Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2018, 6, 770-786
http://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Existence of Ordered Solutions to Quasilinear Schrödinger Equations with General Nonlinear Term

Jia Wu, Gao Jia

College of Science, University of Shanghai for Science and Technology, Shanghai, China
Email: jiawu94wo@163.com

How to cite this paper: Wu, J. and Jia, G. (2018) Existence of Ordered Solutions to Quasilinear Schrödinger Equations with General Nonlinear Term. *Journal of Applied Mathematics and Physics*, **6**, 770-786.
https://doi.org/10.4236/jamp.2018.64068

Received: March 12, 2018
Accepted: April 21, 2018
Published: April 24, 2018

Abstract

In this paper, the existence of a pair of ordered solutions for the following class of equations in $\mathbb{R}^N$

$$-\Delta u+V(x)u-\Delta(u^2)u=g(x,u) \tag{1}$$

was studied. A bounded (PS) (Palais-Smale) sequence was constructed and the related variational principle was used to prove the existence of the positive solution. The existence of the ordered solutions is finally found.

Keywords

Quasilinear Schrödinger Equations, Ordered Solutions, Mountain Pass Lemma, (PS) Sequence

1. Introduction

In recent years, studies about the nontrivial solutions of Schrödinger equations are very popular, involving differential equations, linear algebra and many subjects. The solution of these problems cannot only develop new methods, such as minimizations [1] [2], change of variables [3] [4] [5], Nehari method [6] and perturbation method [7], reveal new laws, but also have important academic value and wide application prospects [8] [9].

In this paper, we consider the existence of ordered solutions for the following quasilinear Schrödinger equations:

$$-\Delta u+V(x)u-\Delta(u^2)u=g(x,u) \quad \text{in } \mathbb{R}^N \tag{2}$$

we make the following assumptions:

(V_1) $V\in C^1(\mathbb{R}^N,\mathbb{R})$;

(V_2) $0<2C_0<V_0\le V(x)\le V_1<\infty$, for all $x\in\mathbb{R}^N$;

(V_3) $V(x)$ is symmetrical radially, which is $V(x)=V(|x|)$;

(V_4) There exists $\gamma\in[1,2)$, such that $(\nabla V(x)\cdot x)^+\in L^{2^*/2^*-\gamma}(\mathbb{R}^N)$, where $(\nabla V(x)\cdot x)^+=\max\{\nabla V(x)\cdot x,0\}$;

(g_1) $g\in C(\mathbb{R},\mathbb{R})$, for any $t\le 0$, $g(x,t)=0$, $g(x,t)=o(t)$ as $t\to 0^+$;

(g_2) There exist $C_0>0$ and $q\in(4,2\cdot 2^*)$, such that $|g(x,t)|\le C_0(t+t^{q-1})$ as $t\in\mathbb{R}^+$;

(g_3) $\lim\limits_{t\to+\infty}\dfrac{g(x,t)}{t}=+\infty$;

(g_4) $G(x,t)=\int_0^t g(x,s)\mathrm{d}s\ge 0$ for all $t\in\mathbb{R}^+$;

(g_5) There exists $K>0$, such that $\left|\int_{\mathbb{R}^N}x\cdot\nabla_x G\mathrm{d}x\right|<K$, where $\nabla_x G=\left(\dfrac{\partial G(x,t)}{\partial x_1},\dfrac{\partial G(x,t)}{\partial x_2},\cdots,\dfrac{\partial G(x,t)}{\partial x_N}\right)$;

(g_6) There exists $C^*>0$, such that $\int_{\mathbb{R}^N}|g_t(x,t)|\mathrm{d}x<C^*$;

(g_7) There exists $\mu\ge 4$, such that $0<\mu G(x,s)<g(x,s)s$ for any $s>0$.

2. Main Results

We are now state the main results of the paper:

Theorem 1.1 Assume conditions (V_1)-(V_4), (g_1)-(g_7) are satisfied, there is at least one positive solution to Equation (2).

Theorem 1.2 Assume conditions (V_1)-(V_4), (g_1)-(g_7) are satisfied, there is at least one pair of ordered positive solutions to Equation (2).

3. Preliminaries

We observe that formally problem (2) is the Euler-Lagrange equation associated of the natural energy functional given by

$$J(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x+\int_{\mathbb{R}^N}u^2|\nabla u|^2\mathrm{d}x-\int_{\mathbb{R}^N}G(x,u)\mathrm{d}x. \tag{3}$$

It is well known that J is not well defined in general in $H^1(\mathbb{R}^N)$. To overcome this difficulty, we make the change of variables developed in [1] by $v=f^{-1}(u)$, where f is defined by

$$f'(t)=\frac{1}{\sqrt{1+2f^2(t)}},\ f(0)=0,\ \text{on } t\in[0,+\infty), \tag{4}$$

and

$$f(t)=-f(-t),\ \text{on } t\in(-\infty,0].$$

Thus we can write $J(u)$ as

$$I(v):=J(f(u))=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2+V(x)f^2(v)\right)\mathrm{d}x-\int_{\mathbb{R}^N}G(x,f(v))\mathrm{d}x, \tag{5}$$

which is well defined in the space

$$H_r^1\left(\mathbb{R}^N\right):=\left\{u\in H^1\left(\mathbb{R}^N\right)\middle|\, u \text{ is symmetrical radially}\right\}.$$

We can see that the nontrivial critical points of $I(v)$ are precise weak solutions for

$$-\Delta v=-V(x)f(v)f'(v)+g(x,f(v))f'(v),\ x\in\mathbb{R}^N. \tag{6}$$

Lemma 3.1 (see in [4]) The function $f(t)$ possesses the following properties.

1) $|f'(t)|\le 1$;

2) $|f(t)|\le|t|$;

3) $|f(t)|\le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$;

4) $\frac{1}{2}f(t)\le tf'(t)\le f(t)$, for all $t\in\mathbb{R}^+$;

5) $\frac{1}{2}f^2(t)\le f(t)f'(t)t\le f^2(t)$;

6) $\lim\limits_{t\to 0}\frac{f(t)}{t}=1$;

7) $\lim\limits_{t\to\infty}\frac{f(t)}{\sqrt{t}}=2^{\frac{1}{4}}$;

8) There exists a positive constant *C*, such that

$$|f(t)|\ge\begin{cases}C|t|, & \text{if } |t|\le 1,\\ C|t|^{\frac{1}{2}}, & \text{if } |t|\ge 1;\end{cases}$$

9) For each $\lambda>1$, we have $f^2(\lambda t)\le\lambda^2 f^2(t)$, for all $t\in\mathbb{R}$.

Proof: The proofs of (1)-(3) and (6) only require the knowledge of calculus. The reader can refer to the literature [1]. The following proofs (4)-(8).

Let $l(t)=2t-\sqrt{1+2f^2(t)}f(t)$, there is $l(0)=0$ obviously, and

$$\begin{aligned}l'(t)&=2-2\left(f'(t)\right)^2f^2(t)-\sqrt{1+2f^2(t)}f'(t)\\&=1-2f^2(t)\left(f'(t)\right)^2=\frac{1}{1+2f^2(t)}>0.\end{aligned}$$

Thus $l(t)\ge 0$ for all $t\in\mathbb{R}^+$, so we have $2tf'(t)\ge f(t)$ ($t\in\mathbb{R}^+$). $tf'(t)\le f(t)$ ($t\in\mathbb{R}^+$) can be proved similarly. (5) can be derived from (4) easily.

From the conclusion (4), we can get $f(t)\ge f(1)\sqrt{t}$ for any $t>1$. Thus we have $\lim\limits_{t\to+\infty}f(t)=+\infty$, and

$$\lim_{t\to+\infty}\frac{f^2(t)}{t}=\lim_{t\to+\infty}2f'(t)f(t)=2\lim_{t\to+\infty}\frac{f(t)}{\sqrt{1+2f^2(t)}}=\sqrt{2}.$$

Therefore

$$\lim_{t\to+\infty}\frac{f(t)}{\sqrt{t}}=2^{\frac{1}{4}}.$$

(8) can be derived from (6) (7).

Finally we prove (9). For any $t>0$, we have the following inequality by (5)

$$\frac{\left(f^2(t)\right)' t}{f^2(t)}=\frac{2f(t)f'(t)t}{f^2(t)}\le 2.$$

Then we have

$$\ln\frac{f^2(\lambda t)}{f^2(t)}=\int_t^{\lambda t}\frac{\left(f^2(s)\right)'}{f^2(s)}\mathrm{d}s\ge\int_t^{\lambda t}\frac{2}{s}\mathrm{d}s=2\ln\frac{\lambda t}{t}=\ln\lambda^2.$$

Thus

$$f^2(\lambda t)\le\lambda^2 f^2(t).$$

For all $t\in\mathbb{R}$, we have $f^2(\lambda t)\le\lambda^2 f^2(t)$, because $f^2(x)$ is even function.

Remark 3.1. To convenience, we note support as supp, and superior as sup.

Proposition 3.1. (Rellich-Kondrachov theorem) Let $\Omega\subseteq\mathbb{R}^N$ be an open, bounded Lipschitz domain, and let $1\le p<n$. Set

$$p^*:=\frac{np}{n-p}.$$

Then the Sobolev space $W^{1,p}(\Omega,\mathbb{R})$ is continuously embedded in the $L^q(\Omega,\mathbb{R})$ space and is compactly embedded in $L^q(\Omega,\mathbb{R})$ for every $1\le q<p^*$. In symbols, $W^{1,p}(\Omega,\mathbb{R})$ embedding in $L^{p^*}(\Omega,\mathbb{R})$, and $W^{1,p}(\Omega)\subset L^q(\Omega)$ for $1\le q<p^*$.

Proposition 3.2. (Hölder inequality) Let (S,Σ,μ) be a measure space and let $p,q\in[1,+\infty)$ with $1/p+1/q=1$. Then, for all measurable real or complex-valued functions *f* and *g* on *S*,

$$\|fg\|_1\le\|f\|_p\|g\|_q.$$

Proposition 3.3. (Sobolev inequality) Assume that *u* is a continuously differentiable real-valued function on $\mathbb{R}^n$ with compact support. Then for $1\le p<n$ there is a constant *C* depending only on *n* and *p* such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)}\le C\|Du\|_{L^p(\mathbb{R}^n)}$$

with

$$1/p^*=1/p-1/n.$$

Lemma 3.2 *X* is Banach space, and $\|\cdot\|_X$ is a norm of this space. $\varsigma\subset\mathbb{R}^+$ is a range. The family of functionals $\{\Phi_\lambda(v)\}_{\lambda\in\varsigma}$ of class C^1 in *X* satisfy:

1) For all $\lambda\in\varsigma$, there is $\Phi_\lambda(v)=A(v)-\lambda B(v)$. There is $A(v)\to+\infty$ or $B(v)\to+\infty$ as $\|v\|_X\to\infty$;

2) For each $\lambda\in\varsigma$ and for all $v\in X$, there is $B(v)\ge 0$;

3) There exist two points $v_1,v_2\in X$, such that

$$c_\lambda=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\gamma(t))>\max\{\Phi_\lambda(v_1),\Phi_\lambda(v_2)\},\text{ for each }\lambda\in\varsigma \tag{7}$$

where $\Gamma=\left\{\gamma\in C([0,1],X)\,\middle|\,\gamma(0)=v_1,\gamma(1)=v_2\right\}$.

Thus there exists a sequence $\{v_n(\lambda)\} \subset X$, for *a.e.* $\lambda \in \varsigma$, we have

1) $\{v_n(\lambda)\}$ is bounded;
2) $\{\Phi_\lambda(v_n(\lambda))\} \to c_\lambda$;
3) $\{\Phi'_\lambda(v_n(\lambda))\} \to 0$.

In order to use Lemma 3.2, in the following discussion, we take

$$X = H_r^1(\mathbb{R}^N) := \left\{u \in H^1(\mathbb{R}^N) \middle| u \text{ is symmetrical radially}\right\},$$

and consider the following family of functional

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x) f^2(v)\right)\mathrm{d}x - \lambda\int_{\mathbb{R}^N} G(x, f(v))\mathrm{d}x,\ v \in X,$$

where $\lambda \in \left[\frac{1}{2}, 1\right]$.

Define

$$A(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x) f^2(v)\right)\mathrm{d}x,$$

$$B(v) = \int_{\mathbb{R}^N} G(x, f(v))\mathrm{d}x,$$

so that

$$I_\lambda(v) = A(v) - \lambda B(v),$$

The following lemma shows that $I_\lambda(v)$ satisfies the conditions of Lemma 3.2.

Lemma 3.3 Assume conditions (V_1) and (g_1)-(g_4) are satisfied, we have

1) $B(v) \ge 0$ for all $v \in X$;
2) $A(v) \to \infty$ as $\|v\| \to \infty$;
3) There exists $v_0 \in X$ independent on λ, such that $I_\lambda(v_0) < 0$ for each $\lambda \in \left[\frac{1}{2}, 1\right]$;
4) $c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v_0)\}$, for each $\lambda \in \left[\frac{1}{2}, 1\right]$, where $\Gamma = \left\{\gamma \in C([0,1], X) \middle| \gamma(0) = 0, \gamma(1) = v_0\right\}$.

Proof: (1) can be directly obtained from (g_4). Let's prove (2) by Lemma 3.1 and embedding theorem, we infer that

$$\begin{aligned}
\|v\|^2 &= \int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x + \int_{\{x||v(x)|\le 1\}} v^2\mathrm{d}x + \int_{\{x||v(x)|>1\}} v^2\mathrm{d}x \\
&\le \int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x + C\int_{\{x||v(x)|\le 1\}} f^2(v)\mathrm{d}x + \int_{\{x||v(x)|>1\}} |v|^{2^*}\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x + C_1\int_{\mathbb{R}^N} V(x) f^2(v)\mathrm{d}x + C_2\left(\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x\right)^{\frac{2^*}{2}} \\
&\le C_3\left(A(v) + A(v)^{\frac{2^*}{2}}\right).
\end{aligned}$$

Therefore, A is convex.

To prove (3), firstly, we let

$$J_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)\mathrm{d}x+\int_{\mathbb{R}^N}u^2|\nabla u|^2\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}G(x,u)\,\mathrm{d}x$$

and $U=\left\{x\in\mathbb{R}^N \,\middle|\, u(x)\neq 0\right\}$ ($measU>0$). Then fixing a non-negative radial symmetry function $u\in C_0^\infty\left(\mathbb{R}^N\right)\setminus\{0\}$, for all $t>0$, we have

$$\begin{aligned}J_{\frac{1}{2}}\left(tu\left(\frac{x}{t}\right)\right)&=\frac{t^N}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\frac{t^{N+2}}{2}\int_{\mathbb{R}^N}V(tx)u^2\mathrm{d}x\\&\quad+t^{N+2}\int_{\mathbb{R}^N}u^2|\nabla u|^2\,\mathrm{d}x-\frac{t^N}{2}\int_{\mathbb{R}^N}G(x,tu)\,\mathrm{d}x\\&\leq\frac{t^{N+2}}{2}\left(\frac{1}{t^2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}V_1u^2\right.\\&\quad\left.+2\int_{\mathbb{R}^N}u^2|\nabla u|^2\,\mathrm{d}x-\int_{\mathbb{R}^N}\frac{G(x,tu)}{t^2}\mathrm{d}x\right).\end{aligned}$$

By (g_3): $\lim\limits_{t\to+\infty}\dfrac{g(x,t)}{t}=+\infty$, we have

$$\lim_{t\to+\infty}\frac{G(x,tu)}{t^2}=\lim_{t\to+\infty}\frac{\int_0^{tu}g(x,s)\,\mathrm{d}s}{t^2}=\lim_{t\to+\infty}\frac{ug(x,tu)}{2t}\to+\infty,\ a.e.\ x\in U.$$

$J_{\frac{1}{2}}\left(tu\left(\frac{x}{t}\right)\right)<0$ when t is large enough.

Thus there exists $v_0=f^{-1}(u_0)\in X$ (v_0 is independent on λ), such that $I_\lambda(v_0)=J_\lambda(u_0)\leq J_{\frac{1}{2}}(u_0)<0$, for each $\lambda\in\left[\frac{1}{2},1\right]$.

Finally, we prove (4). Define $\hat{G}(x,t)=-\frac{V_0}{2}f^2(t)+G(x,f(t))$. By ($g_1$), ($g_2$) and Lemma 3.1, we have $\lim\limits_{t\to0}\dfrac{\hat{G}(x,t)}{t^2}=-\dfrac{V_0}{2}$, $\lim\limits_{t\to0}\dfrac{\hat{G}(x,t)}{|t|^{2^*}}=0$.

Hence, there exists $C>0$, such that $\hat{G}(x,t)\leq-\frac{V_0}{4}t^2+C|t|^{2^*}$, for all $t\in\mathbb{R}$.

Then

$$\begin{aligned}I_\lambda(v)&\geq\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V_0f^2(v)\,\mathrm{d}x-\int_{\mathbb{R}^N}G(x,f(v))\\&\geq\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V_0f^2(v)\,\mathrm{d}x\\&\quad+\frac{1}{4}\int_{\mathbb{R}^N}V_0v^2dx-C\int_{\mathbb{R}^N}|v|^{2^*}\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}V_0f^2(v)\,\mathrm{d}x\\&\geq\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x+\frac{V_0}{4}\int_{\mathbb{R}^N}v^2\mathrm{d}x-C\int_{\mathbb{R}^N}|v|^{2^*}\,\mathrm{d}x\\&\geq\min\left\{\frac{1}{2},\frac{V_0}{4}\right\}\|v\|^2-C|v|_{L^{2^*}}^{2^*}.\end{aligned}$$

It follows that $I_\lambda(v)>0$ with $0\leq\|v\|<\rho$. We also have $I_\lambda(0)=0$, $I_\lambda(v_0)<0$, thus $c_\lambda=\inf\limits_{\gamma\in\Gamma}\max\limits_{t\in[0,1]}I_\lambda(\gamma(t))>0>\max\left\{I_\lambda(0),I_\lambda(v_0)\right\}$.

By Lemma 3.2 and Lemma 3.3, we can construct the (PS) sequence of $I_\lambda(u)$. Specifically, there exists $\varsigma_1 \subset \left[\frac{1}{2},1\right]$ ($meas\,\varsigma_1 = 0$), for each $\lambda \in \left[\frac{1}{2},1\right] \setminus \varsigma_1$, then we have a sequence $\{v_n\} \subset X$, satisfy

1) $\{v_n\} \subset X$ is bounded;
2) $I_\lambda(v_n) \to c_\lambda$;
3) $I'_\lambda(v_n) \to 0$.

Lemma 3.4 If $\{v_n\}$ is a (PS) sequence of I_λ, then there exists a subsequence, still denoted by $\{v_n\}$, which convergence to the positive critical point v_λ of I_λ.

Proof: Since $\{v_n\} \subset X$ is bounded, by Rellich-Kondrachov theorem, there exists $v_\lambda \in X$, such that

i) $v_n \to v_\lambda$ in *X*;
ii) $v_n \to v_\lambda$ in $L^{\frac{q}{2}}(\mathbb{R}^N)$;
iii) $v_n \to v_\lambda$ *a.e.* $x \in \mathbb{R}^N$.

By (i) and (ii), we obtain $I'_\lambda(v_\lambda) = 0$.

Next we prove $v_n \to v_\lambda$ in *X*. Firstly, let

$$H(x,t) = \frac{1}{2}V(x)t^2 - \frac{1}{2}V(x)f^2(t) + \lambda G(x, f(t)).$$

Hence $I_\lambda(v)$ is transformed into

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x)v^2\right)dx - \int_{\mathbb{R}^N} H(x,v)dx.$$

Let $h(x,t) = \frac{dH(x,t)}{dt}$, so that

$$h(x,t) = V(x)t - V(x)f(t)f'(t) + \lambda g(x, f(t))f'(t).$$

By (g_1), (g_2) and Lemma 3.1, there exists $C_1, C_2 > 0$, for every $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$, such that

$$\begin{aligned}|h(x,t)| &\le V(x)|t| + V(x)|f(t)|f'(t) + \lambda|g(x, f(t))|f'(t)\\ &\le 2V_1|t| + \lambda C_0|f(t)|f'(t) + \lambda C_0|f(t)|^{q-1}f'(t) \qquad (8)\\ &\le C_1|t| + C_2|t|^{\frac{q-1}{2}}.\end{aligned}$$

By (8) and $v_n \to v_\lambda$ in $L^{\frac{q}{2}}(\mathbb{R}^N)$, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(h(x,v_n) - h(x,v_\lambda)\right)(v_n - v_\lambda)dx = 0.$$

Thus

$$\begin{aligned}o(1) &= \left\langle I'_\lambda(v_n) - I'_\lambda(v_\lambda), v_n - v_\lambda\right\rangle\\ &= \int_{\mathbb{R}^N}\left(|\nabla(v_n - v_\lambda)|^2 + V(x)(v_n - v_\lambda)^2\right)dx\\ &\quad - \int_{\mathbb{R}^N}\left(h(x,v_n) - h(x,v_\lambda)\right)(v_n - v_\lambda)\\ &\ge \min\{1, V_0\}\|v_n - v_\lambda\|^2 + o(1),\end{aligned}$$

that is $v_n \to v_\lambda$ in *X*. Therefore, $v_\lambda > 0$ is the critical point of functional I_λ, and $I_\lambda(v_\lambda) = c_\lambda$. This completes the proof.

Lemma 3.5 Suppose that the conditions of Theorem 1.1 are satisfied. Then there exists $\{\lambda_n\} \subset \left[\frac{1}{2}, 1\right]$ and corresponding critical point sequence $\{v_n^*\} \subset X \setminus \{0\}$, such that $\lim_{n\to\infty} \lambda_n = 1$ and $v_n^* > 0$, $I_{\lambda_n}(v_n^*) = c_{\lambda_n} \le c_{\frac{1}{2}}$, $I'_{\lambda_n}(v_n^*) = 0,\ n = 1, 2, \cdots$.

Proof: Let $\lambda_1 \in \left[\frac{1}{2}, 1\right] \setminus \varsigma_1$, by Lemma 3.1, there exists (PS) sequence $\{v_{1,m}\} \subset X$, such that $I_{\lambda_1}(v_{1,m}) \to c_{\lambda_1}$, $I'_{\lambda_1}(v_{1,m}) \to 0$ as $m \to \infty$. By Lemma 3.4, we have $v_{1,m} \to v_1^*$, and $I_{\lambda_1}(v_1^*) = c_{\lambda_1}$, $I'_{\lambda_1}(v_1^*) = 0$ in *X* as $m \to \infty$.

Similarly, let $\lambda_2 \in \left[\frac{\lambda_1 + 1}{2}, 1\right] \setminus \varsigma_1$, we have $I_{\lambda_2}(v_{2,m}) \to c_{\lambda_2}$, $I'_{\lambda_2}(v_{2,m}) \to 0$ as $m \to \infty$, and $v_{2,m} \to v_2^*$, $I_{\lambda_2}(v_2^*) = c_{\lambda_2}$, $I'_{\lambda_2}(v_2^*) = 0$ in *X*.

Let $\lambda_n \in \left[\frac{\lambda_{n-1} + 1}{2}, 1\right] \setminus \varsigma_1$, we have $I_{\lambda_n}(v_{n,m}) \to c_{\lambda_n}$, $I'_{\lambda_n}(v_{n,m}) \to 0$ as $m \to \infty$, and $v_{n,m} \to v_n^*$, $I_{\lambda_n}(v_n^*) = c_{\lambda_n}$, $I'_{\lambda_n}(v_n^*) = 0$ in *X*.

Thus we get $\lim_{n\to\infty} \lambda_n = 1$, and since $I_\lambda(v)$ is monotonically decreasing with λ, so that $I_{\lambda_n}(v_n^*) = c_{\lambda_n} \le c_{1/2}$. This completes the proof.

Lemma 3.6 If $u \in X$ is a critical point of I_λ, then

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} V(x) f^2(u)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x)\cdot x f^2(u)\mathrm{d}x$$
$$-\lambda N\int_{\mathbb{R}^N} G(x, f(u))\mathrm{d}x - \int_{\mathbb{R}^N} x\cdot\nabla G_x \mathrm{d}x = 0.$$

Proof: Multiply the two sides of the equation

$$-\Delta v + V(x) f(v) f'(v) = \lambda g(x, f(v)) f'(v)$$

by $x\cdot\nabla v$, we have

$$\begin{aligned} 0 &= \left(\Delta v - V(x) f(v) f'(v) + \lambda g(x, f(v)) f'(v)\right)(x\cdot\nabla v) \\ &= div\left(\nabla v(x\cdot\nabla v) - |\nabla v|^2 - x\cdot\nabla\left(\frac{|\nabla v|^2}{2}\right)\right) + \lambda x\cdot\nabla G(x, f(v)) \\ &\quad + \frac{1}{2} x f^2(v)\nabla V(x) + \frac{N}{2} V(x) f^2(v) \\ &= div\left(\nabla v(x\cdot\nabla v) - x\cdot\frac{|\nabla v|^2}{2} + \lambda x G(x, f(v))\right) + \frac{N-2}{2}|\nabla v|^2 \\ &\quad - \lambda N G(x, f(v)) + \frac{1}{2} x f^2(v)\nabla V(x) + \frac{N}{2} V(x) f^2(v). \end{aligned}$$

Finally, we integrate the equation on $\mathbb{R}^N$, and then the improved Pohozaev type identity can be obtained.

Lemma 3.7 The critical point sequence obtained in Lemma 3.2.7 is bounded in Lemma 3.5.

Proof: For convenience, we let $\{v_n\}$ denote $\{v_n^*\}$ of Lemma 3.5. By $I_{\lambda_n}(v_n)=c_{\lambda_n}\le c_{1/2}$ in Lemma 3.5, Lemma 3.6, Hölder inequality, Sobolev inequality, (g_5) and (V_4), then there exists $\gamma\in[1,2)$, such that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x&\le\frac{1}{2}\int_{\mathbb{R}^N}x\cdot\nabla V(x)f^2(v_n)\mathrm{d}x+Nc_{1/2}-\int_{\mathbb{R}^N}x\cdot\nabla_x G\mathrm{d}x\\&\le\frac{1}{2}\left|(\nabla V(x)\cdot x)^+\right|_{2^*/(2^*-\gamma)}\left(\int_{\mathbb{R}^N}f^{\frac{2\cdot 2^*}{\gamma}}(v_n)\mathrm{d}x\right)^{\frac{\gamma}{2^*}}+Nc_{1/2}+K\\&\le C_1\left(\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x\right)^{\frac{\gamma}{2^*}}+Nc_{1/2}+K\\&\le C_2\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^{\frac{\gamma}{2^*}}+Nc_{1/2}+K.\end{aligned}\tag{9}$$

Therefore, $\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x$ is bounded.

Next we prove $\int_{\mathbb{R}^N}V(x)f^2(v_n)\mathrm{d}x$ is bounded. By (g_1), (g_2) and Lemma 3.1, we have $\lim\limits_{t\to 0}\frac{|g(x,f(t))f'(t)t|}{f^2(t)}=0$ and $\lim\limits_{t\to\infty}\frac{|g(x,f(t))f'(t)t|}{|t|^{2^*}}=0$.

Thus, for any $\varepsilon>0$, there exists $C(\varepsilon)>0$, such that

$$|g(x,f(t))f'(t)t|\le\varepsilon f^2(t)+c(\varepsilon)|t|^{2^*}\ \text{ for all } t\in\mathbb{R}.\tag{10}$$

By using $\langle I'_{\lambda_n}(v_n),v_n\rangle=0$ and Lemma 3.1, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)f^2(v_n)\mathrm{d}x\\&\le\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x+\int_{\mathbb{R}^N}V(x)f(v_n)f'(v_n)v_n\mathrm{d}x\\&=\lambda_n\int_{\mathbb{R}^N}g(x,f(v_n))f'(v_n)v_n\mathrm{d}x\\&\le\varepsilon\int_{\mathbb{R}^N}|f(v_n)|^2\mathrm{d}x+C(\varepsilon)\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x\\&\le\frac{\varepsilon}{V_0}\int_{\mathbb{R}^N}V(x)f^2(v_n)\mathrm{d}x+C'(\varepsilon)\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^{\frac{2^*}{2}}.\end{aligned}$$

Choosing enough small $\varepsilon\left(0<\varepsilon<\frac{V_0}{2}\right)$, we obtain that $\int_{\mathbb{R}^N}V(x)f^2(v_n)\mathrm{d}x$ is bounded.

4. Existence Results

Proof of Theorem 1.1. By Lemma 3.5 and Lemma 3.7, there exist $\{\lambda_n\}\subset\left[\frac{1}{2},1\right]$ and a bounded sequence $\{v_n\}\subset X\setminus\{0\}$, such that $\lim\limits_{n\to\infty}\lambda_n=1$, $I_{\lambda_n}(v_n)=c_{\lambda_n}$, $I'_{\lambda_n}(v_n)=0$.

Then by the fact that the map $\lambda\to c_\lambda$ is left continuous, we have

$$\lim_{n\to\infty}I(v_n)=\lim_{n\to\infty}\left(I_{\lambda_n}(v_n)+(\lambda_n-1)\int_{\mathbb{R}^N}G(x,f(v_n))\mathrm{d}x\right)=\lim_{n\to\infty}c_{\lambda_n}=c_1.$$

Similarly, we obtain $I'(v_n)\to 0$ in space X^* yields that $\{v_n\}$ is a bounded (PS) sequence of functional I and $\lim_{n\to\infty} I(v_n)=c_1$. By Lemma 3.4, a positive critical point v can be obtained.

To prove Theorem 1.2, we need to prove the following lemmas.

Lemma 4.1 The trivial solution of Equation (2) is a local minimizer for $I(v)$ in $H_r^1(\mathbb{R}^N)$, and there exists a constant $C>0$ (dependent on V_0 and embedding constant), such that the every non-negative solution v of Equation (2) satisfies the inequality

$$\|\nabla v\| \ge C. \tag{11}$$

Proof: By (V_2), (g_2), and Lemma 3.1 (3), we have

$$\begin{aligned} I(v) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x) f^2(v)\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(x, f(v))\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V_0 f^2(v)\right)\mathrm{d}x - C_0\int_{\mathbb{R}^N} f^2(v)\mathrm{d}x - C_0\int_{\mathbb{R}^N} f^q(v)\mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x + \frac{V_0-2C_0}{2}\int_{\mathbb{R}^N} f^2(v)\mathrm{d}x - C_0\int_{\mathbb{R}^N} f^q(v)\mathrm{d}x \\ &\ge C_1\|v\|^2 - C_2\|v\|^{\frac{q}{2}}. \end{aligned}$$

Thus $I(v)>0$ as $\|v\|$ is enough small and $q>4$. In conclusion, the trivial solution $v=0$ is a local minimizer for $I(v)$ in $H_r^1(\mathbb{R}^N)$.

$v\in H_r^1(\mathbb{R}^N)$ is the non-negative of Equation (2), so that $\langle I'(v),v\rangle = 0$. By ($V_1$), Lemma 3.1 (2) (4) (5) and the embedding between $L^p(\mathbb{R}^N)$ and $H_r^1(\mathbb{R}^N)$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N}|\nabla v|^2 + \int_{\mathbb{R}^N} V(x) f(v) f'(v) v - \int_{\mathbb{R}^N} g(x, f(v)) f'(v) v \\ &\ge \int_{\mathbb{R}^N}|\nabla v|^2 + \frac{V_1}{2}\int_{\mathbb{R}^N} f^2(v) - \frac{1}{2}\int_{\mathbb{R}^N} g(x, f(v)) f(v) \\ &\ge \int_{\mathbb{R}^N}|\nabla v|^2 + \frac{V_1}{2}\int_{\mathbb{R}^N} f^2(v) - \frac{C_0}{2}\int_{\mathbb{R}^N}\left(f^2(v)+f^q(v)\right) \\ &\ge |\nabla v|_2^2 - \frac{C_0}{2}\int_{\mathbb{R}^N} v^q \ge |\nabla v|_2^2 - \frac{C_0}{2}|\nabla v|_2^q. \end{aligned}$$

This implies that inequality (11) is satisfied. This completes the proof.

Lemma 4.2 Suppose that the conditions of theorem 1.2 are satisfied, Equation (2) admits a positive solution v, and v is a local minimizer for $I(v)$ in $H_r^1(\mathbb{R}^N)$.

Proof: According to the reference [10] and related theories of differential equations, Equation (2) admits sub-solutions and sup-solutions. Let $\underline{u}$ be the sub-solution and $\overline{u}$ be the sup-solution of Equation (2). Define

$$M = \left\{u\in H_r^1 : \underline{u}\le u\le \overline{u}\right\}.$$

Let v be a solution of Equation (2) on M, then

$$I(v) = \inf_{\xi\in M} I(\xi). \tag{12}$$

Next, we prove that v is a local minimizer for $I(v)$ in $H_r^1(\mathbb{R}^N)$ Suppose by contradiction that v is not the local minimizer for $I(v)$ in $H_r^1(\mathbb{R}^N)$. Then there exists a sequence $\{u_n\}$ in $H_r^1(\mathbb{R}^N)$, such that $\|u_n - v\| \to 0$ as $n \to \infty$ and $I(u_n) < I(v)$. Put

$$v_n = \max\{\underline{u}, \min\{\overline{u}, u_n\}\} = \begin{cases} \underline{u}, & u_n < \underline{u}, \\ u_n, & \underline{u} < u_n < \overline{u}, \\ \overline{u}, & u_n > \overline{u}, \end{cases}$$

$$\omega_n = (u_n - \overline{u})^+ = \begin{cases} 0, & u_n < \overline{u}, \\ u_n - \overline{u}, & u_n \ge \overline{u}, \end{cases}$$

$$z_n = (\underline{u} - u_n)^+ = \begin{cases} 0, & u_n > \underline{u}, \\ \underline{u} - u_n, & u_n \le \underline{u}. \end{cases}$$

Therefore, $u_n = v_n - z_n + \omega_n$, $v_n \in M$, and ω_n and z_n have disjoint support.

The following defines some sets and functions:

$$R_n = \{x \in \mathbb{R}^N : \underline{u} \le u_n \le \overline{u}\},$$

$$S_n = \operatorname{supp}(\omega_n),$$

$$T_n = \operatorname{supp}(z_n),$$

$$L(x,u) = -\frac{1}{2}V(x) f^2(u) + G(x, f(u)).$$

And then $I(u_n)$ is transformed into

$$\begin{aligned} I(u_n) = &\frac{1}{2}\int_{S_n} |\nabla u_n|^2 \mathrm{d}x - \int_{S_n} L(x,u_n)\mathrm{d}x + \frac{1}{2}\int_{T_n} |\nabla u_n|^2 \mathrm{d}x \\ &- \int_{T_n} L(x,u_n)\mathrm{d}x + \frac{1}{2}\int_{R_n} |\nabla u_n|^2 \mathrm{d}x - \int_{R_n} L(x,u_n)\mathrm{d}x. \end{aligned} \tag{13}$$

Obviously, $v_n = \overline{u}$ on S_n, so that

$$\begin{aligned} \int_{S_n}\left(\frac{1}{2}|\nabla u_n|^2 - L(x,u_n)\right)\mathrm{d}x &= \int_{S_n}\left(\frac{1}{2}|\nabla(v_n + \omega_n)|^2 - L(x, v_n + \omega_n)\right)\mathrm{d}x \\ &= \int_{S_n}\left(\frac{1}{2}|\nabla(\overline{u} + \omega_n)|^2 - L(x, \overline{u} + \omega_n)\right)\mathrm{d}x. \end{aligned}$$

Similarly, by $v_n = \underline{u}$ on T_n, we have

$$\begin{aligned} \int_{T_n}\left(\frac{1}{2}|\nabla u_n|^2 - L(x,u_n)\right)\mathrm{d}x &= \int_{T_n}\left(\frac{1}{2}|\nabla(v_n + z_n)|^2 - L(x, v_n + z_n)\right)\mathrm{d}x \\ &= \int_{T_n}\left(\frac{1}{2}|\nabla(\underline{u} + z_n)|^2 - L(x, \underline{u} + z_n)\right)\mathrm{d}x. \end{aligned}$$

Since $v_n = u_n$ on R_n, we get

$$\begin{aligned} &\int_{R_n}\left(\frac{1}{2}|\nabla u_n|^2 - L(x,u_n)\right)\mathrm{d}x = \int_{R_n}\left(\frac{1}{2}|\nabla v_n|^2 - L(x,v_n)\right)\mathrm{d}x \\ &= I(v_n) - \int_{S_n}\left(\frac{1}{2}|\nabla \overline{u}|^2 - L(x,\overline{u})\right)\mathrm{d}x - \int_{T_n}\left(\frac{1}{2}|\nabla \underline{u}|^2 - L(x,\underline{u})\right)\mathrm{d}x. \end{aligned}$$

Consequently, we have

$$\begin{aligned} I(u_n) = I(v_n) &+ \int_{S_n} \frac{|\nabla(\overline{u}+\omega_n)|^2 - |\nabla \overline{u}|^2}{2} \mathrm{d}x - \int_{S_n} \left(L(x,\overline{u}+\omega_n) - L(x,\overline{u})\right)\mathrm{d}x \\ &+ \int_{T_n} \frac{|\nabla(\underline{u}-z_n)|^2 - |\nabla \underline{u}|^2}{2} \mathrm{d}x - \int_{T_n} \left(L(x,\underline{u}-z_n) - L(x,\underline{u})\right)\mathrm{d}x. \end{aligned} \tag{14}$$

Since $\underline{u}$ is a sub-solution, we obtain

$$-\Delta \underline{u} \le f'(\underline{u})\left(g(x, f(\underline{u})) - V(x) f(\underline{u})\right) = L_u(x,\underline{u}),$$

Yields that $-\Delta \underline{u} \le L_u(x,\underline{u})$.

Similarly, $\overline{u}$ is a sup-solution, so that $-\Delta \overline{u} \ge L_u(x,\overline{u})$.

Hence,

$$\int_{\mathbb{R}^N} \nabla \underline{u} \nabla(-z_n)\mathrm{d}x \ge \int_{\mathbb{R}^N} L_u(x,\underline{u})(-z_n)\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \nabla \overline{u} \nabla \omega_n \mathrm{d}x \ge \int_{\mathbb{R}^N} L_u(x,\overline{u})\omega_n \mathrm{d}x.$$

In addition, we noticed that

$$\frac{1}{2}|\nabla(\overline{u}+\omega_n)|^2 - |\nabla \overline{u}|^2 = \nabla \overline{u} \nabla \omega_n + \frac{1}{2}|\nabla \omega_n|^2,$$

$$\frac{1}{2}|\nabla(\underline{u}-z_n)|^2 - |\nabla \underline{u}|^2 = \nabla \underline{u} \nabla(-z_n) + \frac{1}{2}|\nabla z_n|^2,$$

So that by (14), we have

$$\begin{aligned} I(u_n) \ge I(v_n) &+ \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \omega_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla z_n|^2 \mathrm{d}x \\ &- \int_{S_n} \left(L(x,\overline{u}+\omega_n) - L(x,\overline{u}) - L_u(x,\overline{u})\omega_n\right)\mathrm{d}x \\ &- \int_{T_n} \left(L(x,\underline{u}-z_n) - L(x,\underline{u}) - L_u(x,\underline{u})(-z_n)\right)\mathrm{d}x. \end{aligned}$$

To complete the Lemma, we still need to prove the following claim: as $n \to \infty$,

$$\int_{S_n} \left(L(x,\overline{u}+\omega_n) - L(x,\overline{u}) - L_u(x,\overline{u})\omega_n\right)\mathrm{d}x \le o(1)\int_{\mathbb{R}^N} |\nabla \omega_n|^2 \mathrm{d}x, \tag{15}$$

$$\int_{T_n} \left(L(x,\underline{u}-z_n) - L(x,\underline{u}) - L_u(x,\underline{u})(-z_n)\right)\mathrm{d}x \le o(1)\int_{\mathbb{R}^N} |\nabla z_n|^2 \mathrm{d}x. \tag{16}$$

Since the proofs of inequalities (15) and (16) are similar, we only prove (15) Firstly, note

$$L_n(x) = L(x,\overline{u}+\omega_n) - L(x,\overline{u}) - L_u(x,\overline{u})\omega_n,$$

Split

$$L_n(x) = L_{0n}(x) - L_{1n}(x),$$

where

$$L_{0n}(x) = -\frac{1}{2}V(x)\left(f^2(\overline{u}+\omega_n) - f^2(\overline{u})\right) + V(x) f(\overline{u}) f'(\overline{u})\omega_n,$$

$$L_{1n}(x) = G\left(x, f(\overline{u}+\omega_n)\right) - G\left(x, f(\overline{u})\right) - g\left(x, f(\overline{u})\right) f'(\overline{u})\omega_n.$$

By the define of f, for any $t\in\mathbb{R}$, we have

$$\left(f(t)f'(t)\right)'\le 1, \tag{17}$$

$$f''(t)\le 2. \tag{18}$$

By differential mean value theorem, Lemma 3.1(1) and (17), we have

$$\begin{aligned}L_{0n}(x)&=-\frac{1}{2}V(x)\left(f^2(\bar{u}+\omega_n)-f^2(\bar{u})\right)+V(x)f(\bar{u})f'(\bar{u})\omega_n\\&=-V(x)f(\bar{u}+\theta_1\omega_n)f'(\bar{u}+\theta_1\omega_n)\omega_n+V(x)f(\bar{u})f'(\bar{u})\omega_n\\&=V(x)\omega_n^2\theta_1\left(f(\bar{u}+\theta_1\theta_2\omega_n)f'(\bar{u}+\theta_1\theta_2\omega_n)\right)'\\&\le V(x)\omega_n^2,\quad (0\le\theta_1\le 1, 0\le\theta_2\le 1).\end{aligned}$$

Then, by Hölder inequality and Sobolev inequality, we obtain

$$\begin{aligned}\int_{S_n}L_{0n}(x)&\le\int_{S_n}\left|L_{0n}(x)\right|\le\int_{S_n}V(x)\left|\omega_n\right|^2\le\beta\int_{S_n}\omega_n^2\\&\le\beta\left|S_n\right|^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}\left(\omega_n^2\right)^{\frac{N}{N-2}}\right)^{\frac{N-2}{N}}\\&=\beta\left|S_n\right|^{\frac{2}{N}}\left|\omega_n\right|_{2^*}^2\le\beta\left|S_n\right|^{\frac{2}{N}}\left|\omega_n\right|_2^2.\end{aligned}$$

Moreover, by the define of S_n, we have $\lim\limits_{n\to\infty}\left|S_n\right|=0$. In fact, for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$, such that $\left|\{\bar{u}\le v+\delta\}\right|<\varepsilon$, since $v<\bar{u}$ in $\mathbb{R}^N$. Thus

$$S_n\subset\{\bar{u}\le v+\delta\}\cup\{v+\delta<\bar{u}<u_n\}.$$

Again since $\left|u_n-v\right|_2\to 0$ as $n\to\infty$, there exists n_0, such that for $n\ge n_0$

$$\varepsilon\delta^2\ge\int_{\mathbb{R}^N}\left(u_n-v\right)^2\ge\int_{\{u_n>v+\delta\}}\left(u_n-v\right)^2\ge\int_{\{u_n>v+\delta\}}\delta^2=\delta^2\left|\{u_n>v+\delta\}\right|$$

Therefore

$$\left|S_n\right|\le\left|\{u_n\le v+\delta\}\right|+\left|v+\delta<\bar{u}<u_n\right|\le 2\varepsilon,$$

and then as $n\to\infty$, we have

$$\int_{\mathbb{R}^N}L_{0n}(x)\le o(1)\left\|\omega_n\right\|^2.$$

Set

$$g_2(x,t)=G_s\left(x,f(t)\right)=g\left(x,f(t)\right)f'(t),$$

It follows from differential mean value theorem that

$$\begin{aligned}L_{1n}(x)&=G\left(x,f(\bar{u}+\omega_n)\right)-G\left(x,f(\bar{u})\right)-g\left(x,f(\bar{u})\right)f'(\bar{u})\omega_n\\&=g\left(x,f\left(\bar{u}+\theta_1(x)\omega_n\right)\right)f'\left(\bar{u}+\theta_2(x)\omega_n\right)\omega_n-g\left(x,f(\bar{u})\right)f'(\bar{u})\omega_n.\end{aligned}$$

To be continue, set

$$P(x,u)=g\left(x,f(u)\right)f'(u),$$

$$p(x,u)=P_s(x,u)=g_s\left(x,f(u)\right)\left(f'(u)\right)^2+g\left(x,f(u)\right)f''(u)$$

Again by differential mean value theorem, we have

$$\begin{aligned} L_{1n} &= \left(P\left(x,\overline{u}+\theta_1(x)\omega_n\right)-P\left(x,\overline{u}\right)\right)\omega_n \\ &= p\left(x,\overline{u}+\theta_1(x)\theta_2(x)\omega_n\right)\theta_1(x)\omega_n^2. \end{aligned}$$

Set $u_0=\overline{u}+\theta_1(x)\theta_2(x)\omega_n$, so that by Lemma 3.1 (3), (g_2), (g_6), (17), (18), Hölder inequality and Sobolev inequality, we get

$$\begin{aligned} &\int_{S_n} L_{1n}(x)\mathrm{d}x \le \int_{S_n} p\left(x,\overline{u}+\theta_1(x)\theta_2(x)\omega_n\right)\theta_1\omega_n^2\mathrm{d}x \\ &\le \int_{S_n}\left(c\left(f'(u_0)\right)^2+g\left(x,f(u_0)\right)f''(u_0)\right)\omega_n^2\mathrm{d}x \\ &\le \int_{S_n}\left(g_u'\left(x,f(u_0)\right)\left(f'(u_0)\right)^2\mathrm{d}x+C_0 f(u_0)f''(u_0)\mathrm{d}x+C_0 f^{q-1}(u_0)f''(u_0)\right)\omega_n^2\mathrm{d}x \\ &\le C_1\int_{S_n}\omega_n^2\mathrm{d}x+C_2\left(\int_{\mathbb{R}^N}\left|\overline{u}\right|^{\frac{q-1}{2}\frac{2^*}{2^*-2}}\mathrm{d}x\right)^{\frac{2^*-2}{2^*}}\left(\int_{\{u_n\ge\overline{u}\}}\omega_n^{2\frac{2^*}{2}}\mathrm{d}x\right)^{\frac{2}{2^*}}+C_3\int_{S_n}\omega_n^{\frac{q+3}{2}}\mathrm{d}x \\ &\le o(1)\left(\int_{\mathbb{R}^N}\left|\overline{u}\right|^{\frac{q-1}{2}\frac{2^*}{2^*-2}}\mathrm{d}x\right)^{\frac{2^*-2}{2^*}}\left(\int_{\mathbb{R}^N}\omega_n^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}\le o(1)\left|\nabla\omega_n\right|_2^2, \end{aligned}$$

which implies that (15) is satisfied.

By (15) and (16), as $n\to\infty$ we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N}\left|\nabla\omega_n\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}\left|\nabla z_n\right|^2\mathrm{d}x \\ &\le I(u_n)-I(v_n)+\int_{S_n}\left(L\left(x,\overline{u}+\omega_n\right)-L\left(x,\overline{u}\right)-L_u\left(x,\overline{u}\right)\omega_n\right)\mathrm{d}x \\ &\quad+\int_{T_n}\left(L\left(x,\underline{u}-z_n\right)-L\left(x,\underline{u}\right)-L_u\left(x,\underline{u}\right)z_n\right)\mathrm{d}x \\ &<\int_{S_n}\left(L\left(x,\overline{u}+\omega_n\right)-L\left(x,\overline{u}\right)-L_u\left(x,\overline{u}\right)\omega_n\right)\mathrm{d}x \\ &\quad+\int_{T_n}\left(L\left(x,\underline{u}-z_n\right)-L\left(x,\underline{u}\right)-L_u\left(x,\underline{u}\right)z_n\right)\mathrm{d}x \\ &\le o(1)\left|\nabla\omega_n\right|_2^2+o(1)\left|\nabla z_n\right|_2^2. \end{aligned}$$

Since ω_n and z_n have disjoint support, as $n\to\infty$

$$\int_{\mathbb{R}^N}\left|\nabla\omega_n\right|^2\mathrm{d}x\le o(1)\left|\nabla\omega_n\right|_2^2,$$

$$\int_{\mathbb{R}^N}\left|\nabla z_n\right|^2\mathrm{d}x\le o(1)\left|\nabla z_n\right|_2^2.$$

Then $z_n(x)=\omega_n(x)=0$ a.e. $x\in\mathbb{R}^N$, which implies $u_n=v_n$ a.e. $x\in\mathbb{R}^N$. By (12), we have

$$I(v)\le I(v_n)=I(u_n).$$

Contradiction. Thus, the proof is complete.

Define a set

$$\Pi=\left\{u\in H_r^1\left(\mathbb{R}^N\right):0\le u\le v,\ a.e.\ x\in\mathbb{R}^N\right\},$$

where v is a positive solution in Lemma 4.1. The critical point in II is also the critical point in $H_r^1\left(\mathbb{R}^N\right)$ of I [3].

Lemma 4.3 Suppose that the conditions of theorem 1.2 are satisfied, then I satisfies (PS) condition on Π.

Proof: Firstly, we need to prove the boundedness of any (PS) sequence $\{v_n\}$ on Π. Assume $\{v_n\} \subset \Pi$ is a (PS) sequence, then

$$I(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v_n)\mathrm{d}x - \int_{\mathbb{R}^N} G(x, f(v_n))\mathrm{d}x = c + o(1), \tag{19}$$

$$\begin{aligned} \langle I'(v_n), \phi\rangle &= \int_{\mathbb{R}^N} \nabla v_n \nabla \phi \mathrm{d}x + \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n)\phi \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} g(x, f(v_n)) f'(v_n)\phi \mathrm{d}x \\ &= o(1)\|\phi\|. \end{aligned} \tag{20}$$

By (g_7) and (19), there exists $\mu \ge 4$, such that

$$\begin{aligned} c + o(1) = I(v_n) &> \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v_n)\mathrm{d}x \\ &\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n)\mathrm{d}x. \end{aligned} \tag{21}$$

Specially, choose $\phi = \frac{f(v_n)}{f'(v_n)} = \sqrt{1+2f^2(v_n)} f(v_n)$. By Lemma 3.1 (2) (3), we have

$$\begin{aligned} |\phi| &= \sqrt{1+2f^2(v_n)}|f(v_n)| \le \sqrt{2}\sqrt{1+f^2(v_n)}|f(v_n)| \\ &\le \sqrt{2}(1+|f(v_n)|)|f(v_n)| \le \sqrt{2}\left(|f(v_n)| + |f(v_n)|^2\right) \\ &\le \sqrt{2}|v_n| + 2|v_n| \le 4|v_n|. \end{aligned}$$

Again since

$$|\nabla \phi| = \left(\frac{1+4f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n| \le 2|\nabla v_n|,$$

implies that $\|\varphi\| \le C\|v_n\|$. (20) is transformed to

$$\int_{\mathbb{R}^N} \frac{1+4f^2(v_n)}{1+2f^2(v_n)}|\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(v_n) - \int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n) = o(1)\|v_n\|, \tag{22}$$

(22) implies

$$\begin{aligned} &\frac{1}{\mu}\int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n)\mathrm{d}x \\ &= \frac{1}{\mu}\int_{\mathbb{R}^N} \frac{1+4f^2(v_n)}{1+2f^2(v_n)}|\nabla v_n|^2 + \frac{1}{\mu}\int_{\mathbb{R}^N} V(x) f^2(v_n) \\ &\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n) + o(1)\|v_n\|. \end{aligned} \tag{23}$$

Substituting (23) into (21), we obtain

$$\begin{aligned} c + o(1) = I(v_n) &> \left(\frac{1}{2} - \frac{1}{\mu}\frac{1+4f^2(v_n)}{1+2f^2(v_n)}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x \\ &\quad + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x) f^2(v_n)\mathrm{d}x - o(1)\|v_n\|. \end{aligned}$$

Thus

$$\|v_n\|^2 \le c + o(1)(1+\|v_n\|),$$

This implies that $\|v_n\|^2$ is bounded.

Next, let's prove (PS) sequence $\{v_n\}$ satisfies (PS) condition. Since $\{v_n\} \subset H_r^1(\mathbb{R}^N)$ is bounded in $H_r^1(\mathbb{R}^N)$, by Rellich-Kondrachov theorem, there exists $v \in H_r^1(\mathbb{R}^N)$, such that

i) $v_n \xrightarrow{\text{Weak convergence}} v$ in $H_r^1(\mathbb{R}^N)$;

ii) $v_n \to v$ in $L^{\frac{q}{2}}(\mathbb{R}^N)$;

iii) $v_n \to v$ a.e. in $\mathbb{R}^N$.

By (i) and (ii), we obtain $I'_\lambda(v_\lambda) = 0$.

The following we prove $v_n \to v$ in $H_r^1(\mathbb{R}^N)$. Set

$$H(x,t) = \frac{1}{2}V(x)t^2 - \frac{1}{2}V(x)f^2(t) + G(x, f(t)).$$

Then $I(v)$ is transformed to

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x)v^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} H(x,v)\mathrm{d}x.$$

Set $h(x,t) = \frac{\mathrm{d}H(x,t)}{\mathrm{d}t}$, then

$$h(x,t) = V(x)t - V(x)f(t)f'(t) + g(x, f(t))f'(t).$$

By (g_1), (g_2) and Lemma 3.1, there exists $C_1, C_2 > 0$, for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, such that

$$|h(x,t)| \le C_1|t| + C_2|t|^{\frac{q-1}{2}}. \tag{24}$$

By (24) and $v_n \to v$ in $L^{\frac{q}{2}}(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(h(x,v_n) - h(x,v)\right)(v_n - v)\mathrm{d}x = 0.$$

Thus

$$\begin{aligned} o(1) &= \langle I'(v_n) - I'(v_\lambda), v_n - v\rangle \\ &= \int_{\mathbb{R}^N}\left(|\nabla(v_n - v)|^2 + V(x)(v_n - v)^2\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N}\left(h(x,v_n) - h(x,v)\right)(v_n - v)\mathrm{d}x \\ &\ge \min\{1, V_0\}\|v_n - v\|^2 + o(1), \end{aligned}$$

Which implies that $v_n \to v$ in $H_r^1(\mathbb{R}^N)$. The proof is complete.

Lemma 4.4. (see ([11], Theorem II.11.8).) Suppose *M* is a closed, convex subset of a Banach space *V*, $E \in C^1(V)$ satisfies (PS) on *M*, and admits two distinct relative minima u_1, u_2 in *M*. Then either $E(u_1) = E(u_2) = \beta$ and u_1, u_2 can be connected in any neighborhood of the set of relative minima $u \in M$ of *E* with $E(u) = \beta$, or there exists a critical point $\tilde{u}$ of *E* in *M* which is not a relative minimizer of *E*.

Proof of Theorem 1.2. Applying Lemma 4.4, we arrive to the following dichotomy

1) $I(v) = I(0)$ and v and 0 may be connected in any neighborhood of the set of local minima of I to II, or

2) I admits a critical point $\tilde{u}$ in II which is not a local minimum.

But Lemma 4.1 ensures that the trivial solution is an isolated solution of problem (2). Hence a second independent solution of problem (2) should exist since the solution found in Lemma 4.2 is a local minimum of I. In conclusion, problem (2) admits one pair of ordered positive solutions to equation. The proof is complete.

References

[1] Liu, J.Q. and Wang, Z.Q. (2008) Symmetric Solutions to a Modified Nonlinear Schrödinger Equation. *Nonlinearity*, **21**, 121-133. https://doi.org/10.1088/0951-7715/21/1/007

[2] Poppenberg, M., Schmitt, K. and Wang, Z.Q. (2002) On the Existence of Solutions to Quasilinear Schrödinger Equations. *Calculus of Variations and Partial Differential Equations*, **14**, 329-344. https://doi.org/10.1007/s005260100105

[3] Colin, M. and Jeanjean, L. (2004) Solutions for a Quasilinear Schrödinger Equations: A Dual Approach. *Nonlinear Analysis*, **56**, 213-226. https://doi.org/10.1016/j.na.2003.09.008

[4] Fang, X.D. and Szulkin, A. (2013) Multiple Solutions for a Quasilinear Schrödinger Equation. *Journal of Differential Equations*, **254**, 2015-2032. https://doi.org/10.1016/j.jde.2012.11.017

[5] Liu, J.Q., Wang, Y.Q. and Wang, Z.Q. (2003) Soliton Solutions for Quasilinear Schrödinger Equations II. *Journal of Differential Equations*, **187**, 473-493. https://doi.org/10.1016/S0022-0396(02)00064-5

[6] Liu, X.Q., Wang, Y.Q. and Wang, Z.Q. (2004) Solutions for Quasilinear Schrödinger Equations via the Nehari Method. *Communications in Partial Differential Equations*, **29**, 879-901. https://doi.org/10.1081/PDE-120037335

[7] Liu, X.Q., Liu, J.Q. and Wang, Z.Q. (2013) Quasilinear Elliptic Equations via Perturbation Method. *Proceedings of the American Mathematical Society*, **141**, 253-263. https://doi.org/10.1090/S0002-9939-2012-11293-6

[8] Kurihura, S. (1981) Large-Amplitude Quasi-Solitons in Superfluids Films. *Journal of the Physical Society of Japan*, **50**, 3262-3267. https://doi.org/10.1143/JPSJ.50.3262

[9] Laedke, E.W., Spatschek, K.H. and Stenflo, L. (1983) Evolution Theorem for a Class of Pertubed Envelop Soliton Solutions. *Journal of Mathematical Physics*, **24**, 2764-2769. https://doi.org/10.1063/1.525675

[10] de Paiva, F.O. (2012) Nonnegative Solutions for Indefinite Sublinear Elliptic Problems. *Communications in Contemporary Mathematics*, **14**, 1-20. https://doi.org/10.1142/S0219199712500216

[11] Struwe, M. (2008) Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamitonian Systems. Springer-Verlag, Berlin, 15-87.