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Poisson (Co)homology of a Class of Frobenius Poisson Algbras

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Abstract

In this paper, we study the truncated polynomial algebra Λ in n variables, and discuss the following four problems in detail: 1) Homology complex and homology group of Poisson algebra Λ; 2) Given a new Poisson bracket by calculation modular derivation of Frobenius Poisson algebra; 3) Calculate the twisted homology group of Poisson algebra Λ; 4) Verify the theorem of twisted Poincaré duality between twisted Poisson homology and Poisson Cohomology.

Keywords

Poisson Algebra, Poisson (Co)homology, Twisted Poisson Module, Twisted Poincaré Duality

1. Introduction

Poisson structures appear in a large variety of different contexts, ranging from string theory, classical/quantum mechanics and differential geometry to abstract algebra, algebraic geometry and representation theory. They play an important role in Poisson geometry, in algebraic geometry and non-commutative. Poisson cohomologies are important invariants of Poisson structures. The set of Casimir elements of the Poisson structure is the 0-th cohomology; Poisson derivations modulo Hamiltonian derivations are the 1-st cohomology. Poisson cohomology appears as one considers deformations of Poisson algebras. Given a Poisson algebra, we can get some vital information about the Poisson algebra structure from calculate its Poisson (Co)homology.

C. Kassel started calculate the (Co)homology of linear Poisson structure (see [1]). Luo J, Wang S.Q (see [2]) get the Twisted Poincare duality between Poisson homology and Poisson cohomology in quadratic Poisson algebra. Roger C and

Vanhaecke P (see [3]) do the research about the Poisson cohomology of the affine plane. Pichereau calculate the Poisson homology in 3 dimension affine space (see [4]). Tagne Pelap calculates the Poisson (Co)homology of polynomial algebra with 3 variable in generalized Jacobian Poisson structure (see [5]). Foramlity theorem has been proved by Kontsevich in 2003 (see [6]), and the results revealed the importance Poisson algebra and its deformation quantization. In general, it is very important to calculate Poisson cohomology from a given Poisson structure. These researches mainly focused on the smooth algebra and the finite dimension algebra.

The homology theorem of the singular algebra is few. Launois S and Richard L [7] calculate the Poisson (Co)homology of truncated polynomial algebras in 2 variables, and established the twisted Poincaré duality between them. [8] and [9] proofed this conclusion stands for all Frobenius Poisson algebra. In this paper, we want to study infinite dimension situation: a truncated polynomial algebra with n variables.

2. Main Results

In this paper, we let k is a number field. We consider the truncated polynomial algebras in n variables $\Lambda = k\langle x_1, x_2, \cdots, x_n\rangle \big/ \langle x_i x_j - x_j x_i, x_i^2\rangle$ with the Poisson bracket

$$\{x_i, x_j\} = \lambda_{ij} x_i x_j,\ \forall 1 \le i < j \le n,\ \lambda_{ij} \in k^+$$

We get some mainly results. In part 4, we get the i-th homology group $HP_i(\Lambda)$ of algebra Λ

$$HP_0(\Lambda) = k \oplus kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n,\quad HP_i(\Lambda) = 0\,(i \ge 1).$$

In part 5, we calculate Poisson modular derivation after we get the Frobenius pairing

$$\begin{aligned} D(x_i) &= \{1, x_i\}_D \\ &= \sigma^{-1}\left(\left(\left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right)\right) x_1^* \cdots \hat{x}_i^* \cdots x_n^*\right) \\ &= \left(\left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right)\right) x_i \end{aligned}$$

then we can get the new Poisson module structure

$$\{x_i, x_j\}_{\Lambda^D} = \begin{cases} \left(\left(\lambda_{j\,j+1} + \cdots + \lambda_{j\,n}\right) - \left(\lambda_{1j} + \lambda_{2j} + \cdots + \lambda_{i-1\,j} + \lambda_{i+1\,j} + \cdots + \lambda_{j-1\,j}\right)\right) x_i x_j, & (i < j) \\ \left(\left(\lambda_{j\,j+1} + \cdots + \lambda_{j\,i-1} + \lambda_{j\,i+1} + \cdots + \lambda_{j\,n}\right) - \left(\lambda_{1j} + \lambda_{2j} + \cdots + \lambda_{j-1\,j}\right)\right) x_i x_j, & (i > j) \\ 0, & (i = j) \end{cases}$$

In part 6, we get the twisted Poisson homology group and we get the results

$$HP_0(\Lambda, \Lambda^D) = k,\quad HP_n(\Lambda, \Lambda^D) = 0$$

the elements in $HP_m(\Lambda, \Lambda^D)(1 \le m \le n-1)$ with the length of $0 \sim m$ in $\ker \partial_m^\pi$.

In part 7, we check the twisted Poincaré duality between Poisson homology and Poisson Cohomology $P.D.: HP^i(\Lambda, \Lambda) \cong \left(HP_i(\Lambda, \Lambda^D)\right)^*$ by calculating the

Twisted Poisson cohomology.

3. Some Preliminary Definition and Proposition

Definition 1 [10]. *A* Poisson algebra is an *k*-vector space *A* equipped with two multiplications $(x,y)\mapsto x\cdot y$ and $(x,y)\mapsto\{x,y\}$ such that

1) $(A,\cdot)$ is a commutative associative algebra over *k*, with unit 1;
2) $(A,\{\cdot,\cdot\})$ is a Lie algebra over F;
3) The two multiplications are compatible in the sense that

$$\{x\cdot y,z\}=x\cdot\{y,z\}+y\cdot\{x,z\}.$$

where *x*, *y* and *z* are arbitrary elements of *A*. The Lie bracket $\{\cdot,\cdot\}$ is then called a Poisson bracket

Example 1. The commutative polynomial algebra in two variables $S=k[x,y]$ is a Poisson algebra for the bracket defined on the generators by

$$\{x,y\}=1,$$

or equivalently for any $P,Q\in S$

$$\{P,Q\}=\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y}-\frac{\partial Q}{\partial x}\frac{\partial P}{\partial y}.$$

More generally, for any $n\geq 1$, $S=k[x_1,\cdots,x_n,y_1,\cdots,y_n]$ is a Poisson algebra for the "symplectic" bracket defined on the generators by

$$\{x_i,y_j\}=\delta_{i,j},\{x_i,y_j\}=\{y_i,y_j\}=0\text{, for all }1\leq i,j\leq n$$

or equivalently for any $P,Q\in S$

$$\{P,Q\}=\sum_{i=1}^{n}\frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial y_i}-\frac{\partial Q}{\partial x_i}\frac{\partial P}{\partial y_i}.$$

We refer to this Poisson algebra as the Poisson-Weyl algebra, denoted by $S_n(k)$.

Example 2. For any $\lambda\in k$, the commutative polynomial algebra $S=k[x,y]$ is a Poisson algebra for the "multiplicative" bracket defined on the generators by

$$\{x,y\}=\lambda xy.$$

More generally, for any $n\geq 2$ and for any $n\times n$ antisymmetric matrix $\lambda=(\lambda_{ij})$ with entries in *k*, $S=k[x_1,\cdots,x_n]$ is a Poisson algebra for the bracket defined on the generators by

$$\{x_i,x_j\}=\lambda_{ij}x_ix_j\text{ for all }1\leq i<j\leq n.$$

We refer to this Poisson algebras as the Poisson-quantum plane and Poisson-quantum space respectively, denoted by $P_2^{\lambda}(k)$ and $P_n^{\lambda}(k)$.

Definition 2 [11]. Let $(A,\{-,-\})$ be a Poisson algebra. A right *A*-module $(M,+,\cdot)$ is called a right Poisson module over *A*, if there is a bilinear map $\{-,-\}_M:M\times R\to M$ such that

1) $(M,\{-,-\}_M)$ is a right Lie-module over the Lie algebra $(R,\{-,-\})$;
2) $\{x\cdot a,b\}_M=\{x,b\}_M\cdot a+x\cdot\{a,b\}$ for any $a,b\in R,x\in M$;

3) $\{x,ab\}_M = \{x,a\}_M \cdot b + \{x,b\}_M \cdot a$ for any $a,b \in R, x \in M$.

Left Poisson modules are defined similarly. In particular, any Poisson algebra R is naturally a right and left Poisson module over itself

For a Poisson algebra A, the space A has a natural right (also, left) Poisson module structure. Given two right Poisson modules $(M,\cdot,\{-,-\}_M)$ and $(N,\cdot,\{-,-\}_N)$ over A, a k-linear map $f: M \to N$ is called a morphism of Poisson modules if

$$f(m \cdot a) = f(m) \cdot a, \quad f(\{m,a\}_M) = \{f(m),a\}_N.$$

for each $m \in M$ and $a \in A$. The following two properties on Poisson modules are straightforward.

Proposition 1 [8]. Suppose that $(M,\cdot,\{-,-\}_M)$ is a right Poisson module over A, Then the following actions define a left Poisson module structure on $M^* := Hom_k(M,k)$

- $a \cdot \alpha : M \to k, (a \cdot \alpha)(m) := \alpha(m \cdot a)$.
- $\{a \cdot \alpha\}_{M^*} : M \to k, \{a \cdot \alpha\}_{M^*}(m) := \alpha(\{m,a\}_M)$.

for each $a \in A, \alpha \in M^*, m \in M$. Similarly, a left Poisson module M yields a right Poisson module M^*.

Proposition 2 [8]. Let $(N,\cdot,\{-,-\}_N)$ be a right Poisson module over A and M be a right A-module. Suppose that $f: M \to N$ is an isomorphism of A-modules. Then there exists a right Poisson module structure on M given by: $\{m,a\}_M := f^{-1}(\{f(m),a\}_N)$. for each $a \in A, m \in M$, such that f is an isomorphism of Poisson modules.

Definition 3 [2]. Let R be a Poisson algebra. In general, let $\Omega^1(R)$ be the Kähler differential module of R and $\Omega^p(R)$ be the p-th Kahler differential forms. Given a right Poisson module M over the Poisson algebra R, there is a canonical chain complex

$$\begin{gathered}\cdots \longrightarrow M \otimes_R \Omega^p(R) \xrightarrow{\partial_p} M \otimes_R \Omega^{p-1}(R) \xrightarrow{\partial_{p-1}} \cdots \\ \xrightarrow{\partial_2} M \otimes_R \Omega^1(R) \xrightarrow{\partial_1} M \otimes_R R \xrightarrow{\partial_0} 0\end{gathered} \tag{1}$$

where for $p \geq 1$, $\partial_p : M \otimes_R \Omega^p(R) \to M \otimes_R \Omega^{p-1}(R)$ is defined as:

$$\begin{aligned}&\partial_p(m \otimes \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_p) \\ &= \sum_{i=1}^{p} (-1)^{i-1} \{m,a_i\}_M \otimes \mathrm{d}a_1 \wedge \cdots \widehat{\mathrm{d}a_i} \cdots \wedge \mathrm{d}a_p \\ &\quad + \sum_{1 \le i < j \le p} (-1)^{i+j} m \otimes \mathrm{d}\{a_i,a_j\} \wedge \mathrm{d}a_1 \wedge \cdots \widehat{\mathrm{d}a_i} \cdots \widehat{\mathrm{d}a_j} \cdots \wedge \mathrm{d}a_p.\end{aligned}$$

It is easily check that ∂_p is well defined, that is $\partial_{p-1}\partial_p = 0$.

The complex (1) is called the Poisson complex of R with values in M, and for $p \geq 0$ its p-th homology is called the p-th Poisson homology of R with values in M, denoted by $HP_p(R,M)$.

Definition 4 [2]. For any $p \in \mathbb{N}$, $\mathcal{X}^p(M)$ be the p-fold polyderivations from R to M. There is also a canonical cochain complex

$$0 \longrightarrow M \xrightarrow{\delta^0} \chi^1(M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} \chi^p(M) \xrightarrow{\delta^p} \chi^{p+1}(M) \longrightarrow \cdots \tag{2}$$

where $\delta^p : \chi^p(M) \to \chi^{p+1}(M)$ is defined as $F \mapsto \delta^p(F)$ with

$$\begin{aligned}
&\delta^p(F)(a_1 \wedge \cdots \wedge a_{p+1}) \\
&= (-1)^{p+1} \sum_{i=1}^{p+1} (-1)^i \left\{ F\left(a_1 \wedge \cdots \hat{a}_i \cdots \wedge a_{p+1}\right), a_i \right\}_M \\
&\quad + (-1)^{p+1} \sum_{1 \le i < j \le p+1} (-1)^{i+j} F\left(\{a_i, a_j\} \wedge a_1 \wedge \cdots \hat{a}_i \cdots \hat{a}_j \cdots \wedge a_{p+1}\right)
\end{aligned}$$

It is easily check that ∂_p is well defined, that is $\delta^p \delta^{p-1} = 0$.

The complex (2) is called the Poisson cochain complex of R with values in M, and for $p \ge 0$ its p-th cohomology is called the p-th Poisson cohomology of R with values in M, denoted by $HP^p(R, M)$.

The elements in $\ker \delta^1$ are called Poisson derivations, and the elements in $\operatorname{Im} \delta^0$ are called Hamiltonian derivations which are of the form $\{m, -\}_M$, for $m \in M$.

Definition 5 [8]. A finite dimensional k-algebra R is frobenius if satisfied: as left R module, $R \cong R^*$, where $R^* = Hom_k(R, k)$.

Definition 6 [8]. Let A be a Frobenius Poisson algebra with defining bilinear form $\langle -, - \rangle$. Define a map $D : A \to A$ via $D(a) = \{1, a\}_D$, for $\forall a \in A$. We call D the modular derivation

Proposition 3 [2]. Let $D \in \chi^1(R)$ be a Poisson derivation, and M be a right Poisson R-module. Define a new bilinear map $\{-,-\}_{M^D} : M \times R \to M$ as

$$\{m, a\}_{M^D} := \{m, a\}_M + m \cdot D(a).$$

Then the R-module M with $\{-,-\}_{M^D}$ is a right Poisson R-module, which is called the twisted Poisson module of M twisted by the Poisson derivation D, denoted by M^D.

Lemma 1 [11]. Let S be a Frobenius Poisson algebra. Then, for all $i \in N$, we have:

$$P.D. : HP^i(A, A) \cong \left(HP_i(A, A_D)\right)^*$$

4. Homology Group

Before we calculate the homology, it is necessary to write the homology complex first. We give the basis in linear space in every point of the complex. Obviously, $\Omega^0(\Lambda) = \Lambda$. $\Omega^1(\Lambda) = C_1 dx_1 \oplus C_2 dx_2 \oplus \cdots \oplus C_n dx_n$, $C_j = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, $i_j = 0$ and $i_k = 0$ or 1, for $i_k = 0$. $\Omega^1(\Lambda) = \oplus C x_1^{i_1} \cdots x_n^{i_n} dx_j, i_j = 0, 0 \le i_1, \cdots, i_n \le 1$, and $C \in k$.

$dx_j^2 = 0$, $\Omega^2(\Lambda) = \oplus C x_1^{i_1} \cdots x_n^{i_n} dx_j \wedge dx_k, \; j < k, i_j = i_k = 0, 0 \le i_1, \cdots, i_n \le 1, C \in k$.

Similarly, we can get

$$\begin{aligned}
&\Omega^3(\Lambda) = \oplus C x_1^{i_1} \cdots x_n^{i_n} dx_p \wedge dx_q \wedge dx_r, \; p < q < r, \\
&i_p = i_q = i_r = 0, 0 \le i_1, \cdots, i_n \le 1, C \in k;
\end{aligned}$$

$$\Omega^4(\Lambda) = \oplus C x_1^{i_1} \cdots x_n^{i_n} \mathrm{d}x_p \wedge \mathrm{d}x_q \wedge \mathrm{d}x_r \wedge \mathrm{d}x_s,\ p<q<r<s,$$
$$i_p = i_q = i_r = i_s = 0,\ 0 \le i_1, \cdots, i_n \le 1, C \in k;$$
$$\cdots$$
$$\Omega^n(\Lambda) = C \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, C \in k;$$
$$\Omega^m(\Lambda) = 0\,(m>n).$$

So we can get the Poisson complex

$$\longrightarrow \Omega_{q+1}(\Lambda) \xrightarrow{\partial_{q+1}^{\pi}} \Omega_q(\Lambda) \xrightarrow{\partial_q^{\pi}} \Omega_{q-1}(\Lambda) \xrightarrow{\partial_{q-1}^{\pi}} \cdots \longrightarrow \Omega_2(\Lambda) \quad (3)$$
$$\xrightarrow{\partial_2^{\pi}} \Omega_1(\Lambda) \xrightarrow{\partial_1^{\pi}} \Lambda \xrightarrow{\partial_0} 0$$

Then we calculate the Poisson homology group and get some results.

Proposition 4.1. $HP_0(\Lambda) = k \oplus kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$

Proof:

$$\partial_1^{\pi}: \Omega_1(\Lambda) \to \Lambda$$
$$\sum_{j=1}^{n} C x_1^{i_1} \cdots x_j^{0} \cdots x_n^{i_n} \mathrm{d}x_j \mapsto \sum_{j=1}^{n} C \left\{ x_1^{i_1} x_2^{i_2} \cdots x_j^{0} \cdots x_n^{i_n}, x_j \right\}$$

With the definition of Poisson bracke, we can write $\left\{ x_{i_1} \cdots x_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n \right\}$ as the results of image $\left\{ x_1^{i_1} \cdots x_j^{0} \cdots x_n^{i_n}, x_j \right\}$. Next, we will prove that, when $i_k \ge 2$, we can find preimage in $\Omega_1(\Lambda)$ for all elements

1) The preimage of image with the length of 2

$$\frac{1}{\lambda_{ij}} x_i \mathrm{d}x_j \mapsto \frac{1}{\lambda_{ij}} \left\{ x_i, x_j \right\} = \frac{1}{\lambda_{ij}} \lambda_{ij} x_i x_j = x_i x_j, (i<j).$$

2) The preimage of image with the length of 3

$$\frac{1}{\lambda_{ip} + \lambda_{jp}} x_i x_j \mathrm{d}x_p \mapsto \frac{1}{\lambda_{ip} + \lambda_{jp}} \left\{ x_i x_j, x_p \right\}$$
$$= \frac{1}{\lambda_{ip} + \lambda_{jp}} \left(x_i \left\{ x_j, x_p \right\} + x_j \left\{ x_i, x_p \right\} \right) = x_i x_j x_p, (i<j<p).$$

3) The preimage of image with the length of n

$$\frac{1}{\lambda_{1n} + \lambda_{2n} + \cdots + \lambda_{n-1\,n}} x_1 x_2 \cdots x_{n-1} \mathrm{d}x_n$$
$$\mapsto \frac{1}{\lambda_{1n} + \lambda_{2n} + \cdots + \lambda_{n-1\,n}} \left\{ x_1 x_2 \cdots x_{n-1}, x_n \right\} \mapsto x_1 x_2 \cdots x_n.$$

4) Generally, The preimage of image with the length of $m\,(1<m<n)$

$$\frac{1}{\lambda_{i_1 i_m} + \cdots + \lambda_{i_{m-1} i_m}} x_{i_1} \cdots x_{i_{m-1}} \mathrm{d}x_{i_m} \mapsto \frac{1}{\lambda_{i_1 i_m} + \cdots + \lambda_{i_{m-1} i_m}} \left\{ x_{i_1} \cdots x_{i_{m-1}}, x_{i_m} \right\}$$
$$= \frac{1}{\lambda_{i_1 i_m} + \cdots + \lambda_{i_{m-1} i_m}} \left(x_{i_1} \left\{ x_{i_2} \cdots x_{i_{m-1}}, x_{i_m} \right\} + \cdots + x_{i_{m-1}} \left\{ x_{i_1} \cdots x_{i_{m-2}}, x_{i_m} \right\} \right)$$
$$= x_{i_1} \cdots x_{i_m}.$$

we can find preimage in $\Omega_1(\Lambda)$ for all elements

Hence, we can find preimage of the image in $\Omega_1(\Lambda)$ when image length i_k satisfied $i_k \ge 2$. However, we can not find the preimage of image with the length of 1 in $\Omega_1(\Lambda)$.

So we can get 0-th homology group
$HP_0(\Lambda) = \ker \partial_0^{\pi} / \operatorname{Im} \partial_1^{\pi} = k \oplus kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$.

The proof is completed.

Proposition 4.2. $HP_1(\Lambda) = 0$

Proof:

$$\Omega_2(\Lambda) \xrightarrow{\partial_2^{\pi}} \Omega_1(\Lambda) \xrightarrow{\partial_1^{\pi}} \Lambda$$

Firstly, we calculate $\ker \partial_1^{\pi}$

$$\partial_1^{\pi} : \Omega_1(\Lambda) \to \Lambda$$

$$\sum_{j=1}^{n} C x_1^{i_1} \cdots x_j^{0} \cdots x_n^{i_n} \mathrm{d}x_j \mapsto \sum_{j=1}^{n} C \left\{ x_1^{i_1} x_2^{i_2} \cdots x_j^{0} \cdots x_n^{i_n}, x_j \right\}$$

the mapping between the elements is

$$\begin{aligned} & x_1^{i_1} x_2^{i_2} \cdots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}} \mathrm{d}x_n + x_1^{j_1} x_2^{j_2} \cdots x_{n-2}^{j_{n-2}} x_n^{j_n} \mathrm{d}x_{n-1} + \cdots + x_1^{p_1} x_3^{p_3} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n} \mathrm{d}x_2 \\ & + x_2^{q_2} x_3^{q_3} \cdots x_{n-1}^{q_{n-1}} x_n^{q_n} \mathrm{d}x_1 \\ & \mapsto k_{i_1,\cdots,i_n} \left\{ x_1^{i_1} x_2^{i_2} \cdots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}}, x_n \right\} + k_{j_1,\cdots,j_n} \left\{ x_1^{j_1} x_2^{j_2} \cdots x_{n-2}^{j_{n-2}} x_n^{j_n}, x_{n-1} \right\} + \cdots \\ & + k_{p_1,\cdots,p_n} \left\{ x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n}, x_2 \right\} + k_{q_1,\cdots,q_n} \left\{ x_2^{q_2} x_3^{q_3} \cdots x_{n-1}^{q_{n-1}} x_n^{q_n}, x_1 \right\} \end{aligned}$$

We have n preimage, each of them has the length of $k(1 \le k \le n-1)$. The image has $k+1$ in length, then we calculate the image in each item.

The first term:
$k_{i_1,\cdots,i_n} \left\{ x_1^{i_1} x_2^{i_2} \cdots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}}, x_n \right\} = k_{i_1,\cdots,i_n} \left(i_1 \lambda_{1n} + i_2 \lambda_{2n} + \cdots + i_{n-1} \lambda_{n-1\,n} \right) x_1^{i_1} x_2^{i_2} \cdots x_{n-1}^{i_{n-1}} x_n$ obviously, image has the different form with different value in $i_1, i_2, \cdots, i_{n-1}$ $(0 \le i_1, \cdots, i_{n-1} \le 1)$. Now we discuss all the situation about the first item in Table 1.

The second item:
$$\begin{aligned} & k_{j_1,\cdots,j_n} \left\{ x_1^{j_1} x_2^{j_2} \cdots x_{n-2}^{j_{n-2}} x_n^{j_n}, x_{n-1} \right\} \\ & = k_{k_1,\cdots,k_n} \left(j_1 \lambda_{1\,n-1} + j_2 \lambda_{2\,n-1} + \cdots + j_n \lambda_{n\,n-1} \right) x_1^{j_1} x_2^{j_2} \cdots x_{n-1} x_n^{j_n} \end{aligned}$$

Similarly, we discuss all the situation about the second item in Table 2.

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The $(n-1)$-th item:
$$k_{p_1,\cdots,p_n} \left\{ x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n}, x_2 \right\} = k_{p_1,\cdots,p_n} \left(p_1 \lambda_{12} + p_3 \lambda_{32} + \cdots + p_n \lambda_{n\,2} \right) x_1^{p_1} x_2 \cdots x_{n-1}^{p_{n-1}} x_n^{p_n}$$

The n-th item:
$$k_{q_1,\cdots,q_n} \left\{ x_2^{q_2} \cdots x_{n-1}^{q_{n-1}} x_n^{q_n}, x_1 \right\} = k_{q_1,\cdots,q_n} \left(q_2 \lambda_{21} + q_3 \lambda_{31} + \cdots + q_n \lambda_{n\,1} \right) x_1 x_2^{q_2} \cdots x_{n-1}^{q_{n-1}} x_n^{q_n}$$

Similarly, we can discuss all the situation about the $(n-1)$-th, n-th item.

Because the Poisson structure of Λ is homogeneous. We can discuss the image of ∂_1^{π} by length.

1) The preimage of image with the length of 1

$$\sum C_{ij} x_i \mathrm{d}x_j \mapsto \sum C_{ij} \lambda_{ij} x_i x_j$$

Let $\sum C_{ij} x_i dx_j \in \ker \partial_1^{\pi}$, then $\sum C_{ij} \lambda_{ij} x_i x_j = 0$. we can get $\left(C_{ij} - C_{ji} \right) \lambda_{ij} = 0$ for $\forall i < j$, that is $C_{ij} = C_{ji}$. we have

Table 1. This table shows different form in image with the different value in $i_1, i_2, \cdots, i_{n-1}$.

Image length	value	Denoted by	image
2	$i_1 = 1, i_2 = i_3 = \cdots = i_{n-1} = 0$	$i_{1,0,0,\cdots,0}$	$\lambda_{1n} x_1 x_n$
	$i_2 = 1, i_1 = i_3 = \cdots = i_{n-1} = 0$	$i_{0,1,0,\cdots,0}$	$\lambda_{2n} x_2 x_n$
	$\cdots$	$\cdots$	$\cdots$
	$i_{n-1} = 1, i_1 = i_2 = \cdots = i_{n-2} = 0$	$i_{0,0,0,\cdots,1}$	$\lambda_{n-1\,n} x_{n-1} x_n$
3	$i_1 = i_2 = 1, i_3 = \cdots = i_{n-1} = 0$	$i_{1,1,0,\cdots,0}$	$(\lambda_{1n} + \lambda_{2n}) x_1 x_2 x_n$
	$i_1 = i_3 = 1, i_2 = \cdots = i_{n-1} = 0$	$i_{1,0,1,\cdots,0}$	$(\lambda_{1n} + \lambda_{3n}) x_1 x_3 x_n$
	$\cdots$	$\cdots$	$\cdots$
	$i_{n-2} = i_{n-1} = 1, i_1 = \cdots = i_{n-3} = 0$	$i_{0,0,\cdots,1,1}$	$(\lambda_{n-2\,n} + \lambda_{n-1\,n}) x_{n-2} x_{n-1} x_n$
		$\cdots$	
$n-1$	$i_1 = 0, i_2 = i_3 = \cdots = i_{n-1} = 1$	$i_{0,1,1,\cdots,1}$	$(\lambda_{2n} + \cdots + \lambda_{n-1\,n}) \hat{x}_1 x_2 \cdots x_{n-1} x_n$
	$i_2 = 0, i_1 = i_3 = \cdots = i_{n-1} = 1$	$i_{1,0,1,\cdots,1}$	$(\lambda_{1n} + \cdots + \lambda_{n-1\,n}) x_1 \hat{x}_2 \cdots x_{n-1} x_n$
	$\cdots$	$\cdots$	$\cdots$
	$i_{n-1} = 0, i_1 = i_2 = \cdots = i_{n-2} = 1$	$i_{1,1,1,\cdots,0}$	$(\lambda_{1n} + \cdots + \lambda_{n-2\,n}) x_1 \cdots x_{n-2} \hat{x}_{n-1} x_n$
n	$i_1 = i_2 = \cdots = i_{n-2} = i_{n-1} = 1$	$i_{1,1,\cdots,1,1}$	$(\lambda_{1\,n} + \cdots + \lambda_{n-1\,n}) x_1 \cdots x_n$

Table 2. This table shows different form in image with the different value in $j_1, j_2, \cdots, j_n$.

Image length	value	Denoted by	image
2	$j_1 = 1, j_2 = \cdots = j_{n-2} = j_n = 0$	$j_{1,0,0,\cdots,0}$	$\lambda_{1\,n-1} x_1 x_{n-1}$
	$j_2 = 1, j_1 = \cdots = j_{n-2} = j_n = 0$	$j_{0,1,0,\cdots,0}$	$\lambda_{2\,n-1} x_2 x_{n-1}$
	$\cdots$	$\cdots$	$\cdots$
	$j_n = 1, j_1 = \cdots = j_{n-2} = 0$	$j_{0,0,\cdots,1,0}$	$\lambda_{n\,n-1} x_{n-1} x_n$
3	$j_1 = j_2 = 1, j_3 = \cdots = j_n = 0$	$j_{1,1,0,\cdots,0}$	$(\lambda_{1n-1} + \lambda_{2n-1}) x_1 x_2 x_{n-1}$
	$j_1 = j_2 = 1, j_3 = \cdots = j_n = 0$	$j_{1,0,1,\cdots,0}$	$(\lambda_{1n-1} + \lambda_{3n-1}) x_1 x_3 x_{n-1}$
	$\cdots$	$\cdots$	$\cdots$
	$j_{n-2} = j_n = 1, j_1 = \cdots = j_{n-3} = 0$	$j_{0,0,\cdots,1,1}$	$(\lambda_{n-2\,n-1} + \lambda_{n\,n-1}) x_{n-2} x_{n-1} x_n$
		$\cdots$	
$n-1$	$j_1 = 0, j_2 = j_3 = \cdots = j_n = 1$	$j_{0,1,1,\cdots,1}$	$(\lambda_{2n-1} + \cdots + \lambda_{n\,n-1}) \hat{x}_1 x_2 \cdots x_{n-1} x_n$
	$j_2 = 0, j_1 = j_3 = \cdots = j_n = 1$	$j_{1,0,1,\cdots,1}$	$(\lambda_{1n-1} + \cdots + \lambda_{n\,n-1}) x_1 \hat{x}_2 \cdots x_{n-1} x_n$
	$\cdots$	$\cdots$	$\cdots$
	$j_n = 0, j_1 = j_2 = \cdots = j_{n-2} = 1$	$j_{1,1,1,\cdots,0}$	$(\lambda_{1n-1} + \cdots + \lambda_{n-2\,n-1}) x_1 x_2 \cdots x_{n-1} \hat{x}_n$
n	$j_1 = j_2 = \cdots = j_{n-2} = j_n = 1$	$j_{1,1,\cdots,1,1}$	$(\lambda_{1\,n-1} + \cdots + \lambda_{n\,n-1}) x_1 \cdots x_n$

$$
\begin{gathered}
x_1 \mathrm{d}x_2 + x_2 \mathrm{d}x_1 \in \ker \partial_1^\pi \\
\vdots \\
x_1 \mathrm{d}x_n + x_n \mathrm{d}x_1 \in \ker \partial_1^\pi \\
x_2 \mathrm{d}x_3 + x_3 \mathrm{d}x_2 \in \ker \partial_1^\pi \\
\vdots \\
x_{n-1} \mathrm{d}x_n + x_n \mathrm{d}x_{n-1} \in \ker \partial_1^\pi
\end{gathered}
$$

Obviously, for $\forall i<j$, the preimage of $x_i \mathrm{d}x_j + x_j \mathrm{d}x_i$ is $\frac{1}{\lambda_{ij}} \mathrm{d}x_i \wedge \mathrm{d}x_j$ under the map ∂_2^π.

2) The preimage of image with the length of 2

$$\sum C_{ijk} x_i x_j \mathrm{d}x_k \mapsto \sum C_{ijk} \left(\lambda_{ik} + \lambda_{jk}\right) x_i x_j x_k$$

Let $\sum C_{ijk} x_i x_j \mathrm{d}x_k \in \ker \partial_1^\pi$, then $\sum C_{ijk} \left(\lambda_{ik} + \lambda_{jk}\right) x_i x_j x_k = 0$. we can get $C_{ijk}\left(\lambda_{ik} + \lambda_{jk}\right) + C_{jki}\left(\lambda_{ji} + \lambda_{ki}\right) = 0$ for $\forall i<j<k$, that is $C_{ijk} = \frac{\lambda_{ik} - \lambda_{ji}}{\lambda_{ik} + \lambda_{jk}} C_{jki}$.

we have:

$$\frac{\lambda_{1n} - \lambda_{21}}{\lambda_{1n} + \lambda_{2n}} x_1 x_2 \mathrm{d}x_n + x_2 x_n \mathrm{d}x_1 \in \ker \partial_1^\pi$$
$$\vdots$$
$$\frac{\lambda_{n-1\,n} - \lambda_{1\,n-1}}{\lambda_{1n} + \lambda_{n-1\,n}} x_1 x_{n-1} \mathrm{d}x_n + x_1 x_n \mathrm{d}x_{n-1} \in \ker \partial_1^\pi$$

Obviously, for $\forall i<j<k$, the preimage of $\frac{\lambda_{ik} - \lambda_{ji}}{\lambda_{ik} + \lambda_{jk}} x_i x_j \mathrm{d}x_k + x_j x_k \mathrm{d}x_i$ is $-x_j \mathrm{d}x_i \wedge \mathrm{d}x_k$ under the map ∂_2^π.

3) Generally, the preimage of image with the length of $m\,(1<m<n-1)$

$$\sum C_{i_1 \cdots i_{m-1} i} x_{i_1} \cdots x_{i_{m-1}} x_i \mathrm{d}x_j \mapsto \sum C_{i_1 \cdots i_{m-1} i} \left(\lambda_{i_1 j} + \cdots + \lambda_{ij}\right) x_{i_1} \cdots x_{i_{m-1}} x_i x_j$$

Let $\sum C_{i_1 \cdots i_{m-1} i} x_{i_1} \cdots x_{i_{m-1}} x_i \mathrm{d}x_j \in \ker \partial_1^\pi$, then
$\sum C_{i_1 \cdots i_{m-1} i} \left(\lambda_{i_1 j} + \cdots + \lambda_{ij}\right) x_{i_1} \cdots x_{i_{m-1}} x_i x_j = 0$.

we have:

$$\begin{aligned} &-\left(\lambda_{i_1 i} + \lambda_{i_2 i} + \cdots + \lambda_{i_{m-1} i} + \lambda_{ij}\right) x_{i_1} x_{i_2} \cdots x_{i_{m-1}} x_i \mathrm{d}x_j \\ &+\left(\lambda_{i_1 j} + \lambda_{i_2 j} + \cdots + \lambda_{i_{m-1} j} + \lambda_{ij}\right) x_{i_1} x_{i_2} \cdots x_{i_{m-1}} x_j \mathrm{d}x_i \in \ker \partial_1^\pi \end{aligned}$$

Obviously, under the map ∂_2^π, for $\forall i<j$, the preimage of

$$\begin{aligned} &-\left(\lambda_{i_1 i} + \lambda_{i_2 i} + \cdots + \lambda_{i_{m-1} i} + \lambda_{ji}\right) x_{i_1} x_{i_2} \cdots x_{i_{m-1}} x_i \mathrm{d}x_j \\ &+\left(\lambda_{i_1 j} + \lambda_{i_2 j} + \cdots + \lambda_{i_{m-1} j} + \lambda_{ij}\right) x_{i_1} x_{i_2} \cdots x_{i_{m-1}} x_j \mathrm{d}x_i \end{aligned}$$ is $-x_{i_1} x_{i_2} \cdots x_{i_{m-1}} \mathrm{d}x_i \wedge \mathrm{d}x_j$.

(4)The preimage of image with the length of $n-1$

$$\sum C_{1 \cdots \hat{k} \cdots n} x_1 \cdots \hat{x}_k \cdots x_n \mathrm{d}x_k \mapsto \sum C_{1 \cdots \hat{k} \cdots n} \left(\lambda_{1k} + \cdots + \lambda_{nk}\right) x_1 \cdots x_k \cdots x_n$$

Let $\sum C_{1 \cdots \hat{k} \cdots n} x_1 \cdots \hat{x}_k \cdots x_n \mathrm{d}x_k \in \ker \partial_1^\pi$, then
$\sum C_{1 \cdots \hat{k} \cdots n} \left(\lambda_{1k} + \cdots + \lambda_{nk}\right) x_1 \cdots x_k \cdots x_n = 0$.

we have:

$$\frac{\lambda_{1n} + \lambda_{2n} + \cdots + \lambda_{n-1\,n}}{\lambda_{12} + \lambda_{13} + \cdots + \lambda_{1\,n}} x_2 x_3 \cdots x_n \mathrm{d}x_1 + x_1 x_2 \cdots x_{n-1} \mathrm{d}x_n \in \ker \partial_1^\pi$$
$$\vdots$$
$$\frac{\lambda_{1n} + \cdots \lambda_{jn} + \cdots + \lambda_{kn} + \cdots + \lambda_{n-1\,n}}{\lambda_{12} + \cdots \lambda_{1j} + \cdots + \lambda_{1k} + \cdots + \lambda_{1\,n}} x_1 \cdots \hat{x}_k \cdots x_n \mathrm{d}x_k + x_1 \cdots \hat{x}_j \cdots x_n \mathrm{d}x_j \in \ker \partial_1^\pi$$

Obviously, under the map ∂_2^π, for $\forall 1\le j<k\le n$, the preimage of

$$\left(\lambda_{1n}+\cdots\lambda_{jn}+\cdots+\lambda_{kn}+\cdots+\lambda_{n-1\,n}\right)x_1\cdots\widehat{x}_k\cdots x_n\mathrm{d}x_k$$
$$+\left(\lambda_{12}+\cdots\lambda_{1j}+\cdots+\lambda_{1k}+\cdots+\lambda_{1\,n}\right)x_1\cdots\widehat{x}_j\cdots x_n\mathrm{d}x_j$$

is

$$-x_1x_2\cdots\widehat{x}_j\cdots\widehat{x}_k\cdots x_n\mathrm{d}x_j\wedge\mathrm{d}x_k.$$

In conclusion, ∂_2^π is surjection. So we have the 1-th homology group

$$HP_1(\Lambda)=\frac{\ker\partial_1^\pi}{\operatorname{Im}\partial_2^\pi}=0.$$

The proof is completed.

Now, we write the general condition in Proposition 4.3, the process of proof is similar with Proposition 4.2.

Proposition 4.3. $HP_p(\Lambda)=0,\ p(1\le p\le n)$.

Proposition 4.4. $HP_n(\Lambda)=0$

Proof: lastly, we calculate the n-th homology group $HP_n(\Lambda)=\dfrac{\ker\partial_n^\pi}{\operatorname{Im}\partial_{n+1}^\pi}$

$$\Omega_{n+1}(\Lambda)\xrightarrow{\partial_{n+1}^\pi}\Omega_n(\Lambda)\xrightarrow{\partial_n^\pi}\Omega_{n-1}(\Lambda)$$

Firstly, we calculate $\ker\partial_n^\pi$,

$$\begin{aligned}
&\partial_n^\pi:\Omega_n(\Lambda)\to\Omega_{n-1}(\Lambda)\\
&\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\mapsto\sum_{1\le i<j<\cdots<k\le n}(-1)^{i+j}\mathrm{d}\{F_i,F_j\}\wedge\mathrm{d}F_1\wedge\cdots\wedge\mathrm{d}\hat{F}_i\\
&\wedge\cdots\wedge\mathrm{d}\hat{F}_j\wedge\cdots\wedge\mathrm{d}\hat{F}_k\wedge\cdots\wedge\mathrm{d}F_n\\
&=(-1)^{1+2}\mathrm{d}\{x_1,x_2\}\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n+(-1)^{1+3}\mathrm{d}\{x_1,x_3\}\wedge\mathrm{d}x_2\wedge\mathrm{d}x_4\\
&\ \wedge\cdots\wedge\mathrm{d}x_n+\cdots+(-1)^{(n-1)+n}\mathrm{d}\{x_{n-1},x_n\}\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_{n-2}
\end{aligned}$$

the image is:

$$\begin{aligned}
&(-1)^{1+2}\mathrm{d}\{x_1,x_2\}\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n+(-1)^{1+3}\mathrm{d}\{x_1,x_3\}\wedge\mathrm{d}x_2\wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n\\
&+\cdots+(-1)^{(n-1)+n}\mathrm{d}\{x_{n-1},x_n\}\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_{n-2}\\
&=-\mathrm{d}\{x_1,x_2\}\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n+\mathrm{d}\{x_1,x_3\}\wedge\mathrm{d}x_2\wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n\\
&\ +\cdots-\mathrm{d}\{x_{n-1},x_n\}\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_{n-2}\\
&=-\lambda_{12}(x_1\mathrm{d}x_2+x_2\mathrm{d}x_1)\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n+\lambda_{13}(x_1\mathrm{d}x_3+x_3\mathrm{d}x_1)\wedge\mathrm{d}x_2\\
&\ \wedge\mathrm{d}x_4\wedge\cdots\wedge\mathrm{d}x_n+\cdots-\lambda_{n-1\,n}(x_{n-1}\mathrm{d}x_n+x_n\mathrm{d}x_{n-1})\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_{n-2}
\end{aligned}$$

We can easily see that this element can not be 0, that is $\ker\partial_n^\pi=0$, so

$$HP_n(\Lambda)=\frac{\ker\partial_n^\pi}{\operatorname{Im}\partial_{n+1}^\pi}=0$$

The proof is completed.

In conclusion, we have the homology group of Λ $HP_i(\Lambda)$

$$HP_i(\Lambda)=\begin{cases}k\oplus kx_1\oplus\cdots\oplus kx_n, & (i=0)\\ 0, & (i\ge 1)\end{cases}\tag{4}$$

5. Modular Derivation and Twisted Poisson

In this part, we should calculate the Frobenius modular derivation and get the twisted Poisson structure. Firstly, we need get the Frobenius isomorphism and Frobenius pairing.

5.1. Frobenius Isomorphism and Frobenius Pairing

The Frobenius algebra Λ has dimension 2^n, with basis $\{1\}\cup\{x_{i_1}\cdots x_{i_k} \mid 1\le i_1<\cdots<i_k<n\}$. The dual space Λ^* has the basis $\{1\}\cup\{x_{i_1}^*\cdots x_{i_k}^* \mid 1\le i_1<\cdots<i_k<n\}$, satisfied

$$\begin{aligned} x_i^*:\Lambda &\to k \\ x_i &\mapsto 1 \\ x_j &\mapsto 0\left(j\ne i\right) \end{aligned}$$

Frobenius isomorphism $\sigma:\Lambda\to\Lambda^*$ is given by

$$\sigma\left(1\right):\sum c_{i_1,\cdots,i_n}x_{i_1}\cdots x_{i_n}\mapsto c_{i_1,\cdots,i_n}.$$

Choosing a basis $\left\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\middle|0\le i_1,i_2,\cdots,i_n\le 1\right\}$.

$$\begin{aligned} \sigma:\Lambda &\to \Lambda^* \\ 1 &\mapsto \sigma\left(1\right). \end{aligned}$$

We can easily get the Frobenius paring is

$$\left\langle x_1^{q_1}\cdots x_n^{q_n},x_1^{p_1}\cdots x_n^{p_n}\right\rangle=\begin{cases}1, & q_i+p_i=1,\forall i\\ 0, & \text{else}\end{cases}$$

5.2. Twisted Poisson Module

Now we calculate the Modular derivation $D\left(x_i\right)$

$$\begin{aligned} D\left(x_i\right) &= \left\{1,x_i\right\}_D \\ &= \sigma^{-1}\left(\left\{\sigma\left(1\right),x_i\right\}_{A^*}\right) \\ &= \sigma^{-1}\left(\sum 1+k_ix_i^*+k_{ij}x_i^*x_j^*+\cdots+k_{1\cdots n}x_1^*\cdots x_n^*\right). \end{aligned}$$

Compute it by each item

$$\left\{\sigma\left(1\right),x_i\right\}\left(1\right)=\sigma\left(1\right)\left\{x_i,1\right\}=0 \tag{5}$$

$$\begin{aligned} \left\{\sigma\left(1\right),x_i\right\}\left(x_j\right) &= \sigma\left(1\right)\left\{x_i,x_j\right\} \\ &= \sigma\left(1\right)\left(\lambda_{ij}x_ix_j\right)=0 \end{aligned} \tag{6}$$

$$\cdots$$

$$\left\{\sigma\left(1\right),x_i\right\}\left(x_1\cdots\hat{x}_j\cdots x_{n-1}\right)=\sigma\left(1\right)\left\{x_i,x_1\cdots\hat{x}_j\cdots x_{n-1}\right\}=0 \tag{7}$$

$$\left\{\sigma\left(1\right),x_i\right\}\left(x_1\cdots\hat{x}_j\cdots x_n\right)=\sigma\left(1\right)\left\{x_i,x_1\cdots\hat{x}_j\cdots x_n\right\} \tag{8}$$

By the definition of $\sigma\left(1\right)$, Equations (5)-(7) equal to 0. Then we discuss (8). when $i=j$,

$$
\begin{aligned}
&\left\{x_i, x_1 \cdots \hat{x}_i \cdots x_n\right\} \\
&= x_2 x_3 \cdots \hat{x}_i \cdots x_n \left\{x_i, x_1\right\} + x_1 x_3 \cdots \hat{x}_i \cdots x_n \left\{x_i, x_2\right\} + \cdots \\
&\quad + x_1 x_2 \cdots x_{i-2} \hat{x}_i \cdots x_n \left\{x_i, x_{i-1}\right\} + x_1 x_2 \cdots \hat{x}_i x_{i+2} \cdots x_n \left\{x_i, x_{i+1}\right\} \\
&\quad + x_1 x_2 \cdots \hat{x}_i x_{i+1} x_{i+3} \cdots x_n \left\{x_i, x_{i+2}\right\} + \cdots + x_1 x_2 \cdots \hat{x}_i \cdots x_{n-1} \left\{x_i, x_n\right\} \\
&= \left(\left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right)\right) x_1 \cdots x_i \cdots x_n .
\end{aligned}
$$

We have

$$
\begin{aligned}
&\left\{\sigma(1), x_i\right\}\left(x_1 \cdots \hat{x}_i \cdots x_n\right) = \left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right) \\
&\Rightarrow \left\{\sigma(1), x_i\right\}_{A^*} = \left(\left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right)\right) x_1^* \cdots \hat{x}_i^* \cdots x_n^* .
\end{aligned}
$$

We can get

$$
\begin{aligned}
D(x_i) &= \left\{1, x_i\right\}_D \\
&= \sigma^{-1}\left\{\left(\left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right)\right) x_1^* \cdots \hat{x}_i^* \cdots x_n^*\right\} \quad (9) \\
&= \left(\left(\lambda_{i\,i+1} + \lambda_{i\,i+2} + \cdots + \lambda_{i\,n}\right) - \left(\lambda_{1i} + \lambda_{2i} + \cdots + \lambda_{i-1\,i}\right)\right) x_i .
\end{aligned}
$$

Then we give the twisted Poisson module of Λ

$$
\left\{x_i, x_j\right\}_{\Lambda^D} = \begin{cases} \left(\left(\lambda_{j\,j+1} + \cdots + \lambda_{j\,n}\right) - \left(\lambda_{1j} + \lambda_{2j} + \cdots + \lambda_{i-1\,j} + \lambda_{i+1\,j} + \cdots + \lambda_{j-1\,j}\right)\right) x_i x_j, & (i < j) \\ \left(\left(\lambda_{j\,j+1} + \cdots + \lambda_{j\,i-1} + \lambda_{j\,i+1} + \cdots + \lambda_{j\,n}\right) - \left(\lambda_{1j} + \lambda_{2j} + \cdots + \lambda_{j-1\,j}\right)\right) x_i x_j, & (i > j) \\ 0, & (i = j) \end{cases} \quad (10)
$$

6. Twisted Poisson Homology

In this part, we calculate the twisted Poisson homology after we get the new poisson module structure $\left\{x_i, x_j\right\}_{\Lambda^D}$. give a right Poisson module Λ^D, we have the new canonical chain complex

$$
\begin{aligned}
&\cdots \longrightarrow \Lambda^D \otimes_\Lambda \Omega^p(\Lambda) \xrightarrow{\partial_p^\pi} \Lambda^D \otimes_\Lambda \Omega^{p-1}(\Lambda) \xrightarrow{\partial_{p-1}^\pi} \cdots \\
&\xrightarrow{\partial_2^\pi} \Lambda^D \otimes_\Lambda \Omega^1(\Lambda) \xrightarrow{\partial_1^\pi} \Lambda^D \otimes_\Lambda \Lambda \xrightarrow{\partial_0^\pi} 0 .
\end{aligned} \quad (11)
$$

where $p \geq 1$, $\partial_p^\pi : \Lambda^D \otimes_\Lambda \Omega^p(\Lambda) \to \Lambda^D \otimes_\Lambda \Omega^{p-1}(\Lambda)$ is defined by:

$$
\begin{aligned}
&\partial_p^\pi\left(m \otimes \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_p\right) \\
&= \sum_{i=1}^{p} (-1)^{i-1} \left\{m, a_i\right\}_{\Lambda^D} \otimes \mathrm{d}a_1 \wedge \cdots \widehat{\mathrm{d}a_i} \cdots \wedge \mathrm{d}a_p \\
&\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} m \otimes \mathrm{d}\left\{a_i, a_j\right\} \wedge \mathrm{d}a_1 \wedge \cdots \widehat{\mathrm{d}a_i} \cdots \widehat{\mathrm{d}a_j} \cdots \wedge \mathrm{d}a_p .
\end{aligned}
$$

Then we calculate the twisted Poisson homology group and get some results.

6.1. 0-th Twisted Poisson Homology

$$
\begin{aligned}
&\partial_1^\pi : \Omega_1(\Lambda) \to \Lambda \\
&\sum x_1^{i_1} \cdots x_j^0 \cdots x_n^{i_n} \mathrm{d}x_j \mapsto \left\{x_1^{i_1} \cdots x_j^0 \cdots x_n^{i_n}, x_j\right\}_{\Lambda^D} .
\end{aligned}
$$

we will prove that we can find preimage in $\Omega_1(\Lambda)$ for all elements when $i_k \geq 1$

1) The preimage of image with the length of 1

$$\frac{1}{\left(\left(\lambda_{i\,i+1}+\cdots+\lambda_{1\,n}\right)-\left(\lambda_{1\,i}+\lambda_{2\,i}\cdots+\lambda_{i-1\,i}\right)\right)}\mathrm{d}x_i \mapsto x_i,\left(1\leq i\leq n\right).$$

2) The preimage of image with the length of 2

$$\frac{1}{\left(\lambda_{j\,j+1}+\cdots+\lambda_{j\,n}\right)-\left(\lambda_{1\,j}+\cdots+\lambda_{i-1\,j}+\lambda_{i+1\,j}+\cdots+\lambda_{j-1\,j}\right)}x_i\mathrm{d}x_j \mapsto x_ix_j,\left(i<j\right).$$

3) The preimage of image with the length of 3

$$\frac{1}{2\left(\lambda_{pp+1}+\cdots+\lambda_{pn}\right)-\left(\lambda_{1p}+\cdots+\lambda_{j-1p}+\lambda_{j+1p}+\cdots+\lambda_{p-1p}\right)-\left(\lambda_{1p}+\cdots+\lambda_{i-1p}+\lambda_{i+1p}+\cdots+\lambda_{p-1p}\right)}x_ix_j\mathrm{d}x_p$$
$$\mapsto x_ix_jx_p,\left(\forall 1\leq i<j<p\leq n\right)$$

......

4) The preimage of image with the length of n

$$\frac{x_1x_2\cdots x_{n-1}}{-\left(\lambda_{2n}+\cdots+\lambda_{n-1n}\right)-\cdots-\left(\lambda_{1n}+\cdots+\lambda_{n-2n}\right)}\mathrm{d}x_n \mapsto x_1x_2\cdots x_{n-1}x_n.$$

Hence, we can find preimage of the image $x_1\cdots x_m\left(1\leq m\leq n\right)$ in $\Omega_1(\Lambda)$, the preimage is

$$\frac{x_1x_2\cdots x_{m-1}}{(m-1)\left(\lambda_{m\,m+1}+\cdots+\lambda_{mn}\right)-\left(\lambda_{2m}+\cdots+\lambda_{m-1m}\right)-\cdots-\left(\lambda_{1m}+\cdots+\lambda_{m-2m}\right)}\mathrm{d}x_m \mapsto x_1\cdots x_m.$$

But for the constant term k (with the length of 0), we can not find the preimage. So we can get 0-th twisted homology group

$$HP_0\left(\Lambda,\Lambda^D\right)=\frac{\ker\partial_0^\pi}{\operatorname{Im}\partial_1^\pi}=k.$$

6.2. 1-th Twisted Poisson Homology

$$\partial_1^\pi:\Omega_1(\Lambda)\to\Lambda$$
$$\sum_{j=1}^n Cx_1^{i_1}\cdots x_j^0\cdots x_n^{i_n}\mathrm{d}x_j \mapsto \left\{x_1^{i_1}x_2^{i_2}\cdots x_j^0\cdots x_n^{i_n},x_j\right\}_{\Lambda^D}.$$

the mapping between the elements is

$$\begin{aligned}&Cx_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}}\mathrm{d}x_n+x_1^{j_1}x_2^{j_2}\cdots x_{n-2}^{j_{n-2}}x_n^{j_n}\mathrm{d}x_{n-1}+\cdots+x_2^{k_2}x_3^{k_3}\cdots x_{n-1}^{k_{n-1}}x_n^{k_n}\mathrm{d}x_1\\&\mapsto k_{i_1,\cdots,i_n}\left\{x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}},x_n\right\}_{\Lambda^D}+k_{j_1,\cdots,j_n}\left\{x_1^{j_1}x_2^{j_2}\cdots x_{n-2}^{j_{n-2}}x_n^{j_n},x_{n-1}\right\}_{\Lambda^D}+\cdots\\&+k_{p_1,\cdots,p_n}\left\{x_1^{p_1}x_3^{p_3}\cdots x_{n-1}^{p_{n-1}}x_n^{p_n},x_2\right\}_{\Lambda^D}k_{p_1,\cdots,p_n}+\left\{x_2^{p_2}x_3^{p_3}\cdots x_{n-1}^{p_{n-1}}x_n^{p_n},x_1\right\}_{\Lambda^D}.\end{aligned}$$

We have n preimage, and each of them has the length of $k\left(1\leq k\leq n-1\right)$, and the image has $k+1$ in length. Similarly, we can calculate the image in each item like in part 3, we do not repeat.

Because the Poisson structure of Λ is homogeneous. We can discuss the image of ∂_1^π by length.

1) The preimage of image with the length of 1

$$\sum C_{ij} x_i \mathrm{d}x_j \mapsto \sum C_{ij} \left(\left(\lambda_{j\,j+1} + \cdots + \lambda_{j\,j+n} \right) - \left(\lambda_{1\,j} + \cdots + \lambda_{i-1\,j} + \lambda_{i+1\,j} + \lambda_{j-1\,j} \right) \right) x_i x_j$$

Let $\sum C_{ij} x_i \mathrm{d}x_j \in \ker \partial_1^\pi$, then

$$\sum C_{ij} \left(\left(\lambda_{j\,j+1} + \cdots + \lambda_{j\,j+n} \right) - \left(\lambda_{1\,j} + \cdots + \lambda_{i-1\,j} + \lambda_{i+1\,j} + \lambda_{j-1\,j} \right) \right) x_i x_j = 0 .$$

for $\forall i < j$, $C_{ij} \dfrac{\lambda_{j\,j+1} + \cdots + \lambda_{ji-1} + \cdots + \lambda_{ji+1} + \cdots + \lambda_{jn} - \left(\lambda_{1j} + \cdots + \lambda_{j-1\,j} \right)}{\lambda_{j\,j+1} + \cdots + \lambda_{j\,n} - \left(\lambda_{1\,j} + \cdots + \lambda_{i-1\,j} + \cdots + \lambda_{i+1\,j} + \cdots + \lambda_{j-1\,j} \right)} = -C_{ji}$.

We have

$$\frac{\lambda_{12} + \cdots + \lambda_{1\,n-1}}{\lambda_{2\,n} + \cdots + \lambda_{n-1\,n}} x_1 \mathrm{d}x_n + x_n \mathrm{d}x_1 \in \ker \partial_1^\pi$$

$$\vdots$$

$$\frac{\left(\lambda_{j\,j+1} + \cdots + \lambda_{ji-1} + \cdots + \lambda_{ji+1} + \cdots + \lambda_{jn} \right) - \left(\lambda_{1j} + \cdots + \lambda_{j-1\,j} \right)}{\lambda_{j\,j+1} + \cdots + \lambda_{j\,n} - \left(\lambda_{1\,j} + \cdots + \lambda_{i-1\,j} + \cdots + \lambda_{i+1\,j} + \cdots + \lambda_{j-1\,j} \right)} x_i \mathrm{d}x_j + x_j \mathrm{d}x_i \in \ker \partial_1^\pi$$

$$\vdots$$

$$\frac{\left(\lambda_{1\,i} + \cdots + \lambda_{i-1\,i} \right) - \lambda_{i\,i+1} + \cdots + \lambda_{i\,n-1}}{\lambda_{1\,n} + \cdots + \lambda_{i-1\,n} + \cdots + \lambda_{i+1\,n} + \lambda_{n-1\,n}} x_i \mathrm{d}x_n + x_n \mathrm{d}x_i \in \ker \partial_1^\pi$$

We can see that the image $x_1^{i_1} x_2^{i_2} \cdots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}} \mathrm{d}x_n$ and $x_1^{i_1} x_2^{i_2} \cdots x_{n-2}^{i_{n-2}} x_n^{i_n} \mathrm{d}x_{n-1}$ in $\Omega_2(\Lambda)$ has the same coefficient, it is conflict with our results. So the image with length 1 in $\ker \partial_1^\pi$ can not find the preimage in $\Omega_2(\Lambda)$. In the same time, the constant term k (with the length of 0) also can not find the preimage. So we can get 1-th twisted homology group.

2) The preimage of image with the length of 2

$$\sum C_{ijk} x_i x_j \mathrm{d}x_k \mapsto \sum C_{ijk} \Big(2\left(\lambda_{k\,k+1} + \cdots + \lambda_{k\,k+n} \right) - 2\big(\lambda_{1\,k} + \cdots + \widehat{\lambda}_{ik}$$
$$+ \widehat{\lambda}_{jk} + \cdots + \lambda_{k-1\,j} \big) - \lambda_{ik} - \lambda_{jk} \Big) x_i x_j x_k$$

Let $\sum C_{ijk} x_i x_j \mathrm{d}x_k \in \ker \partial_1^\pi$, then

$$\sum C_{ijk} \left(2\left(\lambda_{k\,k+1} + \cdots + \lambda_{k\,k+n} \right) - 2\left(\lambda_{1\,k} + \cdots + \widehat{\lambda}_{ik} + \widehat{\lambda}_{jk} + \cdots + \lambda_{k-1\,j} \right) - \lambda_{ik} - \lambda_{jk} \right) x_i x_j x_k$$
$$= 0 .$$

for $\forall i < j < k$, that is

$$C_{jik} = -\frac{\left(2\left(\lambda_{k\,k+1} + \cdots + \lambda_{k\,n} \right) - 2\left(\lambda_{1\,k} + \cdots + \widehat{\lambda}_{ik} + \widehat{\lambda}_{jk} + \cdots + \lambda_{k-1\,k} \right) - \lambda_{ik} - \lambda_{jk} \right)}{\left(2\left(\lambda_{k\,k+1} + \cdots + \lambda_{k\,n} \right) - 2\left(\lambda_{1\,k} + \cdots + \widehat{\lambda}_{jk} + \widehat{\lambda}_{ik} + \cdots + \lambda_{k-1\,k} \right) - \lambda_{jk} - \lambda_{ik} \right)} C_{ijk} .$$

We have

$$\frac{\lambda_{1\,2} + \lambda_{13} \cdots + \lambda_{1\,n-1}}{\lambda_{1\,n} + \cdots + \lambda_{n-1\,n}} x_1 x_2 \mathrm{d}x_n + x_2 x_n \mathrm{d}x_1 \in \ker \partial_1^\pi$$

$$\vdots$$

$$-\frac{\lambda_{2\,n-1} + \cdots + \lambda_{n-2\,n-1}}{\lambda_{2\,n} + \cdots + \lambda_{n-1\,n}} x_1 x_{n-1} \mathrm{d}x_n + x_1 x_n \mathrm{d}x_{n-1} \in \ker \partial_1^\pi$$

Obviously, for $\forall i < j < k$, the preimage of this image is $-x_j \mathrm{d}x_i \wedge \mathrm{d}x_k$ under the map ∂_2^π.

3) Generally, the preimage of image with the length of $m\,(1<m<n-1)$

$$\begin{aligned}&\sum C_{i_1\cdots i_{m-1}\,i}x_{i_1}x_{i_2}\cdots x_{i_{m-1}}x_i\mathrm{d}x_j\\&\mapsto\sum C_{i_1\cdots i_{m-1}\,i}\Big(2\big(\lambda_{j\,j+1}+\cdots+\lambda_{j\,j+n}\big)-2\big(\lambda_{1\,j}+\cdots+\widehat{\lambda}_{i_1j}+\widehat{\lambda}_{i_2j}\\&+\cdots+\lambda_{j-1\,i_m}+\lambda_{j-1\,i}\big)-\lambda_{i_1j}-\cdots-\lambda_{ij}\Big)x_{i_1}\cdots x_{i_{m-1}}x_ix_j\end{aligned}$$

Let $\sum C_{i_1\cdots i_{m-1}\,i}x_{i_1}x_{i_2}\cdots x_{i_{m-1}}x_i\mathrm{d}x_j\in\ker\partial_1^\pi$, then

$$\begin{aligned}&\sum C_{i_1\cdots i_{m-1}\,i}\Big(2\big(\lambda_{j\,j+1}+\cdots+\lambda_{j\,j+n}+\lambda_{1\,j}+\cdots+\widehat{\lambda}_{i_1j}+\widehat{\lambda}_{i_2j}\\&+\cdots+\lambda_{j-1\,i_m}+\lambda_{j-1\,i}\big)-\lambda_{i_1j}-\cdots-\lambda_{ij}\Big)x_{i_1}\cdots x_{i_{m-1}}x_ix_j=0\end{aligned}$$

We have,

$$\begin{aligned}&x_{i_1}\cdots x_{i_{m-1}}x_i\mathrm{d}x_j\\&+\frac{2\big(\lambda_{i\,i+1}+\cdots+\lambda_{i\,i+n}\lambda_{1\,i}+\cdots+\lambda_{i-1\,i_m}+\lambda_{i-1\,j}\big)-\lambda_{i_1i}-\cdots-\lambda_{ji}}{2\big(\lambda_{j\,j+1}+\cdots+\lambda_{j\,j+n}\lambda_{1\,j}+\cdots+\lambda_{j-1\,i_m}+\lambda_{j-1\,i}\big)-\lambda_{i_1j}-\cdots-\lambda_{ij}}x_{i_1}\cdots x_{i_{m-1}}x_j\mathrm{d}x_i\in\ker\partial_1^\pi\end{aligned}$$

for $\forall i_1<\cdots<i_{m-1}<i<j$, the preimage of this element is $-x_{i_1}x_{i_2}\cdots x_{i_{m-1}}\mathrm{d}x_i\wedge\mathrm{d}x_j$.

4) The preimage of image with the length of $n-1$

$$\sum C_1x_1\cdots x_{n-1}\mathrm{d}x_n\mapsto\sum C_1\Big(-(n-2)\big(\lambda_{1\,n}+\cdots+\lambda_{n-1\,1}\big)-\lambda_{1n}-\cdots-\lambda_{n-1n}\Big)x_1\cdots x_n$$

Let $\sum C_1x_1\cdots x_{n-1}\mathrm{d}x_n\in\ker\partial_1^\pi$,

then

$$\begin{aligned}&C_1\Big(-(n-2)\big(\lambda_{1\,n}+\cdots+\lambda_{n-1\,1}\big)-\lambda_{1n}-\cdots-\lambda_{n-1n}\Big)x_1\cdots x_n\\&+C_n\Big(-(n-2)\big(\lambda_{n1}+\cdots+\lambda_{1n-1}\big)-\lambda_{n1}-\cdots-\lambda_{n-1n}\Big)x_1\cdots x_n=0\end{aligned}.$$

We have

$$\begin{aligned}&-\frac{\lambda_{1n}+\lambda_{2n}+\cdots+\lambda_{n-1\,n}}{\lambda_{12}+\lambda_{13}+\cdots+\lambda_{1\,n}}x_2x_3\cdots x_n\mathrm{d}x_1+x_1x_2\cdots x_{n-1}\mathrm{d}x_n\in\ker\partial_1^\pi\\&\qquad\vdots\\&-\frac{\lambda_{1n}+\cdots\lambda_{jn}+\cdots+\lambda_{kn}+\cdots+\lambda_{n-1\,n}}{\lambda_{12}+\cdots\lambda_{1j}+\cdots+\lambda_{1k}+\cdots+\lambda_{1\,n}}x_1\cdots\widehat{x}_k\cdots x_nx_j\mathrm{d}x_k+x_1\cdots\widehat{x}_j\cdots x_nx_k\mathrm{d}x_j\in\ker\partial_1^\pi\end{aligned}$$

for $\forall 1\le j<k\le n$, the preimage of this element is

$$\begin{aligned}&x_2x_3\cdots x_{n-1}\mathrm{d}x_n\wedge\mathrm{d}x_1-\frac{\lambda_{1n}+\lambda_{2n}+\cdots+\lambda_{n-1\,n}}{\lambda_{12}+\lambda_{13}+\cdots+\lambda_{1\,n}}x_2x_3\cdots x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_n\\&\qquad\vdots\\&-x_1\cdots\widehat{x}_j\cdots\widehat{x}_k\cdots x_n\mathrm{d}x_j\wedge\mathrm{d}x_k-\frac{\lambda_{1k}+\lambda_{2k}+\cdots+\lambda_{n-1\,k}}{\lambda_{j2}+\lambda_{j3}+\cdots+\lambda_{j\,n}}-x_1\cdots\widehat{x}_j\cdots\widehat{x}_k\cdots x_n\mathrm{d}x_j\wedge\mathrm{d}x_k\end{aligned}$$

In conclusion, ∂_2^π is not surjection. So we have the 1-th homology group

$$HP_1\left(\Lambda,\Lambda^D\right)=\frac{\ker\partial_1^\pi}{\operatorname{Im}\partial_2^\pi}=\begin{cases} k \\ \dfrac{\lambda_{12}+\cdots+\lambda_{1\,n-1}}{\lambda_{2\,n}+\cdots+\lambda_{n-1\,n}}x_1\mathrm{d}x_n+x_n\mathrm{d}x_1 \\ \dfrac{\lambda_{21}+\cdots+\lambda_{2\,n-1}}{\lambda_{1\,n}+\lambda_{3\,n}\cdots+\lambda_{n-1\,n}}x_2\mathrm{d}x_n+x_n\mathrm{d}x_2 \\ \vdots \\ \dfrac{\left(\lambda_{j\,j+1}+\cdots+\widehat{\lambda}_{ji}+\cdots+\lambda_{jn}\right)-\left(\lambda_{1j}+\cdots+\lambda_{j-1\,j}\right)}{\lambda_{j\,j+1}+\cdots+\lambda_{j\,n}-\left(\lambda_{1\,j}+\cdots+\widehat{\lambda}_{i\,j}+\cdots+\lambda_{j-1\,j}\right)}x_i\mathrm{d}x_j+x_j\mathrm{d}x_i \\ \vdots \\ \dfrac{\left(\lambda_{1\,i}+\cdots+\lambda_{i-1\,i}\right)-\lambda_{i\,i+1}+\cdots+\lambda_{i\,n-1}}{\lambda_{1\,n}+\cdots+\lambda_{i-1\,n}+\cdots+\lambda_{i+1\,n}+\lambda_{n-1\,n}}x_i\mathrm{d}x_n+x_n\mathrm{d}x_i \end{cases}$$

6.3. 2-th Twisted Poisson Homology

$$\begin{aligned}&\partial_2^\pi:\Omega_2(\Lambda)\to\Omega_1(\Lambda)\\ &\sum_{1\le j<k\le n}Cx_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\mathrm{d}x_j\wedge\mathrm{d}x_k\mapsto\left\{x_1^{i_1}\cdots x_n^{i_n},x_j\right\}_{\Lambda^D}\mathrm{d}x_k\\ &-\left\{x_1^{i_1}\cdots x_n^{i_n},x_k\right\}_{\Lambda^D}\mathrm{d}x_j-x_1^{i_1}\cdots x_n^{i_n}\mathrm{d}\left\{x_j,x_k\right\}\end{aligned}$$

That is

$$\begin{aligned}&\sum_{1\le j<k\le n}Cx_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\mathrm{d}x_j\wedge\mathrm{d}x_k\\ &=C_1x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n+C_2x_1^{i_1}x_2^{i_2}\cdots x_{n-3}^{i_{n-3}}x_{n-1}^{i_{n-1}}\mathrm{d}x_{n-2}\wedge\mathrm{d}x_n\\ &\quad+\cdots+C_{c_n^2}x_3^{i_3}x_4^{i_4}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n}\mathrm{d}x_1\wedge\mathrm{d}x_2\end{aligned}$$

we calculate the image in each item

The first term

$$\begin{aligned}&x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n\mapsto\left\{x_1^{i_1}\cdots x_{n-2}^{i_{n-2}},x_{n-1}\right\}_{\Lambda^D}\mathrm{d}x_n\\ &-\left\{x_1^{i_1}\cdots x_{n-2}^{i_{n-2}},x_n\right\}_{\Lambda^D}\mathrm{d}x_{n-1}-x_1^{i_1}\cdots x_{n-2}^{i_{n-2}}\mathrm{d}\left\{x_{n-1},x_n\right\}\\ &=x_2^{i_2}x_3^{i_3}\cdots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}}\left\{x_1^{i_1},x_{n-1}\right\}_{\Lambda^D}+x_1^{i_1}x_3^{i_3}\cdots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}}\left\{x_2^{i_2},x_{n-1}\right\}_{\Lambda^D}\\ &\quad+\cdots+x_1^{i_1}x_2^{i_2}\cdots x_{n-3}^{i_{n-3}}\left\{x_{n-2}^{i_{n-2}},x_{n-1}\right\}_{\Lambda^D}-x_2^{i_2}x_3^{i_3}\cdots x_{n-2}^{i_{n-2}}\left\{x_1^{i_1},x_n\right\}_{\Lambda^D}\\ &+x_1^{i_1}x_3^{i_3}\cdots x_{n-2}^{i_{n-2}}\left\{x_2^{i_2},x_n\right\}_{\Lambda^D}+\cdots+x_1^{i_1}x_2^{i_2}\cdots x_{n-3}^{i_{n-3}}x_{n-1}^{i_{n-1}}\left\{x_{n-2}^{i_{n-2}},x_n\right\}_{\Lambda^D}\\ &-x_1^{i_1}\cdots x_{n-2}^{i_{n-2}}\mathrm{d}\left\{x_{n-1},x_n\right\}\\ &=-\left(\lambda_{n\,n-1}+\lambda_{n\,n-2}+\lambda_{n\,n1}\right)\left(x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}}\mathrm{d}x_n+x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}x_n^{i_n}\mathrm{d}x_{n-1}\right)\end{aligned}$$

......

The C_n^2-th term

$$\begin{aligned}&x_3^{i_3}x_4^{i_4}\cdots x_n^{i_n}\mathrm{d}x_1\wedge\mathrm{d}x_2\mapsto\left\{x_3^{i_3}\cdots x_n^{i_n},x_1\right\}_{\Lambda^D}\mathrm{d}x_2\\ &-\left\{x_3^{i_3}\cdots x_n^{i_n},x_2\right\}_{\Lambda^D}\mathrm{d}x_1-x_3^{i_3}\cdots x_n^{i_n}\mathrm{d}\left\{x_1,x_2\right\}\\ &=x_4^{i_4}x_5^{i_5}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n}\left\{x_3^{i_3},x_1\right\}_{\Lambda^D}+x_3^{i_3}x_5^{i_5}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n}\left\{x_4^{i_4},x_1\right\}_{\Lambda^D}\\ &\quad+\cdots+x_3^{i_3}x_4^{i_4}\cdots x_{n-1}^{i_{n-1}}\left\{x_n^{i_n},x_1\right\}_{\Lambda^D}-x_4^{i_4}x_5^{i_5}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n}\left\{x_3^{i_3},x_2\right\}_{\Lambda^D}\end{aligned}$$

$$+x_3^{i_3}x_5^{i_5}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n}\left\{x_4^{i_4},x_2\right\}_{\Lambda^D}+\cdots+x_3^{i_3}x_4^{i_4}\cdots x_{n-1}^{i_{n-1}}\left\{x_n^{i_n},x_2\right\}_{\Lambda^D}$$
$$-x_3^{i_3}\cdots x_n^{i_n}\mathrm{d}\left\{x_1,x_2\right\}$$
$$=-\left(\lambda_{1n}+\lambda_{2n}+\lambda_{n-1\,n}\right)\left(x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}x_{n-1}^{i_{n-1}}\mathrm{d}x_n+x_1^{i_1}x_2^{i_2}\cdots x_{n-2}^{i_{n-2}}x_n^{i_n}\mathrm{d}x_{n-1}\right)$$

Next, For every element in $\Omega_2(\Lambda)$, we try to find the preimage in $\Omega_3(\Lambda)$.

$$\partial_3^\pi:\Omega_3(\Lambda)\to\Omega_2(\Lambda)$$
$$\sum_{1\le p<q<r\le n}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\mathrm{d}x_p\wedge\mathrm{d}x_q\wedge\mathrm{d}x_r$$
$$\mapsto\left\{x_1^{i_1}\cdots x_n^{i_n},x_p\right\}_{\Lambda^D}\mathrm{d}x_q\wedge\mathrm{d}x_r-\left\{x_1^{i_1}\cdots x_n^{i_n},x_q\right\}_{\Lambda^D}\mathrm{d}x_p\wedge\mathrm{d}x_r$$
$$+\left\{x_1^{i_1}\cdots x_n^{i_n},x_r\right\}_{\Lambda^D}\mathrm{d}x_p\wedge\mathrm{d}x_q-x_1^{i_1}\cdots x_n^{i_n}\mathrm{d}\left\{x_p,x_q\right\}\wedge\mathrm{d}x_r$$
$$+x_1^{i_1}\cdots x_n^{i_n}\mathrm{d}\left\{x_p,x_r\right\}\wedge\mathrm{d}x_q-x_1^{i_1}\cdots x_n^{i_n}\mathrm{d}\left\{x_q,x_r\right\}\wedge\mathrm{d}x_p$$

We just take one item for example

$$\sum_{1\le p<q<r\le n}x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\mathrm{d}x_p\wedge\mathrm{d}x_q\wedge\mathrm{d}x_r$$
$$\mapsto\left\{x_1^{i_1}\cdots x_{n-3}^{i_{n-3}},x_{n-2}\right\}_{\Lambda^D}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n-\left\{x_1^{i_1}\cdots x_{n-3}^{i_{n-3}},x_{n-1}\right\}_{\Lambda^D}\mathrm{d}x_{n-2}\wedge\mathrm{d}x_n$$
$$+\left\{x_1^{i_1}\cdots x_{n-3}^{i_{n-3}},x_n\right\}_{\Lambda^D}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_{n-2}-x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\mathrm{d}\left\{x_{n-2},x_{n-1}\right\}\wedge\mathrm{d}x_n$$
$$+x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\mathrm{d}\left\{x_{n-2},x_n\right\}\wedge\mathrm{d}x_{n-1}-x_1^{i_1}\cdots x_n^{i_n}\mathrm{d}\left\{x_{n-1},x_n\right\}\wedge\mathrm{d}x_{n-2}$$
$$=\left(i_1\lambda_{1\,n-2}+i_2\lambda_{2\,n-2}+\cdots+i_{n-3}\lambda_{n-3\,n-2}\right)x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n$$
$$-\left(i_1\lambda_{1\,n-1}+i_2\lambda_{2\,n-1}+\cdots+i_{n-3}\lambda_{n-3\,n-1}\right)x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\mathrm{d}x_{n-2}\wedge\mathrm{d}x_n$$
$$+\left(i_1\lambda_{1\,n}+\cdots+i_{n-3}\lambda_{n-3\,n}\right)x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_{n-2}$$
$$-\lambda_{n-2\,n-1}x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\left(x_{n-2}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n+x_n\mathrm{d}x_{n-2}\wedge\mathrm{d}x_{n-1}\right)$$
$$+\lambda_{n-2\,n}x_1^{i_1}\cdots x_{n-3}^{i_{n-3}}\left(x_{n-2}\mathrm{d}x_n\wedge\mathrm{d}x_{n-1}+x_n\mathrm{d}x_{n-2}\wedge\mathrm{d}x_{n-1}\right)$$
$$-\lambda_{n-1\,n}x_1^{i_1}\cdots x_n^{i_n}\left(x_{n-1}\mathrm{d}x_n\wedge\mathrm{d}x_{n-2}+x_n\mathrm{d}x_{n-1}\wedge\mathrm{d}x_{n-2}\right)$$

Similarly, because the Poisson structure of Λ is homogeneous. We can discuss the image of ∂_2^π by length.

1) The preimage of image with the length of 1

$$\sum C_{pq}x_i\mathrm{d}x_p\wedge\mathrm{d}x_q$$
$$\mapsto\sum C_{pq}\Big(\Big(2\left(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\right)-2\left(\lambda_{1p}+\cdots+\widehat{\lambda}_{ip}+\cdots+\lambda_{p-1p}\right)-\widehat{\lambda}_{ip}-\widehat{\lambda}_{pq}\Big)x_ix_p\mathrm{d}x_q$$
$$-\Big(2\left(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\right)-2\left(\lambda_{1q}+\cdots+\cdots+\lambda_{q-1q}\right)-\widehat{\lambda}_{iq}-\widehat{\lambda}_{pq}\Big)x_ix_q\mathrm{d}x_p\Big)$$

Let $\sum C_{pq}x_i\mathrm{d}x_p\wedge\mathrm{d}x_q\in\ker\partial_2^\pi$, then

$$\sum C_{pq}\Big(\Big(2\left(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\right)-2\left(\lambda_{1p}+\cdots+\widehat{\lambda}_{ip}+\cdots+\lambda_{p-1p}\right)-\widehat{\lambda}_{ip}-\widehat{\lambda}_{pq}\Big)x_ix_p\mathrm{d}x_q$$
$$-\Big(2\left(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\right)-2\left(\lambda_{1q}+\cdots+\cdots+\lambda_{q-1q}\right)-\widehat{\lambda}_{iq}-\widehat{\lambda}_{pq}\Big)x_ix_q\mathrm{d}x_p\Big)=0$$

Then for $\forall i<p<q$,

$$C_{pq}\Big(\Big(2\left(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\right)-2\left(\lambda_{1p}+\cdots+\widehat{\lambda}_{ip}+\cdots+\lambda_{p-1p}\right)-\widehat{\lambda}_{ip}-\widehat{\lambda}_{pq}\Big)x_ix_p\mathrm{d}x_q$$
$$-\Big(2\left(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\right)-2\left(\lambda_{1q}+\cdots+\cdots+\lambda_{q-1q}\right)-\widehat{\lambda}_{iq}-\widehat{\lambda}_{pq}\Big)x_ix_q\mathrm{d}x_p\Big)$$
$$C_{iq}\Big(\Big(2\left(\lambda_{i\,i+1}+\cdots+\lambda_{in}\right)-2\left(\lambda_{1i}+\cdots+\widehat{\lambda}_{pi}+\cdots+\lambda_{i-1i}\right)-\widehat{\lambda}_{pi}-\widehat{\lambda}_{iq}\Big)x_px_i\mathrm{d}x_q$$

$$-\left(2\left(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\right)-2\left(\lambda_{1q}+\cdots+\cdots+\lambda_{q-1q}\right)-\hat{\lambda}_{iq}-\hat{\lambda}_{iq}\right)x_p x_q \mathrm{d}x_i\right)$$

$$C_{ip}\left(\left(2\left(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\right)-2\left(\lambda_{1p}+\cdots+\hat{\lambda}_{ip}+\cdots+\lambda_{p-1p}\right)-\hat{\lambda}_{ip}-\hat{\lambda}_{pi}\right)x_q x_p \mathrm{d}x_i\right.$$

$$\left.-\left(2\left(\lambda_{i\,i+1}+\cdots+\lambda_{in}\right)-2\left(\lambda_{1i}+\cdots+\cdots+\lambda_{i-1i}\right)-\hat{\lambda}_{qi}-\hat{\lambda}_{pi}\right)x_q x_i \mathrm{d}x_p\right)=0$$

That is

$$C_{ij}\frac{\lambda_{j\,j+1}+\cdots+\lambda_{ji-1}+\cdots+\lambda_{ji+1}+\cdots+\lambda_{jn}-\left(\lambda_{1j}+\cdots+\lambda_{j-1\,j}\right)}{\lambda_{j\,j+1}+\cdots+\lambda_{j\,n}-\left(\lambda_{1\,j}+\cdots+\lambda_{i-1\,j}+\cdots+\lambda_{i+1\,j}+\cdots+\lambda_{j-1\,j}\right)}=-C_{ji}\,.$$

we have

$$\frac{\lambda_{1n}+\cdots+\lambda_{n-1n}}{\lambda_{n1}+\cdots+\lambda_{21}}x_1\mathrm{d}x_2\wedge\mathrm{d}x_n-x_n\mathrm{d}x_1\wedge\mathrm{d}x_2+x_2\mathrm{d}x_1\wedge\mathrm{d}x_n\in\ker\partial_2^{\pi}$$

$$\vdots$$

$$\frac{\lambda_{iq}+\cdots+\lambda_{pq}}{\lambda_{qi}+\cdots+\lambda_{pi}}x_i\mathrm{d}x_p\wedge\mathrm{d}x_q-x_q\mathrm{d}x_i\wedge\mathrm{d}x_p+x_p\mathrm{d}x_i\wedge\mathrm{d}x_q\in\ker\partial_2^{\pi}$$

$$\vdots$$

$$\frac{\lambda_{n-11}+\cdots+\lambda_{n-1n}}{\lambda_{n1}+\cdots+\lambda_{n-11}}x_1\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n-x_n\mathrm{d}x_1\wedge\mathrm{d}x_{n-1}+x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_n\in\ker\partial_2^{\pi}$$

2) The preimage of image with the length of 2

$$\sum C_{pq}x_i x_j\mathrm{d}x_p\wedge\mathrm{d}x_q$$

$$\mapsto\sum C_{pq}\left(\left(2\left(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\right)-2\left(\lambda_{1p}+\cdots+\lambda_{p-1p}\right)-\hat{\lambda}_{ip}-\hat{\lambda}_{jp}-\hat{\lambda}_{pq}\right)x_i x_j x_p\mathrm{d}x_q\right.$$

$$\left.-\left(2\left(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\right)-2\left(\lambda_{1q}+\cdots+\lambda_{q-1q}\right)-\hat{\lambda}_{iq}-\hat{\lambda}_{pq}-\hat{\lambda}_{jp}\right)x_i x_j x_q\mathrm{d}x_p\right)$$

Let $\sum C_{pq}x_i x_j\mathrm{d}x_p\wedge\mathrm{d}x_q\in\ker\partial_2^{\pi}$, then

$$\sum C_{pq}\left(\left(2\left(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\right)-2\left(\lambda_{1p}+\cdots+\lambda_{p-1p}\right)-\hat{\lambda}_{ip}-\hat{\lambda}_{jp}-\hat{\lambda}_{pq}\right)x_i x_j x_p\mathrm{d}x_q\right.$$

$$\left.-\left(2\left(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\right)-2\left(\lambda_{1q}+\cdots+\lambda_{q-1q}\right)-\hat{\lambda}_{iq}-\hat{\lambda}_{pq}-\hat{\lambda}_{jp}\right)x_i x_j x_q\mathrm{d}x_p\right)=0$$

for $\forall i<j<p<q$,

$$C_{ijk}\left(2\left(\lambda_{k\,k+1}+\cdots+\lambda_{k\,n}\right)-2\left(\lambda_{1\,k}+\cdots+\hat{\lambda}_{ik}+\hat{\lambda}_{jk}+\cdots+\lambda_{k-1\,k}\right)-\lambda_{ik}-\lambda_{jk}\right)$$

$$C_{jik}\left(2\left(\lambda_{k\,k+1}+\cdots+\lambda_{k\,n}\right)-2\left(\lambda_{1\,k}+\cdots+\hat{\lambda}_{jk}+\hat{\lambda}_{ik}+\cdots+\lambda_{k-1\,k}\right)-\lambda_{jk}-\lambda_{ik}\right)=0$$

that is

$$C_{jik}=-\frac{\left(2\left(\lambda_{k\,k+1}+\cdots+\lambda_{k\,n}\right)-2\left(\lambda_{1\,k}+\cdots+\hat{\lambda}_{ik}+\hat{\lambda}_{jk}+\cdots+\lambda_{k-1\,k}\right)-\lambda_{ik}-\lambda_{jk}\right)}{\left(2\left(\lambda_{k\,k+1}+\cdots+\lambda_{k\,n}\right)-2\left(\lambda_{1\,k}+\cdots+\hat{\lambda}_{jk}+\hat{\lambda}_{ik}+\cdots+\lambda_{k-1\,k}\right)-\lambda_{jk}-\lambda_{ik}\right)}C_{ijk}$$

We have

$$\frac{\lambda_{31}+\cdots+\lambda_{n1}-\lambda_{42}}{\lambda_{34}-\lambda_{41}+\cdots+\lambda_{4n}}x_3x_4\mathrm{d}x_1\wedge\mathrm{d}x_2-x_3x_1\mathrm{d}x_4\wedge\mathrm{d}x_2+x_3x_2\mathrm{d}x_4\wedge\mathrm{d}x_1\in\ker\partial_2^{\pi}$$

$$\vdots$$

$$\frac{\lambda_{iq}+\cdots+\lambda_{nq}-\lambda_{pr}}{\lambda_{ip}-\lambda_{pq}+\cdots+\lambda_{pn}}x_i x_p\mathrm{d}x_q\wedge\mathrm{d}x_r-x_i x_q\mathrm{d}x_p\wedge\mathrm{d}x_r+x_i x_r\mathrm{d}x_p\wedge\mathrm{d}x_q\in\ker\partial_2^{\pi}$$

We can see that the image $x_1^{i_1}x_2^{i_2}\cdots x_{n-3}^{i_{n-3}}x_{n-2}^{i_{n-2}}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n$ and must have the same coefficient, it is conflict with our results. So the image with length 1 and 2 in $\ker\partial_2^\pi$ can not find the preimage in $\Omega_3(\Lambda)$. In the same time, the constant term k (with the length of 0) also can not find the preimage. Now we discuss the elements in $\ker\partial_2^\pi$ when the length ≥ 3, we will prove that we can get the preimage in $\Omega_3(\Lambda)$

3) The preimage of image with the length of 3

$$\sum C_{pqr}x_ix_jx_r\mathrm{d}x_p\wedge\mathrm{d}x_q\mapsto\sum C_{pq}\Big(\Big(2\big(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\big)$$
$$-2\big(\lambda_{1p}+\cdots+\lambda_{p-1p}\big)-\hat{\lambda}_{ip}-\hat{\lambda}_{jp}-\hat{\lambda}_{pq}-\hat{\lambda}_{rq}\Big)x_ix_jx_p\mathrm{d}x_q$$
$$-\Big(2\big(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\big)-2\big(\lambda_{1q}+\cdots+\lambda_{q-1q}\big)-\hat{\lambda}_{iq}-\hat{\lambda}_{pq}-\hat{\lambda}_{jp}-\hat{\lambda}_{rp}\Big)x_ix_jx_q\mathrm{d}x_p\Big)$$

Let $\sum C_{pqr}x_ix_j\mathrm{d}x_p\wedge\mathrm{d}x_q\wedge\mathrm{d}x_r\in\ker\partial_2^\pi$, then

$$\sum C_{pq}\Big(\Big(2\big(\lambda_{p\,p+1}+\cdots+\lambda_{pn}\big)-2\big(\lambda_{1p}+\cdots+\lambda_{p-1p}\big)-\hat{\lambda}_{ip}-\hat{\lambda}_{jp}-\hat{\lambda}_{pq}-\hat{\lambda}_{rq}\Big)x_ix_jx_p\mathrm{d}x_q$$
$$-\Big(2\big(\lambda_{q\,q+1}+\cdots+\lambda_{qn}\big)-2\big(\lambda_{1q}+\cdots+\lambda_{q-1q}\big)-\hat{\lambda}_{iq}-\hat{\lambda}_{pq}-\hat{\lambda}_{jp}-\hat{\lambda}_{rp}\Big)x_ix_jx_q\mathrm{d}x_p\Big)=0$$

We have

$$\frac{\lambda_{31}+\cdots+\lambda_{n1}-\lambda_{42}}{\lambda_{34}-\lambda_{41}+\cdots+\lambda_{n2}}x_3x_4x_5\mathrm{d}x_1\wedge\mathrm{d}x_2-x_3x_5x_1\mathrm{d}x_4\wedge\mathrm{d}x_2+x_3x_5x_2\mathrm{d}x_4\wedge\mathrm{d}x_1\in\ker\partial_2^\pi$$
$$\vdots$$
$$\frac{\lambda_{iq}+\cdots+\lambda_{nq}-\lambda_{pr}}{\lambda_{ip}-\lambda_{pq}+\cdots+\lambda_{nr}}x_ix_px_j\mathrm{d}x_q\wedge\mathrm{d}x_r-x_ix_qx_j\mathrm{d}x_p\wedge\mathrm{d}x_r+x_ix_jx_r\mathrm{d}x_p\wedge\mathrm{d}x_q\in\ker\partial_2^\pi$$

Obviously, for $\forall i<j<k$, the preimage of this image is

$$-x_5\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_4$$
$$\vdots$$
$$-x_j\mathrm{d}x_q\wedge\mathrm{d}x_r\wedge\mathrm{d}x_p$$

under the ∂_2^π.

4) The preimage of image with the length of $n-2$

$$\sum C_{n-1\,n}x_1\cdots x_{n\text{-}2}\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n$$
$$\mapsto\sum C_1\Big(\Big(-(n-2)\big(\lambda_{1\,n}+\cdots+\lambda_{n-11}\big)-\lambda_{1n}-\cdots-\lambda_{n-1n}\Big)x_1\cdots x_{n-1}\mathrm{d}x_n$$
$$+\Big(-(n-2)\big(\lambda_{1\text{-}1n}+\cdots+\lambda_{11}\big)-\lambda_{1n\text{-}1}-\cdots-\lambda_{1n\text{-}1}\Big)x_1\cdots x_n\mathrm{d}x_{n-1}\Big)$$

Let $\sum C_{n-1\,n}x_1\cdots x_{n\text{-}2}\mathrm{d}x_{n\text{-}1}\wedge\mathrm{d}x_n\in\ker\partial_1^\pi$, then

$$\sum C_1\Big(\Big(-(n-2)\big(\lambda_{1\,n}+\cdots+\lambda_{n-11}\big)-\lambda_{1n}-\cdots-\lambda_{n-1n}\Big)x_1\cdots x_{n-1}\mathrm{d}x_n$$
$$+\Big(-(n-2)\big(\lambda_{1\text{-}1n}+\cdots+\lambda_{11}\big)-\lambda_{1n\text{-}1}-\cdots-\lambda_{1n\text{-}1}\Big)x_1\cdots x_n\mathrm{d}x_{n-1}\Big)=0$$

We have

$$\frac{\lambda_{12}+\lambda_{32}+\cdots+\lambda_{n2}}{\lambda_{21}+\lambda_{31}+\cdots+\lambda_{n1}} x_1\hat{x}_2 x_3\cdots x_{n-1}\mathrm{d}x_2\wedge \mathrm{d}x_n - x_2x_3\cdots x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_n + x_3x_4\cdots x_n\mathrm{d}x_1\wedge\mathrm{d}x_2$$

$$\frac{\lambda_{13}+\lambda_{23}+\cdots+\lambda_{n3}}{\lambda_{21}+\lambda_{31}+\cdots+\lambda_{n1}} x_1x_2\hat{x}_3\cdots x_{n-1}\mathrm{d}x_3\wedge \mathrm{d}x_n - x_2x_3\cdots x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_n + x_2x_4\cdots x_n\mathrm{d}x_1\wedge\mathrm{d}x_3$$

$$\vdots$$

$$\frac{\lambda_{1n-1}+\lambda_{2n-1}+\cdots+\lambda_{n\,n-1}}{\lambda_{21}+\lambda_{31}+\cdots+\lambda_{n1}} x_1x_2x_3\cdots x_{n-2}\mathrm{d}x_{n-1}\wedge \mathrm{d}x_n$$

$$- x_2x_3\cdots x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_n + x_2x_4\cdots x_n\mathrm{d}x_1\wedge\mathrm{d}x_{n-1}$$

Obviously, the preimage of this image is

$$\begin{gathered} x_3x_4\cdots x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_n \\ x_2x_4\cdots x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\mathrm{d}x_n \\ \vdots \\ x_2x_3\cdots x_{n-2}\mathrm{d}x_1\wedge\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n \end{gathered}$$

under the ∂_2^π.

In conclusion, ∂_2^π is not surjection. So we have the 2-th twisted homology group

$$HP_2\left(\Lambda,\Lambda^D\right)=\frac{\ker\partial_2^\pi}{\operatorname{Im}\partial_3^\pi}$$

$$=\begin{cases} k \\ \left.\begin{array}{l} \dfrac{\lambda_{1n}+\cdots+\lambda_{n-1n}}{\lambda_{n1}+\cdots+\lambda_{21}} x_1\mathrm{d}x_2\wedge\mathrm{d}x_n - x_n\mathrm{d}x_1\wedge\mathrm{d}x_2 + x_2\mathrm{d}x_1\wedge\mathrm{d}x_n \\ \vdots \\ \dfrac{\lambda_{iq}+\cdots+\lambda_{pq}}{\lambda_{qi}+\cdots+\lambda_{pi}} x_i\mathrm{d}x_p\wedge\mathrm{d}x_q - x_q\mathrm{d}x_i\wedge\mathrm{d}x_p + x_p\mathrm{d}x_i\wedge\mathrm{d}x_q \\ \vdots \\ \dfrac{\lambda_{n-11}+\cdots+\lambda_{n-1n}}{\lambda_{n1}+\cdots+\lambda_{n-11}} x_1\mathrm{d}x_{n-1}\wedge\mathrm{d}x_n - x_n\mathrm{d}x_1\wedge\mathrm{d}x_{n-1} + x_{n-1}\mathrm{d}x_1\wedge\mathrm{d}x_n \end{array}\right\}\text{length of 1} \\ \left.\begin{array}{l} \dfrac{\lambda_{31}+\cdots+\lambda_{n1}-\lambda_{42}}{\lambda_{34}-\lambda_{41}+\cdots+\lambda_{4n}} x_3x_4\mathrm{d}x_1\wedge\mathrm{d}x_2 - x_3x_1\mathrm{d}x_4\wedge\mathrm{d}x_2 + x_3x_2\mathrm{d}x_4\wedge\mathrm{d}x_1 \\ \vdots \\ \dfrac{\lambda_{iq}+\cdots+\lambda_{nq}-\lambda_{pr}}{\lambda_{ip}-\lambda_{pq}+\cdots+\lambda_{pn}} x_ix_p\mathrm{d}x_q\wedge\mathrm{d}x_r - x_ix_q\mathrm{d}x_p\wedge\mathrm{d}x_r + x_ix_r\mathrm{d}x_p\wedge\mathrm{d}x_q \end{array}\right\}\text{length of 2} \end{cases}$$

In general situation, for $1\le m\le n-1$, the m-th twisted homology $HP_m\left(\Lambda,\Lambda^D\right)$ has the elements in $\ker\partial_m^\pi$ with length of $0\sim m$.

6.4. *n*-th Twisted Poisson Homology

$$\Omega_{n+1}(\Lambda)\xrightarrow{\partial_{n+1}^\pi}\Omega_n(\Lambda)\xrightarrow{\partial_n^\pi}\Omega_{n-1}(\Lambda)$$

$$\partial_n^{\pi} : \Omega_n(\Lambda) \to \Omega_{n-1}(\Lambda)$$

$$C\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \mapsto \sum_{1 \le i < j < \cdots < k \le n} (-1)^{i+j} \mathrm{d}\{F_i, F_j\} \wedge \mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}\hat{F}_i \wedge \cdots \wedge \mathrm{d}\hat{F}_j$$

$$\wedge \cdots \wedge \mathrm{d}\hat{F}_k \wedge \cdots \wedge \mathrm{d}F_n = (-1)^{1+2} \mathrm{d}\{x_1, x_2\} \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n$$

$$+(-1)^{1+3} \mathrm{d}\{x_1, x_3\} \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n + \cdots$$

$$+(-1)^{(n-1)+n} \mathrm{d}\{x_{n-1}, x_n\} \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{n-2}$$

We calculate the Poisson bracket

$$(-1)^{1+2} \mathrm{d}\{x_1, x_2\} \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n$$

$$+(-1)^{1+3} \mathrm{d}\{x_1, x_3\} \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n + \cdots$$

$$+(-1)^{(n-1)+n} \mathrm{d}\{x_{n-1}, x_n\} \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{n-2}$$

$$= -\mathrm{d}\{x_1, x_2\} \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n + \mathrm{d}\{x_1, x_3\} \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n$$

$$+ \cdots - \mathrm{d}\{x_{n-1}, x_n\} \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{n-2}$$

$$= -\lambda_{12}(x_1 \mathrm{d}x_2 + x_2 \mathrm{d}x_1) \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n$$

$$+ \lambda_{13}(x_1 \mathrm{d}x_3 + x_3 \mathrm{d}x_1) \mathrm{d}x_2 \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n$$

$$+ \cdots - \lambda_{n-1\,n}(x_{n-1} \mathrm{d}x_n + x_n \mathrm{d}x_{n-1}) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{n-2}$$

We can easily see that this element can not be 0, that is $\ker \partial_n^{\pi} = 0$, so

$$HP_n(\Lambda, \Lambda^D) = \frac{\ker \partial_n^{\pi}}{\operatorname{Im} \partial_{n+1}^{\pi}} = 0$$

7. Twisted Poincaré Duality between Poisson Homology and Cohomology

In this part, we will check the Twisted Poincaré duality between Poisson homology and cohomology, that is for $i \in N$, we have

$$P.D.: HP^i(A, A) \cong (HP_i(A, A_D))^*$$

Next, we need to calculate the cohomology group.

obviously, for all i, j, the equation established

$$\begin{aligned} &\{x_1^{i_1} \cdots x_k^{i_k} \cdots x_n^{i_n}, x_k\} \\ &= (\lambda_{1k} i_1 + \cdots + \lambda_{k-1k} i_{k-1} + \lambda_{k+1k} i_{k+1} + \cdots + \lambda_{nk} i_n) x_1^{i_1} \cdots x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}. \end{aligned} \tag{12}$$

For every $p \in N$, p-fold polyderivations from Λ to Λ, denoted $\chi^p(\Lambda)$, $\chi^i(\Lambda) = Hom(\Omega^i, \Lambda)$. recall the canonical cochain complex

$$0 \longrightarrow \Lambda \xrightarrow{\delta^0} \chi^1(\Lambda) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{k-1}} \chi^k(\Lambda) \xrightarrow{\delta^k} \chi^{k+1}(\Lambda) \longrightarrow \cdots$$

For every $P \in \chi^k$ and $f_0, \cdots, f_k \in \Lambda$. $\delta_k : \chi^k \to \chi^{k+1}$ is defined by

$$\delta^p(P)(f_0, \cdots, f_k) = \sum_{i=0}^{k} (-1)^i \{f_i, P(f_0, \cdots, \hat{f}_i, \cdots, f_k)\}$$
$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} P(\{f_i, f_j\}, f_0, \cdots, \hat{f}_i, \cdots, \hat{f}_j \cdots, f_k)$$

obviously, $\delta_k(P) \in \chi^{k+1}$ and $\delta_{k+1} \circ \delta_k = 0$.

Let $\lambda=\sum C_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}\in\Lambda$, and assume it satisfies

$$\{\lambda,x_k\}=\left\{\sum C_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n},x_k\right\}=0\ \ (k=1,\cdots,n).$$

Then Equation (12) lead successively to

For $i_k\neq 1$ and $i_1=i_2=\cdots=i_{k-1}=i_{k+1}=\cdots=i_n\neq 0$, we have $C_{i_1\cdots i_n}=0$.

Hence, we can get the Proposition 7.1.

Proposition 7.1. $HP^0(\Lambda)=C$.

Proof: for every $d\in\mathcal{X}^1$, d is uniquely determined by the values of

$$d(x_1)=\sum C^1_{i_1\cdots i_n}x_1^{i_1}x_2^{i_2}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n},$$

$$d(x_2)=\sum C^2_{i_1\cdots i_n}x_1^{i_1}x_2^{i_2}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n},$$

$$\cdots$$

$$d(x_{n-1})=\sum C^{n-1}_{i_1\cdots i_n}x_1^{i_1}x_2^{i_2}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n},$$

$$d(x_n)=\sum C^n_{i_1\cdots i_n}x_1^{i_1}x_2^{i_2}\cdots x_{n-1}^{i_{n-1}}x_n^{i_n}.$$

Moreover, d must satisfy the relations $d(x_1^2)=d(x_2^2)=\cdots=d(x_n^2)=0$, since d is a derivation, so we have $x_1d(x_1)=x_2d(x_2)=\cdots=x_nd(x_n)=0$.

So we have the that for $\forall i_1,\cdots,i_j$, $C^1_{0\,i_2\cdots i_n}=C^2_{i_1\,0\cdots i_n}=\cdots=C^n_{i_1\cdots i_{n-1}\,0}=0$. Hence the space $\mathcal{X}^1=Der(\Lambda,\Lambda)$ has basis $\{d^1_{i_1\cdots i_n}\}\cup\{d^2_{i_1\cdots i_n}\}\cup\cdots\cup\{d^n_{i_1\cdots i_n}\}$, where

1) for $1\le i_1<2$, $0\le i_2,i_3,\cdots,i_n<2$, the derivation $d^1_{i_1\cdots i_n}$ is defined by

$$d^{(1)}_{i_1\cdots i_n}(x_1)=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\text{ and }d^{(1)}_{i_1\cdots i_n}(x_k)=0\,(\text{when }k\neq 1);$$

2) for $1\le i_2<2$, $0\le i_1,i_3,\cdots,i_n<2$, the derivation $d^2_{i_1\cdots i_n}$ is defined by

$$d^{(2)}_{i_1\cdots i_n}(x_2)=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\text{ and }d^{(2)}_{i_1\cdots i_n}(x_k)=0\,(\text{when }k\neq 2);$$

......

(n) for $1\le i_n<2$, $0\le i_1,i_2,\cdots,i_{n-1}<2$, the derivation $d^n_{i_1\cdots i_n}$ is defined by

$$d^{(n)}_{i_1\cdots i_n}(x_n)=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\text{ and }d^{(n)}_{i_1\cdots i_n}(x_k)=0\,(\text{when }k\neq n);$$

In particular, $\dim(\mathcal{X}^1)=n\cdot 2^n$. Let

$$d=\sum_{i_1\neq 0}\alpha^{(1)}_{i_1\cdots i_n}d^{(1)}_{i_1\cdots i_n}+\sum_{i_2\neq 0}\alpha^{(2)}_{i_1\cdots i_n}d^{(2)}_{i_1\cdots i_n}+\cdots+\sum_{i_n\neq 0}\alpha^{(n)}_{i_1\cdots i_n}d^{(n)}_{i_1\cdots i_n}\in\mathcal{X}^1$$

$d\in\ker\delta_1$ if and only if d satisfied:

$$d(\{x_k,x_1\cdots\hat{x}_k\cdots x_n\})=\{d(x_k),x_1\cdots\hat{x}_k\cdots x_n\}+\{x_k,d(x_1\cdots\hat{x}_k\cdots x_n)\}.\qquad(13)$$

In Equation (13), we have:

$$LHS=d(\{x_k,x_1\cdots\hat{x}_k\cdots x_n\})$$
$$=(n-1)\left(\alpha^{(1)}_{i_1\cdots i_n}x_1^{i_1}x_2^{i_2+1}\cdots x_n^{i_n+1}+\alpha^{(2)}_{i_1\cdots i_n}x_1^{i_1+1}x_2^{i_2}\cdots x_n^{i_n+1}+\cdots+\alpha^{(n)}_{i_1\cdots i_n}x_1^{i_1+1}x_2^{i_2+1}\cdots x_n^{i_n}\right)$$

$$\begin{aligned}RHS&=\{d(x_k),x_1\cdots\hat{x}_k\cdots x_n\}+\{x_k,d(x_1\cdots\hat{x}_k\cdots x_n)\}\\&=\{x_1^{i_1}\cdots x_n^{i_n},x_1\cdots\hat{x}_k\cdots x_n\}+\{x_k,d(x_1\cdots\hat{x}_k\cdots x_n)\}\\&=(\lambda_{12}+\cdots+\lambda_{1n}+\lambda_{23}+\cdots+\lambda_{2n}+\cdots+\lambda_{n-1\,n})x_1^{i_1+1}\cdots x_k^{i_k}\cdots x_n^{i_n+1}\\&\quad+\{x_k,x_2\cdots x_nd(x_1)+\cdots+x_1\cdots x_{n-1}d(x_n)\}\end{aligned}$$

$$=\left(\lambda_{12}+\cdots+\lambda_{1n}+\lambda_{23}+\cdots+\lambda_{2n}+\cdots+\lambda_{n-1\,n}\right)x_1^{i_1+1}\cdots x_k^{i_k}\cdots x_n^{i_n+1}$$
$$+\left\{x_k,\alpha^{(1)}_{i_1\cdots i_n}x_1^{i_1}x_2^{i_2+1}\cdots x_n^{i_n+1}+\cdots+\alpha^{(n)}_{i_1\cdots i_n}x_1^{i_1+1}x_2^{i_2+1}\cdots x_n^{i_n}\right\}$$

From the coefficient of $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ we can get for $\forall 0\le i_1,\cdots,i_n\le 1$, we have

$$d\in\ker\delta_1$$
$$\Leftrightarrow\left(n-2-\left(\lambda_{12}+\lambda_{13}+\cdots+\lambda_{1\,n}\right)\right)\alpha^{(1)}_{i_1\cdots i_n}$$
$$+\left(n-2-\left(\lambda_{21}+\lambda_{23}+\cdots+\lambda_{2\,n}\right)\right)\alpha^{(2)}_{i_1\cdots i_n}$$
$$+\cdots+\left(n-2-\left(\lambda_{n-11}+\lambda_{n-12}+\cdots+\lambda_{n-1\,n}\right)\right)\alpha^{(n-1)}_{i_1\cdots i_n}$$
$$+\left(n-2-\left(\lambda_{n1}+\lambda_{n2}+\cdots+\lambda_{n\,n-1}\right)\right)\alpha^{(n)}_{i_1\cdots i_n}=0$$

The proof is completed.

Proposition 7.2. $HP^1(\Lambda)=C\oplus Cd^{(1)}_{1,1,\cdots,0}\oplus Cd^{(2)}_{0,1,1,\cdots,0}\oplus\cdots\oplus Cd^{(n)}_{0,0,\cdots,1,1}$.

Proof: let

$d\left(\left\{x_k,x_1\cdots\hat{x}_k\cdots x_n\right\}\right)=\left\{d\left(x_k\right),x_1\cdots\hat{x}_k\cdots x_n\right\}+\left\{x_k,d\left(x_1\cdots\hat{x}_k\cdots x_n\right)\right\}\in\ker\delta_1$.

Set $\lambda:=\sum_{i_2,i_3,\cdots,i_n\neq 0}\alpha^{(1)}_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}\in\Lambda$, from Equation (12) we can get $d_1=d+\{\lambda,-\}$ is a Poisson derivation and satisfied $d_1\left(x_1\right)=\sum_{i_1\ge 0}\alpha^{(1)}_{i_1,0\cdots 0}x_1^{i_1}$. From Equation (13) we can get that for all $i_1\neq 0$, we have $\alpha^{(1)}_{i_1,0\cdots 0}$. That is $d_1\left(x_1\right)=\alpha^{(1)}_{i_1,0\cdots 0}x_1$. Similarly,

Let $d_2\left(x_2\right)=\sum_{i_1=0}^{1}\alpha^{(1)}_{i_1,0\cdots 0}x_1^{i_1}+\cdots+\sum_{i_n=0}^{1}\alpha^{(n)}_{i_1,0\cdots 0}x_n^{i_n}$.

Let a new set

$$u:=\sum_{i_1=0}{}^{1}\alpha^{(1)}_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}+\sum_{i_2=0}{}^{1}\alpha^{(1)}_{i_1\cdots i_n}x_1^{i_1}\cdots x_n^{i_n}+\cdots\in\Lambda.$$

$d_2\left(x_2\right)=d_1\left(x_1\right)+\cdots+d_n\left(x_n\right)-\{u,-\}$, so we have

$$d_2\left(x_2\right)=\alpha^{(1)}_{i_1\cdots i_n}d_{1,0,\cdots,0}+\cdots+\alpha^{(2)}_{i_1\cdots i_n}d_{0,0,\cdots,1}.$$

That is the image of $Cd^{(1)}_{1,1,\cdots,0}\oplus Cd^{(2)}_{0,1,1,\cdots,0}\oplus\cdots\oplus Cd^{(n)}_{0,0,\cdots,1,1}$ in $HP^1(\Lambda)$.

The proof is completed.

Similarly, we can calculate the m-th cohomology $HP^m(\Lambda)$.

From the propostion 7.1 and propostion 7.2 we can easily get that the demension of m-th cohomology group $HP^m(\Lambda)$ equal to the demension of m-th Twisted Poisson homology, $HP_m\left(\Lambda,\Lambda^D\right)$, we check the twisted Poincaréduality between them.

8. Conclusion

In this paper, we successfully calculate the i-th homology group of the algebra $\Lambda=k\left\langle x_1,x_2,\cdots,x_n\right\rangle/\left\langle x_ix_j-x_jx_i,x_i^2\right\rangle$ in part 4. In part 5, after getting Frobenius pairing, we calculate the modular derivation and then have the twisted Poisson module structure. Furthermore, we calculate the twisted Poisson homology group in part 6. Lastly, we verified the twisted Poincaré duality between Poisson homology and Poisson Cohomology through the cohomology group of Λ in part

7. In future studies, we will discuss whether all these conclusions hold up for general algebra.

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