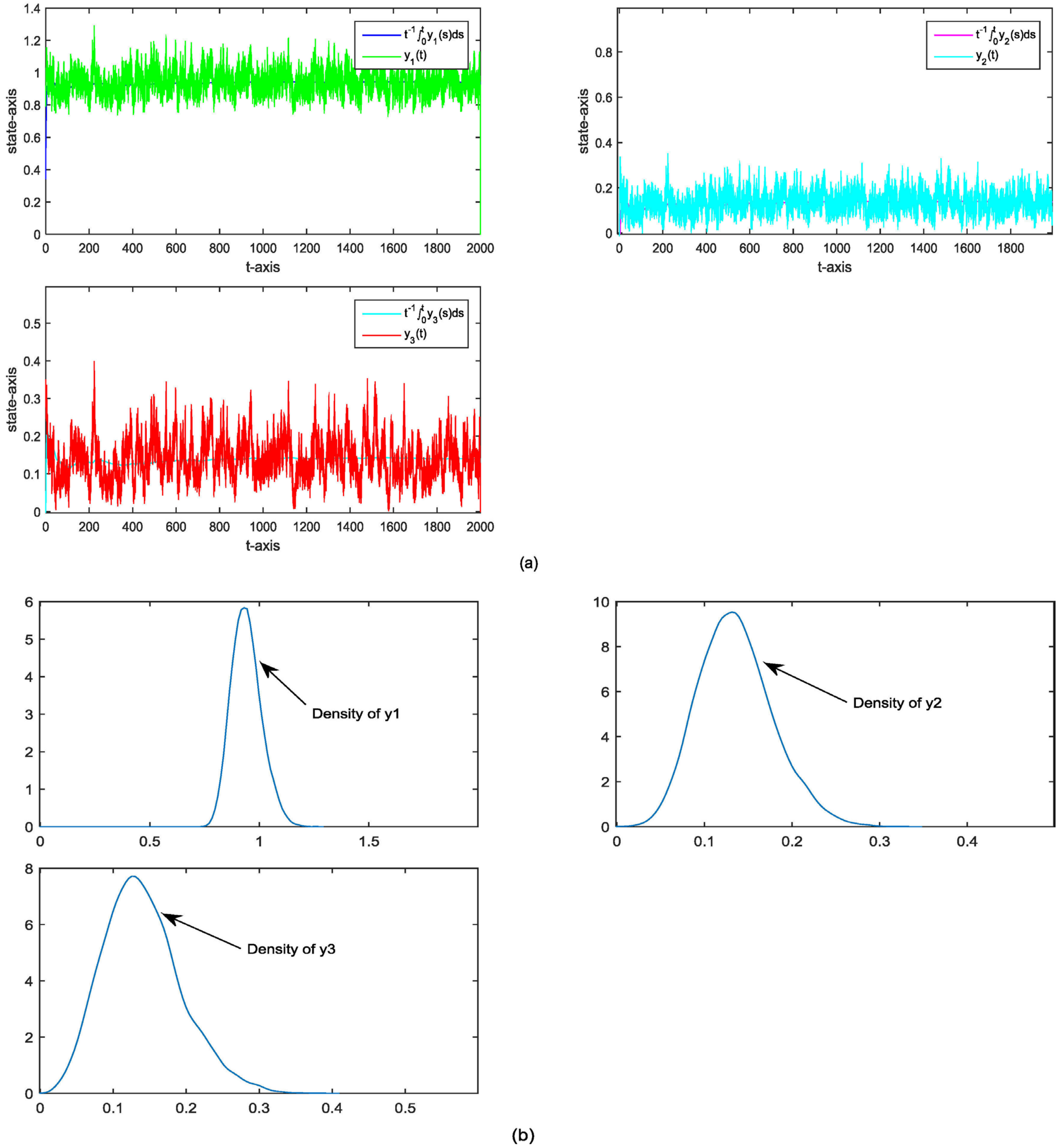

Figure 2. (a) is path of $y_1(t), y_2(t), y_3(t), \overline{y_1(t)}, \overline{y_2(t)}, \overline{y_3(t)}$; (b) is probability density functions of $y_1(t), y_2(t), y_3(t)$. The figure is with $\sigma_1^2 = 0.44$, $\sigma_2^2 = 1.13$, $\sigma_3^2 = 0.26$.

(3) Set $\sigma_1^2 = 2.10$, $\sigma_2^2 = 0.34$, $\sigma_3^2 = 2.22$, we can easy to count that $b_2 = 1.3310 > 0$, $\eta = -0.2208 < 0$, $\Lambda_{11} = -0.085 < 0$. In view of (III) of Theorem 3.1, y_1, y_3 are extinction, and

$$\lim_{t\to\infty} \overline{y_2(t)} = \frac{b_2}{a_{22}} = 1.3178.$$

See Figure 3.

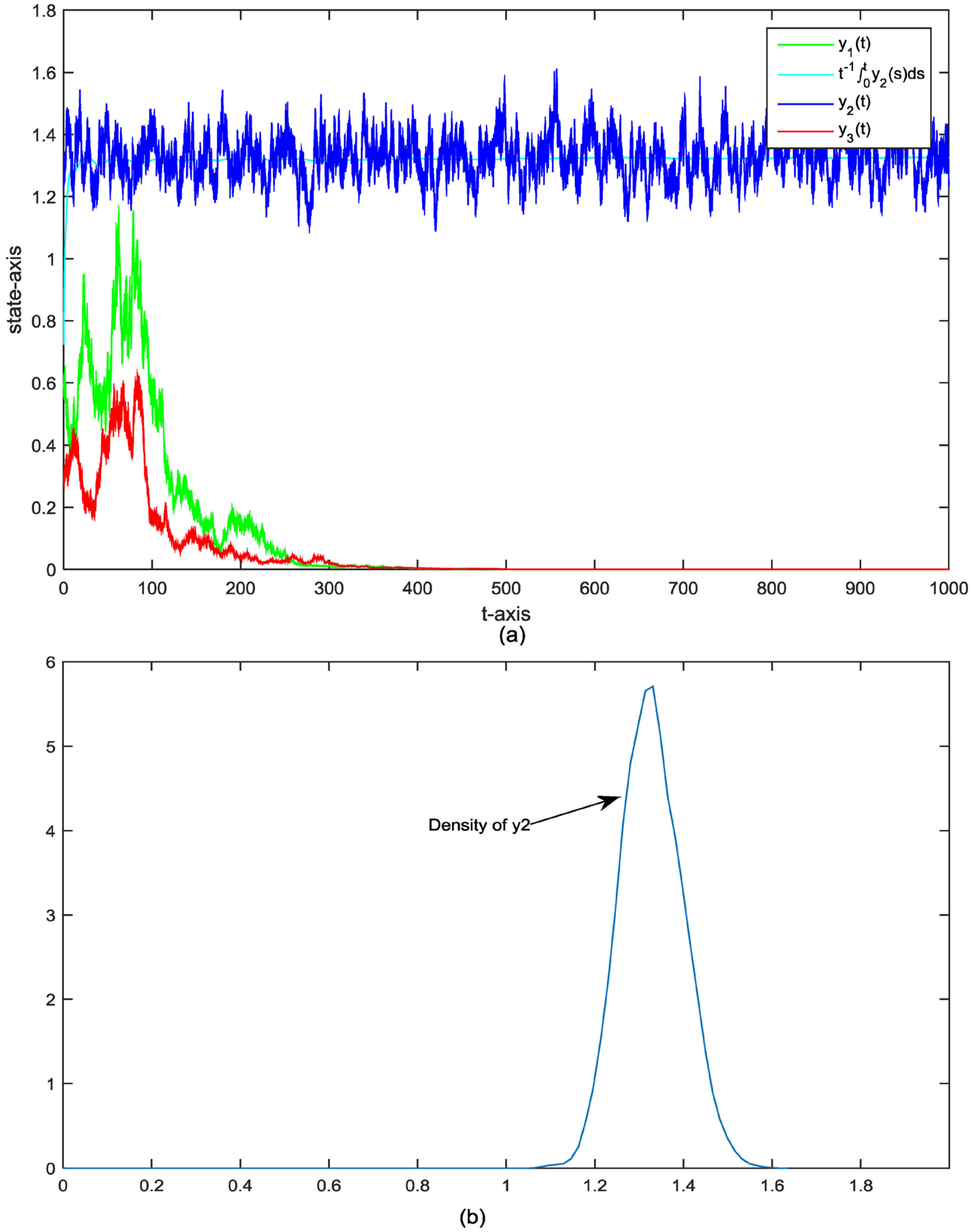

Figure 3. (a) is path of y_1, y_2, y_3; (b) is probability density function of y_2. The figure is with $\sigma_1^2 = 2.10$, $\sigma_2^2 = 0.34$, $\sigma_3^2 = 2.22$.

(4) Set $\sigma_1^2 = 0.06$, $\sigma_2^2 = 2.78$, $\sigma_3^2 = 0.36$, we can count that $b_1 = 1.13.1 > 0$, $\eta = -0.4702$, $\Lambda_{12} = -0.1215 < 0$, according to (IV) of Theorem 3.1, we obtain y_2, y_3 are extinction, and

$$\lim_{t\to\infty} \overline{y_1(t)} = \frac{b_1}{a_{11}} = 1.3454$$

See Figure 4.

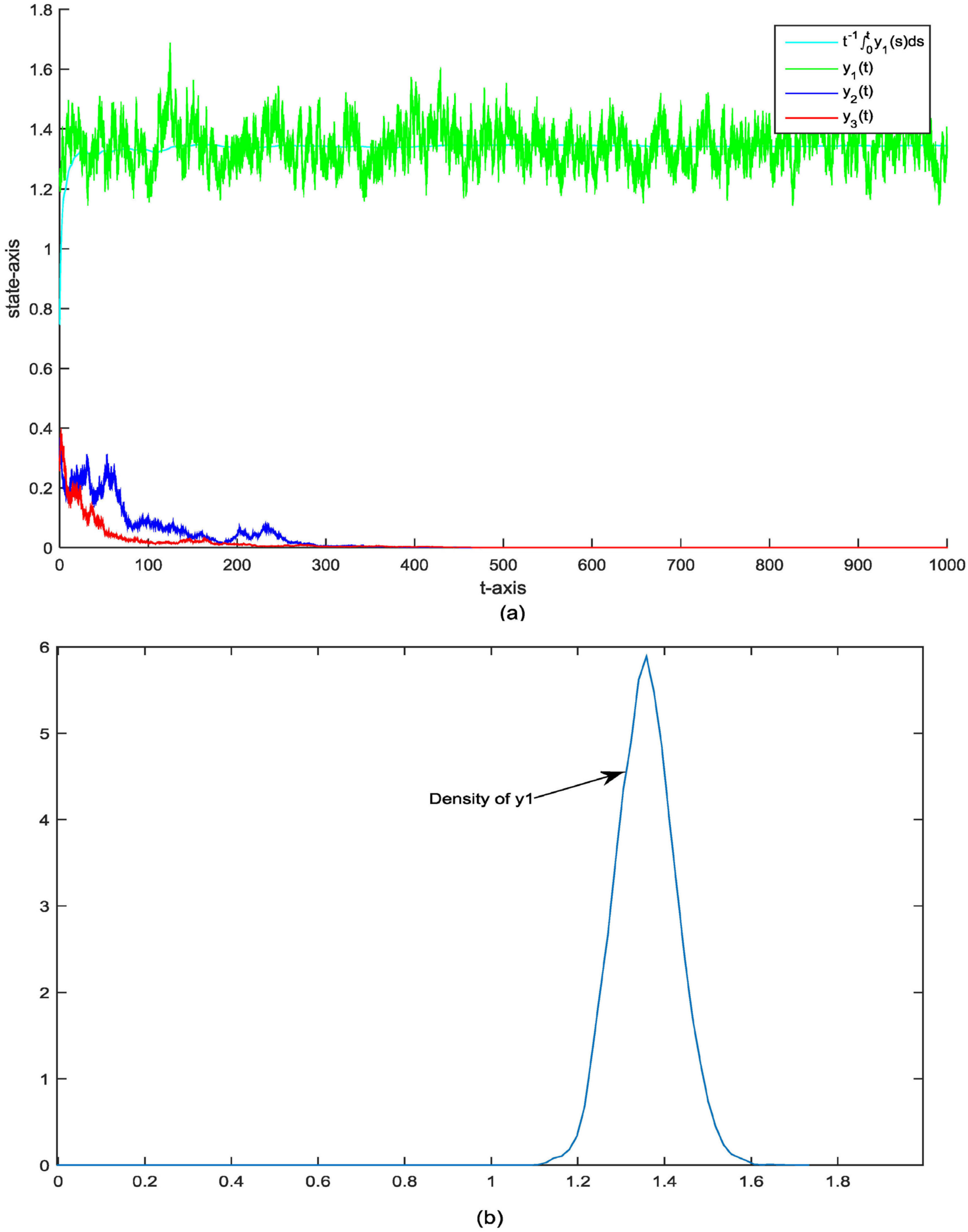

Figure 4. (a) is path of y_1, y_2, y_3; (b) is probability density function of y_1. The figure is with $\sigma_1^2 = 0.06$, $\sigma_2^2 = 2.78$, $\sigma_3^2 = 0.36$.

(5) Set $\sigma_1^2 = 0.21$, $\sigma_2^2 = 0.96$, $\sigma_3^2 = 1.64$, we can easily count that $\eta = -0.3033 < 0$, $\Lambda_{11} = 0.9125 > 0$, $\Lambda_{12} = 0.6572 > 0$, $\Delta_{33} = 0.8199$, according to (V) of Theorem 3.1, we obtain

$$\lim_{t\to\infty}\overline{y_1(t)} = \frac{\Lambda_{11}}{\Delta_{33}} = 1.1129, \lim_{t\to\infty}\overline{y_2(t)} = \frac{\Lambda_{11}}{\Delta_{33}} = 0.8016,$$

and y_3 is extinction. See Figure 5.

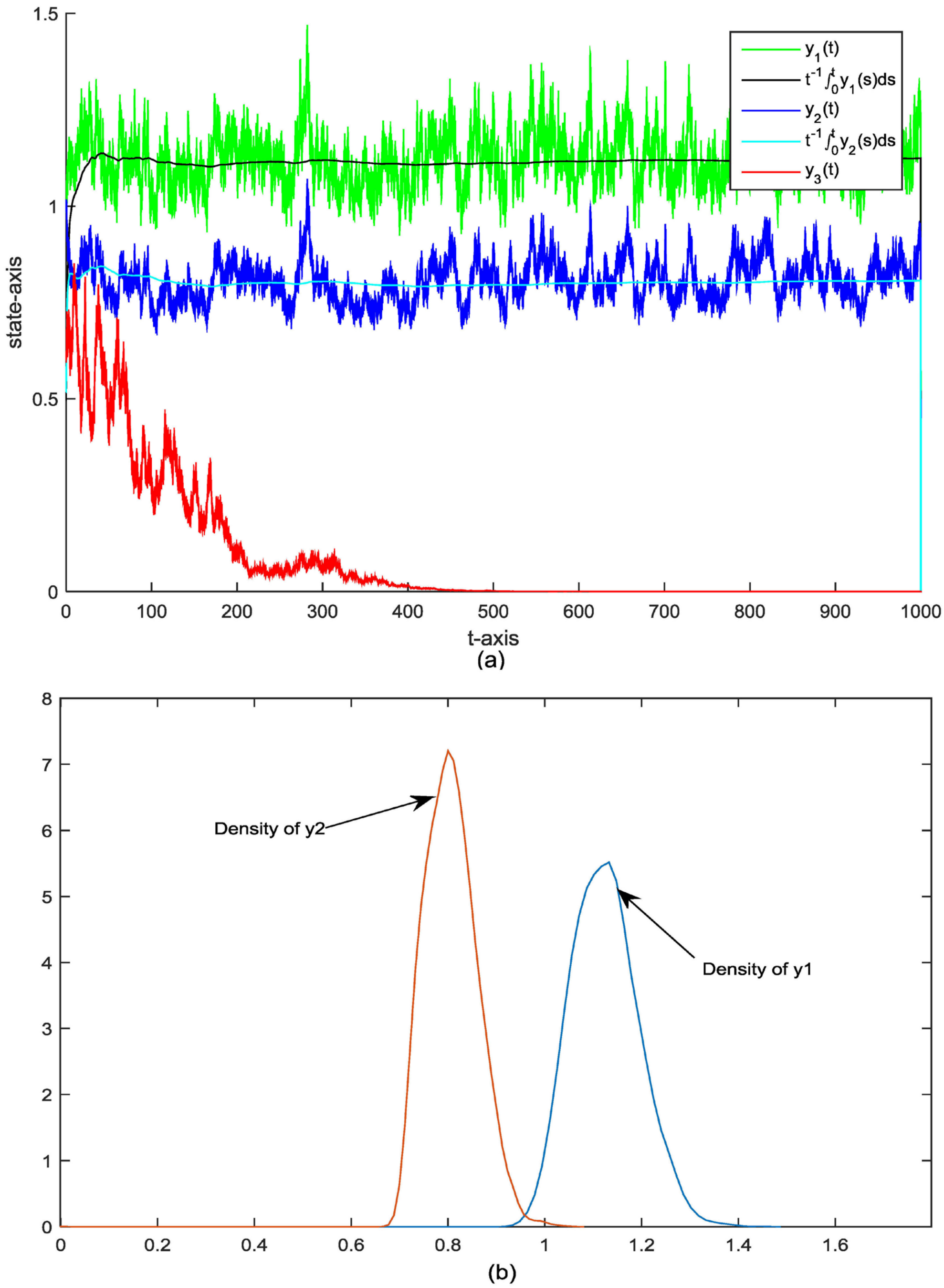

Figure 5. (a) is path of $y_1(t), y_2(t), y_3(t), \overline{y_1(t)}, \overline{y_2(t)}$; (b) is probability density functions of $y_1(t), y_2(t)$. The figure is with $\sigma_1^2 = 0.21, \sigma_2^2 = 0.96, \sigma_3^2 = 1.64$.

5. Conclusions

In this paper, based on a three-species model with traditional time delays and white noise, the S-type distributed time delays and Lévy noises are considered in our model. Different from traditional models, this paper is different in the following aspects: Firstly, in the case of two kinds of noise, we obtain that there exists a unique positive solution in model (4). Secondly, the permanence of model (4) is investigated. Sufficient conditions for the model to be permanent in mean are given. This provides an idea for the management strategy of biological resources. In the end, the correctness of the results is confirmed by numerical simulation.

Moreover, our main results reveal that:

1) White noise can lead to the change of species quantity. When the intensity of white noise is too large, the population may go extinct.

2) The S-type distributed time delays and Lévy noises have important effects on species persistence and extinction. These effects can also lead to dynamic changes in species.

3) Different intensity of white noise and Lévy noises can lead to different outcomes, such as the extinction of predators while prey is permanent. Or only one prey will survive and the other will die out. We can see that from Theorem 3.1.

Recently, the telephone noises in the model have been proposed by many scholars [37] [38]. In fact, there are many other factors that affect the population system. We will incorporate more real-world factors into our models, such as the telephone noise, pulse process and environment pollution for future work.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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